REPRESENTATION THEORY OF FINITE GROUPS

Proseminar Wintersemester 2019/20
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1. Lecture 1: Basics

Define a representation of a group with examples. Define subrepresentations and irreducible representation with examples. If time permits, discuss tensor product (and symmetric squares) of two representations. For examples see [St, Chapter 3].

2. Lecture 2: Mascke's theorem

Define unitary representations and prove that for a finite group, all (finite dimensional) representations are unitary. Use this to prove the Mascke's theorem, i.e., representations of a finite group is completely reducible. Use [St, Section 3.2].

Alternate approach: Prove [S, Theorem 1 and Theorem 2].

3. Lecture 3: Character theory

Definition of the character of a representation. Calculate the character of the sum (tensor product and symmetric squares) of representations. Calculate the characters of the examples discussed in previous lectures. Prove Schur's lemma and it's corollaries from [St, Corollary 4.1.8-4.1.10].

4. lecture 4: Schur's orthogonality relations

Introduce $\mathbb{C}[G]$ as an inner product space over $\mathbb{C}$ and prove the Schur's orthogonality relations. Basically prove [St, Theorem 4.2.8]. (Compare this with [S, Corollary 2 and Corollary 3].)

5. Lecture 5: Number of irreducible characters I

Define class functions. Prove the first orthogonality relation, i.e., orthogonality of the irreducible characters. As a corollary, conclude that the number of irreducible representations is finite and bounded by the number of conjugacy classes. Explain [St, Example 4.3.17] (which will motivate (depend on) the next Lecture). Follows [St, Section 4.3] and [S, Theorem 3].

6. Lecture 6: Multiplicity of irreducible components and a criterion of irreducibility

As a corollary of the first orthogonality relation, prove that the multiplicity of an irreducible component in a representation is determine by the inner product of the characters of respective representations [St, Theorem 4.3.14] and [S, Theorem 4]. Also prove that a representation is irreducible if and only if self inner product of its character is 1. Discuss [St, Example 4.3.17] in full details.

7. Lecture 7: Regular representation and Number of irreducible characters II*

Define regular representation and its character. Discuss the decomposition the regular representation in terms of the irreducible representations and its corollaries [S, Section 2.4] and [St, Section 4.4]. Prove that irreducible characters forms a basis of the space of class functions. (Note that we have already proved that they are orthogonal, so here we have to show that they span this space) [S, Section 2.5] and [St, Section 4.4]. Conclude that the number of irreducible representations is same as the number of conjugacy classes.

8. Lecture 8: Character table, second orthogonality relation and Representation of Abelian groups*

Define character table and prove the second orthogonality relation, i.e., the columns of the table are also orthogonal [St, Section 4.4]. Discuss the character table of S_3 and $\mathbb{Z}/4\mathbb{Z}$.

9. Lecture 9: Representation of Abelian groups

Show that a finite group is abelian if and only if all its irreducible representations are of degree 1. Calculate all irreducible representations of cyclic groups and classify the characters of the product of two groups in terms of the characters of the groups [St, Corollary 4.4.9, Example 4.4.10 and Proposition 4.5.1] and [S, Section 3.1 and Section 3.2]. Assuming that every finite abelian group is a product of cyclic groups, mention that the above calculations determine all irreducible representations of an abelian group. If time permits, compute the character table of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

10. Lecture 10: Induced Representation and its characters

Discuss induced representation and its characters from [S, Section 3.3] and [St, Section 8.1 and Section 8.2]. Discuss the [St, Example 8.2.1 and Example 8.2.3].

11. Lecture 11: Examples

Recall (or state) the properties of characters of a finite groups [JL, Chapter 24]. Discuss examples A_4, D_4 and the group of cubes from [S, Chap 5].

12. Lecture 12: Basics of Non-commutative Algebra

Define non-commutative rings and left modules over them. Define division rings, simple modules and prove Schur's lemma. Define semi-simple modules and semi-simple rings. Follow text from [M, Chapter 1 and Chapter 2].

13. Lecture 13: Semi-simple rings

Prove that a ring is semi-simple if and only if every left module over it is semi-simple [M, Theorem 2.4.7 ((1) iff (2))]. Define isotypical modules. Show that the matrix rings over a division ring is semi-simple. State (without proof) Artin-Molien-Wedderburn Theorem [M, Theorem 2.4.10]. Describe the terms in details.

14. Lecture 14: Group Algebra, Maschke's theorem

For a field K, define the group algebra $K[G]$. (Calculate its center.) Prove that $K[G]$ is semi-simple if $|G|$ is invertible in K [M, Theorem 3.1.7 (if part)]. Prove that G-representations (on K-vector spaces) are equivalent to $K[G]$-modules and conclude the complete irreducibility of G-representations on $\mathbb{C}$-vector spaces. Assuming the fact that there are no non-trivial finite dimensional division algebra over an algebraically closed field and Artin-Molien-Wedderburn Theorem, discuss (without proof) [M, Theorem 3.3.10].

15. Lecture 15: Representation theory of S_n

Discuss the representation and character theory of symmetric groups S_n [St, Chapter 10] and [M, Chapter 5]. More precisely, following [St, Chapter 10], define Young diagram, Conjugate partition (with [St, Example 10.1.7]), Young tableaux, Column stabilizer, Tabloid (denoted by T^λ), Polytabloid (state [St, Proposition 10.2.7]). Lastly define Sprecht representation and state [St, Theorem 10.2.17]. If time permits, do the calculations for S_3 and S_5.

References

[JL] G. James and M. Liebeck, *Representation and Characters of Groups*, Second edition, Cambridge University Press, New York, 2001.

[M] C. Musili, *Representation of finite groups* , Hindustan book agency, 2011.

[S] J.-P. Serre, *Linear representations of finite groups*, Graduate Texts in Mathematics, Vol. **42**. Springer-Verlag, 1977.

[St] B. Steinberg, *Representation theory of finite groups*, Universitext, Springer, 2012.