# §2 Wirtinger Calculus on forms

## §2.1 Classical Wirtinger Calculus

Recall from complex analysis:

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i \cdot \frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i \cdot \frac{\partial}{\partial y}\right)$$

They satisfy

$$\frac{\partial}{\partial z} z = 1 \qquad \frac{\partial}{\partial z} \bar z = 0$$

$$\frac{\partial}{\partial \bar z} z = 0 \qquad \frac{\partial}{\partial \bar z} \bar z = 1$$

~~pss~~ together w/ usual properties of derivations.

We can recover the "usual" partial derivatives from them and since every smooth map in x and y can be expressed by z and $\bar z$ we do not loose any information.

No information loss using $\partial_z$ and $\partial_{\bar z}$.

Advantage: Adapted to cpx analysis b/c
f holomorphic $\Leftrightarrow \partial_{\bar z} f = 0$!

Alternatively:

$$f : \mathbb{R}^2 \longrightarrow \mathbb{C} \qquad \text{differential}$$

$$d_p f \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^2; \mathbb{C}) = \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}, \mathbb{C})$$

$$\shortparallel$$

$$\partial_x f\, dx + \partial_y f\, dy$$

Every $\mathbb{R}$-linear map can be decomposed into $\mathbb{C}$-linear and $\mathbb{C}$-anti linear maps

$$\mathrm{Hom}_{\mathbb{R}}(\mathbb{C}, \mathbb{C}) = \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, \mathbb{C}) \oplus \mathrm{Hom}_{\bar{\mathbb{C}}}(\mathbb{C}, \mathbb{C})$$

$$A \longmapsto \Big( \underbrace{\frac{A - IAI}{2}}_{P^{1,0}(A)} , \underbrace{\frac{A + IAI}{2}}_{P^{0,1}(A)} \Big)$$

<u>Hence:</u>

$$d_p f = (d_p f)^{1,0} + (d_p f)^{0,1}$$

$$= \frac{\partial}{\partial z} f\, dz + \frac{\partial}{\partial \bar{z}} f\, d\bar{z} \qquad \begin{array}{l} dz = dx + i dy \\ d\bar{z} = dx + -i dy \end{array}$$

In higher dim.

$$f : \mathbb{R}^{2n} = \mathbb{C}^n \longrightarrow \mathbb{C}$$

$$df = \sum_{j=1}^{n} \partial_{x_j} f\, dx_j + \partial_{y_j} f\, dy_j$$

$$= d^{1,0} f + d^{0,1} f$$

$$= \sum_{j=1}^{n} \partial_{z_j} f\, dz_j + \sum_{j=1}^{n} \partial_{\bar{z}_j} f\, d\bar{z}_j$$

$$=: \partial f + \bar{\partial} f$$

and of course, this can be generalised to complex manifolds:

$$\cancel{T^* X_{\mathbb{C}} = \mathrm{Hom}(TX, \mathbb{C}) =}$$

$$f : X \longrightarrow \mathbb{C} \qquad \text{smooth}$$

$$\Rightarrow df : TX_{\mathbb{R}} \longrightarrow \mathbb{C} \qquad \text{hence } d_p f \in \mathrm{Hom}_{\mathbb{R}}(TX; \mathbb{C})$$

$$= \mathrm{Hom}_{\mathbb{C}}(TX, \mathbb{C}) \oplus \mathrm{Hom}_{\bar{\mathbb{C}}}(TX; \mathbb{C}) \qquad \cancel{T^*X} \;{}^{1,0} \quad {}^{0,1}$$

It follows:

$$\Omega^\bullet(X;\mathbb{C}) = \Gamma(\Lambda^\bullet \mathrm{Hom}_{\mathbb{R}}(TX;\mathbb{C}))$$

$$= \Gamma\left( \bigoplus_{k=0}^{2n} \bigoplus_{p+q=k} \Lambda^p \mathrm{Hom}_{\mathbb{C}}(TX;\mathbb{C}) \otimes \Lambda^q \mathrm{Hom}_{\bar{\mathbb{C}}}(TX;\mathbb{C}) \right)$$

$$= \Gamma\left( \bigoplus_{k=0}^{2n} \bigoplus_{p+q=k} \Lambda^{p,q} \right)$$

Locally:

$$\Omega^k \ni \omega = \sum_{p+q=k} \omega_{p,q}\, dz_{i_1} \wedge \dots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \dots \wedge d\bar{z}_{j_q}$$

Furthermore, $d = \partial + \bar{\partial}$ b/c it is true on functions

$$\partial : \Omega^{p,q}(X) = \Gamma(\Lambda^{p,q}) \longrightarrow \Omega^{p+1,q}(X)$$
$$\bar{\partial} : \Omega^{p,q}(X) \longrightarrow \Omega^{p,q+1}(X)$$
$$\partial \omega_{p,q}\, dz_{I_p} \wedge d\bar{z}_{J_q} = \sum_{j=1}^{n} \partial_{z_j} \omega_{p,q}\, dz_j \wedge dz_{I_p} \wedge d\bar{z}_{J_q}$$
$$\bar{\partial} \omega_{p,q}\, dz_{I_p} \wedge d\bar{z} = \sum_{j=1}^{n} \partial_{\bar{z}_j} \omega_{p,q}\, d\bar{z}_j \wedge$$

Note: $0 = d^2 = (\partial + \bar{\partial})^2 = \underbrace{\partial^2}_{=0} \oplus \underbrace{\partial\bar{\partial} + \bar{\partial}\partial}_{=0} \oplus \underbrace{\bar{\partial}^2}_{=0}$

b/c different targets

Recall:

de Rham cohomology groups

$$H^k(X;\mathbb{C}) = \frac{\ker d : \Omega^k(X) \to \Omega^{k+1}(X)}{\operatorname{im} d : \Omega^{k-1}(X) \to \Omega^k(X)}$$

Dolbeault cohomology

$$H^{p,q}(X;\mathbb{C}) = \frac{\ker \bar{\partial} : \Omega^{p,q} \longrightarrow \Omega^{p,q+1}}{\operatorname{im} \bar{\partial} : \Omega^{p,q-1} \longrightarrow \Omega^{p,q}}$$

Task: Relate Dolbeault cohomology groups that contain "holomorphic" information to de Rham cohomology that (only) contains topological information

# § 3 Hermitian and Kähler Geometry

## § 3.1 Hermitian Metrics and Forms - Linear Algebra

<u>Definition</u>

A hermitian metric ~~is~~ is a $\mathbb{C}$-linear bundle map

$$TX \otimes \overline{TX} \longrightarrow \underline{\mathbb{C}} = X \times \mathbb{C}$$

s.t. it restricts to a hermitian ~~vector~~ inner product on each fibre.

In other words: ~~we~~ we have a family of hermitian inner products that depend smoothly on the base point.

<u>Fact</u>: Every cpx mfld has a hermitian metric

locally this is always true b/c we can trivialise. Then use a partition of unity to glue them together.

<u>Lemma</u>:

Let $h$ be a hermitian metric on $X$. Then

(i) $g := \operatorname{Re} h$ is a Riem. metric

(ii) $\omega := \operatorname{Im} h \in \Omega^{1,1}(X) \subseteq \Omega^2(X)$

(iii) $\omega$ is non-degenerate

(iv) $\omega^n = c \cdot n! \operatorname{vol}_g$ $\quad (c>0)$

<u>Proof</u>:

(i) Clear ✓

(ii) $\cancel{\omega(iX,Y) = \mathrm{Im}\, ih}$

$\overline{\omega} = \overline{\mathrm{Im}\, h} = \mathrm{Im}\, h = \omega$

$\Rightarrow \omega$ is real and hence $\omega \in \Omega^{1,1}(X)$

b/c $\Lambda^{1,1} \subseteq \Lambda^2 T^*X_{\mathbb{C}}$ is the only subspace that is inv. under cpx conj.

(iii) $X \in T_pX$, then $\omega(X, iX) = \mathrm{Im}\, h(X, iX)$

$= -\mathrm{Im}\, i\, h(X,X) = -\mathrm{Re}\, h(X,X) < 0$

(iv) Let $e_1, \dots, e_n$ be an orth. basis of $T_pX$ for $h_p$.

Use that $\omega = -\sum_{j=1}^{n} e_j \wedge i e_j$, ~~and calc~~

whence

$$\omega^n = (-1)^n \left( \sum_{j=1}^{n} e_j \wedge i e_j \right)^n$$

$$= (-1)^n n!\, (e_1 \wedge ie_1) \wedge \dots \wedge (e_n \wedge ie_n)$$

$$= (-1)^n n!\, \mathrm{vol}_g$$

□

**Recall:** $(V, \underset{\langle\cdot,\cdot\rangle}{h})$ euclidean/herm ~~hermitian~~ v.s. w/ ONB $e_1, \dots, e_n$

$\Lambda V$ is an euclidean/herm. ~~herm.~~ v.s. s.t.

- $\Lambda^k V$ are mutually orth.
- $e_{i_1} \wedge \dots \wedge e_{i_k}$ is an ONB

**Definition:** Hodge $*$-operator

$(V, h, \mathrm{vol}_{\mathrm{Re}\, h = g})$ hermitian v.s. w/ orientation

$*: \Lambda^k V \longrightarrow \Lambda^{2n-k} V$ $\mathbb{C}$-linear s.t.

$$\alpha \wedge *\overline{\beta} = h(\alpha, \beta)\, \mathrm{vol}_g$$

[$*$ only depends on $h$ and the orientation.

Hence it can be generalised to hermitian mfds b/c it is a point wise construction

Definition: $(V, h, \text{orSy})$ oriented herm. v.s

(i) $L : \Lambda^k V \longrightarrow \Lambda^{k+2} V$ Lefschetz operator

$\varphi \longmapsto \frac{i}{2} \varphi \wedge \omega_h$

has bidegree $(1,1)$ (i.e. $L : \Lambda^{p,q} \to \Lambda^{p+1,q+1}$)

(ii) $\Lambda := L^* = \bar{*}^{-1} L \bar{*}$ has bidegree $(-1,-1)$

(iii) $H : \Lambda^{\bullet} V \longrightarrow \Lambda^{\bullet} V$ counting operator

$$H|_{\Lambda^k V} := (k-n) \cdot id$$

Lemma:

$$[H, L] = 2L, \quad [H, \Lambda] = -2\Lambda, \quad [L, \Lambda] = H$$

hence $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mapsto L$ $\quad \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \mapsto H$

$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \mapsto \Lambda$

is an $sl_2 \mathbb{C}$ - representation on $\Lambda^{\bullet} V$

□

## §3.2 Hermitian Geometry on closed mflds

$X$ closed, complex manifold w/ hermitian metric $h$.
and orientation $\mathrm{vol}_g$.

$$(\varphi, \psi)_{L^2} := \int_X \varphi \wedge \underbrace{* \bar{\psi}}_{= \bar{*} \psi}$$ defines an hermitian metric on $\Omega^\bullet(X; \mathbb{C})$

$L^2(X) := \overline{\Omega^\bullet(X)}^{\|\cdot\|_{L^2}}$ denote its completion
(Hilbert space)

**Lemma** (identification of formal adjoints)

(1) $d^* = -\bar{*} d \bar{*}$

(2) $\partial^* = -\bar{*} \partial \bar{*}$ and $\bar{\partial}^* = -* \bar{\partial} *$

**Proof:** (uniquely)
$d^*$ defined by $(d^*\varphi, \psi) = (\varphi, d\psi)$ $\forall \varphi, \psi \in \Omega^\bullet(X)$
$\partial^*, \bar{\partial}^*$ analogously. [if we ignore domain issues]
but I do not want to dive
into the theory of unbounded
operators

$\cancel{(\varphi, d\psi)_{L^2} = \int_X \varphi \wedge \bar{*} d\psi =}$

Stokes: $$0 = \int_X d(\varphi \wedge \bar{*}\psi) = \int d\varphi \wedge \bar{*}\psi + (-1)^{|\varphi|} \varphi \wedge d\bar{*}\psi$$
$$= \int d\varphi \wedge \bar{*}\psi + (-1)^{|\varphi|} \varphi \wedge \bar{*} \bar{*}^{-1} d\bar{*}\psi$$

$$\Leftrightarrow \int d\varphi \wedge \bar{*}\psi = (-1)^{|\varphi|+1} \int \varphi \wedge \bar{*}\, \bar{*}^{-1} \underbrace{d\bar{*}\psi}_{\deg = 2n - |\varphi| + 1}$$ $\bar{*}^{-1} = (-1)^{|\cdot|} \cdot *$

$$= \int \varphi \wedge \bar{*}\, d^* \psi$$

The second statement follows from dec. $\square$

Corollary:

$$H^k(X;\mathbb{C}) \cong \mathcal{H}^k \overset{*}{\cong} \mathcal{H}^{n-k} \cong H^k(X;\mathbb{C}) \cong$$ Poincaré duality

$$H^{p,q}(X;\mathbb{C}) \cong \mathcal{H}^{p,q} \underset{*}{\cong} \mathcal{H}^{n-p,n-q} \cong H^{n-p,n-q}(X;\mathbb{C})$$ (Serre-Kodaira)

not a top. result.

In fact: These isomorphisms have a very geometric interpretation: one maps a cohomology class to its minimiser of the volume functional!

## §4 The Kählerian Case

Definition:

$h$ is a Kähler metric, if $d\omega_H = 0$.

$X$ is a Kähler mfd, if it has a Kähler metric

Examples:

1| $\mathbb{C}^n, \langle\cdot,\cdot\rangle_{st}$ $\qquad \omega_H = \frac{i}{2}\sum_{j=1}^{n} dz_j \wedge d\bar{z}_j$

2| Complex Tori: $\mathbb{C}^n/\Lambda \qquad \Lambda = \mathrm{span}_{\mathbb{Z}}\{\omega_1, \dots, \omega_{2n}\}$ $\mathbb{R}$-lin. indep.

($\langle\cdot,\cdot\rangle_{st}$ descends b/c it is translation inv.)

3| $(\mathbb{CP}^n, h_{FS})$ Fubini Study metric

$$\omega_{FS} = i\,\partial\bar\partial \log \|z\|^2 \qquad \mathbb{CP}^n = \mathbb{C}^{n+1}\setminus\{0\}/\mathbb{C}^*$$

$$= i\left[\frac{\|z\|^2 \cdot \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j - \sum_{j,k} z_k \bar{z}_j\, dz_j \wedge d\bar{z}_k}{\|z\|^4}\right]$$

It is easy to see that it is closed b/c $d\omega = 0$ $\Leftrightarrow \partial\omega = \bar\partial\omega = 0$

On each $z \in \mathbb{C}^{n+1}\setminus\{0\}$ this gives a non-deg. form on $(\mathbb{C}z)^\perp \subseteq \mathbb{C}^{n+1}$, hence it descends to $\mathbb{CP}^{n+1}$.

4| ~~#~~ Submanifolds of Kähler manifolds are Kähler.

Indeed $\iota : Y \hookrightarrow X$, then $\omega_{\iota^* h} = \iota^* \omega_h$ and $d\,\iota^*\omega_H = \iota^* d\omega_H = 0$

5| Non-Example: Hopf Manifold

Let $\rho > 1$. Then $\mathbb{Z} \curvearrowright \mathbb{C}^n \setminus \{0\}$ via $k . \underline{z} = \rho^k \underline{z}$
freely and properly

$\Rightarrow H := \mathbb{C}^n \setminus \{0\} / \mathbb{Z}$ is a complex mfd,

and $H \cong S^1 \times S^{2n-1}$

$$[z] \mapsto \left( \exp\left(2\pi i \frac{\ln \|z\|}{\ln \rho}\right), \frac{z}{\|z\|} \right)$$

~~Later: H is no Kähler mfd b/c dim $H^1(H) = 1$~~

$\Rightarrow H^2(H; \mathbb{C}) = \{0\}$, so it cannot be Kähler
b/c $[\omega_H] \neq 0 \in H^2(X; \mathbb{C})$

Theorem: (Kähler Identities)

(1) $[\partial, L] = [\bar\partial, L] = 0$ ; $[\partial^*, \Lambda] = [\bar\partial^*, \Lambda] = 0$

(2) $[\bar\partial^*, L] = i\,\partial$ $\qquad [\partial^*, L] = -i\bar\partial$

$[\bar\partial^*, \Lambda] = -i\,\partial^*$ $\qquad [\partial, \Lambda] = i\,\bar\partial^*$

Postpone proof until end of talk

only one is interesting

The other ones follow from conjugation and adjunction.

(2) is the heart of Kähler Theory
But the proof is a bit longish!

**Definition:** $(X, h, \mathrm{vol}_g)$ cpt closed hermitian mfd

$\Delta := dd^* + d^*d$ Hodge-Laplace operator

$\Box := \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ Dolbeault-Laplace operator

**Theorem:** Hodge - Kodaira

$$\begin{array}{ccccccccc} L^2(X) & = & \overline{\mathrm{im}\, d} & \oplus & \overline{\mathrm{im}\, d^*} & \oplus & \ker\Delta & =: & \mathcal{H}^k \\ \cup\!\mid & & \cup\!\mid & & \cup\!\mid & & \| & & \\ \Omega^*(X) & = & \mathrm{im}\, d & \oplus & \mathrm{im}\, d^* & \oplus & \ker\Delta & & \end{array}$$

$$\begin{array}{ccccccccc} L^2(\Lambda^{p,q}) & = & \overline{\mathrm{im}\,\bar\partial_{p,q-1}} & \oplus & \overline{\mathrm{im}\,\bar\partial^*_{p,q+1}} & \perp & \ker\Box_{p,q} & =: & \mathcal{H}^{p,q} \\ \cup\!\mid & & \cup\!\mid & & \cup\!\mid & & \| & & \\ \Omega^{p,q}(X) & = & \mathrm{im}\,\bar\partial_{p,q-1} & \oplus & \mathrm{im}\,\bar\partial^*_{p,q+1} & \oplus & \ker\Box_{p,q} & & \end{array}$$

**Proof - Sketch**

Not hard to see that the space are perpendicular to each other! For example

- $(d\varphi, d^*\psi) = (d^2\varphi, \psi) = 0$
- $0 = (\underset{0}{\Delta\varphi}, \varphi) = (d\varphi, d\varphi) + (d^*\varphi, d^*\varphi) \Rightarrow \begin{array}{l} d\varphi = 0 \\ d^*\varphi = 0 \end{array}$

Problem. In uncomplete space, we don't have a good notion of an orth. complement.
We wish to pursue the strategy to show $\Omega^* =$ ... by showing that the orth. complement $= 0$ - a useful trick in Hilbert space theory.
This is why we go into the $L^2$-completion.

$\Delta\varphi \overset{\text{weak}}{=} 0 :\Leftrightarrow (\Delta\varphi, \psi) = 0 \;\; \forall \psi \in \Omega^*(X)$

To show that the ker $\Delta$ does not increase, we use the concept of elliptic regularity
This implies that weak solution are actual solutions and, moreover, that the kernel is finite dim

If questions come

$\square$

The Kähler identities have important consequences on the Dolbeault operators:

<u>Proposition</u> Let $(X,h)$ be a closed Kähler manifold.

(1) $\partial\bar\partial^* = -\bar\partial^*\partial$ and $\bar\partial\partial^* = -\partial^*\bar\partial$

(2) $\Box = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial = \partial\partial^* + \partial^*\partial$ in part. the operator is real

(3) $\Box = \frac{1}{2}\Delta$

<u>Proof</u>: (1) + (2) exercise! Use Kähler identities

(3) $\Delta = dd^* + d^*d = (\partial+\bar\partial)(\partial+\bar\partial)^* + (\partial+\bar\partial)^*(\partial+\bar\partial)$

$$\cancel{= (\partial+\bar\partial)(\partial+\bar\partial)^* +}$$

$$\cancel{= \partial\partial^* + \partial}$$

$$\overset{(1)}{=} (\partial\partial^* + \partial^*\partial) + (\bar\partial\bar\partial^* + \bar\partial^*\bar\partial)$$

$$\overset{(2)}{=} 2\Box$$

∎

(1) $\partial\bar\partial^* = -i\partial[\partial,\Lambda] = i\partial\Lambda\partial$

$\bar\partial^*\partial = -i[\partial,\Lambda]\partial = -i\partial\Lambda\partial$

The 2nd follows from adj.

(2) $\Box = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial = \bar\partial i[\partial,\Lambda] + i[\partial,\Lambda]\bar\partial$

$$= i[\bar\partial\partial\Lambda - \bar\partial\Lambda\partial + \partial\Lambda\bar\partial - \Lambda\partial\bar\partial]$$

$\partial\bar\partial = -\partial\bar\partial$

$$= -i[\partial\bar\partial\Lambda - \partial\Lambda\bar\partial + \bar\partial\Lambda\partial - \Lambda\bar\partial\partial]$$

$$= -i\partial[\bar\partial,\Lambda] - i[\bar\partial,\Lambda]$$

$$= \partial\partial^* + \partial^*\partial$$

Corollary: Hodge-Kodaira decomposition

(1) $\mathcal{H}^r(X) = \bigoplus_{p+q=r} \mathcal{H}^{p,q}(X)$

(2) $\overline{\mathcal{H}^{p,q}(X)} = \mathcal{H}^{q,p}(X)$

(1') $H^r(X;\mathbb{C}) \cong \bigoplus_{p+q=r} H^{p,q}(X)$

(2') $\overline{H^{p,q}(X)} \cong H^{q,p}(X)$

These iso's are metric independent!

Proof:

(1)
$$\begin{array}{ccc} \Omega^r(X;\mathbb{C}) & \xrightarrow{\Delta} & \Omega^r(X;\mathbb{C}) \\ \| \wr & & \| \wr \\ \bigoplus_{p+q=r} \Omega^{p,q}(X) & \xrightarrow{\oplus 2\square_{p,q}} & \bigoplus_{p+q=r} \Omega^{p,q}(X) \end{array}$$
$\Rightarrow \Delta\varphi = 0 \iff \square_{p,q}\varphi^{(p,q)} = 0$

(2) $\overline{\Omega^{p,q}} = \Omega^{q,p}$ and ~~$\square_{p,q}$~~ since $\square$ is real
$$\bar{\ } \circ \square_{p,q} \circ \bar{\ } = \square_{q,p}$$

(1') The isomorphism $\cancel{\mathcal{H}^r \cong}\ H^r \cong \mathcal{H}^r \quad H^{p,q} \cong \mathcal{H}^{p,q}$ is given by $[\varphi] \mapsto \min\{\|\varphi'\|^2_{L^2} \mid \varphi \underset{\text{cohomologous}}{\sim} \varphi'\}$

(2') follows from (1') and (2)

$\square$

Top. Cons.

Corollary: Let $X$ be a closed Kähler mfd, $\beta_i = \dim H^i(X;\mathbb{C})$ $i$-th Betti-number

(1) $\beta_i = \sum_{p+q=i} \beta_{p,q} = \sum_{p<\frac{i}{2}} 2\beta_{p,i-p} + \underbrace{\beta_{\frac{i}{2},\frac{i}{2}}}_{=0 \text{ if } i/2 \notin \mathbb{Z}}$

In part. $i$ odd $\Rightarrow \beta_i$ even

(2) $i$ even $\Rightarrow \beta_i \neq 0 \quad (i \leq 2\dim X)$

b/c $L^r(1) \neq 0 \in H^{r,r}(X)$ for all $r \leq \dim X$

# Classes of ~~Complex~~ closed Complex manifolds

$\{\text{closed cpx mfds}\} \supsetneq \{\text{closed Kähler}\}$ Siegel. Torus $T_{Siegel} := \mathbb{C}^2 / \begin{pmatrix} 1 & 0 & \sqrt{2} & \sqrt{-5} \\ 0 & 1 & \sqrt{3} & \sqrt{-7} \end{pmatrix}$ (*)

Hopf mfd

$\supsetneq \{\text{closed Hodge mfds}\}$ $[\omega_H] \in \mathrm{im}[H^2(X;\mathbb{Z}) \to H^2(X;\mathbb{C})]$

$\overset{=}{\supseteq}$ {proj. algebraic}

Kodaira

(*) The reason why the Siegel torus is not proj is because it does not have non-constant meromorphic function.

If $T_{Siegel} \hookrightarrow \mathbb{C}P^N$, then $\Gamma(\mathbb{C}P^N, [T_{Siegel}])$

$$\cong \Gamma(\mathbb{C}P^N, \mathcal{O}(r)) \quad r>0$$

$$\cong \mathbb{C}[z_0 : \dots : z_N]_{\deg = r}$$

~~By Bertini's thm we find a linear polynom s.t. $H \cap T_{Siegel}$ is a smooth codim 1 submfd (in $T_{Siegel}$)~~

We find a ~~hypersurf~~ hyperplane s.t. $\emptyset \neq T_{Siegel} \cap H \subseteq T_{Siegel}$

The r. th power of generating functional yields a non-zero holomorphic section on $[T_{Siegel}]|_{T_S}$ which yields a non-zero meromorphic fct.