Exercises for the lecture
"Commutative Algebra and Algebraic Geometry"
SS 2019 Sheet 5,
Submission Date: 04.06.2019

Exercise 1.

1. Show that 2 and 3 are invertible in $\mathbb{Z}_6 = S^{-1}\mathbb{Z}$, where $S = \{1, 6, 6^2, \dots\}$. Conclude that $\mathbb{Z}_6 = T^{-1}\mathbb{Z}$, where $T = \{2^n 3^m : n, m \geq 0\}$. (See Exercise 5 for a general statement.)

2. Let $S = \mathbb{Z} \setminus \{0\}$. Then show that S is a multiplicatively closed subset and $S^{-1}\mathbb{Z} \cong \mathbb{Q}$.

3. Let k be a field and let $S = k[x] \setminus \{0\}$. Then show that S is a multiplicatively closed subset and $S^{-1}k[x] \cong k(x)$. (See Exercise 6 for a general statement.)

(6 Points)

Exercise 2.

1. Let S be a multiplicatively closed subset of a ring A and let M be a finitely generated A-module. Show that $S^{-1}M = 0$ if and only if there exists $s \in S$ such that $sM = 0$.

2. Let M be an A-module and let I be an ideal of A. If $M_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m} \supset I$, then show that $M = IM$.

3. Consider $M = \mathbb{Z}/m\mathbb{Z}$ as an $\mathbb{Z}$ module and let $S_p = \{1, p, p^2, \dots\}$ for a prime number p. Determine all $m \in \mathbb{N}$ such that $(\mathbb{Z}/m\mathbb{Z})_p = S_p^{-1}M = 0$.

(6 Points)

Exercise 3.

1. Let A be a ring. Suppose that for each prime ideal $\mathfrak{p} \subset A$, the local ring $A_{\mathfrak{p}}$ has no non-zero nilpotent elements. Then show that A has no non-zero nilpotent element.
2. Give an example of a ring A such that A is not an integral domain but $A_{\mathfrak{p}}$ is integral domain for all prime ideals $\mathfrak{p} \subset A$.

(4 Points)

Exercise 4.

1. Show that $\mathbb{Z}_2$ is not finitely generated as an $\mathbb{Z}$-module but it is finitely generated as an $\mathbb{Z}$-algebra.
2. Let B be an A-algebra and let $f \in B$. Then show that there exists an isomorphism $B_f \xrightarrow{\cong} \frac{B[t]}{\langle ft-1 \rangle}$ (which sends $1/f \mapsto t$). In particular, conclude that if B is finitely generated as an A-algebra, then so is B_f.
3. Let k be a field. Then show that $k(t)$ is not finitely generated as a k-algebra. In particular, localisation of a finitely generated A-algebra at a general multiplicatively closed subset is not always finitely generated as an A-algebra.

(6 Points)

Class Exercises(no points)

Remark 1 *These exercises are for fun and to learn the subject without worrying about the marks. You should not submit the solutions of these exercises but should discuss the same in the exercise classes.* □

Exercise 5.

A multiplicative subset $S \subset A$ is set to be saturated if $xy \in S \iff x \in S$ and $y \in S$.

1. Show that S is saturated iff $A \setminus S$ is a union of prime ideals of A.
2. Given a multiplicatively closed subset S of A, prove that there exists a unique smallest multiplicatively closed subset $\overline{S}$ containing S.
3. Prove that there exists a natural isomorphism $A_S \xrightarrow{\cong} A_{\overline{S}}$.
4. If S and T are as in Exercise 1, then $T = \overline{S}$.

Exercise 6.

Let A be an integral domain and let $S = A \setminus \{0\}$. Then show that S is a saturated multiplicatively closed set and $S^{-1}(A)$ is the smallest field containing A, i.e., $S^{-1}(A)$ is a field and if L is a field such that $A \subset L$, then $S^{-1}(A) \subset L$.

Exercise 7.

Compute $(\mathbb{Z}/m\mathbb{Z})_n$ for general m and n (in terms of the gcd of m and n).