GK seminar: vector bundles

winter term 2012/13

July 25, 2012

wednesdays, 2-4 p.m., SR404, Eckerstr. 1

The seminar tries to give an introduction to various aspects of vector bundles. There are three blocks of talks. The first block (talks 1-6) mostly deals with topological/smooth vector bundles, classification, Grassmannians and characteristic classes. The second block (talks 7-12) deals with holomorphic vector bundles, comparison holomorphic vs. algebraic and holomorphic vs. continuous, moduli spaces etc. The last block (talks 13-15) deals with more algebraic topics, centered around projective modules over polynomial rings.

1. **definitions and examples**

   24.10.2012

   definition vector bundles, locally free sheaves, projective modules, translation bundles vs. sheaves. description via coverings and cocycles.

   continuous/smooth/holomorphic vector bundles

   demonstrate the transition between the worlds for one example, e.g. the tangent bundle of S^2 (this is also an example of a non-trivial but stably trivial vector bundle)

   possible literature e.g. [Hus94], [Hat09], [MS74, Section 2]. it seems there is no single source that would cover all of the above, the literature might need some assembly.

2. **Grassmannians**

   31.10.2012

   construction of Grassmannians, the universal bundle, and cell structure of the Grassmannian

   you are free to choose your point of view, you might want to talk about the structure of smooth manifold for $\mathrm{Gr}_{n,k}(\mathbb{R})$, complex manifold $\mathrm{Gr}_{n,k}(\mathbb{C})$ or projective variety for $\mathrm{Gr}_{n,k}(k)$.

   same for the description of the cell structure

   e.g. for topological viewpoint [MS74, Sections 5,6], for complex viewpoint [GH94, Chapter 1.5],

3. **topological classification**

   07.11.2012

   topological vector bundles as functor (pullback), homotopy invariance for vector bundles (bundles over $B \times [0,1]$ are isomorphic to bundles pulled back from B if B is paracompact)

topological classification: for any paracompact Hausdorff space X, there is a bijection between isomorphism classes of rank n vector bundles on X and homotopy classes of maps $[X, \mathrm{Gr}_n]$

(explain the bijection)

[Hus94, Chapter 4] or [MS74]

4. **characteristic classes**

   14.11.2012

   definition of characteristic classes. again there are different points of view to choose from: real vector bundles and Stiefel-Whitney classes resp. complex vector bundles and Chern classes

   one possible definition is via a computation of cohomology rings of Grassmannians [MS74], another one (for Chern classes) is via curvature forms, yet another axiomatically reduces everything to line bundles [Gro58, Ful98], choose one point of view

5. **characteristic classes, ctd.**

   21.11.2012

   characteristic classes have concrete interpretations as obstruction classes, e.g. [MS74, Section 12]

   there are nice topological results related to vector bundles and characteristic classes e.g. Adams' theorem on the maximal number of linearly independent vector fields on spheres or the Adams-Atiyah solution to the Hopf invariant 1 problem [Hus94] (only mention these, the proof is beyond the scope of the seminar)

6. **outlook: vector bundles in differential topology**

   28.11.2012, Sebastian Goette

7. **GAGA-principle**

   05.12.2012

   GAGA-principle refers to the comparison of holomorphic data vs. algebraic data. a particular case is the comparison of holomorphic vs. algebraic vector bundles.

   formulate the statement, discuss why it is true (this can be done nicely for $\mathbb{P}^n$), sketch the counterexample in the non-proper case [Har77, Example B.2.0.1]

8. **outlook: Oka principle**

   12.12.2012, Daniel Greb

9. **moduli of vector bundles**

   19.12.2012

   moduli spaces of semistable vector bundles, with particular view of vector bundles on $\mathbb{P}^n$. the study of moduli spaces of vector bundles shows that on projective there is a huge difference between topological and holomorphic vector bundles. one of the things that one can do in one talk is a

discussion of the Serre-Hartshorne correspondence between codimension 2 subvarieties and rank two vector bundles on $\mathbb{P}^n$

[OSS11] or [Sch80]

10. **outlook: positivity I**

    09.01.2012, Stefan Kebekus, Alex Küronya

11. **outlook: positivity II**

    16.01.2012, Stefan Kebekus, Alex Küronya

12. **outlook: vector bundles and induced representations**

    23.01.2012, Wolfgang Soergel

13. **the theorem of Serre-Swan**

    30.01.2012

    the theorem of Serre-Swan refers to the comparison of continuous vector bundles vs. projective modules

    [Ser55], [Swa62]

14. **projective modules over polynomial rings**

    05.02.2012

    the theorem of Quillen and Suslin states that projective modules over polynomial rings are free, this is an algebraic version of homotopy invariance of vector bundles

    [Qui76], [Sus76], [Lam06]

15. **outlook: vector bundles in $\mathbb{A}^1$-homotopy theory**

    12.02.2012, Matthias Wendt

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