# GALOIS GROUPS AND FUNDAMENTAL GROUPS

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## Introduction

The ultimate goal of Algebraic Number Theory is to study properties of the ring of integers $\mathbb{Z}$ and the rational number field $\mathbb{Q}$. One of the most general strategies consists in studying equations with integer or rational coefficients. Given a polynomial

$$f(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n \in \mathbb{Q}[x]$$

we can construct its splitting field $E_f$: it is a finite Galois extension of $\mathbb{Q}$, and every such extension arises in this way. The arithmetic properties of $f(x)$ are encoded in the Galois group $\mathrm{Gal}(E_f/\mathbb{Q})$. Given a finite Galois extension $E/\mathbb{Q}$, the main theorem of Galois theory yields an order-reversing bijection

$$\begin{aligned} \{\text{subfields of } E\} &\leftrightarrow \{\text{subgroups of } \mathrm{Gal}(E/\mathbb{Q})\} \\ K &\to \mathrm{Gal}(E/K) := \{g \in \mathrm{Gal}(E/\mathbb{Q}) \mid g|_K = id_K\} \\ E^H := \{a \in E \mid h(a) = a\ \forall h \in H\} &\leftarrow H. \end{aligned}$$

The following questions arise naturally:

(1) given a finite group $G$, is there a finite Galois extension $E/\mathbb{Q}$ such that $\mathrm{Gal}(E/\mathbb{Q}) \simeq G$?
(2) if so, in how many different ways can we find such an extension $E$, and how are they related to each other?

This is the so-called *inverse Galois problem* for $\mathbb{Q}$, an extremely difficult open problem. In order to study it properly one has to consider at once the system of all finite Galois extensions of $\mathbb{Q}$, or better the corresponding system of all finite Galois groups over $\mathbb{Q}$. It is convenient to glue all these finite groups into a single object, called the absolute Galois group of $\mathbb{Q}$ and denoted $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$: it is constructed as the "limit" of the above system of finite Galois groups and has a natural structure ot topological group; its topology records the whole collection of finite Galois groups as well as their relations. Now the problem becomes: how to get information about the group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ concretely? This will be the main motivating question of the present seminar.

The abstract framework of Galois theory occurs not only in the algebraic theory of fields and field extensions, but also in several other mathematical areas. In all these cases, Galois theory studies the groups $\mathrm{Aut}(Y/X)$ of automorphisms of a given object $Y$ over another object $X$; it provides a bijection of the form

$$\begin{aligned}\{\text{objects intermediate between } X \text{ and } Y\} &\leftrightarrow \{\text{subgroups of } \mathrm{Aut}(Y/X)\}\\ Z &\to \mathrm{Aut}(Y/Z)\\ Y^H &\leftarrow H\end{aligned}$$

analogous to the one of classical Galois theory. One of the most important examples of such "Galois phenomena" occurs in Topology with the theory of *covering spaces*. It is intimately related to the theory of the *fundamental group*, which describes one of the main algebraic invariants of topological spaces. In the end, Galois groups and fundamental groups should two examples of the same theory. But does such a theory exist in nature?

The answer is affirmative, in a suitable sense. The crucial task is to construct a world where where fields and topological spaces naturally interact with each other; typically, this means having certain topological spaces endowed with certain fields of functions on them. In the present seminar we will introduce two main examples:

- Riemann surfaces with their fields of meromorphic functions;
- algebraic curves with their fields of rational functions.

As we will see, finite covering spaces of a Riemann surface correspond to suitable finite extensions of its meromorphic function field, and the Galois theory of coverings corresponds to the Galois theory of meromorphic functions. Compact Riemann surfaces are examples of smooth proper algebraic curves: namely, they correspond to smooth proper algebraic curves defined over $\mathbb{C}$. In the world of algebraic curves it is possible to define a purely algebraic notion of covering spaces. This leads to an algebraic version of the fundamental group, called the *étale fundamental group*, which agrees with the topological fundamental group in the case of Riemann surfaces.

But the étale fundamental group makes sense for curves defined over arbitrary fields: not only $\mathbb{C}$, but also $\mathbb{Q}$ or $\bar{\mathbb{Q}}$ for example. Given a curve $X$ defined over $\mathbb{Q}$, the étale fundamental group of $X$ interacts with the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ in a very interesting way. This general fact, combined with the Hilbert Irreducibility Theorem, allows to construct Galois extensions of $\mathbb{Q}$ starting from covering spaces of $X$, thus providing new information around the inverse Galois problem for $\mathbb{Q}$. This is the final goal of the seminar.

## Preparing your talk

- Start early. Find out if you will get away with reading the texts specified in the references or if you need additional literature to cover basics.
- Ask us. If you run into problems, aks the assistent or the professor. It is our job to support you. This can concern mathematical questions, advice on literature or the contents and structure of your talk.
- Find out what you are supposed to do. In doubt, ask. Sometimes your talk provides preparation for subsequent talks that cannot be left out. It is a good idea to coordinate with other speakers.
- We want to have 10 minutes for feedback, so your talk should end after 80 minutes. Expect that there are questions or a discussion during your talk. You do not want this to derail your presentation. If you do a test without audience, 70 minutes is a good aim.
- If the program contains more material than you can cover (actually this is the norm), it is part of your job to make choices. Do not speed up instead.
- Think about timing. What will you leave out if time is running out? Are there additional examples that you can add if there is still time? One option is leaving a long proof for the end of the talk, so they can be cut short. Make sure that the main message is not lost due to time problems.
- You are very welcome to use interactive ingredients, and overall do a better job than an average lecture.
- We expect some kind of hybrid format for the seminar. It may be necessary to switch on short notice.
- If you are giving a black board talk or a hand written talk on a tablet, please prepare a hand out with the main points. If you are preparing a presentation, we can use that instead.
- A week before your talk at the latest meet the assistent to discuss a draft of your talk.
- Start early.

**Contacts.**


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# PART 1: GALOIS GROUPS

## 1. Profinite groups

Reference: [RibZal, § 1,2].

**Aim.** The aim of this talk is to introduce *profinite groups*, a certain class of topological groups generalising finite groups. Profinite groups can be defined as inverse limits of finite groups: this yields a natural topology on them. It turns out that they are characterised by certain topological properties.

### 1.1. **Inverse limits.** [RibZal, § 1.1]

Introduce the notion of *inverse limit* of an inverse systems of sets (or groups, or topological spaces, ...) in terms of a universal property. Explain first how to construct inverse limits of sets concretely, and give a sufficient consition in order for the inverse limit to be non-empty (Proposition 1.1.4).

Then consider the special cases of groups and topological spaces:

- given an inverse system of groups and group homomorphisms, the inverse limit (endowed with the component-wise multiplication and inversion maps) is an inverse limit of groups;
- given a system of topological spaces and continuous maps, the inverse system (endowed with the subspace topology of the product topology) is an inverse system of topological spaces.

### 1.2. **Profinite groups as inverse limits.** [RibZal, §§ 2.1, 2.4, 3.2]

Recall the general notion of *topological group*; as a basic example, point out that every group can be regarded as a discrete topological group. Define the notion of *topological generators* of a topological group.

Introduce the notion of *profinite group* as inverse limit of a system of finite groups. By the previous discussion, they are topological groups; moreover, every finite group is canonically a profinite group with the discrete topology.

In order to give more examples, introduce the *profinite completion* of a group. Discuss the following examples:

- the profinite completion of $\mathbb{Z}$;
- the free profinite group on a set.

### 1.3. **Topological characterisation of profinite groups.** [RibZal, §§ 1-2]

Prove that every profinite group is compact, Hausdorff, and completely disconnected (Proposition 1.1.3); and conversely, every topological group with these properties is profinite (Theorem 2.1.3 (a) $\iff$ (b)).

Deduce the following consequences: first, in a profinite group, a subgroup is open if and only if it is closed and has finite index (Lemma 2.1.2); second, every closed subgroup of a profinite group is itself a profinite group with the induced topology.

### 1.4. **Continuous actions of profinite groups.** [RibZal, § 5]

Recall the notion of *action* of a group on a set.

Then introduce the notion of *continuous action* of a topological group on a set (regarded as a discrete topological group). Explain the characterisation of continuous actions in terms of openness of all stabilizers.

Deduce that, whenever a profinite group acts continuously on a set, all orbits are finite. As a key example, discuss the action of a profinite group on the cosets of an open subgroup.

## 2. Infinite Galois theory

Reference: [Sza, § 1]

**Aim.** We assume that the participants know about the main results of finite Galois theory. The aim of this talk is to generalize these results to infinite field extensions. This leads to the construction of the *absolute Galois group* of a field, which is naturally a profinite group. The absolute Galois group will be our main object of interest in this seminar.

### 2.1. **Separable field extensions.** [Sza, § 1.1]

Start by recalling the basic definitions about algebraic field extensions and algebraically closed fields, as well as the main properties of the algebraic closure (Proposition 1.1.3).

Then recall the notion of *separable field extension* and the *separable closure* of a field. Explain how to characterise separable extensions in terms of embeddings in an algebraic closure (Lemma 1.1.6).

Recall what a perfect field is. Point out that all fields of characteristic 0 are perfect - we will be mostly interested in this case.

### 2.2. **Galois extensions.** [Sza, § 1.2]

Recall the general definition of a *Galois extension*: notice that this makes sense for finite as well as infinite extensions. Discuss the main properties of Galois extensions (Proposition 1.2.4).

Recall the main theorem of finite Galois theory (Theorem 1.2.5) and the main ideas behind the proof.

Finally, introduce the so-called *inverse Galois problem*: given a field $k$, which finite groups $G$ occur as Galois groups of finite Galois extensions of $k$? A first general piece of information about this question is given in Example 1.2.11.

### 2.3. **The absolute Galois group of a field.** [Sza, § 1.3]

Prove that the Galois group of a Galois extension is a profinite group (Proposition 1.3.5). Then prove the main theorem of infinite Galois theory (Theorem 1.3.11).

Define the *absolute Galois group* of a field $k$ (p. 13, end of Example 1.2.3). Point out that the Galois group of every other Galois extension of $k$ is a quotient of it by a closed subgroup.

Point out that not every subgroup of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ is closed (Remark 1.3.12), hence the closure condition in the main theorem is crucial. In fact, it turns out that not every subgroup of finite index of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ is open (Remark 1.3.10(2)).

In order to see why absolute Galois groups are fundamental in the study of fields, state the Neukirch–Pop Theorem (Remark 1.3.13); you can focus on the case of fields of characteristic 0, when the result is more explicit. Deduce that, for example, every automorphism of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ is induced by an automorphism of $\bar{\mathbb{Q}}$.

## 3. Grothendieck's formulation of Galois theory

Reference: [Sza, § 1].

**Aim.** The aim of this talk is to give an alternative description of the absolute Galois group $\mathrm{Gal}(k_s/k)$ of a field as an automorphism group. This allows to express the main theorem of Galois theory in terms of continuous actions of $\mathrm{Gal}(k_s/k)$ and characterise these actions concretely in terms of finite étale $k$-algebras.

### 3.1. **Categories and functors.** [Sza, § 1.4]

Introduce the general language of *categories* and *functors*, and give examples. Define *morphisms of functors*, explain how to compose them, and define isomorphisms of functors.

Then introduce the notion of *representable functor* and prove the Yoneda Lemma (Lemma 1.4.2). This result is fundamental and will be used many times in the seminar.

Finally, introduce equivalence of categories and explain how to characterise them (Lemma 1.4.9).

### 3.2. **The category of Galois sets.** [Sza, § 1.5]

The results of this section give a first reformulation of Galois theory in categorical terms. Show that the functor represented by the separable closure $k_s$ yields an equivalence between finite extensions of $k$ and finite continuous transitive $\mathrm{Gal}(k_s/k)$-sets (Theorem 1.5.2).

### 3.3. **Finite étale algebras.** [Sza, § 1.5]

Introduce *finite étale $k$-algebras* as a natural generalisation of finite separable field extensions of $k$, and explain how to characterise them (Proposition 1.5.6).

Finally, generalise the result of Theorem 1.5.2 to non-necessarily transitive actions using étale $k$-algebras (Theorem 1.5.4).

# PART 2: FUNDAMENTAL GROUPS IN TOPOLOGY

## 4. Real surfaces and their classification

Reference: [Mas, § I]

**Aim.** This is a survey talk about *real surfaces* their structure. The main results are the classification theorem and the existence of canonical forms of compact real surfaces. The notion of *triangulation* of a compact real surface plays a key role: it allows to define numerical invariants such as the *Euler characteristic* and the *genus*. It is also important for our purposes to extend the previous constructions to those real surfaces obtained by removing a finite number of points from a compact surface.

For sake of completeness, in this talk we will treat both orientable and non-orientable surfaces; in the rest of the seminar we will only need orientable surfaces.

### 4.1. **Generalities on compact real surfaces.** [Mas, § I.1-I.4]

Recall the general notion of *$n$-dimensional (real) manifold*. The talk is only about *surfaces*, i.e. 2-manifolds. Recall the notion of *orientable* surface.

Give examples of orientable and non-orientable compact connected surfaces, including:

- the 2-sphere;
- the torus;
- the real projective plane.

Then define the *connected sum* of two disjoint surfaces. Explain what a connected sum of tori looks like.

### 4.2. **Triangulations and classification.** [Mas, § I.5-I.7]

State the classification theorem for compact surfaces (Theorem 5.1).

Describe first the canonical form for a connected sum of tori or projective planes. Then introduce triangulations of compact real surfaces and discuss the examples of the torus and the projective plane.

Finally, sketch the main steps of the proof of the classification theorem. You do not have to include detailed arguments, but rather explain the general ideas and constructions stressing the role of triangulations.

It might be useful to illustrate everything with pictures, as done in the book.

### 4.3. **Euler characteristic and genus.** [Mas, §§ I.8]

Introduce the *Euler characteristic* of a compact real surface and show that it is well defined.

Discuss how the Euler characteristic changes under connected sums (Proposition 8.1). Prove that a compact real surface is completely described by orientability and Euler characteristic (Theorem 8.2).

Finally, define the *genus* of a compact real surface. In the orientable case, explain how the genus changes under connected sums and interpret it geometrically.

### 4.4. **Non-compact real surfaces.** [Mas, §§ I.9-I.11]

Introduce briefly *manifolds with boundary.* Discuss how to extend the previous constructions and results to surfaces with boundary (see Theorems 10.1 and 11.1).

## 5. The topological fundamental group

Reference: [Mas, § II,IV]

**Aim.** The aim of this talk is to introduce an important algebraic invariant of topological spaces called the *fundamental group*, and compute this invariant in the case of orientable real surfaces. The definition of the fundamental group in terms of homotopy classes of closed paths is not directly amenable to computations; in practice, one has to use the Seifert–Van Kampen Theorem. This allows to find a canonical presentation of the fundamental group of compact real surfaces (possibly with some points removed) in terms of the canonical form described in talk 4.

### 5.1. **Construction of the fundamental group.** [Mas, §§ II.1-II.5]

Recall the notion of *homotopy of paths* in a topological space. Introduce the *concatenation* of paths and show that it is associative up to homotopy.

Then define the *fundamental group* of a topological space with respect to a chosen base-point. Show that, if the space is path-connected, the definition does not depend on the chosen base-point up to isomorphism.

Discuss the functoriality of the fundamental group with repsect to continuous maps preserving the base-points.

Discuss the following basic examples:

- contractible spaces;
- the circle.

### 5.2. **The Seifert–Van Kampen Theorem.** [Mas, §§ IV.1-IV.4]

State the Seifert–Van Kampen Theorem (Theorem 2.1) and explain its concrete meaning. Discuss the first applications of this result. If time permits, explain the main ideas behind the proof.

### 5.3. **The fundamental group of real surfaces.** [Mas, § IV.5]

Combining the Seifert–Van Kampen Theorem with the canonical form of compact real surfaces, compute the fundamental group of an orientable compact real surface of genus $g$. Extend the computation to the case of a compact surface with $n \geq 0$ points removed.

## 6. Covering spaces

Reference: [Sza, § 2].

**Aim.** The aim of this talk is to introduce the topological notion of *covering spaces* and show that it gives rise to the same formalism as in the Galois theory of fields. As we will explain in the following talks, this is by no means coincidence.

### 6.1. **Generalities on covering spaces.** [Sza, § 2.1]

Introduce the general notion of *covering space*; explain the local triviality condition and the structure of the fibres.

An important non-trivial situation is given by the even action of a group on a space (Lemma 2.1.7). This allows to give many examples (Examples 2.1.8).

### 6.2. **Galois covers.** [Sza, § 2.2]

[From now on, assume all spaces are locally connected - this assumption is always verified in the cases we are interested in.]

Discuss the rigidity properties of covering spaces with respect to continuous maps (Proposition 2.2.2) and automorphisms (Proposition 2.2.3).

Determine the automorphisms of a covering space arising from an even group action (Proposition 2.2.4). This leads to the general notion of Galois covers.

### 6.3. **Galois theory of covering spaces.** [Sza, § 2.2]

Prove the main theorem of Galois covers (Theorem 2.2.10) and point out the formal analogy with the Galois theory of fields.

## 7. Fundamental groups and covering spaces

Reference: [Sza, § 2]

**Aim.** The aim of this talk is to establish a connection between the theory of the fundamental group and the theory of covering spaces. This result provides a description of the fundamental group as the automorphism group of a fibre functor, therefore strengthening the analogy between fundamental groups and Galois groups.

### 7.1. **The monodromy action.** [Sza, § 2.3]

Define the *monodromy action* of the fundamental group of a topological space on the fibres of a covering space (Construction 2.3.3).

State the main theorem on the equivalence between covering spaces of $X$ and $\pi_1(X, x)$-sets (Theorem 2.3.4). Explain how to prove it assuming the representability of the corresponding fibre functor. Point out the analogy with Grothendieck's formulation of Galois theory.

### 7.2. **The universal cover.** [Sza, § 2.4]

Construct the *universal cover* of a topological space (Construction 2.4.1). Show that, if the space if path-connected, the result is independent of the choice of a base-point up to isomorphism.

Then discuss the two intermediate results used in the proof of the main theorem (Theorems 2.3.5 and 2.3.7).

### 7.3. **The case of real surfaces.** [Mas, § V]

To give a concrete example, construct the universal cover of some (connected) compact real surfaces, possibly with some points removed following the procedure of Construcion 2.3.3. Include the following examples:

- the 2-sphere;
- the torus;
- the complex plane with $n$ points removed.

### 7.4. **The profinite completion of the fundamental group.** [Sza, § 2.3]

Discuss the variant of the main theorem for finite covers of $X$ (Corollary 2.3.9). This yields a profinite version of the topological fundamental group that will be interesting to consider in the following talks.

# PART 3: RIEMANN SURFACES

## 8. Riemann surfaces

Reference: [Mir].

**Aim.** This is a survey talk on the general theory of *Riemann surfaces.* It will provide the necessary background material for the next two talks.

### 8.1. **Basic definitions.** [Mir, § I]

Introduce the notion of *complex chart* on a topological space and compatible charts. Introduce transition functions and show that they have non-vanishing derivative (Lemma 1.7). Discuss some examples.

Introduce the notion of *complex atlas* as well as the equivalence relation on complex atlases and the notion of *complex structure.*

Introduce the notion of *Riemann surface.* Remark that, from a topological viewpoint, every Riemann surface is an orientable real surface. In particular, every compact Riemann surface has a well-defined genus $g \geq 0$.

### 8.2. **Basic examples.** [Mir, § I]

Discuss the following examples of non-compact Riemann surfaces:

- open subsect of the complex plane;
- graphs of holomorphic functions;
- smooth affine algebraic curves over $\mathbb{C}$.

Discuss then the following examples of compact Riemann surfaces:

- the Riemann sphere, alias the projective line over $\mathbb{C}$;
- complex tori;
- smooth projective plane curves.

### 8.3. **Holomorphic and meromorphic functions.** [Mir, § II]

Introduce the notion of *holomorphic function* on a Riemann surface and give some examples.

Introduce the classification of singularities of a function at a point. Introduce the notion of *meromorphic function* and give some examples.

Define the order of a meromorphic function at a point and discuss its main properties. Discuss the case of the Riemann sphere in detail.

List the results on holomorphic and meromorphic functions inherited from the theory of holomorphic functions of one variable (Theorem 1.33, Corollary 1.34, Theorems 1.35, 1.36, 1.37).

### 8.4. **Morphisms of Riemann surfaces.** [Mir, § II]

Recall the basic facts about the topology of Riemann surfaces already discussed in talk in the setting of real surfaces: in particular the results of Theorem 3.6.1, Remark 3.6.2, Theorem 3.6.3.

## 9. Complex Elliptic Curves

References: [Kob], [Mir], [Silv].

**Aim.** The aim of this talk is to study in more detail an important class of Riemann surfaces, namely *complex tori*, and realize them as particular algebraic curves known as *elliptic curves*. We can classify complex tori up to isomorphism as Riemann surfaces. On the side of elliptic curves, this classification is related to an algebraic invariant called the $j$-invariant. This invariant has extremely important arithmetic applications connected to Belyi's Theorem.

### 9.1. **Complex tori as elliptic curves.** [Silv, § VI.1-VI.2]

Introduce the general *Weierstrass equation* of an elliptic curve over $\mathbb{C}$ and express it in Legendre form. Define the *holomorphic differential* $\omega$ and the associated elliptic integral. Explain how this gives rise to a complex torus.

Conversely, start from a complex torus. Introduce *elliptic functions* and the fundamental parallelogram associated to its lattice. Prove the basic properties of elliptic functions (Theorem 2.2).

Define the *Weierstrass $\mathcal{P}$-function* and show that it is an even elliptic function (Theorem 3.1 (b) and (c)). Express the ring of elliptic functions in terms of the Weierstrass function and its derivative (Theorem 3.2) and state the algebraic relation between them (Theorem 3.5(b)).

In the end, we can realised every complex torus as a complex elliptic curve (Proposition 3.6(b)).

### 9.2. **Morphisms of complex tori.** [Mir, § III]

Discuss the structure of holomorphic maps between complex tori (Propositions 1.11, 1.12, 1.13).

### 9.3. **The space of complex elliptic curves.** [Kob, § III.1]

Introduce the action of the group $\mathrm{SL}_2(\mathbb{R})$ on the Riemann sphere and its restriction to the *Poincaré upper half-plane*. Explain how this is related to lattices and maps of complex tori.

Explain the structure of a *fundamental domain* for this action (Proposition 1). If time permits, deduce some consequences on the structure of the group $\mathrm{SL}_2(\mathbb{Z})$ (Propositions 2,3,4).

### 9.4. **The $j$-invariant.** [Silv, § III.1]

Define the *$j$-invariant* associated to a Weierstrass equation. This quantity classifies complex elliptic curves up to isomorphism (Proposition 1.4(b)).

Moreover, the $j$-invariant plays an important arithmetic role: given a complex elliptic curve $E$ defined by a Weiestrass equation with coeffieints in $\bar{\mathbb{Q}}$, the $j$-invariant $j_E$ is algebraic and the minimal field of definition of $E$ is precisely $\mathbb{Q}(j_E)$ (Proposition 1.4(d)).

## 10. Local structure of holomorphic maps

Reference: [Mir, § II.4].

**Aim.** This talk focuses on the local structure of holomorphic maps between Riemann surfaces. The main result is that every such map is a ramified covering, so it restricts to a topological covering map outside a finite set. The ramification datum associated to a map of Riemann surfaces provides important global information about the two surfaces involved: this is summarized by the Riemann–Hurwitz formula relating their genera.

### 10.1. **Local normal form and ramification.** [Mir, § II.4]

Prove the *local normal form* of a non-constant holomorphic map of Riemann surfaces (Proposition 4.1). Define the notion of *multiplicity* at a point and use it to define *ramification points* and *branch points*.

As a first example, explain how to compute multiplicities in the case of a non-constant meromorphic function, seen as a map to the Riemann sphere (Lemma 4.7).

Define the degree of a map of Riemann surfaces and show that it is well-defined (Proposition 4.8). Prove that degree behave multiplicatively under composition of maps, and in particular isomorphisms have degree 1. Deduce a criterion for a Riemann surface to be isomorphic to the Riemann sphere (Proposition 4.11).

### 10.2. **The Riemann–Hurwitz formula.** [Mir, § II.4]

Using the technique of triangulations introduced in 4, prove the Riemann–Hurwitz formula on the variation of genera along holomorphic maps (Theorem 4.16).

### 10.3. **Examples.** [Mir, § II.4]

In this seminar, we are especially interested in unramified coverings of Riemann surfaces. As an application of the Riemann–Hurwitz formula, discuss the following results:

- the only unramified covering of the Riemann sphere is the Riemann sphere itself;
- the only unramified coverings of a complex torus come from complex tori.

## 11. Coverings of Riemann surfaces

References: [Sza, § 3].

**Aim.** In this talk we apply the local analysis of holomorphic maps in order to study covering spaces of Riemann surfaces. The first main result is that every topological covering space of a Riemann surface is itself a Riemann surface; even better, the categories of topological and holomorphic covers of a Riemann surfaces are equivalent. The second main result is that the topological fundamental group of a Riemann surface is related to the absolute Galois group of the corresponding meromorphic function field. This allows to solve some inverse Galois problems with topological methods.

### 11.1. **Holomorphic and topological covers.** [Sza, § 3.2]

Deduce from the results of the previous talk that proper holomorphic maps of Riemann surfaces yield topological covers outside the branch points (Proposition 3.2.6).

Then prove the equivalence between holomorphic covers and finite topological covers of a Riemann surface (Theorem 3.2.7).

The main theorem allows to introduce the notion of Galois holomorphic cover of a Riemann surface. Prove the properties of the automorphism groups of such covers (Proposition 3.2.10).

### 11.2. **Galois theory of meromorphic functions.** [Sza, § 3.3-3.4]

Discuss the relation between non-constant holomorphic maps of Riemann surfaces and extensions of meromorphic function fields (Proposition 3.3.5). Show that this yields an equivalence of categories, with Galois covers corresponding to Galois extensions (Theorem 3.3.7).

Deduce a precise description of Riemann surfaces in terms of Galois theory (Corollaries 3.3.10 and 3.3.12).

Finally, discuss the relation between Galois groups of meromorphic functions and fundamental groups of Riemann surfaces (Theorem 3.4.1)

State Theorem 3.4.1 and prove it in detail, including Lemma 3.4.2. Discuss Remark 3.4.3.

11.3. **The absolute Galois group of $\mathbb{C}(t)$.** [Sza, § 3.4]

Prove Douady's Theorem on the absolute Galois group of $\mathbb{C}(t)$ (Theorem 3.4.8). Deduce the answer to the inverse Galois problem for $\mathbb{C}(t)$ (Corollary 3.4.4).

# PART 4: THE ETALE FUNDAMENTAL GROUP OF CURVES

## 12. Dedekind domains and Algebraic curves

Reference: [Sza, § 4].

**Aim.** This is a survey talk on the language of *Dedekind domains* and *valuation rings*. Dedekind domains correspond to *smooth affine curves*; these can be glued together to form *smooth proper curves*. These notions and results will be exploited to define étale coverings of algebraic curves and the étale fundamental group in the next talk.

12.1. **Dedekind domains and prime factorisation.** [Sza, § 4.1]

Introduce the notion of *integral ring extension* and *integral closure*, and state their main properties (Facts 4.1.1 and 4.1.4). Introduce *decomposition groups* and *inertia subgroups* associated to a Galois extension and state their main properties (Facts 4.1.3).

Introduce *Dedekind domains* and give some examples, including principal ideal domains. Explain why the integral closure of a Dedekind domain in a finite separable field extension is again a Dedekind domain.

State the fundamental property of prime decomposition in Dedekind domains (Facts 4.1.5). Explain how prime decompose along an extension of Dedekind domains (Proposition 4.1.6) and define the ramification indices. Discuss the case of Galois extensions (Corollary 4.1.7).

12.2. **Affine curves.** [Sza, § 4.3]

Introduce the abstract notion of *ringed space*. Explain how to construct the ringed space associated to a ring (Construction 4.3.2).

Define morphisms of ringed spaces and explain their relation with ring homomorphisms (Proposition 4.3.6).

Introduce the notion of *base-change* of curves and define *geometrically integral* curves (Construction 4.3.7).

Discuss the correspondence between maps of normal affine curves and extensions of rational function fields (Theorem 4.3.10).

12.3. **Proper normal curves.** [Sza, § 4.4]

Explain the characterisation of local rings of an affine normal curve (Lemma 4.4.1).

Discuss the example of the affine line (Example 4.4.2); using this as a motivation, construct the *proper normal curve* attached to a field (Construction 4.4.3).

Extend the results of the previous section (Propositions 4.4.5 and 4.4.6)

Introduce the notion of *finiteness* and explain its role in the theory of proper normal curves (Proposition 4.4.7 and Corollary 4.4.8).

## 13. The fundamental group of an algebraic curve

Reference: [Sza, § 4].

**Aim.** The aim of this talk is to introduce the *étale fundamental group* of an algebraic curve. This notion generalizes on the one hand the absolute Galois group of a field, on the other hand the (profinite completion of the) topological fundamental group of a topological space.

### 13.1. **Finite branched covers of normal curves.** [Sza, § 4.5]

Introduce the notion of *separable morphism* and *étale morphism* of integral affine curves, and characterise them (Lemma 4.5.2). Discuss the example of the $n$-th power map (Example 4.5.4 and Remark 4.5.5).

Prove that every finite separable morphism of curves is étale outside a finite number of points (Proposition 4.5.9).

This motivates the notion of *finite branched cover* and *Galois cover* of curves. Characterise étaleness over a point in terms of inertia subgroups (Proposition 4.5.11 and Corollary 4.5.12).

### 13.2. **The étale fundamental group of algebraic curves.** [Sza, § 4.6]

Establish the fundamental relation between étale covers of an algebraic curve and certain Galois extensions of the corresponding function field (Proposition 4.6.1). Using this result as a motivation, define the algebraic fundamental group of an algebraic curve.

Prove the description of étale morphisms to a curve in terms of its étale fundamental group (Theorem 4.6.4 and Corollary 4.6.5). Stress the analogy with previous similar results obtained in the setting of Galois groups of fields and fundamental groups of topological spaces.

Point out that there is no such notion as universal cover in the world of algebraic curves. However, one can partially remedy this issue using the formalism of propoints and pro-étale covers (Remark 4.6.6).

### 13.3. **Structure theorem for complex algebraic curves.** [Sza, § 4.6-4.7]

In the case of a curve over $\mathbb{C}$, discuss the equivalence between algebraic covers, holomorphic covers, and extensions of the function field. Using this, we can give a presentation of the étale fundamental group of such curves in terms of the underlying topological space (Theorem 4.6.7).

## 14. Arithmetic and geometric aspects of the fundamental group

Reference: [Sza, § 4].

**Aim.** The aim of this talk is to explore some key properties of the étale fundamental group of an algebraic curve. It is important to understand what happens changing the field of definition of the curve: this leads to important connections with number theory in the case of curves defined over $\bar{\mathbb{Q}}$.

### 14.1. **Invariance under extensions of algebraically closed fields.** [Sza, § 4.6]

Explain the base-change construction for separable and étale covers of curves (Construction 4.6.9). State the invariance property of the étale fundamental group under extensions of algebraically closed fields (Theorem 4.6.10).

Combining this with the information already available for complex curves, it is possible to deduce a presentation of the étale fundamental group of general curves in characteristic 0 (Corollary 4.6.11).

A crucial example is that of the projective line with $n$ points removed (Example 4.6.12).

### 14.2. **The arithmetic and the geometric fundamental group.** [Sza, § 4.7]

Introduce the *arithmetic and geometric fundamental groups* of a geometrically integral curve $U/k$. Establish the fundamental short exact sequence relating them with the absolute Galois group of $k$ (Proposition 4.7.1).

Define the outer action of $\mathrm{Gal}(\bar{k}/k)$ on $\pi_1(U_{\bar{k}})$. Define the inertia groups of propoints, and describe them as stabilizers (Lemma 4.7.2 and Corollary 4.7.3).

### 14.3. **Belyi's Theorem.** [Sza, § 4.7]

Belyi's Theorem (Theorem 4.7.6) is a truly surprising result connecting the arithmetic properties of algebraic curves with the existence of morphism to the projective line with three ramification points. Prove it carefully and introduce the notion of *Belyi functions*.

### 14.4. **Faithfulness of the outer Galois action.** [Sza, § 4.7]

As an important application of Belyi's Theorem, prove that the outer action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on the fundamental group of the projective line minus three points is faithful (Theorem 4.7.7). This uses the facts about elliptic curves and the $j$-invariant discussed in talk 9 (see Facts 4.7.8).

## 15. The inverse Galois problem for $\mathbb{Q}(t)$

References: [Ser, § 7], [Sza, § 4].

**Aim.** In the course of talk 11 we solved the inverse Galois problem for $\mathbb{C}(t)$. The inverse Galois problem for $\mathbb{Q}$ is much more difficult. The aim of this talk is to consider an intermediate case, namely the field $\mathbb{Q}(t)$. The machinery developed in the previous talks allows to construct interesting Galois extensions of $\mathbb{Q}(t)$ starting from certain covers of the projective line: the main purpose of this talk is to compute some explicit examples. One can pass from $\mathbb{Q}(t)$ to $\mathbb{Q}$ using Hilbert's Irreducibility Theorem.

15.1. **Regular extensions of $\mathbb{Q}(t)$ and $\mathbb{Q}$-covers of $\mathbb{P}^1$.** [Sza, § 4.8]

Introduce the notion of *regular Galois extension* of $\mathbb{Q}(t)$, and state the modified version of the inverse Galois problem (Problem 4.8.1). Explain how to exploit an answer to this problem in the direction of the inverse Galois problem for $\mathbb{Q}$ (Fact 4.8.2).

15.2. **The criterion of rigid conjugacy classes.** [Sza, § 4.8]

Discuss a first technical result about extension of continuous homomorphisms (Lemma 4.8.3). Using this as a motivation, introduce the notion of *rigid tuples of conjugacy classes* and of *rational conjugacy* class.

Prove the main theorem on regular Galois extensions (Theorem 4.8.7). Discuss carefully the role of rationality (Remark 4.8.8).

15.3. **Examples.** [Sza, § 4.8]

Discuss some examples of application of the main theorem. Besides the case of $\mathrm{PSL}_2(\mathbb{F}_p)$ (Example 4.8.10) there are other interesting cases in [Ser], such as the symmetric group $S_n$ ([Ser, § 7.4.1]) and the alternating group $A_n$ ([Ser, § 7.4.2]).

Notice that in the example of $\mathrm{PSL}_2(\mathbb{F}_p)$ one cannot apply the criterion of Theorem 4.8.7 directly but has to use the variant of Remark 4.8.8.

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