COBORDISM

Seminar of the Graduiertenkolleg WS 2017/18

Introduction

Bordism or cobordism starts out quite naively as an equivalence relation of manifolds: two closed manifolds X, Y are *cobordant* if there is a manifold with boundary Z whose boundary is $X \amalg Y$. There are alternative versions taking the orientation into account or allowing only complex manifolds. However, the set of equivalence classes turns out to have a lot of structure, e.g., it is in fact a group, even a ring. The definition can be made relative to a base, so that we have a bordism group for every manifold. This turns out to define a generalized cohomology theory and hence is very closely related to the construction of characteristic classes. In particular, vector bundles also have cobordism classes. The information in cobordism in finer than in singular cohomology. One extra bit of information in complex cobordism is the formal group law that describes the formula for the first Chern class of the tensor product of line bundles.

In the last decade, an analogous theory was also developed in algebraic geometry, mostly by Levine and Morel. There are two approaches: there is an explicit geometric construction mimicking the topological situation. Alternatively, one uses the setting of motivic homotopy theory and defines cobordism as the cohomology of the spectrum MGL. The theory is compatible with the topological one under the analytification functor. As in the topological case, it gives insights in the nature of characteristic classes for algebraic varieties. As a corollary, we obtain surprising divisibility properties of algebraic cycle classes. These were needed in the acclaimed proof of the generalized Milnor conjecture (also called Bloch–Kato conjecture) by Voevodsky and Rost.

In the seminar, we want to learn about the basics of the theory and then concentrate on some aspects. Important applications will have to be left out. The first talks treat the original case, i.e., differential topology. We look in more detail into the case of complex corbordism and present the work of Quillen. There is a nice application to the integral Hodge conjecture that already shows the significance for algebraic geometry. In the second part of the program, we explain the geometric definition of Levine and Morel. A selected number of properties will be proved, while others will have to be taken for granted. Finally, the application to the Bloch–Kato conjecture will be explained in a survey talk.

Talks

Talks are 90 minutes (two times 45). Recall that we would like to have plenty of time for questions and discussions. Hence we recommend that speakers prepare a 60 minute talk.

1. The Pontryagin–Thom construction [13, 19] 18.10.17 (Anja Wittmann)

This talk should introduce the homotopy-theoretical view on topological cobordism. Reserve much time for the last two items below.

(1) Definition of unoriented Ω^O and oriented cobordism Ω^{SO}, ring structure.
(2) Recall Whitney's embedding and transversality theorem (without proofs).
(3) Recall the classifying spaces $BO(k)$ and $BSO(k)$. We only need the universal property. Introduce a model only if there is time.
(4) Define the Thom spaces $MO(k)$ and $MSO(k)$.
(5) Explain the Pontryagin–Thom homomorphisms

$$\pi_{n+k}(MO(k)) \to \Omega_n^O \text{ and } \pi_{n+k}(MSO(k)) \to \Omega_n^{SO}.$$

(6) Explain why the Pontryagin–Thom homomorphisms become isomorphisms for large k.

2. Computation of cobordism up to finite groups [13, 19] 25.10.17 (Ben McDonnel, Fabian Kertels)

We recapitulate the classical computations of cobordism groups.

(1) Introduce complex cobordism and the Pontryagin–Thom homomorphism $\pi_{n+2k}(MU(k)) \to \Omega_n^U$ [18].
(2) Explain the Thom isomorphism theorem.
(3) Explain the Serre–Hurewicz theorem for homotopy and homology up to finite groups [13, Sect.18].
(4) Recall Chern classes and numbers. Note that Chern numbers are bordism invariants [13, Sect. 16].
(5) Give (multiplicative) generators of the rational cobordism ring $\Omega^U \otimes \mathbb{Q}$: for ideas and references see the "complex bordism" page [9]. discuss Milnor manifolds and Milnor hypersurfaces.
    Explicitly mention divisibility of characteristic number ([13, 16.6,16-E]): for d-dimensional manifold, $s_d(M) \equiv 0 \bmod \ell$. Milnor manifolds: manifolds of dimension $d = \ell^i - 1$ with $s_d(M) \equiv \pm\ell \bmod \ell^2$. This will reappear in the last talk.
(6) If time remains, also give generators of $\Omega^{SO} \otimes \mathbb{Q}$. We will deal with Ω^O later.

3. h-cobordism theorem and handle decomposition [12, 3] 8.11.17 (Simone Murro, Ksenia Fedosova)

This talk considers aspects of bordism that belong to another branch of the story as the rest of the talks in this program. We will see a method how to split a cobordism into a sequence of elementary cobordisms (= handle attachments).

(1) [3, Sect. 1.2], [12] Introduce handles and the basics of handlebody decomposition.
(2) [3, Sect. 1.4], [12] Present some results on reducing the handlebody decomposition.
(3) [12, §9] Formulate the h-cobordism theorem. The proof uses the methods from above. But instead of proving the theorem sketch some of the applications.

4. Bordism as a multiplicative cohomology theory [1, 2] 15.11.17 (Eva-Maria Müller, Yuhang Hou)

This talk introduces multiplicative cohomology theories, orientations of vector bundles. Oriented and complex cobordism are nice examples.

(1) Stabilising the Pontryagin-Thom homomorphisms in direction of k leads to Thom spectra MO, MSO, MU [1].
(2) Give a general definition of a spectrum E and the associated generalised homology and cohomology theories (don't talk about smash products or maps of spectra, really, don't).
(3) Define multiplicative cohomology theories E and E-orientations (Thom classes) of vector bundles [5, Def 7.19, 8.10]
(4) Starting from (2), give geometric descriptions of (unoriented/ oriented/complex) bordism homology and cohomology of smooth manifolds [2]. If there is time, explain products.
(5) Exhibit Poincaré duality and explain the pushforward in bordism cohomology.

5. Formal group laws and the Lazard ring 22.11.17 (Nelvis Fornasin, Jorgen Lye)

(1) From (commutative) Lie groups to formal group laws. (examples: additive, multiplicative, maybe elliptic?)
(2) Explain universality for group laws and show that there is a universal one-dimensional formal commutative group law. Introduce the Lazard ring $\mathbb{L}$ by giving generators and relations (maybe compute the first nontrivial relation explicitly).
(3) Introduce complex oriented cohomology theories and explain characteristic classes (for example, Conner–Floyd Chern classes for complex bordism [4]).
(4) Explain how to associate a formal group law to a complex-oriented cohomology theory.

For literature references, start searching from the Wikipedia page. Coordinate closely with the next talk to know what statements about the formal group laws and Lazard ring are required (and for appropriate choice of notation).

6. Quillen's theorem on complex cobordism [15, 16] 29.11.17 (René Recktenwald)

This talk finally describes the complex cobordism ring Ω^U (and its real analogue Ω^O) completely.

(1) Explain Quillen's theorem that complex cobordism gives rise to the universal formal (associative and commutative) group law.
(2) Sketch the proof following either [15] or [16], without going into too much details.
(3) Explain the analogous statement on Ω^O.
(4) Maybe explain how to get from complex cobordism to integral cohomology and topological K-theory (if you want you can state Landweber exact functor theorem)
(5) Maybe (i.e., if you want) describe Ω^U at a prime p and introduce Brown–Peterson cohomology.

7. Counterexamples to the integral Hodge conjecture [20] 6.12.17 (Elmiro Vetere)

We want to present Totaro's counterexamples. His argument uses a factorisation of the cycle class map via cobordism. (The talk should focus on explaining the Chow groups and the factorization of the cycle class map.)

(1) Introduce Chow groups (up to rational equivalence) for smooth varieties over a field (e.g. using Fulton's intersection theory book)
(2) Sketch the construction of the cycle class map to cohomology, formulate the Hodge conjecture and its integral version
(3) explain factorisation of cycle class map through complex cobordism, sections 3 and 4 of [20] (for time reasons, the product structure can be sketchy)
(4) explain the application to the integral Hodge conjecture [20, Section 7], but since it's not in the focus of the seminar, we don't need to see all details for the construction of the varieties approximating the classifying spaces.

8. Oriented theories over a base field k 13.12.17 (Fritz Hörmann)

[7, 8, 10] This talk introduces the notion of oriented cohomology theory for smooth varieties over a field k. This definition is directly inspired by Quillen's ideas in 5. and 6.

(1) Definition
(2) Examples: CH^*, singular and etale cohomology, $K_0[\beta, \beta^{-1}]$, $\mathrm{MGL}^{2*,*}$, etc.
(3) The formal group law of an oriented theory
(4) Discuss the role of different notions of orientability in topology and algebraic geometry

9. Algebraic cobordism: basic properties 20.12.17 (Giovanni Zaccanelli)

This is a summary of the basic properties and structures of algebraic cobordism [10, Introduction, Ch. 1]. Note that the proofs of some of these properties will be given in the next few talks whereas this talks is about presenting the results

(1) Algebraic cobordism as the universal oriented theory
(2) Extra structure: localization sequence
(3) $\Omega^*(k) \cong \mathbb{L}^*$
(4) Conner–Floyd and $\Omega^* \otimes \mathbb{Z} \cong \mathrm{CH}^*$
(5) The analogue of Quillen's theorem: degrees and generalized degree formulas. Examples.

10. The construction of algebraic cobordism 10.1.17 Annette Huber

This talk gives the construction of algebraic cobordism as a Borel-Moore functor Ω_*. Cf [10, 2.1 and 2.2]

11. Basic properties: localization sequence [10, 3.2] 17.1.18 Frédéric Deglise

This lecture proves the fundamental localization sequence for a closed embedding $i : Z \to X$ of smooth schemes, with open complement $j : U \to X$.

$$\Omega_*(Z) \xrightarrow{i_*} \Omega_*(X) \xrightarrow{j^*} \Omega_*(U) \to 0.$$

As a preliminary step, the class of a normal crossing divisor is constructed. (If you want, you can discuss the relation with the excision sequence in topology.)

Mention without proof that homotopy invariance and the projective bundle formula are true.

12. $\Omega^*(k)$ and the Lazard ring
24.1.18 Annette Huber

The main theorem: $\Omega_*(k) \cong \mathbb{L}_*$ for fields of characteristic zero, [10, Ch. 4.3].

The injectivity is rather easy, either by relying on topology or by using characteristic numbers.

The surjectivity reduces to showing that the additive theory $\Omega_* \otimes_{\mathbb{L}} \mathbb{Z}$ on $X = \mathrm{Spec} k$ is just $\mathbb{Z}$. One starts by using the computations of the classes of projective space bundles from 8(a) to reduce to a birational statement, and then using the generic projection of a smooth projective variety to reduce to the case of hypersurfaces, then finally deforming to a union of hyperplanes.

13. Ω^* and CH [10, Ch. 4.5]
31.1.18 Brad Drew

This lecture shows that CH^* is the universal additive theory, thus identifying CH^* with $\Omega^* \otimes \mathbb{Z}$. Additional computations and a discussion of the topological filtration on Ω_* are discussed.

(1) $\Omega^* \otimes \mathbb{Z} \cong \mathrm{CH}^*$ [10, Section 14.1]
(2) The topological filtration [10, Section 14.2].
(3) Computations [10, Section 14.3].

Mention without proof the relation to $K_0[\beta, \beta^{-1}]$ and multiplicative formal group law[10, 4.2].

14. Bloch–Kato conjecture
6.2.18 (Matthias Wendt)

Mainly a discussion of [10, Ch. 4.4]. The main results are the "generalized degree formula" in algebraic cobordism and its application to Rost's degree formula for the characteristic numbers constructed from the Newton class. (the statement of Rost's degree formula concerns the characteristic number s_d)

Explain the formulation of the Bloch–Kato conjecture. Sketch the proof. Make sure that the degree formulas are mentioned.

Contact

For questions on the topological part of the program: Sebastian Goette, Nadine Große.
For questions on the algebraic part of the program: Annette Huber, Matthias Wendt.

References

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