Exercises for the lecture
"Commutative Algebra and Algebraic Geometry"
SS 2019 Sheet 6,
Submission Date: 11.06.2019

Exercise 1.

Let $f : A \to B$ be a ring homomorphism. Let I and J be the ideals of A and B respectively. Let $I^e = f(I)B$, i.e., I^e is the ideal of B generated by the subset $f(I)$. In other word, I^e is the set of all finite sums $\sum y_i f(x_i)$, where $y_i \in B$ and $x_i \in A$. Moreover, let $J^c = f^{-1}(J)$.

1. Consider the natural inclusion $f : \mathbb{Z} \to \mathbb{Z}[i]$, where $i = \sqrt{-1}$ and let $\mathfrak{p} = \langle 2 \rangle \subset \mathbb{Z}$. Then show that the image $f(\mathfrak{p})$ of $\mathfrak{p}$ is not an ideal of $\mathbb{Z}[i]$ and $\mathfrak{p}^e = \langle (1+i)^2 \rangle$.

2. Show that in general, $I \subset I^{ec}$ and $J \supset J^{ce}$.

3. Show that $I^e = I^{ece}$ and $J^c = J^{cec}$.

4. Conclude that there is a bijection between the sets $\{I^e : I \text{ is an ideal of } A\}$ and $\{J^c : J \text{ is an ideal of } B\}$.

(See Exercise 6 for more properties of these operations)

(8 Points)

Exercise 2.

1. Let I be an ideal of A. Show that there is a bijection between the ideals J of A which contain I and the ideals $\overline{J}$ of A/I given by $J = \phi^{-1}(\overline{J})$, where $\phi : A \to A/I$ is the quotient map.

2. Show that in above exercise, J is a prime ideal of A containing I if and only if $\overline{J}$ is a prime ideal of A/I.

3. Determine the prime ideals of $\mathbb{C}[T]/\langle T^2 - z \rangle$ for $z \in \mathbb{C}$.

(6 Points)

Exercise 3.
Let S be a multiplicatively closed subset of A.

1. Show that all ideals of $A_S = S^{-1}A$ are of the form IA_S, where I is an ideal of A.

2. Show that $\mathfrak{p} \mapsto \mathfrak{p}A_S$ defines a bijection between the prime ideals of A which are disjoint from S with the prime ideals of A_S. Conclude that for a prime ideal $\mathfrak{q}$, the prime ideals of $A_{\mathfrak{q}}$ correspond bijectively with the prime ideals of A contained in $\mathfrak{q}$.

3. Let $S = \{1, 3, 3^2, \dots\}$. Recall that $\mathbb{Z}_3 = S^{-1}\mathbb{Z}$. Give an example of two ideals I and J of $\mathbb{Z}$ such that they are disjoint from S and $I\mathbb{Z}_3 = J\mathbb{Z}_3$. Also determine all the prime ideals of the ring $\mathbb{Z}_3$.

(6 Points)

Exercise 4.
Let S be a multiplicatively closed subset of A and let I be an ideal of A. Let $\overline{S}$ denote the image of S under the quotient map $A \to A/I$. Then show that $A_S/IA_S \cong (A/I)_{\overline{S}}$.

(4 Points)

Class Exercises(no points)

Remark 1 *These exercises are for fun and to learn the subject without worrying about the marks. You should not submit the solutions of these exercises but should discuss the same in the exercise classes.* □

Exercise 5.
1. Let $f : A \to B$ be a ring homomorphism and let $\mathfrak{p} \subset A$ be a prime ideal of A. Then show that $S = f(A \smallsetminus \mathfrak{p})$ is a multiplicatively closed subset of B. Let $B_{\mathfrak{p}} = S^{-1}B$. Then show that the prime ideals of $B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}$ are in bijection with the prime ideals Q of B such that $Q^c = \mathfrak{p}$.

2. Determine all the prime ideals of $B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}$, where $A = \mathbb{C}[X]$, $B = \mathbb{C}[X,Y]/\langle Y^2 - X^3 \rangle$, $f : A \to B$ is the natural ring homomorphism and $\mathfrak{p} = \langle T - z \rangle$, where $z \in \mathbb{C}$. Discuss a geometric picture of this problem.

Exercise 6.
With notations as in Exercise 1, show that

1. $(I_1 + I_2)^e = I_1^e + I_2^e$ and $(J_1 + J_2)^c \supset J_1^c + J_2^c$.

2. $(I_1 \cap I_2)^e \subset I_1^e \cap I_2^e$ and $(J_1 \cap J_2)^c = J_1^c \cap J_2^c$.

3. $(I_1 I_2)^e = I_1^e I_2^e$ and $(J_1 J_2)^c \supset J_1^c J_2^c$.

4. $r(I)^e \subset r(I^e)$ and $r(J)^c = r(J^c)$, where $r(I) = \{a \in A : a^n \in I \text{ for some } n \geq 0\}$ is the radical of I and $r(J) \subset B$ is the radical of J.

5. For a prime ideal Q of B, show that Q^c is a prime ideal of A. Conclude from Exercise 1 that for a prime ideal $\mathfrak{p}$ of A, the ideal $\mathfrak{p}^e$ may not be a prime ideal of B.