Exercises for the lecture
"Commutative Algebra and Algebraic Geometry"
SS 2019 Sheet 8,
Submission Date: 02.07.2019

Exercise 1.

1. Give a detailed proof that a unique factorization domain is integrally closed in its field of fractions. In particular, conclude that $\mathbb{Z}$ is integrally closed in $\mathbb{Q}$ and $k[t]$ is integrally closed in $k(t)$, where k is a field.

2. Find the integral closure of $\mathbb{Z}$ in $\mathbb{Q}[\sqrt{-1}]$.

3. Let A be a ring and let $f(t) \in A[t]$ be a monic polynomial (i.e, the leading coefficient of f is a unit in A). Then show that $A[t]/\langle f(t) \rangle$ is integral over A.

(4 + 2 + 2 Points)

Exercise 2.
Let $A \hookrightarrow B$ be an injective homomorphism of rings and let B be integral over A. Then show that

1. If $a \in A$ is a unit in B, then it is a unit in A.

2. The Jacobson radical ideal of A is the contraction of the Jacobson radical of B.

3. Show that the induced map $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is surjective.

(6 Points)

Exercise 3.
Let A be an integrally closed integral domain. Let $b, c \in A$ such that $b^2 = c^3$. Then show that there exists a unique element $a \in A$ such that $b = a^3$ and $c = a^2$.

(3 Points)

Exercise 4.

Let A be a ring and let G a finite group of automorphisms of A. Let $A^G = \{a \in A : \sigma(a) = a \text{ for all } \sigma \in G\}$. Show that A^G is subring of A. Moreover, show that for $a \in A$, the coefficients of the polynomial $\prod_{\sigma \in G}(t - \sigma(a))$ lie in A^G. Conclude that A integral over A^G.

(3 Points)

Class Exercises(no points)

Remark 1 *These exercises are for fun and to learn the subject without worrying about the marks. You should not submit the solutions of these exercises but should discuss the same in the exercise classes.* □

Exercise 5.

[A generalisation of Exercise 1]

1. Let A be a subring of B such that $B \setminus A$ is multiplicatively closed. Then show that B is integral over A.
2. Find the integral closure of $\mathbb{Z}$ in $\mathbb{Q}[\sqrt{-5}]$.

Exercise 6.

Let G a finite group of automorphisms of a ring A. We have seen in Exercise 4 that A is integral over A^G. Let $\mathfrak{p}$ be a prime ideal of A^G and let P be a set of prime ideals of A whose contraction is $\mathfrak{p}$. Show that G acts transitively on P. In particular, P is finite. (Compare this with results in algebraic number theory.)