Introduction to Mixed Hodge Structures

Sommersemester 2023

The cohomology of a an algebraic variety over $\mathbb{C}$ carries a "mixed Hodge structure" and all natural map between these cohomology groups are compatible with this mixed Hodge structure. This is arguably the most important tool we have in the study of algebraic varieties in characteristic 0. Envisioned by Grothendieck, the theory was developed by Deligne in the 1970s. Generalisations to families and even a full 6 functor formalsim came later.

In the seminar, we want to go through the basic details of theory. The stress is on the step from the case of a smooth projective variety (an analytic result in Kähler geometry) to the case of general varieties. The program relies mostly on the original references [Del2], [Del3], but the material is also contained in the book by Peters–Steenbrink [PS]. Talks 2-9 are supposed to develop the theory carefully. The others have a survey flavour.

Conventions: Unless specifically required all algebraic varieties are over $\mathbb{C}$, Hodge structures are $\mathbb{R}$-Hodge structures. The letter Ω^n is used to denote the sheaf of holomorphic n-forms. All cohomology groups are sheaf cohomology on the associated complex manifolds.

Prerequisites: Sheaves and sheaf cohomology, basics of homological algebra like the long exact cohomology sequence, language of complex manifolds and quasi-projective varieties.

1. **18.4.23** The Theorem of Hodge (analysis) [Thorsten Hertl]
   Formulate the Theorem of Hodge for Kähler manifolds and sketch as much of the proof as will fit into the talk. A proof can be found in [Wa], but feel free to use something else.

   Explain why projective complex manifolds (i.e., smooth projective algebraic varieties) are Kähler.

2. **25.4.23** The category of pure polarisable Hodge structures [Roumain Farthoat]
   abstract definition and formal properties, Tate Hodge structures, polarisations, Hodge structure of a smooth projective variety. Explain why it is independent from the embedding into projective space. Chern class of a line bundle.
   [PS] Chapter 2, until Corollary 2.12, [Del2] 2.1-2.2, but omit the discussion of the Weil torus S

3. **2.5.23** The category of mixed Hodge structures (linear algebra) [Francesco Gallinaro]
   Give the definition of the category of mixed Hodge structures [Del2] (2.3.1)

and (2.3.2). Then turn to [Del2] Chapter 1.1 and 1.2 and prove Theorem 1.2.10 Deduce Theorem 2.3.5

4. **9.5.23** Spectral sequences (homological algebra) [Paul Meffle]
   Introduce the notion of a spectral sequence (both E_1 and E_2). Use algebraic geometry grading: complexes are cohomological, differentials $d_r : E_r^{pq} \to E_r^{p+r,q-r+1}$. We will only need cases where the first page is bounded. Discussion degeneration. Discuss the example of the spectral sequence for a filtered complex of vector spaces; [Go], [We] Chapter 5

5. **16.5.23** Hypercohomology [Annette Huber]
   Discuss the hypercohomology spectral sequence, see [Del2] 1.4. Instead of abstract acyclic resololutions in abelian categories use directy sheaves and the Godement resolution, see [Hu] Chapter 5. We only need the case of an ordinary topological space.

   Apply to the "Hodge spectral sequence" for the studid filtration of the complex of holomorphic differential forms on a projective complex manifold. Using the Hodge decomposition, deduce that it degenerates at E_2.

6. **23.5.23** Differentials with log poles [Jonas Schnitzer]
   We consider a smooth projective algebraic variety X and a simple normal crossings divisor. Explain the terminology. Introduce the notion of sheaves of differential forms on X with logarithmic poles along Y and their properties (weight filtration, residues), [Del2] Chapter 3 until (3.1.8). The orientation sheaf can be chosen trivial in our case. Do the example of a curve in detail.

7. **6.6.23** Construction of the MHS on cohomology of smooth varieties I [Christoph Brackenhofer]
   Let X be a smooth quasi-projective variety. Define the weight and the Hodge filtration on $H^n(X, \mathbb{C})$ as in [Del2] 3.2. Note that there is a shift in the definition of the weight filtration. Construct the spectral sequences in (3.2.4) Formulate Theorem 3.2.5 and start with the first steps of the proof

8. **13.6.23** Opposed filtrations (linear algebra)[Martin Kalck]
   We investigate what happens to the spectral sequence of a filtered complex if it is equipped with a second filtration. [Del2] Section 1.3

9. **20.6.23** Construction of the MHS on cohomology of smooth varieties II [Leonardo Patimo]
   Apply the results of the last talk to finish the proof of [Del2] Theorem 3.2.5. Establish functoriality of the Hodge structure and independence of the choice of compactification. Do the example of a smooth curve in detail

10. **27.6.23** Singular varieties [Thomas Agugliaro]
    Sketch the construction of the MHS on the cohomology of a singular variety either as via simplicial hypercovers in [Del3] or via the blow-up sequence.

11. **4.7.23** 1-motives [Johan Commelin]
Introduce the notion of a 1-motive and explain the equivalence of categories between 1-motives and Hodge structures of curve type, see [Del3]

12. **11.7.23** Variations of pure Hodge structure [Pedro Nunez]
Introduce the notion of a variation of pure Hodge structure on a smooth S. Explain how a smooth projective family $\pi : X \to S$ gives rise to such a variation on $R^n\pi_*\mathbb{Z}$, in particular the Gauss-Manin connection and Griffiths transversality, see [GS] §3a, [S] §2

13. **18.7.23** Period domains and the period map
Period domains are certain open subsets of flag manifolds. Every variation of pure Hodge structure gives rise to map to such a period domain (modulo some discrete group). Explain the notions and discuss why the map is holomorphic. [GS] §3b, [S]§3, [CMP] §12

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