Exercises for the lecture
“Commutative Algebra and Algebraic Geometry”
SS 2019 Sheet 9,
Submission Date: 09.07.2019

Remark 1 *Let A be a subring of B such that B is integral over A. Recall that B is a field if and only if A is a field.* □

Exercise 1.
Let A be a subring of B. Let K be an algebraically closed field and $\phi : A \to K$ be a ring homomorphism.

1. If B is finitely generated as an A-module, then (by induction on number of generators) show that ϕ extends to a homomorphism $\phi : B \to K$.

2. Using Zorn's Lemma to show that if B is integral over A, then ϕ extends to a homomorphism $\phi : B \to K$.

(6 Points)

Exercise 2.

1. Let $f : B \to B'$ be a homomorphism of A-algebras and let C be an A-algebra. Show that if f is integral, then so is the induced homomorphism $f \otimes \mathrm{id} : B \otimes_A C \to B' \otimes_A C$.

2. Let $B_1, \dots, B_n$ be integral A-algebras. Then show that $\prod_i B_i$ is also integral over A.

(4 Points)

Exercise 3.
[Weak Nullstellensatz] Let k be a field and $r \geq 1$.

1. For $(a_1, \dots, a_n) \in k^n$, show that the ideal $\langle t_1 - a_1, \dots, t_n - a_n \rangle$ is a maximal ideal of $k[t_1, \dots, t_n]$.

2. For a maximal ideal $\mathfrak{m} \subset k[t_1, \ldots, t_n]$, let $A = k[t_1, \ldots, t_n]/\mathfrak{m}$. Using Noether's Normalization and Remark 1 show that A is integral over k (i.e, show that r in the statement of Noether Normalization is 0 in this case).

3. Let k be an algebraically closed field. Show that every maximal ideal of $k[t_1, \ldots, t_n]$ is of the type $\langle t_1 - a_1, \ldots, t_n - a_n \rangle$ for some $(a_1, \ldots, a_n) \in k^n$.

4. Give an example of a maximal ideal of $\mathbb{R}[t]$ which is not of the form $\langle t - a \rangle$ for any $a \in \mathbb{R}$.

(8 Points)

Exercise 4.
Using [Assignment-7, Exercise 3.1] and Exercise 3, draw the set of closed points of the following topological spaces.

1. $\mathrm{Spec}\,(\mathbb{C})$.
2. $\mathrm{Spec}\,(\mathbb{C}[t])$ and $\mathrm{Spec}\,(\mathbb{C}[t_1, t_2])$.
3. $\mathrm{Spec}\,(\mathbb{C}[t_1, t_2]/\langle t_1 - t_2 \rangle)$.
4. $\mathrm{Spec}\,(\mathbb{C}[t_1, t_2]/\langle t_1^2 + t_2^2 - 1 \rangle)$.

(8 Points)

Class Exercises(no points)

Remark 2 *These exercises are for fun and to learn the subject without worrying about the marks. You should not submit the solutions of these exercises but should discuss the same in the exercise classes.* □

Exercise 5.
[Hilbert Nullstellensatz] Let k be a field and let B be a finitely generated k-algebra. Prove the following steps to show that if B is a field then B is finitely generated field extension of k (i.e., B is finitely generated as k-module).

1. Let $x_1, \ldots, x_n$ generate B as a k-algebra. Prove the result for $n = 1$.

2. Now assume that $n > 1$ and that B is a field over k. Let $A = k[x_1] \subset B$ and $K = k(x_1) \subset B$. By induction hypothesis, conclude that B is finite algebraic extension of K.

3. Show that there exists $f \in A$ such that B is integral over A_f.

4. Since $K \subset B$, it follows that K is integral over A_f.

5. If x_1 is integral over k, then we are done. Let, if possible, x_1 is transcendental over k. Then show that $A_f = K$ and obtain a contradiction.

Exercise 6.
Deduce the result of Exercise 3 from Exercise 5.