Exercises for the lecture
“Commutative Algebra and Algebraic Geometry”
SS 2019 Sheet 7,
Submission Date: 25.06.2019

In this Exercise sheet we construct a topological space from a ring and study basic properties of the topological space.

Exercise 1.

Let A be a ring and let X denote the set of all prime ideals of A. For an ideal $I \subset A$, let $V(I) \subset X$ denote the set of all prime ideals of A containing I. Then show that

1. $V(I) = V(\mathrm{rad}(I))$, where $\mathrm{rad}(I) = \{a \in A : a^n \in I \text{ for some } n \geq 0\}$ is the radical ideal of I.
2. $V(\langle 0 \rangle) = X$ and $V(\langle 1 \rangle) = \phi$.
3. $\cap_{i \in \mathcal{I}} V(I_i) = V(\sum_{i \in \mathcal{I}} I_i)$, where $\mathcal{I}$ is an indexing set which might be infinite.
4. $V(I_1) \cup V(I_2) = V(I_1 \cap I_2) = V(I_1 I_2)$.

In particular, the sets $V(I)$ satisfy the axioms of closed sets of a topological space. The induced topology is called Zariski topology and the topological space X is said to be prime spectrum of A which we denote by $\mathrm{Spec}\,(A)$.

(8 Points)

Exercise 2.

For each $f \in A$, let X_f denote the complement of $V(\langle f \rangle)$. By definition, X_f is an open subset of X.

1. Show that X_f forms a basis of open sets of the Zariski topology (introduced in Exercise 1).
2. For $f, g \in A$, show that $X_f \cap X_g = X_{fg}$.
3. $X_f = \varnothing$ if and only if f is a nilpotent element.
4. $X_f = X$ if and only if f is a unit.

The sets X_f are called basic open sets.

(8 Points)

Exercise 3.
Let $\mathfrak{p}$ and $\mathfrak{q}$ denote prime ideals of A. Then show that

1. $\{\mathfrak{p}\} \subset \operatorname{Spec}(A)$ is closed if and only if $\mathfrak{p}$ is a maximal ideal of A.

2. $\overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$.

3. $\mathfrak{q} \in \overline{\{\mathfrak{p}\}}$ if and only if $\mathfrak{p} \subset \mathfrak{q}$.

4. If A is an integral domain, then $\langle 0 \rangle \in \operatorname{Spec}(A)$ and $\overline{\{\langle 0 \rangle\}} = \operatorname{Spec}(A)$.

(8 Points)

Exercise 4.
Let $\phi : A \to B$ be a ring homomorphism. Let $X = \operatorname{Spec}(A)$ and $Y = \operatorname{Spec}(B)$. Recall from [Assignment 6, Exercise 6.5] that if $\mathfrak{q}$ is a prime ideal of B then $q^c = \phi^{-1}(\mathfrak{q})$ is a prime ideal of A. Let $\phi^* : Y \to X$ be the map such that $\phi^*(\mathfrak{q}) = \mathfrak{q}^c$. Then show that

1. For $f \in A$, $(\phi^*)^{-1}(X_f) = Y_{\phi(f)}$. Conclude that ϕ^* is continuous.

2. If I is an ideal of A then $(\phi^*)^{-1}V(I) = V(I^e)$, where recall from [Assignment 6, Exercise 6.5] that I^e is the ideal of B generated by the set $\phi(I)$.

3. If J is an ideal of B then $\overline{\phi^*(V(J))} = V(J^c)$.

(6 Points)

Class Exercises(no points)

Remark 1 *These exercises are for fun and to learn the subject without worrying about the marks. You should not submit the solutions of these exercises but should discuss the same in the exercise classes.* □

Exercise 5.
More topological properties of $\operatorname{Spec}(A)$.

1. Show that $X = \operatorname{Spec}(A)$ is a quasi-compact topological space (i.e., every open covering of X has a finite subcovering).

2. Show that $X = \operatorname{Spec}(A)$ is a T_0-space (i.e., given two distinct points $x, y \in X$, either there exists an open subset containing x but not y or there exists an open subset containing y but not x.

3. Give an example of a ring A such that $\operatorname{Spec}(A)$ is not a Hausdorff-space.
4. Show that the induced topology on the set of closed points of $\operatorname{Spec}(\mathbb{C}[t])$ is the cofinite topology (i.e., a proper subset is closed if and only if it is a finite set).

Exercise 6.
Draw following topological spaces $\operatorname{Spec}(\mathbb{Z}), \operatorname{Spec}(\mathbb{R}), \operatorname{Spec}(\mathbb{Z}[t])$ and $\operatorname{Spec}(\mathbb{R}[t])$.

Exercise 7.
Show that as topological spaces there is a homeomorphism $X_f \xrightarrow{\cong} \operatorname{Spec}(A_f)$.

Remark 2 *A basic idea of the construction of* $\operatorname{Spec}(A)$ *is as follows. We want to study the set of solutions of a number of polynomials (with say* $\mathbb{Q}$*-coefficients in* n*-variables). The point is that rational solutions of these polynomials may not exists. So we would like to study solutions in various fields (for example, number fields,* $\mathbb{R}$*,* $\mathbb{C}$*,* $\mathbb{Q}(t)$*) and more generally in different rings. For many purposes, it suffices to study the set of closed points of* $\operatorname{Spec}(A)$*. This set is denoted by* $\operatorname{Specm}(A)$*. But note that for a ring homomorphism* $A \to B$ *and for a maximal ideal* $\mathfrak{m}$ *of* B*, the ideal* $\mathfrak{m}^c \subset A$ *is not always a maximal ideal (but it is a prime ideal), we therefore study* $\operatorname{Spec}(A)$*.*

$\square$