Foundations of the theory of (φ, Γ)-modules over the Robba ring

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Contents

Introduction

The goal of this thesis is to work out the definition and some properties of the Robba ring $\mathcal{R}$. We want to give some insight into the cohomology of certain chain-complexes of (φ, Γ)-modules and characterize the one dimensional (φ, Γ)-modules over $\mathcal{R}$.

Let L be a finite extension of $\mathbb{Q}_p$, the field of p-adic numbers. One of the important problems in number theory is to understand the absolute Galois group $G_L = \mathrm{Gal}(\overline{L}/L)$. Instead of looking at the group G_L itself we can look at its representations. In his paper [Fon90] Fontaine defined an equivalence of categories between the category of such representations of G_L and the category of étale (φ, Γ_L)-modules for $L = \mathbb{Q}_p$. Later a sketch of how to extend this theory to any finite extension L of $\mathbb{Q}_p$ was given by Kisin and Ren in [KR09].

A (φ, Γ_L)-module is a free module over a ring R that is equipped with a semilinear operation φ such that $\varphi(M) = M$ and a semilinear action of Γ_L that commutes with φ. In this thesis we will work with (φ, Γ)-modules over the Robba ring $\mathcal{R}_L$, the ring of all Laurent series over a finite extension L of $\mathbb{Q}_p$, that converge on some annulus $0 < v_p(T) \leq r$ for $r \in \mathbb{R}_{>0}$. With this equivalence we can find out something about the Galois group G_L by looking at the (φ, Γ)-modules, which gives us a new perspective on the category of representations of G_L and hence on G_L itself.

The Robba ring is an important tool in the theory of p-adic differential equations and p-adic Galois representations. One aim of this thesis will be to understand this ring and some of its properties. The main question of this thesis was to characterize the rank 1 (φ, Γ)-modules over $\mathcal{R}_L$. With the correspondence mentioned above one could then also characterize the 1 dimensional Galois representations.

Unfortunately in answering the main question, a faulty line of argumentation was used which was discovered too late. This leaves a hole in the last part of the last chapter of the thesis where we attempt to characterize the rank (φ, Γ)-modules over $\mathcal{R}$. There are other lines of arguments by Colmez which can be used to answer the central question. However this thesis is still a valuable reference for understanding underlying theory of (φ, Γ)-modules.

We work mostly with different papers of Colmez, but also with papers of Berger, Kedlaya and Lazard. Moreover we refer to a bit number of books from various authors.

Summary of chapters

The first chapter will give all the necessary background knowledge about p-adic analysis. This chapter is intended to give the reader a summary of the results that are used in the rest of the thesis and not as a complete introduction to p-adic analysis. Most results will not be proven here, but the proofs can be found in the cited literature.

The second chapter gives an introduction to φ-modules. These are modules over a commutative unitary ring A equipped with a semi-linear map φ. We will show that the category of φ-modules is abelian and define the tensor product of two φ-modules. Moreover we define étale φ-modules and show that they form an abelian category. At last we give a brief definition of (φ, Γ)-modules over a ring A, originally given by Fontaine. Later we will use a slightly different notion of (φ, Γ)-modules.

In the third chapter we define the Robba ring. We first discuss what it means for a p-adic Laurent series to converge on some annulus and define a valuation on the vector space of Laurent series. We define the Robba ring $\mathcal{R}$ and give a Fréchet topology on $\mathcal{R}$ that is given by a metric induced by the valuations mentioned before, such that $\mathcal{R}$ is complete with respect to this metric.

In the fourth chapter we work out some properties of the Robba ring. We define a Frobenius operator φ and an action of some group Γ on $\mathcal{R}$, that give the (φ, Γ)-modules over $\mathcal{R}$ their name. We give a decomposition of the elements of $x \in \mathcal{R}$ with bounded coefficients, introduce the notion of the differential operator ∂ and the residue Res on $\mathcal{R}$ and show that for certain elements the logarithm function is well defined.

In the last chapter we define the cohomology over a chain complex of (φ, Γ)-modules and show that the first cohomology group $H^1(M)$ is isomorphic to the group of extensions of M by $\mathcal{R}$. We want to use the properties of chapter 4 and some cohomology to classify the 1 dimensional (φ, Γ)-modules over the Robba ring. The result should be that any one dimensional (φ, Γ)-module over $\mathcal{R}$ is isomorphic to a module $\mathcal{R}(\delta)$, where $\delta\colon \mathbb{Q}_p^* \to L^*$ is a continuous character that twists the Frobenius operator and the action of Γ on $\mathcal{R}$. With help of the 0-th cohomology group we show that two different characters give rise to different (φ, Γ)-modules. In the last section I wanted to prove that all rank 1 (φ, Γ)-module are isomorphic to $\mathcal{R}(\delta)$ for some continuous character δ. The proof is based on an unpublished paper by Colmez, available on his personal homepage at the time of writing. Too late I found out that the proof has a mistake in it. The statement is true, but it seems like another technique is necessary to prove this theorem. I left the proof inside this thesis, but marked the steps where the mistake is happening.

Further research

The foundations laid out in this thesis could be useful for answering further questions. They can serve as a starting point for working out 2 dimensional Galois representations, corresponding to rank 2 (φ, Γ)-modules. The tensor product of $\mathbb{Q}_p$ with the Tate module $T_p(E)$ of an elliptic curve E over L is such a 2-dimensional $\mathbb{Q}_p$-representation of G_L. It would be an interesting example to calculate the rank 2 (φ, Γ)-module corresponding to this representation.

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Chapter 1

Background on p-adic analysis

In this chapter we want to give a brief introduction to the field of p-adic numbers $\mathbb{Q}_p$ and some p-adic analysis. In the following chapters we will work with functions over some extension of $\mathbb{Q}_p$. This chapter can be used as a reference in case of ambiguities in the later chapters. Most theorems and propositions will remain unproven. Proofs can be found in the cited literature. For some of the propositions I will give proofs, mostly because I could not find them in the literature.

1.1 Norms and valuations

We fix a field k.

Definition 1.1 (Definition on page 31 of [Lan94])**.** An *absolute value* on k is a real valued function $|-|: x \mapsto |x|$ on k such that for all $x, y \in k$:

(i) $|x| \geq 0$ with equality if and only if $x = 0$

(ii) $|xy| = |x| \cdot |y|$

(iii) $|x + y| \leq |x| + |y|$.

An absolute value $|-|$ is called *non-Archimedean* if the ultrametric triangle inequality

(iv) $|x + y| \leq max(|x|, |y|)$

holds for all $x, y \in k$.

Definition 1.2 (Definition on page 3 of [Sch02])**.** The field k is called *non-Archimedean* if we can define a non-Archimedean absolute value on it such that k is complete with respect to this absolute value.

From now on let k be equipped with an absolute value $|-|$.

Definition 1.3 (Definition on page 6 and 9 of [Sch02])**.** Let V be a k-vector space. A *seminorm* on V is a function $q\colon V \to \mathbb{R}$ such that for all $x, y \in V$, $a \in k$:

(i) $q(ax) = |a| \cdot q(x)$,

(ii) $q(x + y) \leq q(x) + q(y)$,

(iii) $q(x) \geq 0$.

A seminorm on V is called a *norm* if additionally

(iv) $q(x) = 0 \iff x = 0$.

If k is non-Archimedean, then the seminorm (resp. norm) is called *non-Archimedean* if

(v) $q(x + y) \leq \max(q(x), q(y))$.

Definition 1.4. A *valuation of* k is a map $v\colon k \to \mathbb{R} \cup \infty$ such that for any $a, b \in k$:

(i) $v(a) = \infty \iff a = 0$,

(ii) $v(ab) = v(a) + v(b)$,

(iii) $v(a + b) \geq \min(v(a), v(b))$.

A field k with a valuation defined on it is called *valued field.*

Remark. If $v(a) > v(b)$, then

$$v(a + b) \geq \min(v(a), v(b)) = v(b) = v(b + a - a) \geq \min(v(b + a), v(a))$$

hence $v(a+b) = v(b)$. So if $v(a) \neq v(b)$ in (iii) we can write "$=$" instead of "$\geq$".

Proposition 1.5. *From any valuation* $v\colon k \to \mathbb{R} \cup \infty$ *we can construct a non-Archimedean absolute value on* k *by setting* $|a| = q^{-v(a)}$ *for a fixed* $q \in \mathbb{R}_{>1}$.

Proof. The axioms of a non-Archimedean absolute value follow quickly from the axioms of a valuation. $\square$

Definition 1.6. Let k be a valued field with valuation v and let V be a k-vector space. We will call a function $w\colon V \to \mathbb{R} \cup \infty$ *valuation of* V if for any $x, y \in V$, $a \in k$:

(i) $w(x) = \infty \iff x = 0$,

(ii) $w(ax) = v(a) + w(x)$,

(iii) $w(x + y) \geq \min(w(x), w(y))$.

Remark. A valuation of a vector space V induces a non-Archimedean norm, just as a valuation of a field induced a non-Archimedean absolute value in Proposition 1.5.

Proposition 1.7 (Statement on page 12 of [Sch02])**.** *In a vector space with non-Archimedean norm* $|-|$*, which is complete with respect to this norm, a series* $\sum_{i\in\mathbb{N}} a_i$ *converges if and only if the coefficients* $(a_i)_{i\in\mathbb{N}}$ *form a null-sequence (i.e.* $a_i \to 0$*).*

Proof. If a_i is a null-sequence, then for any $m > n$:

$$|\sum_{i=0}^{m} a_i - \sum_{i=0}^{n} a_i| = |\sum_{i=n}^{m} a_i| \leq \max_{n \leq i \leq m} |a_i|$$

hence for $n, m \to \infty$ we have $|\sum_{i=0}^{m} a_i - \sum_{i=0}^{n} a_i| \to 0$ and $\sum_{i=0}^{m} a_i$ is a Cauchy sequence and therefore convergent. The other direction is clear. $\square$

Proposition 1.8 (See chapter 4 of [Neu92])**.** *Let* $(k, |-|)$ *be a valued field. Let*

$$\mathcal{C} \coloneqq \{(x_n)_n \colon (x_n)_n \textit{ is a Cauchy sequence in } k \textit{ with respect to } |-|\}$$

be equipped with component wise addition and multiplication. This makes $\mathcal{C}$ *a commutative ring with unity. Furthermore the null sequences*

$$\mathcal{N} \coloneqq \{(x_n)_n \in \mathcal{C} \colon \lim_{n\to\infty} |x_n| = 0\}$$

form a maximal ideal of $\mathcal{C}$*. The* completion $(\hat{k}, |-|)$ of k *is defined by the quotient* $\hat{k} = \mathcal{C}/\mathcal{N}$ *and the extension of the absolute value by setting* $|a| = \lim_{n\to\infty} |a_n|$.

1.2 Fréchet spaces

Let k be a with absolute value $|-|$.

Definition 1.9 (Definition IV 1.1 of [Con85])**.** A *topological k-vector space* is a k-vector space V together with a topology such that the addition $+\colon V \times V \to V$ and the scalar multiplication $\cdot\colon k \times V \to V$ are continuous maps, where $V \times V$ and $k \times V$ are equipped with the product topology.

Proposition 1.10. *Any normed vector space is a topological vector space.*

Proof. Let V be a vector space over k with norm $\| - \|$. If in $V \times V$ the sequence (x_n, y_n) converges to (x, y), then with the triangle inequality we get

$$\|x_n + y_n - (x + y)\| \leq \|x_n - x\| + \|y_n - y\| \to 0 + 0 = 0$$

for $n \to \infty$. Similary for $(\alpha_n, x_n) \to (\alpha, x)$ in $k \times V$, with the triangle inequality and the second norm axiom we get

$$\begin{aligned}\|\alpha_n x_n - \alpha x\| &= \|\alpha_n x_n - \alpha_n x + \alpha_n x - \alpha x\| \\ &\leq \|\alpha_n x_n - \alpha_n x\| + \|\alpha_n x - \alpha x\| \\ &= |\alpha_n| \|x_n - x\| + |\alpha_n - \alpha| \|x\| \to 0 + 0 = 0\end{aligned}$$

for $n \to \infty$. $\square$

Proposition 1.11 ([Con85] after Definition IV 1.1)**.** *Let V be a vector space and $\mathcal{P}$ a family of seminorms on V. We can define a topology $\mathcal{T}$ by taking as subbase the sets $\{x \in V : p(x - x_0) < \epsilon\}$ for $p \in \mathcal{P}$, $x_0 \in V$ and $\epsilon > 0$. This means $\mathcal{T}$ is the smallest topology containing all sets of this form.*

Thus as [Con85] states a subset U of V is open if and only if for every x_0 in U there are $p_1, ..., p_n \in \mathcal{P}$ and $\epsilon_1, ..., \epsilon_n > 0$ such that $\bigcap\{x \in V : p_j(x - x_0) < \epsilon_j\} \subseteq U$.

Definition 1.12 (Definition IV 1.2 of [Con85])**.** A *locally convex space* is a topological vector space, whose topology is defined by a family of seminorms $\mathcal{P}$ such that $\bigcap_{p \in \mathcal{P}}\{x : p(x) = 0\} = \{0\}$.

Proposition 1.13 (Proposition IV 2.1 of [Con85])**.** *A locally convex space is metrizable if and only if its topology is given by a countable family of seminorms $\{p_1, p_2, ...\}$. For $x, y \in V$ the metric is then given by*

$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n} \frac{p_n(x - y)}{1 + p_n(x - y)}.$$

Proposition 1.14. *Let V be a locally convex space, whose topology $\mathcal{T}$ defined by a countable family of seminorms $\mathcal{P}$. Then a sequence converges with respect to $\mathcal{T}$ this if and only if it converges with respect to each of the seminorms.*

Proof. We denote $P \subset_{fin} \mathcal{P}$ for a finite subset of $\mathcal{P}$. Every point $y \in V$ has a basis of neighborhoods consisting of open balls around y, that is $\mathcal{B}_y = \{\{x : p(x-y) < \epsilon \quad \forall p \in P\} : \forall P \subset_{fin} \mathcal{P}, \epsilon > 0\}$. With this basis of neighborhoods, a sequence $x_n \in V$ converges to $y \in V$ if for any $\epsilon > 0$ and finite subset of seminorms $P \subset_{fin} \mathcal{P}$, there is a $N \in \mathbb{N}$ such that for all $n > N$ and all seminorms $p \in P$ we have $p(x_n - y) < \epsilon$.

"$\Rightarrow$" Suppose that $x_n \to y$ for $\mathcal{T}$. For any $p \in \mathcal{P}$ take $P = \{p\} \subset_{fin} \mathcal{P}$, then we find for any $\epsilon > 0$ a $N \in \mathbb{N}$ such that $p(x_n - y) < \epsilon$.

"$\Leftarrow$" Now suppose that $x_n \to y$ with respect to each of the seminorms in $\mathcal{P}$. Let $\epsilon > 0$ and $P \subset_{fin} \mathcal{P}$ any finite subset of seminorms. For each $p \in P$ we can find N_p such that $p(x_n - y) < \epsilon$ for any $n > N_p$. Take $N = \max_{p \in P}\{N_p\}$ then for all $p \in P$ we get $n > N \geq N_p$ and so $p(x_n - y) < \epsilon$. $\square$

Definition 1.15 ([Con85] after Proposition IV 2.1)**.** A metric d on V is called *translation invariant*, if for all $x, y, z \in V$

$$d(x+z, y+z) = d(x, y).$$

Proposition 1.16. *The metric defined in Proposition 1.13 is translation invariant.*

Proof. Immediate by definition. $\square$

Definition 1.17 (Definition IV 2.4 in [Con85])**.** A *Fréchet space* is a topological vector space V, whose topology is defined by a translation invariant metric d such that (V, d) is complete.

Proposition 1.18. *If V is a Fréchet space defined by a countable set of seminorms $\mathcal{P}$, then a function $f : V \to W$ into another topological space W is continuous if and only if it is sequentially continuous with respect to every seminorm $p \in \mathcal{P}$.*

Proof. A Fréchet space V is a metric space and hence first countable. Therefore the function $f : V \to W$ is continuous if and only if it is sequentially continuous. By Proposition 1.14 a sequence x_n converges towards x if and only if x_n converges towards x for all of the seminorms. Therefore a function $f : V \to W$ is sequentially continuous if and only if it is sequentially continuous with respect to every seminorm $p \in \mathcal{P}$. $\square$

1.3 Definition of the p-adic numbers

We fix a prime number p.

Definition 1.19 ([Kob84] Section 1.2, page 2)**.** Define the *p-adic valuation* of $a \in \mathbb{Z}$ as follows:

$$v_p(a) = \begin{cases} \text{the greatest integer } m \text{ such that } p^m | a \text{ for } a \neq 0 \\ \infty \text{ for a=0} \end{cases}.$$

For a rational number $x = \frac{a}{b}$ with $a, b \in \mathbb{Z}$ coprime define the p-adic valuation by

$$v_p(x) = v_p(a) - v_p(b).$$

This defines a valuation of $\mathbb{Q}$. Furthermore define a map $|-|_p$ on $\mathbb{Q}$ by

$$|x|_p = \begin{cases} p^{-v_p(x)} \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}.$$

Proposition 1.20 ([Kob84] Section 1.2, page 2)**.** $|-|_p$ *defines a non-Archimedean absolute value on* $\mathbb{Q}$.

Definition 1.21 (Definition 3.2.9 of [Gou97])**.** We define the field of *p-adic numbers* $(\mathbb{Q}_p, |-|_p)$ as the completion of the valued field $(\mathbb{Q}, |-|_p)$.

Definition 1.22 (Definition 3.3.3 of [Gou97])**.** We define the ring of *p-adic integers* $\mathbb{Z}_p$ to be the valuation ring

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p \colon |x|_p \leq 1\} .$$

Proposition 1.23 (Corollaries 3.3.11 and 3.3.12 of [Gou97])**.** *Any element x of $\mathbb{Z}_p$ can be uniquely written in the form*

$$x = \sum_{i=0}^{\infty} b_i p^i \textit{ for } b_i \in \{0, 1, ..., p-1\}$$

and any element y of $\mathbb{Q}_p$ can uniquely be written in the form

$$y = \sum_{i=N}^{\infty} b_i p^i \textit{ for } b_i \in \{0, 1, ..., p-1\} \textit{ and some } N \in \mathbb{Z}.$$

Proposition 1.24 (Proposition on page 5 of [Rob00])**.** *The group $\mathbb{Z}_p^*$ of invertible elements of $\mathbb{Z}_p$ is given by the elements of absolute value zero,*

$$\mathbb{Z}_p^* = \{x \in \mathbb{Q}_p : |x|_p = 1\} = \left\{ \sum_{i=0}^{\infty} a_i p^i : a_0 \neq 0 \right\}.$$

Proposition 1.25 (Theorem 3.2.13 of [Gou97]). *$\mathbb{Q}_p$ has the following properties:*

(i) There is an embedding $\mathbb{Q} \to \mathbb{Q}_p$ and an embedding $\mathbb{Z} \to \mathbb{Z}_p$,

(ii) $\mathbb{Q}$ is dense in $\mathbb{Q}_p$,

(iii) $\mathbb{Q}_p$ is complete with respect to the absolute value $|-|_p$.

1.4 The completion of the algebraic closure of $\mathbb{Q}_p$

Let L be a finite extension of $\mathbb{Q}_p$.

Proposition 1.26. *The absolute value $|-|_p$ can be uniquely extended to an absolute value on L.*

Proof. See [Rob00] chapter 2.3.3 for uniqueness and 2.3.4 for existence. □

Remark. If we extend the norm $|-|_p$ from $\mathbb{Q}_p$ to L, then the extended norm agrees on $\mathbb{Q}_p$, in particular $|p|_p = p^{-1}$.

Proposition 1.27 (Proposition 5.3.1 of [Gou97]). *A finite extension L of $\mathbb{Q}_p$ is complete with respect to the extended norm of $|-|_p$.*

Definition 1.28. Define $\overline{\mathbb{Q}}_p$ to be an algebraic closure of $\mathbb{Q}_p$.

Proposition 1.29 (See subsection 3.1.1 of [Rob00]). *The norm $|-|_p$ on $\mathbb{Q}_p$ uniquely extends to $\overline{\mathbb{Q}}_p$.*

Proposition 1.30 (Theorem 12 of [Kob84]). *$\overline{\mathbb{Q}}_p$ is not complete.*

Definition 1.31 (See page 72 of [Kob84]). Define C_p as the completion of $\overline{\mathbb{Q}}_p$ with respect to the norm $|-|_p$.

Theorem 1.32 (Theorem 13 of [Kob84]). *C_p is algebraically closed.*

1.5 Laurent series of p-adic numbers

In the following we will give some propositions regarding power series and Laurent series over $\mathbb{Q}_p$. Any of these propositions holds as well for power series and Laurent series over any finite extension of $\mathbb{Q}_p$ with the same kind of proof.

Proposition 1.33 (Corollary 4.1.2 of [Gou97]). *An infinite series $\sum_{n=0}^{\infty} a_n$ with $a_n \in \mathbb{Q}_p$ is convergent if and only if $|a_n|_p \to 0$ as $n \to \infty$.*

Proposition 1.34 (Proposition 4.3.1 of [Gou97]). *Let $f = \sum_{n=0}^{\infty} a_n T^n$ be a power series in $\mathbb{Q}_p$. We then can find some $0 \leq \rho \leq \infty$ such that for $a \in C_p$ the series $f(a) = \sum_{n=0}^{\infty} a_n a^n$ converges if and only if $|a|_p < \rho$. We call ρ the radius of convergence of f.*

Notation. Since we mainly want to work with valuations, whenever we talk about the radius of convergence in the context of valuations we mean the to ρ corresponding $r \in \mathbb{R}$ with $\rho = p^{-r}$. So f converges for any a with $v_p(a) > r$.

Proposition 1.35 (Lemma 3.3 of [Sch11]). *Let k be a non-Archimedean field. Let $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ be convergent series in k. Then the series $\sum_{n=1}^{\infty} w_n$ with $w_n = \sum_{l+m=n}^{\infty} a_l b_m$ is convergent and*

$$\sum_{n=1}^{\infty} w_n = (\sum_{n=1}^{\infty} a_n)(\sum_{n=1}^{\infty} b_n).$$

Proposition 1.36 (See chapter 4.5 of [Gou97]). *The logarithm power series*

$$\log(1+T) = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{T^n}{n}$$

has radius of convergence $\rho = 1$ and the exponential power series

$$\exp(T) = \sum_{n=0}^{\infty} \frac{T^n}{n!}$$

has radius of convergence $\rho = p^{-1/(p-1)}$.

Proposition 1.37 (Proposition 4.5.8 of [Gou97]). *For $a \in C_p$ with $|a| < p^{-1/(p-1)}$ we have that $|\exp(a) - 1| < 1$ so that $\exp(a)$ is in the domain of $\log$ and*

$$\log(\exp(a)) = a.$$

Conversely for $a \in C_p$ with $|a| < p^{-1/(p-1)}$ we have that $|\log(1+a)| < p^{-1/(p-1)}$ so that $\log(1+a)$ is in the domain of $\exp$ and

$$\exp(\log(1+a)) = 1 + a.$$

Definition 1.38. Let $c \in C_p$ with $v_p(c) > 0$, $b = \sum_{i=0}^{\infty} a_i p^i \in \mathbb{Z}_p$ and $(b_n)_{n \in \mathbb{N}}$ with $b_n = \sum_{i=0}^{n} a_i p^i \in \mathbb{N}$. We can define $(1+c)^b$ by setting

$$(1+c)^b := \lim_{n \to \infty} (1+c)^{b_n}.$$

Proposition 1.39. *Let c, b and $(b_n)_{n\in\mathbb{N}}$ be as in Definition 1.38. Then $(1+c)^b$ is well-defined.*

Proof. We need to show that the sequence $(1+c)^{b_n}$ converges.

$$(1+c)^{b_n} = (1+c)^{a_0+a_1p+\ldots a_np^n} = (1+c)^{a_0}(1+c)^{a_1p}\ldots(1+c)^{a_np^n}$$

and therefore

$$(1+c)^{b_{n+1}} = (1+c)^{b_n}(1+c)^{a_{n+1}p^{n+1}}.$$

If $v_p(c) = r > 0$ we have

$$\begin{aligned}
v_p((1+c)^{b_{n+1}} - (1+c)^{b_n}) &= v_p((1+c)^{a_{n+1}p^{n+1}}(1+c)^{b_n} - (1+c)^{b_n}) \\
&= v_p((1+c)^{b_n}((1+c)^{a_{n+1}p^{n+1}} - 1)) \\
&= v_p((1+c)^{b_n}) + v_p(a_{n+1}p^{n+1}c + \ldots + c^{n+1}) \\
&\geq v_p((1+c)^{b_n}) + \min((n+1)+r, (n+1)r)
\end{aligned}$$

so the sequence $(1+c)^{b_n}$ defines a Cauchy-sequence and therefore converges, since C_p is complete. $\square$

Remark. Instead of taking (b_n) as in Definition 1.38 we could have defined $(1+a)^b$ as the limit of $(1+a)^{b_n}$ for any sequence (b_n) in $\mathbb{Z}$, converging towards b. For $v_p(b_n - \tilde{b}_n) \geq m$ we have $b_n = c + p^m d$, $\tilde{b}_n = c + p^m\tilde{d}$, where $c, d, \tilde{d} \in \mathbb{Z}_p$, and we can conclude

$$\begin{aligned}
v_p((1+a)^{b_n} - (1+a)^{\tilde{b}_n}) &= v_p((1+a)^{c+p^m d} - (1+a)^{c+p^m\tilde{d}}) \\
&= v_p\big((1+a)^c(((1+a)^{p^m})^d - ((1+a)^{p^m})^{\tilde{d}})\big) \\
&\geq v_p(((1+a)^{p^m})^d - ((1+a)^{p^m})^{\tilde{d}}) \\
&= v_p(1+p^m(\ldots) - 1 + p^m(\ldots)) \geq m.
\end{aligned}$$

Therefore if b_n and $\tilde{b}_n$ converge to the same element in $\mathbb{Z}_p^*$ also $(1+a)^{b_n}$ and $(1+a)^{\tilde{b}_n}$ converge towards the same element in L and $(1+a)^{b_n}$ is well-defined.

Remark. Equivalently we could define

$$(1+a)^b := \sum_{k=0}^{\infty}\binom{b}{k}a^k \quad \text{with} \quad \binom{b}{k} = \prod_{i=0}^{k-1}\frac{b-i}{i+1}.$$

For $b \in \mathbb{Z}$ this definition equals the usual definition of $(1+a)^b = \sum_{k=0}^{b-1}\binom{b}{k}a^k$.

Definition 1.40. Let $b \in \mathbb{Z}_p$, and $(b_n)_{n\in\mathbb{N}}$ a sequence in $\mathbb{N}$ converging to b. We define

$$(1+T)^b \coloneqq \lim_{n\to\infty}(1+T)^{b_n}$$

where the limit is taken coefficient wise.

Proposition 1.41. *The series $(1+T)^b$ is a well-defined element of $\mathbb{Z}_p[\![T]\!]$.*

Proof. Let $(b_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{N}$ such that $\lim_{n\to\infty} b_n = b$. In other words for any $c \in \mathbb{N}$ there is a $N > 0$ such that for any $n, m > N \in \mathbb{N}$ we have $v_p(b_n - b_m) > c$. We look at the coefficients of $(1+T)^{b_n}$ and want to show that they converge to an element of $\mathbb{Z}_p$.

$$(1+T)^{b_n} = \sum_{k=0}^{\infty}\binom{b_n}{k}T^k = \sum_{k=0}^{\infty}\left(\prod_{i=0}^{k-1}\frac{b_n - i}{i+1}\right)T^k$$

The coefficient of the k-th term then forms a Cauchy-sequence:

$$\begin{aligned}v_p\left(\binom{b_n}{k} - \binom{b_m}{k}\right) &= v_p\left(\prod_{i=0}^{k-1}\frac{b_n - i}{i+1} - \prod_{i=0}^{k-1}\frac{b_m - i}{i+1}\right)\\ &= v_p\left(\prod_{i=0}^{k-1}\frac{1}{i+1}\right) + v_p\left(\prod_{i=0}^{k-1}(b_n - i) - \prod_{i=0}^{k-1}(b_m - i)\right)\end{aligned}$$

and since b_m and b_n agree modulo p^c, so do the products $\prod_{i=0}^{k-1}(b_n - i)$ and $\prod_{i=0}^{k-1}(b_m - i)$, hence

$$v_p\left(\binom{b_n}{k} - \binom{b_m}{k}\right) \geq -v_p(k!) + v_p(b_n - b_m) = -v_p(k!) + c.$$

Since the sequence $\left(\binom{b_n}{k}\right)_{n\in\mathbb{N}}$ lives in $\mathbb{Z}$ and is a Cauchy-sequences with respect to the norm induced by v_p the limit is an element of $\mathbb{Z}_p$. Hence $(1+T)^b$ is an element of $\mathbb{Z}_p[\![T]\!]$.

It remains to show that this is independent of the choice of b_n. So let $\tilde{b}_n$ be another sequence with $\lim_{n\to\infty}\tilde{b}_n = b$. Then by the same kind of argumentation as we have used before (by replacing b_m by $\tilde{b}_n$) we can show that the coefficients $\binom{b_n}{k}$ and $\binom{\tilde{b}_n}{k}$ converge towards the same element of $\mathbb{Z}_p$. Hence $(1+T)^b$ is well-defined. □

Proposition 1.42 (Chapter 6.1.7 of [Rob00])**.** *Let $f = \sum_{n=-\infty}^{\infty} a_n T^n$ be a Laurent series in $\mathbb{Q}_p$ and $f^+ \in \mathbb{Q}_p[\![T]\!], f^- \in \mathbb{Q}_p[\![T^{-1}]\!]$ with $f = f^+ + f^-$. Then f^+ converges on $|x|_p < \rho_+$ for some ρ_+ as defined in Proposition 1.34, while f^- converges on $|x|_p > \rho_-$ for some $\rho_- \in \mathbb{R}$. If $\rho_- < \rho_+$ then f converges precisely on the interval (ρ_-, ρ_+).*

1.6 The composition of Laurent series

We fix a finite extension L of $\mathbb{Q}_p$.

Proposition 1.43 (See page 287-288 of [Rob00])**.** *Let $f = \sum_{i\in\mathbb{N}} a_i T^i, g \in L[\![T]\!]$ be two power series. If $g(0) = 0$ then we can define the composition*

$$(f \circ g)(T) = \sum_{i\in\mathbb{N}} a_i (g(T))^i.$$

Proposition 1.44 (Proposition on page 288 of [Rob00])**.** *Let $f, g, h \in L[\![T]\!]$ be power series with $g(0) = h(0) = 0$. Then*

$$(f \circ g) \circ h = f \circ (g \circ h).$$

Proposition 1.45 (See page 289-290 of [Rob00])**.** *If we compose the series* $\log(1 + T)$ *and* $\exp(T)$ *as in Proposition 1.36 we get*

$$(\exp \circ \log)(1 + T) = 1 + T, \qquad (\log \circ \exp)(T) = T.$$

Notation. We will now assume that the definition of $|-|^{\{r\}}$ and $v^{\{r\}}$ from Definition 3.16 resp. 3.4 is known. In [Rob00] we have a different notation, there $M_r(f) := |f|^{\{r\}}$ is called *growth modulus of f*. We state the proposition in terms of the valutation instead of the norm as in [Rob00].

Proposition 1.46 (Theorem on page 294 of [Rob00])**.** *Let $f, g \in L[\![T]\!]$ be two power series with $g(0) = 0$ that are convergent on $v_p(T) > r_f$, resp. $v_p(T) > r_g$. If $v_p(a) > r_g$ and $v^{\{v_p(a)\}}(g) > r_f$ then the radius of convergence is $r_{f\circ g} < v_p(a)$ and we can evaluate $f \circ g$ at a by setting $(f \circ g)(a) = f(g(a))$.*

1.7 The cyclotomic character χ

Let K be a finite extension of $\mathbb{Q}_p$. We fix an algebraic closure $\overline{K}$ and denote by μ_m the set of m-th roots of unity in this closure, $\mu_m = \{x \in \overline{K} : x^m = 1\}$. We fix a sequence of primitive, p^n-th roots of unity $(\epsilon_n)_{n\in\mathbb{N}}$ such that $\epsilon_{n+1}^p = \epsilon_n$ and let $K_n = K(\epsilon_n)$ and $K_\infty = \bigcup_{n=0}^{\infty} K_n$. The set of all p^n-th roots of unity for $n \in \mathbb{N}$ we denote by $\mu_{p^\infty} = \bigcup_{n\in\mathbb{N}} \mu_{p^n}$. Furthermore let $G_K = \mathrm{Gal}(\overline{K}/K)$.

Definition 1.47 (See page 4 of [Ber04])**.** The *cyclotomic character*

$$\chi\colon G_K \to \mathbb{Z}_p^*$$

is defined by $\sigma(\zeta) = \zeta^{\chi(\sigma)}$ for every $\sigma \in G_K$ and $\zeta \in \mu_{p^\infty}$.

Proposition 1.48 (Mentioned for example in section 0.1 of [Col10]). *The kernel of the cyclotomic character is $H_K = Gal(\overline{K}/K_\infty)$, and χ identifies $\Gamma_K = Gal(K_\infty/K) = G_K/H_K$ with an open subgroup of $\mathbb{Z}_p^*$. If $K = \mathbb{Q}_p$ the cyclotomic character gives an isomorphism $\chi\colon \Gamma_{\mathbb{Q}_p} \tilde{\to} \mathbb{Z}_p^*$.*

We will use this without proof or reference, however the statement can be found in a lot of papers, like [Col08] and [Col10]. Since we will only use, that χ induces the isomorphism $G_{\mathbb{Q}_p} \tilde{\to} \mathbb{Z}_p^*$, no more knowledge about it will be necessary. By replacing every $\chi(\gamma)$ for $\gamma \in \Gamma_{\mathbb{Q}_p}$ with $a \in \mathbb{Z}_p^*$ we can avoid talking about this character all together. Since for the theory of (φ, Γ)-modules working with $\Gamma_{\mathbb{Q}_p}$ is common, we keep on using this notation in this thesis. We will also use without reference that $\Gamma_{\mathbb{Q}_p}$ (resp. Z_p^*) is topologically cyclic. This is mentioned for example in chapter 2.1. of [Col08].

Chapter 2

φ-modules

This chapter will give an introduction to the category of φ-modules over a commutative unitary ring A. This chapter is largely independent of the previous chapter and builds a purely algebraic theory of φ-modules. In chapter 3 and 4 we will define the (unital) Robba ring and its associated φ action. The (φ, Γ)-modules over this Robba ring will be the focus of chapter 5 where we return to the theory of (φ, Γ)-modules. In this chapter we want to get a feeling for the structure of the φ-modules, which will help us to get used to the structure of (φ, Γ)-modules. Because of lack of time the part concerning the category theory for (φ, Γ)-modules is left out, although this would as well be of great interest for the rest of this thesis.

We follow parts of the first chapter of [Fon90] and work out the details.

2.1 Modules over a ring A

All rings will be unitary but not necessarily commutative. We will follow chapter 1.1 of [Fon90].

Notation. The category of left A-modules we will denote by $\mathfrak{AMod}$. For the ring multiplication we will write

$$A \times A \to A, \quad (r, s) \mapsto r \cdot s$$

and for any module operation

$$A \times M \to M, \quad (r, m) \mapsto r.m$$

The axioms for an A left module we will denote by

$$\begin{array}{ll} \text{[M1]} & r.(x+y) = r.y + r.x \\ \text{[M2]} & (r \cdot s).x = r.(s.x) \\ \text{[M3]} & (r+s).x = r.x + s.x \\ \text{[M4]} & 1_A.x = x \end{array}$$

If A is commutative any left A-module is also a right A-module and hence $\mathfrak{AMod}$ is the category of A-modules.

Whenever we write $|-|$ we mean the absolute value on $\mathbb{R}$.

Example 2.1. Given a ring homomorphism $\alpha \colon A \to B$ we can give B the structure of an left A-module by setting $a.b = \alpha(a) \cdot b$. The axiom [M1] follows directly from the distributivity of the ring multiplication, and the other axioms are quickly shown using the fact that α is a ring homomorphism. We denote this A-module by B_α.

Example 2.2. We now can consider the tensor-product $B_\alpha \otimes_A M$ as a B-module with module operation $b.(\sum b_i \otimes m_i) = \sum (b \cdot b_i) \otimes m_i$. This operation is well-defined, since it commutes with the addition of the tensor product, so for example we have

$$\begin{aligned} b.(b_1 \otimes m + b_2 \otimes m) &= b \cdot b_1 \otimes m + b \cdot b_2 \otimes m = (b \cdot b_1 + b \cdot b_2) \otimes m \\ &= b \cdot (b_1 + b_2) \otimes m = b.((b_1 + b_2) \otimes m) \end{aligned}$$

The module axiom [M1] follows directly from the definition of the operation, [M2] and [M3] hold because of the associativity of the multiplication and distributivity in the ring B and [M4] holds, since we just multiply with the unit element.

2.2 Definition of a φ-module

We fix A a commutative ring with identity and $\sigma : A \to A$ a ring endomorphism. Let M be an A-module.

Definition 2.3 (See subsection 1.1.1 of [Fon90])**.** A map $\varphi : M \to M$ will be called *σ-semi-linear*, if it is additive and

$$\varphi(\lambda.m) = \sigma(\lambda).\varphi(m)$$

for all $\lambda \in A$ and $m \in M$.

Definition 2.4 (See subsection 1.1.1 of [Fon90])**.** A *φ-module over* (A, σ) is an A-module M endowed with a σ-semi-linear morphism

$$\varphi : M \to M.$$

Remark. In later chapters the endomorphism $\sigma : A \to A$ will be called φ, better justifying the name φ-module. The associated σ-semi-linear map for a module M will then be written φ_M or even just φ as well. The reason for this current notation is that the φ map compatible with σ is not unique for a module M, making the later notation ambiguous.

In Example 2.2 we have seen how, given a ring homomorphism $\alpha : A \to B$, we can define an B-module $B_\alpha \otimes_A M$. By taking $B = A$ and $\alpha = \sigma$ we get an A-module, $A_\sigma \otimes_A M$.

Definition 2.5 (See subsection 1.1.1 of [Fon90])**.** We define M_σ to be the A-module $A_\sigma \otimes_A M$.

Proposition 2.6 (Statement in subsection 1.1.1 of [Fon90])**.** *Given a map $\varphi : M \to M$, define*

$$\Phi\colon M_\sigma \to M, \quad (\lambda \otimes x) \mapsto \lambda.\varphi(x).$$

Then φ is a σ-semi-linear if and only if Φ is an A-linear map.

Proof. For $\lambda \in A$ and $x \in M$ we have

$$\begin{aligned}\varphi(\lambda.x) = 1_A.\varphi(\lambda.x) = \Phi(1_A \otimes \lambda.x) = \Phi(\lambda.1_A \otimes x)\\ = \Phi(\sigma(\lambda) \otimes x) = \sigma(\lambda).\varphi(x)\end{aligned}$$

and the linearity of Φ gives us for $x, y \in M$

$$\begin{aligned}\varphi(x + y) &= 1_A.\varphi(x + y) = \Phi(1_A \otimes x + y)\\ &= \Phi(1_A \otimes x + 1_A \otimes y) = \Phi(1_A \otimes x) + \Phi(1_A \otimes y)\\ &= 1_A.\varphi(x) + 1_A.\varphi(y) = \varphi(x) + \varphi(y).\end{aligned}$$

Similarly, assuming the σ-semi-linearity of φ we get that Φ is linear. □

2.3 The category of φ-modules

In the rest of this chapter we will work over a fixed ring A with morphism σ. If we say that M is a φ-module, then we mean that M is an A-module with σ-semi-linear map, which we will denote φ_M.

Definition 2.7 (See subsection 1.1.1 of [Fon90])**.** A *morphism of φ-modules* $\eta\colon M \to N$ is an A-linear map that commutes with φ, i.e. $\eta \circ \varphi_M = \varphi_N \circ \eta$.

Proposition 2.8 (Statement in subsection 1.1.1 of [Fon90])**.** *The φ-modules form a category, that we will denote by $\Phi\mathfrak{M}_A$.*

Proof. The composition of two morphisms, $\eta_1 \colon M_1 \to M_2$ and $\eta_2 \colon M_2 \to M_3$ is again a morphism of φ-modules, since

$$\eta_2 \circ \eta_1 \circ \varphi_{M_1} = \eta_2 \circ \varphi_{M_2} \circ \eta_1 = \varphi_{M_3} \circ \eta_2 \circ \eta_1.$$

We know that morphisms of A-modules are associative, thus morphisms of φ-modules are associative as well. The identity morphism $\mathrm{id}_M \colon M \to M$ of A-modules is as well the identity morphism of φ-modules, since it clearly commutes with φ. $\square$

To show that this category is abelian we use another perspective that we get by considering the ring $A_\sigma[\varphi]$ of formal sums

$$\sum_{i=0}^{l} a_{i,1}\varphi^{n_{i,1}} \cdot a_{i,2}\varphi^{n_{i,2}} \cdot \ldots \cdot a_{i,j_i}\varphi^{n_{i,j_i}}$$

for $a_{i,j} \in A$, $n_{i,j} \in \mathbb{N}$ with the relation $\varphi a = \sigma(a)\varphi$ for $a \in A$. With this relation we can push all the φ to the right hand side and so can write any element of $A_\sigma[\varphi]$ uniquely in the form $\sum_{i=0}^{m} a_i\varphi^i$ for $m \in \mathbb{N}$.

Remark. This ring is commutative if and only if σ is the identity morphism. If we have $a \in A$ with $\sigma(a) \neq a$, then $\varphi, a \in A_\sigma[\varphi]$ do not commute, since $a \cdot \varphi \neq \varphi \cdot a = \sigma(a)\varphi$.

Proposition 2.9 (Statement in subsection 1.1.2 of [Fon90])**.** *Any φ-module over A can be seen as left-module over $A_\sigma[\varphi]$ and vice versa.*

Proof. [M1] of the $A_\sigma[\varphi]$ left-module implies [M1] of the φ-module as well as that φ is additive. [M2] for the $A_\sigma[\varphi]$ left-module implies [M2] of the φ module and the semi-linearity of φ. The third axioms are equivalent to each other and the unit of A is also the unit of $A_\sigma[\varphi]$. On the other hand all the axioms of a φ module over A also imply the axioms for a left module over $A_\sigma[\varphi]$. $\square$

Corollary 2.10 (See subsection 1.1.2 of [Fon90])**.** *The category $\Phi\mathfrak{M}_A$ is abelian.*

Proof. In [Wei94, 1.2.2] it is shown that the category of left-modules over any ring A is abelian. With the identification from above it thus follows that the category $\Phi\mathfrak{M}_A$ is as well abelian. $\square$

Lemma 2.11. *The kernel in the category of φ-modules is just the kernel in the category of A-module.*

Proof. Let M, N be φ-modules and $\nu\colon M \to N$ be a morphism of φ-modules. Let (K, k) be the Kernel of ν as morphism of A-modules, ie.

$$\begin{tikzcd} M \arrow[dr, "\nu"] & \\ K \arrow[u, "k"] \arrow[r, "0_{KN}"'] & N \end{tikzcd}$$

commutes and has the universal property.

First note that (K, k) is given by the submodule of M defined by $K = \{x \in M | \nu(x) = 0\}$ and the natural embedding $k\colon K \to M$. φ structure on M therefore as well gives us a φ structure on K. Moreover k clearly commutes with φ, therefore k is a morphism of φ-modules.

Now let (K', k') be another pair of φ-module and φ-module morphism making the diagram above commuting as well. Since (K, k) has the universal property in $\mathfrak{AMod}$ we get a unique $u\colon K' \to K$ that makes the following diagram commute in $\mathfrak{AMod}$.

$$\begin{tikzcd} & M \arrow[dr, "\nu"] & \\ & K \arrow[u, "k"] \arrow[r, "0_{KN}"'] & N \\ K' \arrow[uur, "k'"] \arrow[ur, "u"] \arrow[urr, "0_{K'N}"'] & & \end{tikzcd}$$

It remains to show that u commutes with φ.

So suppose φ does not commute with u, i.e. there is some $a \in K'$ with $\varphi_k \circ u(a) \neq u \circ \varphi_{k'}(a)$. Then composing with k and using that k commutes with φ we get $\varphi_m \circ k \circ (a) \neq k \circ u \circ \varphi_{k'}(a)$. Therefore, with the universal property $k \circ u = k'$, we have $\varphi_m \circ k'(a) \neq k' \circ \varphi_m(a)$ which is a contradiction to k' being a φ-module morphism. $\square$

2.4 The tensor product of φ-modules

Again we fix a ring A with a morphism σ. If we say that M is a φ-module, then we mean that M is an A-module with σ-semi-linear map which we will denote φ_M and sometimes just φ.

Definition 2.12 (See subsection 1.1.3 of [Fon90])**.** The category $\Phi\mathfrak{M}_A$ is equipped with a tensor product. For two objects M and N we define the tensor product $M \otimes N$ by $M \otimes_A N$, the tensor product in A-modules, together with

$$\varphi(\sum_{i=1}^{k} m_i \otimes n_i) = \sum_{i=1}^{k} \varphi(m_i) \otimes \varphi(n_i).$$

Proposition 2.13. *This morphism is well-defined and σ-semi-linear.*

Proof. For $r \in A$ we have

$$\begin{aligned}\varphi(r \cdot m \otimes n) &= \varphi(r \cdot m) \otimes \varphi(n) = (\sigma(r)\varphi(m)) \otimes \varphi(n) = \sigma(r)(\varphi(m) \otimes \varphi(n)) \\ &= \varphi(m) \otimes \sigma(r)\varphi(n) = \varphi(m \otimes r \cdot n).\end{aligned}$$

and that

$$\begin{aligned}\varphi((m + m') \otimes n) &= \varphi(m + m') \otimes \varphi(n) = (\varphi(m) + \varphi(m')) \otimes \varphi(n) \\ &= \varphi(m) \otimes \varphi(n) + \varphi(m') \otimes \varphi(n)\end{aligned}$$

is equal to

$$\varphi(m \otimes n + m' \otimes n) = \varphi(m) \otimes \varphi(n) + \varphi(m') \otimes \varphi(n).$$

With the same argument for the addition in the second argument we find therefore that this morphism is well-defined and σ-semi-linear. □

Proposition 2.14 (See subsection 1.1.3 of [Fon90])**.** *The tensor product in $\Phi\mathfrak{M}_A$ is associative, abelian and has unit object A, seen as an A-module, together with $\varphi = \sigma$.*

Proof. The tensor product in the category of A-modules is associative and abelian. Therefore the same holds in $\Phi\mathfrak{M}_A$. For any φ-module M if we tensor with A we get for any $r \otimes m \in A \otimes_A M$

$$r \otimes m = 1_A \otimes r.m$$

and

$$\varphi(r \otimes m) = \sigma(r) \otimes \varphi(m) = 1_A \otimes \sigma(r).\varphi(m)$$

so $M \otimes_A A$ identifies with M. □

2.5 Étale φ-modules

Definition 2.15 (See subsection 1.1.4 of [Fon90])**.** If A is Noetherian, we call a φ-module M *étale*, if it is finitely generated as A-module and the corresponding $\Phi\colon M_\sigma \to M$ is a bijection. Morphisms of étale φ-modules are just morphisms of φ-modules.

Lemma 2.16 (Statement in subsection 1.1.4 of [Fon90])**.** *If σ is an automorphism then Φ is bijective if and only if φ is bijective.*

Proof. First note that if σ is bijective, we have $\lambda \otimes x = 1 \otimes \sigma^{-1}(\lambda)x$ for $\lambda \in A$, $x \in M$ and any element of $A_\sigma \otimes_A M$ has a unique representative of the form $1 \otimes x$ (for $\lambda = 0$ we have $0 \otimes x = 0 = 1 \otimes 0$). Therefore there is a bijection between the elements of $A_\sigma \otimes_A M$ and M.

If now Φ is bijective, then φ is surjective, since for all $m \in M$ we can find $\lambda \in A$ and $x \in M$ such that

$$m = \Phi(\lambda \otimes x) = \lambda.\varphi(x) = \sigma(\sigma^{-1}(\lambda)).\varphi(x) = \varphi(\sigma^{-1}(\lambda)x)$$

and injective, since using the bijection between $A_\sigma \otimes_A M$ and M we get

$$\varphi(x) = \Phi(1 \otimes x) = 0 \iff 1 \otimes x = 0 \iff x = 0$$

The other direction is immediate. $\square$

It is clear that the étale φ-module build a subcategory of the category of φ-modules, that we denote by $\Phi\mathfrak{M}_A^{\text{ét}}$.

Definition 2.17 (Definition 3.2.1 of [Wei94])**.** An A-module M is called *flat* if for any exact sequence of A-modules $N_1 \to N_2 \to N_3$ the sequence $M \otimes_A N_1 \to M \otimes_A N_2 \to M \otimes_A N_3$ is exact as well. The ring A is called *σ-flat* if the module A_σ is flat.

Proposition 2.18 (Proposition 1.1.5 of [Fon90])**.** *Let A be Noetherian and σ-flat. Then the category $\Phi\mathfrak{M}_A^{\acute{e}t}$ is abelian.*

Proof. Following the proof of Proposition 1.1.5 in [Fon90]. The zero object is given by the zero module together with the trivial morphism, which clearly is étale. Also products and coproducts, together with componentwise defined φ are easily seen to be étale. Let $\eta\colon M \to N$ be a morphism of étale φ-modules. We want to use the σ-flatness to show that the kernel K and the cokernel L of η are étale. So we have the exact sequence

$$0 \longrightarrow K \longrightarrow M \xrightarrow{\eta} N \longrightarrow L \longrightarrow 0$$

A is σ-flat, therefore the sequence that we get by taking the tensor product with A_σ is also exact. We get two exact sequences

$$\begin{tikzcd}
0 \arrow[r] & K_\sigma \arrow[r] \arrow[d] & M_\sigma \arrow[r] \arrow[d] & N_\sigma \arrow[r] \arrow[d] & L_\sigma \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r] & M \arrow[r, "\eta"] & N \arrow[r] & L \arrow[r] & 0
\end{tikzcd}$$

where the vertical arrows are given by the linear maps Φ. The φ-modules M and N are étale, so Φ_M and Φ_N are bijective. By exactness of the upper sequence we have $K_\sigma = Ker(M_\sigma \to N_\sigma) \cong Ker(M \to N) = K$ and $L_\sigma = Coker(M_\sigma \to N_\sigma) \cong Coker(M \to N) = L$. Therefore Φ_L and Φ_K are also bijective and K and L are étale. □

2.6 The functor $\alpha^*\colon \Phi\mathfrak{M}_{A_1} \to \Phi\mathfrak{M}_{A_2}$

Let A_1 and A_2 be two commutative rings equipped with endomorphisms σ_1 and σ_2. Given a ring-homomorphism $\alpha\colon A_1 \to A_2$ that commutes with the endomorphisms we can construct a functor $\alpha^*\colon \Phi\mathfrak{M}_{A_1} \to \Phi\mathfrak{M}_{A_2}$ as follows:

Definition 2.19 (See subsection 1.1.8 of [Fon90])**.** For an (A_1, φ)-module M define $\alpha^*(M) = (A_2)_\alpha \otimes_{A_1} M$ and $\varphi\colon M \to M$ by

$$\varphi(\sum a \otimes m) = \sum \sigma_2(a) \otimes \varphi(m).$$

Proposition 2.20. *φ is well-defined on $\alpha^*(M)$ and this defines a φ-module.*

Proof. φ commutes with addition

$$\begin{aligned}\varphi((a+\tilde{a}) \otimes m) &= \sigma_2(a+\tilde{a}) \otimes \varphi(m)\\ &= \sigma_2(a) \otimes \varphi(m) + \sigma_2(\tilde{a}) \otimes \varphi(m)\\ \varphi(a \otimes m + \tilde{a} \otimes m) &= \sigma_2(a) \otimes \varphi(m) + \sigma_2(\tilde{a}) \otimes \varphi(m)\end{aligned}$$

and is compatible with the structure of the modules, since the following two expressions are equal:

$$\begin{aligned}\varphi(\alpha(a_1)a_2 \otimes m) &= \sigma_2(\alpha(a_1)a_2) \otimes \varphi(m) = \alpha(\sigma_1(a_1))\sigma_2(a_2) \otimes \varphi(m)\\ &= \sigma_2(a_2) \otimes \sigma_1(a_1).\varphi(m)\\ \varphi(a_2 \otimes a_1.m) &= \sigma_2(a_2) \otimes \varphi(a_1.m) = \sigma_2(a_2) \otimes \sigma_1(a_1).\varphi(m).\end{aligned}$$

Moreover this map is by definition additive and σ_2-semi-linear

$$\begin{aligned}\varphi(\tilde{a}_2(a_2 \otimes m)) &= \varphi(\tilde{a}_2 \cdot a_2 \otimes m) = \sigma_2(\tilde{a}_2 \cdot a_2) \otimes \varphi(m)\\ &= \sigma_2(\tilde{a}_2).\varphi(a_2 \otimes m)\end{aligned}$$

and therefore this defines a φ-module over the ring A_2. □

2.7 Fontaines (φ, Γ)-modules

Let (A, σ) be defined as before and let A be equipped with an action of some group Γ, that is compatible with the structure of the ring and commutes with σ.

We now will give the definition that Fontaine is giving in [Fon90]. Later in chapter 6 we will give another, slightly different definition, which we will be working with.

Definition 2.21 (See subsection 3.3.1 of [Fon90])**.** A (φ, Γ)-*module over* A is given by a φ-module that is equipped with a semi-linear action of Γ that commutes with the action of φ.

Suppose now that A and Γ are both equipped with Hausdorff and complete topologies. Furthermore suppose that A is σ-flat and Noetherian. We can then define the notion of an étale (φ, Γ)-module.

Definition 2.22 (See subsection 3.3.2 of [Fon90])**.** We call a (φ, Γ)-module *étale over* A if the underlying φ-module is étale and if its action of Γ is continuous.

Remark. It can be shown that the (φ, Γ)-modules form an abelian category equipped with a tensor product. [Statement in 3.3.2 of [Fon90]]

Chapter 3

The Robba ring $\mathcal{R}$

In the following chapter we want to give an introduction to the theory of Laurent series and more specifically to the Robba ring. We will follow the first few chapters in [Laz62] and [Col08]. In [Laz62] there is a detailed overview of the ring of Laurent series in one variable over a complete valuation field. [Col08] gives the definitions of the Robba ring and some more specific properties but without much detail. We will work out what it means for a series to converge on some annulus in $\mathbb{R}$. After that we will define the Robba ring and equip it with a Fréchet topology. Furthermore will show that the Robba ring is complete with respect to the metric inducing the Fréchet topology.

3.1 Convergent Laurent series in one variable

Most of the definitions are from [Laz62]. We fix a prime number $p \neq 2$ and L, a finite extension of $\mathbb{Q}_p$ and we define v_p to be the p-adic valuation on L with $v_p(p) = 1$.

Notation. Let $f = \sum_{n\in\mathbb{Z}} a_n T^n$ be a formal series with coefficients $a_n \in L$. We will write $f^{(i)}$ for the i-th coefficient of the series, $f^{(i)} := a_i$. The set of all formal Laurent series with coefficients in L we will denote by $\mathcal{L}_L$. We say that f converges at $a \in C_p$ if the evaluation of f at a converges to some element in C_p and we will write $f(a) := \sum_{n\in\mathbb{Z}} a_n a^n$ for this evaluation. We will write $|-|$ for the normal absolute value on $\mathbb{R}$.

L is a valued field with non-Archimedean absolute value $|-|_p$. The series in $\mathcal{L}_L$ form a infinite dimensional vector space with basis $\{T^i\}_{i\in\mathbb{Z}}$.

Proposition 3.1 (Statement in subsection I.1.2 of [Col10])**.**
Let $f = \sum_{n\in\mathbb{Z}} a_n T^n \in \mathcal{L}_L$ and $a \in C_p$ with $v_p(a) = r \in \mathbb{R}$. Then f is convergent on a if and only if $v_p(a_n) + nr \to \infty$ as $|n| \to \infty$.

Proof. We can split f into two series, one negative degree power series f_-, and one positive degree power series f_+. Then f is convergent on a if and only if both f_+ and f_- are convergent on a. By Proposition 1.33 the series converges if and only if the norm of the n-th term, $|a_n a^n|_p$ goes to 0 for $|n| \to \infty$. This is by definition of the p-adic norm equivalent with $v_p(a_n) + nr \to \infty$ as $|n| \to \infty$. □

Remark. Proposition 3.1 tells us that whether a function converges on an element of $a \in C_p$ only depends on the p-adic valuation of a. Therefore convergence on $a \in C_p$ implies the convergence on the circle centered around 0 with radius $v_p(a)$.

Definition 3.2. We will say that $f \in \mathcal{L}_L$ *converges on radius* r with $r \in \mathbb{R}$ if for any $a \in C_p$ with $v_p(a) = r$ we have that f converges on a. We say that f is convergent on radius ∞ (resp. $-\infty$) if $a_n = 0$ for all $n < 0$ (resp. $n > 0$).

Notation. For the set of all $r \in \mathbb{R} \cup \{\pm\infty\}$ such that f converges on all $a \in C_p$ with $v_p(a) = r$ we will write $\mathrm{Conv}(f)$.

The following Lemma will show that the elements of C_p on which f converges form some annulus around 0.

Lemma 3.3 (Mentioned after 1.2 of [Laz62])**.** *The set* $\mathrm{Conv}(f)$ *is an interval* $I \subset \mathbb{R} \cup \{\infty\}$.

Proof. Suppose f converges on elements with valuation $r_1 \in \mathbb{R}$ as well as on elements with valuation $r_2 \in \mathbb{R}$ with $r_1 < r_2$ and let $r \in \mathbb{R}$ with $r_1 \le r \le r_2$. We clearly have

$$\begin{aligned} v_p(f^{(k)}) + r_1 k &< v_p(f^{(k)}) + rk \quad \text{for } k > 0 \\ v_p(f^{(k)}) + r_2 k &< v_p(f^{(k)}) + rk \quad \text{for } k < 0. \end{aligned}$$

Since for both $i \in \{1, 2\}$

$$v_p(f^{(n)}) + nr_i \to \infty \text{ as } |n| \to \infty$$

the left hand side of the inequalities goes to ∞ as $|n| \to \infty$, hence so does the right hand side. Therefore also $v_p(f^{(n)}) + nr \to \infty$ as $|n| \to \infty$. □

Therefore the formal series we are considering can be seen as functions on some annulus $A_I = \{a \in C_p \mid v_p(a) \in I\}$ for some interval I. If $I \subset \mathrm{Conv}(f)$ we can evaluate f at any $a \in A_I$, hence these functions are defined on this annulus.

Definition 3.4 (See subsection I.1.2 of [Col10])**.** For each $r \in \mathbb{R}$ define $v^{\{r\}}(f)$ as follows:

$$v^{\{r\}}(f) = \inf_{n \in \mathbb{Z}} (v_p(f^{(n)}) + nr) \in \mathbb{R} \cup \{\pm\infty\}.$$

Definition 3.5 (See subsection 1.7 of [Laz62])**.** Let $f \neq 0$ and $r \in \mathrm{Conv}(f)$. For $r \neq \infty, -\infty$ define $n(f, r)$ (resp. $N(f, r)$) to be the smallest (resp. biggest) integer i such that $v^{\{r\}}(f) = v_p(f^{(i)}) + ir$, i.e.

$$n(f, r) = \inf \left\{ i \in \mathbb{Z} \mid v^{\{r\}}(f) = v_p(f^{(i)}) + ir \right\}$$
$$N(f, r) = \sup \left\{ i \in \mathbb{Z} \mid v^{\{r\}}(f) = v_p(f^{(i)}) + ir \right\}.$$

These are well-defined, since for elements in $\mathrm{Conv}(f)$ we have that $v_p(f^{(n)}) + nr \to \infty$ as $|n| \to \infty$ so the infimum and the supremum are attained.

Definition 3.6. We call $f \in \mathcal{L}_L$ *bounded at* $r \in \mathbb{R}$ if for $a \in C_p$ with $v_p(a) = r$ the sequences $(|f^{(n)}a^n|_p)_{n \in \mathbb{N}}$ and $(|f^{(n)}a^n|_p)_{-n \in \mathbb{N}}$ are bounded, i.e. if there exists $b \in \mathbb{N}$ such that

$$v_p(f^{(n)}a^n) = v_p(f^{(n)}) + nr > b$$

for all $n \in \mathbb{N}$. f is bounded on some interval $I \subset \mathbb{R}$, if it is bounded at any $r \in I$.

Remark. If f converges on radius r, we have $v_p(f^{(n)}) + nr \to \infty$ as $|n| \to \infty$ so f is bounded at r.

Proposition 3.7. *A series $f \in \mathcal{L}_L$ is bounded at r if and only if $v^{\{r\}}(f) > -\infty$. Furthermore if f converges on radius r and $v_p(a) = r$ we have $v_p(f(a)) \geq v^{\{r\}}(f)$.*

Proof. Clear by definition. □

Notation. From now if we say that a series f is convergent on $v_p(T) = r$, we mean that f converges for elements $a \in C_p$ with $v_p(a) = r$. So $v_p(T) = r$ means we substitute T by a with $v_p(a) = r$.

Proposition 3.8.

(i) If $x_+ = \sum_{i=0}^{\infty} a_i T^i$ is bounded at r, then x_+ converges on $v_p(T) > r$.

(ii) If $x_- = \sum_{i=-\infty}^{0} a_i T^i$ is bounded at r, then x_- converges on $v_p(T) < r$.

(iii) If $x = \sum_{i=-\infty}^{\infty} a_i T^i$ is bounded at r_1 and r_2 with $r_1 < r_2$ then x converges on $r_1 < v_p(T) < r_2$.

Proof. (i) x_+ is bounded at r if there is a $c \in \mathbb{R}$ such that for all $i \in \mathbb{N}$ we have $v_p(a_i) + ir > c$. Then for any $\tilde{r} > r$:

$$v_p(a_i) + i\tilde{r} - i(\tilde{r} - r) > c \iff v_p(a_i) + i\tilde{r} > i(\tilde{r} - r) + c$$

Since $\tilde{r} - r > 0$ if we let $i \to \infty$ the right hand side of the inequality goes to ∞ and therefore also $v_p(a_i) + i\tilde{r} \to \infty$.

(ii) Similary if $\tilde{r} < r$:

$$v_p(a_i) + i\tilde{r} + i(r - \tilde{r}) > c \iff v_p(a_i) + i\tilde{r} > -i(r - \tilde{r}) + c$$

so if $i \to \infty$ also $v_p(a_i) + i\tilde{r}$ goes to ∞.

(iii) We can split $x = x_+ + x_-$ and use that x_+ converges for $v_p(T) > r_1$, while x_- converges for $v_p(T) < r_2$. $\square$

Example 3.9. We want to calculate the $v^{\{r\}}(f)$ for $f(T) = (1+T)^p - 1$. This series will play an important role later on.

$$\begin{aligned} v^{\{r\}}((1+T)^p - 1) &= v^{\{r\}}(\sum_{i=1}^{p} \binom{p}{i} T^i) \\ &= \inf_{1<i\leq p} \{v_p(\binom{p}{i}) + ir\} \\ &= \min(v_p(p) + r, v_p(1) + pr) \\ &= \min(1 + r, pr) \\ &= \begin{cases} r+1 & \text{if } r \geq \frac{1}{(p-1)} \\ pr & \text{if } r < \frac{1}{(p-1)} \end{cases} \end{aligned}$$

since $v_p(\binom{p}{i}) = 1$ for all $1 \leq i \leq p-1$ and hence $v_p(p) + r < v_p(\binom{p}{i}) + ir$ for all $2 \leq i \leq p-1$.

Example 3.10. Let $r > 0$. For $\gamma(T) = (1+T)^a - 1$ with $a \in \mathbb{Z}_p^*$ we calculate $v^{\{r\}}(\gamma(T)) = r$ and $n(\gamma(T), r) = N(\gamma(T), r) = 1$:

$$\begin{aligned} v^{\{r\}}(\gamma(T)) &= v^{\{r\}}(\sum_{i=1}^{\infty} \binom{a}{i} T^i) \\ &= \inf\{v_p(\binom{a}{i}) + ir\} = r \\ n(\gamma(T), r) &= N(\gamma(T), r) = 1 \end{aligned}$$

Definition 3.11 (See subsection I.1.2 of [Col10]). For $r_1 < r_2$ let

$$\mathcal{L}_L[r_1, r_2] \coloneqq \{f \in \mathcal{L}_L \text{ convergent on } r_1 \leq v_p(T) \leq r_2\}$$
$$\mathcal{L}_L]r_1, r_2] \coloneqq \{f \in \mathcal{L}_L \text{ convergent on } r_1 < v_p(T) \leq r_2\}$$
$$\mathcal{L}_L(r_1, r_2] \coloneqq \{f \in \mathcal{L}_L \text{ convergent on } r_1 < v_p(T) \leq r_2 \text{ and bounded at } r_1\}.$$

Sometimes we will write $\mathcal{L}\{r\}$ instead of $\mathcal{L}[r, r]$.

Notation. We use a mix of the notation in [Col10], who is using $\mathcal{L}_L(r_1, r_2]$ for what we call $\mathcal{L}_L]r_1, r_2]$ and does not regard bounded series, and [Col04], where $\mathcal{E}^{(r_1,r_2]}$ is the equivalent of our $\mathcal{L}_L(r_1, r_2]$ and $\mathcal{E}^{]r_1,r_2]}$ is the equivalent of $\mathcal{L}_L]r_1, r_2]$.

Proposition 3.12 (Statement in subsection I.1.2 of [Col10]). *The sets given in Definition 3.11 together with formal addition and multiplication are rings.*

Proof. Let $f = \sum_{i\in\mathbb{Z}} a_i T^i, g = \sum_{i\in\mathbb{Z}} b_i T^i \in \mathcal{L}_L$. Clearly if f and g converge on radius r, also their sum $f + g$ and the additive inverse $-f$ are elements of $\mathcal{L}_L$ that converge converge on radius r. Moreover the unit element 1 converges anywhere.

Let $f \cdot g$ be the formal product $\sum_{i\in\mathbb{Z}}(\sum_{j\in\mathbb{Z}} a_j b_{i-j})T^i$. We first need to show that the coefficients are well-defined elements of L. Suppose that both f and g converge on some radius $r > 0$. For $r = 0$ the statement is immediate and for $r < 0$ we can make a symmetric argument.

By Proposition 3.7 $v^{\{r\}}(f)$ is an element of $\mathbb{R}$ and $v_p(a_n) \geq v^{\{r\}}(f)-nr$, so $v_p(a_n) \to \infty$ as $n \to -\infty$, since $r > 0$. In the same way $v_p(b_n) \to \infty$ as $n \to -\infty$.

We want to show that $v_p(a_j b_{i-j}) \to \infty$ if $|j| \to \infty$.

$$\begin{aligned} v_p(a_j b_{i-j}) + ir &= v_p(a_j) + jr + v_p(b_{i-j}) + (i - j)r \\ &\geq v^{\{r\}}(f) + v_p(b_{i-j}) + (i - j)r \end{aligned}$$

If we take $j \to \infty$ the right hand side goes to ∞, hence also the left hand side, which implies $v^{\{r\}}(a_j b_{i-j}) \to \infty$ if $j \to \infty$. Similarly

$$\begin{aligned} v_p(a_j b_{i-j}) + ir &= v_p(a_j) + jr + v_p(b_{i-j}) + (i - j)r \\ &\geq v_p(a_j) + jr + v^{\{r\}}(g) \end{aligned}$$

implies $v_p(a_j b_{i-j}) \to \infty$ if $j \to -\infty$. So the coefficients of the series are well-defined by Proposition 1.33.

By Proposition 1.35 the product $f \cdot g$ converges on r, if f and g converge on r.

If $f, g \in \mathcal{L}_L$ are bounded in r by $c \in \mathbb{R}$ resp. $d \in \mathbb{R}$, then the additive inverse $-f$ is bounded by c and the sum $f+g$ is bounded by $\min(c,d)$, since

$$v_p(a_i + b_i) + ir \geq \min(v_p(a_i), v_p(b_i)) + ir \geq \min(c,d).$$

We have

$$\begin{aligned}
v^{\{r\}}(fg) &= \inf_{i\in\mathbb{Z}}\{v_p(\sum_{j\in\mathbb{Z}} a_j b_{i-j}) + ir\} \\
&\geq \inf_{i\in\mathbb{Z}}\{\inf_{j\in\mathbb{Z}}\{v_p(a_j b_{i-j})\} + ir\} \\
&= \inf_{i\in\mathbb{Z}}\{\inf_{j\in\mathbb{Z}}\{v_p(a_j) + jr + v_p(b_{i-j}) + ir - jr\}\} \\
&\geq \inf_{j\in\mathbb{Z}}\{v_p(a_j) + jr\} + \inf_{i,j\in\mathbb{Z}}\{v_p(b_{i-j}) + r(i-j)\} \\
&= v^{\{r\}}(f) + v^{\{r\}}(g)
\end{aligned}$$

hence the product $f \cdot g$ is bounded by $c + d$. □

Proposition 3.13 (Proposition 1 of [Laz62]). *Let $r_1 < r_2$. Then $\mathcal{L}_L[r_1, r_2]$, (resp. $\mathcal{L}_L]r_1, r_2]$) together with the addition and multiplication of formal series defines an integral domain. Moreover for $r \in [r_1, r_2]$ (resp. $r \in]r_1, r_2]$), $f, g \neq 0 \in \mathcal{L}_L I$ we have*

$$\begin{aligned}
v^{\{r\}}(fg) &= v^{\{r\}}(f) + v^{\{r\}}(g); \\
n(fg, r) &= n(f, r) + n(g, r); \\
N(fg, r) &= N(f, r) + N(g, r)
\end{aligned}$$

Proof. See proof of Proposition 1 of [Laz62]. □

Proposition 3.14. *For $f, g \in \mathcal{L}_L$ bounded at r we have:*

(i) $v^{\{r\}}(f) = \infty \iff f = 0$

(ii) $v^{\{r\}}(f+g) \geq \min(v^{\{r\}}(f), v^{\{r\}}(g))$

(iii) $v^{\{r\}}(\lambda f) = v_p(\lambda) + v^{\{r\}}(f)$ *for* $\lambda \in L$

Proof. (i) Let $f, g \in \mathcal{L}_L$ bounded at r. For $r = \pm\infty$ we can use that that v_p defines a valuation on L. So assume now $r \in \mathbb{R}$. Then $v^{\{r\}}(f) = \infty$ if and only if $v_p(f^{(n)}) + nr = \infty$ for all $n \in \mathbb{Z}$. This is only the case if $v_p(f^{(n)}) = \infty$ for all $n \in \mathbb{Z}$, in other words if $f = 0$.

(ii) We have

$$\begin{aligned}
v^{\{r\}}(f+g) &= \inf_{n\in\mathbb{Z}} \Big(v_p(f^{(n)} + g^{(n)}) + rn \Big) \\
&\geq \inf_{n\in\mathbb{Z}} \Big(\min(v_p(f^{(n)}), v_p(g^{(n)})) + rn \Big) \\
&= \inf_{n\in\mathbb{Z}} \Big(\min(v_p(f^{(n)}) + rn, v_p(g^{(n)}) + rn) \Big) \\
&= \min \left(\inf_{n\in\mathbb{Z}} \Big(v_p(f^{(n)}) + rn \Big), \inf_{n\in\mathbb{Z}} \Big(v_p(g^{(n)}) + rn \Big) \right) \\
&= \min \Big(v^{\{r\}}(f), v^{\{r\}}(g) \Big).
\end{aligned}$$

(iii) Let $\lambda \in L$.

$$\begin{aligned}
v^{\{r\}}(\lambda f) &= \inf_{n\in\mathbb{Z}} (v_p(\lambda f^{(n)}) + rn) \\
&= \inf_{n\in\mathbb{Z}} (v_p(\lambda) + v_p(f^{(n)}) + rn) \\
&= v_p(\lambda) + \inf_{n\in\mathbb{Z}} (v_p(f^{(n)}) + rn) \\
&= v_p(\lambda) + v^{\{r\}}(f)
\end{aligned}$$

□

Remark. Suppose now f and g converge on radius r. Then for $v^{\{r\}}(f) \neq v^{\{r\}}(g)$ we have $v^{\{r\}}(f+g) = \min(v^{\{r\}}(f), v^{\{r\}}(g))$. To show this let without loss of generality $v^{\{r\}}(f) < v^{\{r\}}(g)$. Suppose first that the infimum in the valuations is attained for the same n. As v_p is a valuation, the equality follows easily. If it is not attained at the same n, so if $v^{\{r\}}(f) = v_p(f^{(n)}) + rn < v^{\{r\}}(g) = v_p(g^{(m)}) + rm$, then the infimum in $v^{\{r\}}(f+g)$ is attained at n, and $v_p(f^{(n)}) < v_p(g^{(n)})$, so the equality again follows immediately.

Proposition 3.15. *Let $I \subset \mathbb{R} \cup \{\pm\infty\}$ be a nonempty interval. For any $r \in I$ the map $v^{\{r\}}$ defines a valuation on $\mathcal{L}_L I$.*

Proof. If f converges on radius r we clearly have $v^{\{r\}}(f) > -\infty$. Together with Proposition 3.13 and 3.14 this shows that $v^{\{r\}}$ defines a valuation on $\mathcal{L}_L I$. □

Definition 3.16. Let $I \subset \mathbb{R} \cup \{\pm\infty\}$ be a nonempty interval. For any $r \in I$ the valuation $v^{\{r\}}$ induces a norm $|-|^{\{r\}}$ on $\mathcal{L}_L I$ by setting $|f|^{\{r\}} = p^{-v^{\{r\}}(f)}$.

Notation. What we call $|-|^{\{r\}}$ is called *growth modulus* in [Rob00].

Remark. We will usually work with the valuation rather than the norm. This might feel a bit odd, but since we have the norm defined via the valuation this will usually save us one step.

Lemma 3.17 (Statement in I.1.2 of [Col10])**.** *The topology induced by $v^{\{r\}}$ makes $\mathcal{L}_L\{r\}$ a Banach space.*

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{L}_L\{r\}$ with respect to $|-|^{\{r\}}$. Then $v^{\{r\}}(f_n - f_m)$ tends to ∞ as $n, m \to \infty$.

$$\begin{aligned} v^{\{r\}}(f_n - f_m) = &\inf_{i\in\mathbb{Z}}\{v_p((f_n - f_m)^{(i)}) + ir\} \to \infty \\ &\Longrightarrow v_p((f_n)^{(i)} - (f_m)^{(i)}) + ir \to \infty \quad \forall i \in \mathbb{Z} \\ &\Longrightarrow v_p((f_n)^{(i)} - (f_m)^{(i)}) \to \infty \quad \forall i \in \mathbb{Z} \end{aligned}$$

So for any $i \in \mathbb{Z}$ the sequence $((f_n)^{(i)})_{n\in\mathbb{N}}$ is a Cauchy sequence in L. But since L is complete this means that $\lim_{n\to\infty}(f_n)^{(i)} \in L$. Therefore $(f_n)_{n\in\mathbb{N}}$ converges to a Laurent series with entries in L. Furthermore since f_n converges for $a \in C_p$ with $v_p(a) = r$ for all n we have $f_n(a) \in C_p$. Then

$$\begin{aligned} v_p((f_n - f_m)(a)) &= v_p(\sum_{i\in\mathbb{Z}}(f_n^{(i)}a^i - f_m^{(i)}a^i)) \\ &\geq \inf\{v_p(f_n^{(i)} - f_m^{(i)}) + ir\} \to \infty \end{aligned}$$

so $(f_n(a))_{n\in\mathbb{N}}$ defines a Cauchy sequence in C_p and we get $\lim_{n\to\infty} f_n(a) \in C_p$, so the limit of f_n converges on radius r. □

Definition 3.18 (See subsection I.1.2 of [Col10])**.** Let $r_1 < r_2 \in \mathbb{R} \cup \{\pm\infty\}$. For $f \in \mathcal{L}_L(r_1, r_2]$ define

$$v^{[r_1,r_2]}(f) = \min(v^{\{r_1\}}(f), v^{\{r_2\}}(f)).$$

Lemma 3.19. *For $f \in \mathcal{L}_L(r_1, r_2]$ the function $v^{[r_1,r_2]}(f)$ is well-defined.*

Proof. This follows immediately from the fact that f is bounded at r_1 and converges on radius r_2. □

Remark. In the literature we can also find the definition

$$v^{[r_1,r_2]}(f) = \inf_{r\in]r_1,r_2]} v^{\{r\}}(f)$$

. Let $r \in\,]r_1, r_2]$. Then if the infimum in $v^{\{r\}}(f) = \inf_{n\in\mathbb{Z}}(v_p(f^{(n)})+nr)$ is attained for positive n, we have that $v_p(f^{(n)})+r_1n < v_p(f^{(n)})+rn$, so $v^{\{r_1\}}(f) < v^{\{r\}}(f)$. If it (possibly simultaneously) is attained for some negative n we have $v_p(f^{(n)})+r_2n < v_p(f^{(n)})+rn$, so $v^{\{r_2\}}(f) < v^{\{r\}}(f)$. Therefore the infimum $\inf_{s\in]r_1,r_2]} v^{\{s\}}(f)$ is attained either at r_1 or at r_2.

Proposition 3.20 (Statement in subsection I.1.2 of [Col10]). *Let $f, g \in \mathcal{L}_L(r_1, r_2]$. Then $v^{[r_1,r_2]}$ is a valuation as in Definition 1.6, so it has the following properties:*

(i) $v^{[r_1,r_2]}(f) = \infty \iff f = 0$,

(ii) $v^{[r_1,r_2]}(f+g) \geq \min(v^{[r_1,r_2]}(f), v^{[r_1,r_2]}(g))$,

(iii) $v^{[r_1,r_2]}(\lambda f) = v_p(\lambda) + v^{[r_1,r_2]}(f)$ *for* $\lambda \in L$.

Proof. (i), (ii) and (iii) follow immediately from the corresponding statements (i), (ii) and (iii) from Proposition 3.14. □

Remark. Let $f, g \in \mathcal{L}_L(r_1, r_2]$. Note that $v^{[r_1,r_2]}$ does not have the stronger property $v^{[r_1,r_2]}(fg) = v^{[r_1,r_2]}(f) + v^{[r_1,r_2]}(g)$. For example $v^{[0,1]}(TT^{-1}) = 0$, but $v^{[0,1]}(T) + v^{[0,1]}(T^{-1}) = \min(0,1) + \min(0,-1) = -1$.

Remark. Since $\mathcal{L}_L[r_1, r_2] \subset \mathcal{L}_L(r_1, r_2]$ the same function $v^{[r_1,r_2]}$ also defines a valuation on $\mathcal{L}_L[r_1, r_2]$.

Definition 3.21. We define a function $|-|^{[r_1,r_2]} : \mathcal{L}_L(r_1, r_2] \to \mathbb{R}$ by

$$|f|^{[r_1,r_2]} := p^{-v^{[r_1,r_2]}(f)}.$$

Lemma 3.22. *For $|-|^{[r_1,r_2]}$ the ultrametric triangle inequality holds, i.e.*

$$|f+g|^{[r_1,r_2]} \leq \max(|f|^{[r_1,r_2]}, |g|^{[r_1,r_2]}).$$

Proof.

$$\begin{aligned} |f+g|^{[r_1,r_2]} &= p^{-v^{[r_1,r_2]}(f+g)} \\ &\leq p^{-\min(v^{[r_1,r_2]}(f), v^{[r_1,r_2]}(g))} = \max(p^{-v^{[r_1,r_2]}(f)}, p^{-v^{[r_1,r_2]}(g)}) \\ &= \max(|f|^{[r_1,r_2]}, |g|^{[r_1,r_2]}) \end{aligned}$$

□

Corollary 3.23. *The function $|-|^{[r_1,r_2]}$ defines a non-Archimedean norm on $\mathcal{L}_L(r_1, r_2]$.*

Proof. $|f|^{[r_1,r_2]} = 0$ iff $f = 0$ follows directly from Proposition 3.20 (i). For $\lambda \in L^*$ we have $|\lambda f|^{[r_1,r_2]} = |\lambda|_p |f|^{[r_1,r_2]}$ by Proposition 3.20 (iii). The ultrametric triangle-inequality holds by Lemma 3.22. □

Remark. Again since $\mathcal{L}_L[r_1, r_2] \subset \mathcal{L}_L(r_1, r_2]$ this also gives us a norm on this subspace.

Lemma 3.24 (Statement in chapter 1.1.2 of [Col04])**.** *The topology induced by* $v^{[r_1,r_2]}$ *makes* $\mathcal{L}_L[r_1, r_2]$ *a Banach space.*

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{L}_L$. Then $v^{[r_1,r_2]}(f_n - f_m)$ tends to ∞ as $n, m \to \infty$, which is equivalent with $v^{\{r\}}(f_n - f_m)$ tending to ∞ for every $r \in [r_1, r_2]$, by the remark after Lemma 3.19. As we have seen in the proof of Lemma 3.17 this implies that $((f_n)^{(i)})_n$ is a Cauchy sequence in L for all $i \in \mathbb{Z}$ and hence f_n converges to an element in $\mathcal{L}_L$. Moreover in the same proof we see that the limit of f_n converges for every $a \in C_p$ with $v_p(a) \in [r_1, r_2]$. □

3.2 The Fréchet topology on $\mathcal{L}_L]0, r]$

We now want to define a Fréchet topology on $\mathcal{L}_L]0, r]$. Recall the notion of a locally convex vector-space and of the Fréchet space from chapter 1.2.

Proposition 3.25. $\mathcal{L}_L]0, r]$ *together with the topology induced by the norms* $\mathcal{P} = \{| - |^{[1/n,r]} \mid n \in \mathbb{N}, n > 1/r\}$ *is a locally convex space.*

Proof. The norms $| - |^{[1/n,r]}$ for $\mathbb{N} \ni n > 1/r$ form a countable family of norms (hence seminorms) on $\mathcal{L}_L]0, r]$. By Proposition 1.11 this family of norms induces a topology on $\mathcal{L}_L]0, r]$. In Proposition 1.10 we have seen that the addition and the scalar multiplication are sequentially continuous for the norms $| - |^{[1/n,r]}$. With Proposition 1.14 therefore the addition and scalar multiplication are also continuous for the topology defined by $\mathcal{P}$. Hence $\mathcal{L}_L]0, r]$ is a topological vector-space. Since $| - |^{[1/n,r]}$ are not only seminorms but even norms we have $\bigcap_{n>1/r}\{x \colon |x|^{[1/n,r]} = 0\} = \{0\}$ by Definition 1.12 this defines a locally convex space. □

Proposition 3.26. $\mathcal{L}_L]0, r]$ *is complete with respect to the metric given by* $\mathcal{P} = \{| - |^{[1/n,r]} \mid n \in \mathbb{N}, n > 1/r\}$ *(for the metric see Proposition 1.13).*

Proof. In Proposition 3.24 we have seen that $\mathcal{L}_L[1/n, r]$ is complete with respect to the norm $| - |^{[1/n,r]}$. With the same argument as in the proof of Proposition 1.14 we find that a sequences is Cauchy with respect to the metric if and only if it is Cauchy with respect to every of the seminorms. Hence with the same Proposition we find that $\mathcal{L}_L]0, r]$ is complete with respect to the metric. □

Proposition 3.27 (See Definition 2.5.1 of [Ked06])**.** *The space* $\mathcal{L}_L]0,r]$ *is a Fréchet space with a topology induced by* $\mathcal{P} = \{|-|^{[1/n,r]} \mid n \in \mathbb{N}, n > 1/r\}$*. We will call this topology the* Fréchet-topology*.*

Proof. In Proposition 3.25 we show that $\mathcal{L}_L]0,r]$ is a locally convex vector-space, which implies that it is a topological vector-space. Since the topology is given by countable many norms the space is metrizable by Proposition 1.13 and by Proposition 1.16 this metric is translation invariant. Finally by Proposition 3.26 the space $\mathcal{L}_L]0,r]$ is complete with respect to this metric. □

3.3 Definition of the Robba ring

We again fix a finite extension L of $\mathbb{Q}_p$. We want to define the Robba ring $\mathcal{R}_L$ over L. We will often just write $\mathcal{R}$ instead of $\mathcal{R}_L$ if L can be inferred from context.

Let $\mathcal{E}$ be the set of Laurent series $\sum_{k\in\mathbb{Z}} a_k T^k$ with $a_k \in L$ that are bounded at 0 and with $\lim_{k\to-\infty} v_p(a_k) = \infty$.

Definition 3.28 (See subsection I.1.3 of [Col10])**.** The *ring of superconvergent elements of* $\mathcal{E}$, denoted by $\mathcal{E}^\dagger$, and the *Robba ring* $\mathcal{R}$ are defined as the unions

$$\mathcal{E}^\dagger = \bigcup_{r>0} \mathcal{L}_L(0,r] \qquad \mathcal{R} = \bigcup_{r>0} \mathcal{L}_L]0,r].$$

We denote by $\mathcal{E}^+$ the intersection of $\mathcal{E}^\dagger$ with $L[\![T]\!]$ and by $\mathcal{R}^+$ the intersection of $\mathcal{R}$ with $L[\![T]\!]$.

Definition 3.29 (See Definition 2.5.1 of [Ked06])**.** We equip $\mathcal{R}$ and $\mathcal{E}^\dagger$ with the direct limit of the Fréchet topologies on the $\mathcal{L}]0,r]$.

In other words a sequence (x_n) in $\mathcal{R}$ converges if for $r' \in \mathbb{R}_{>0}$ sufficiently small (x_n) converges with respect to $v^{\{r\}}$ for all $r \in]0,r']$.

Chapter 4

Properties of the Robba Ring

In this chapter we will discuss some properties of the Robba ring, that will used in proofs later. We define the Frobenius operator and an action of $\Gamma = \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$ on $\mathcal{R}$ and give a decomposition of the elements $x \in \mathcal{E}^\dagger$ into $x = x^0 T^{k(x)} x^+ x^-$ with $x^0 \in L^*$, $k(x) \in \mathbb{Z}$, $x^+ \in 1 + T\mathcal{O}_L[\![T]\!]$ and $x^- \in 1 + \mathfrak{m}_L[\![T^{-1}]\!] \cap \mathcal{E}^t$. Also we define the differential operator ∂, and the notion of the logarithm and the residue on $\mathcal{R}$.

Again fix a finite extension L of $\mathbb{Q}_p$, and denote $\mathcal{O}_L$ the ring of integers of L and $\mathfrak{m}_L$ its maximal ideal.

Let $\Gamma = \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$ and $\chi\colon \Gamma \to \mathbb{Z}_p^*$ be the cyclotomic character (see chapter 1.7).

4.1 The Frobenius operator φ and the action of Γ on $\mathcal{R}$

We want to equip $\mathcal{R}$ with a Frobenius function and some action of Γ. Later this we will use this to define the semi-linear structure of (φ, Γ)-modules over $\mathcal{R}$. The cited papers of Colmez, but also other papers about (φ, Γ)-modules over the Robba ring are giving the definition of this operator and action but do not explicitly show why they are well defined. We will work out in much detail why they are well defined, which will be at some points a bit technical.

Definition 4.1. Let $\gamma \in \Gamma$ be any element. By abuse of notation let

$$\varphi(T) = (1+T)^p - 1 \in L[T], \quad \gamma(T) = (1+T)^{\chi(\gamma)} - 1 \in L[\![T]\!].$$

We want to define linear operators $\varphi\colon \mathcal{R} \to \mathcal{R}$ with $T \mapsto (1+T)^p - 1$ and $\gamma\colon \mathcal{R} \to \mathcal{R}$ with $T \mapsto (1+T)^{\chi(\gamma)} - 1$. In other words for $x \in \mathcal{R}$ we let $\varphi(x) = x \circ \varphi(T)$ and $\gamma(x) = x \circ \gamma(T)$ for the just defined polynomials. For positive power series this is easily seen to be well defined. For negative power series we first need to define $\varphi(T^{-1}) = \varphi(T)^{-1}$. Since the inverse of a power series is not necessarily unique this will need some work.

Lemma 4.2. *The polynomial $\varphi(T) = (1+T)^p - 1$ has $p-1$ roots of valuation $v_p(T) = \frac{1}{p-1}$ and one root at $T = 0$.*

We will use Newton polygons to proof this statement. The Theory of Newton polygons can be found in [Gou97] in Chapter 6.4.

Proof. We will compute the Newton polygon of

$$(1+T)^p - 1 = T^p + pT^{p-1} + ... + pT = pT(p^{-1}T^{p-1} + T^{p-2} + ... + 1).$$

Clearly the valuation of the coefficients $v_p(\binom{p}{i}/p)$ equals -1 for $i = p$ and $i = 0$. Therefore we get a polygon that consists of only one line segment with breaks $(0,0)$ and $(p-1,-1)$ and Newton slope $m = -1/(p-1)$ of length $p-1$. By Proposition 6.4.6 in [Gou97] the polynomial $\varphi(T)$ has exactly $p-1$ roots of valuation $v_p(T) = -m = 1/(p-1)$ and one root at $T = 0$. $\square$

We want to show that φ is well-defined on the whole of $\mathcal{R}$. However, as we will see, something that seems odd occurs when we try to compute $\varphi(\frac{1}{T})$, i.e. when we want to invert the polynomial $\varphi(T)$ as a Laurent-series. There are different ways to invert a polynomial as a Laurent series, that are defined on different annulus. We will first make another example to make the case clear.

Example 4.3. We would like to invert the series $1-T$. We can do this on $\mathcal{L}_L]0,\infty]$ by using the geometric series because for $|T| < 1$ we have $\frac{1}{1-T} = \sum_{i=0}^{\infty} T^i$. But we can also transform $\frac{1}{1-T}$ into a Laurent series that converges on $|T| > 1$ namely $\frac{1}{1-T} = -T^{-1}\frac{1}{1-T^{-1}} = -\sum_{i=1}^{\infty} T^{-i} \in \mathcal{L}_L]-\infty, 0[$. So there are two different ways how we can regard $\frac{1}{1-T}$ as a Laurent series.

Proposition 4.4. *Let $p(T) \in L[T]$ be a polynomial in L with roots $a_i \in C_p$, $v_p(a_i) = \alpha_i$. Then we can find two Laurent series that define* *an inverse of* $p(T)$*, each on a different annulus of convergence* $v_p(T) > \max\{\alpha_i\}$ *resp.* $v_p(T) < \min\{\alpha_i\}$.

Proof. This proof is the result of a discussion with Torsten Schoeneberg on "math.stackexchange.com". Let $p(T) = T^{n_0}\prod_{i=1}^{k}(T-\alpha_i)^{n_i}$ over C_p with $\alpha_i \neq \alpha_j$ for $i \neq j$. With partial fractions we can then write

$$\frac{1}{p(T)} = \frac{p_{0,1}(T)}{T} + ... + \frac{p_{0,n_0}(T)}{T^{n_0}} + \sum_{i=1}^{k}\left(\frac{p_{i,1}(T)}{(T-\alpha_i)} + ... + \frac{p_{i,n_i}(T)}{(T-\alpha_i)^{n_i}}\right)$$

with $p_{i,j} \in L[T]$. If all of the summands are converging on $r \in \mathbb{R}$ also the sum is converging on r. So it remains to check where $\frac{p_{i,j}(T)}{(T-\alpha_i)^j}$ is convergent for $1 \leq j \leq n_i$. On the annulus $v_p(T) > v_p(\alpha_i)$ we have $v_p(T/\alpha_i) > 0$, so there we can transform

$$\begin{aligned}\frac{p_{i,j}(T)}{(T-\alpha_i)^j} &= p_{i,j}(T)\frac{(-1/\alpha_i)^j}{(1-T/\alpha_i)^j}\\ &= p_{i,j}(T)(-1/\alpha_i)^j\left(\sum_{k=0}^{\infty}\left(\frac{T}{\alpha_i}\right)^k\right)^j\\ &= p_{i,j}(T)(-1/\alpha_i)^j\sum_{k=j-1}^{\infty}\binom{k}{j-1}\left(\frac{T}{\alpha_i}\right)^k.\end{aligned}$$

Therefore if $n_0 = 0$ on the annulus $v_p(T) > \max_{i\in\{1,...,k\}}\{v_p(\alpha_i)\}$, the Laurent series with summands as defined above is convergent. If $n_0 \neq 0$ we also have to invert T^{n_0}, so the annulus where this series converges is $\infty > v_p(T) > \max_{i\in\{1,...,k\}}\{v_p(\alpha_i)\}$.

On the other hand we can also express $\frac{p_{i,j}(T)}{(T-\alpha_i)^j}$ in the following way for $v_p(T) < v_p(\alpha_i)$:

$$\begin{aligned}\frac{p_{i,j}(T)}{(T-\alpha_i)^j} &= p_{i,j}(T)T^{-j}\frac{1}{(1-\alpha_i T^{-1})^j}\\ &= p_{i,j}(T)T^{-j}\left(\sum_{k=0}^{\infty}(\alpha_i T^{-1})^k\right)^j\\ &= p_{i,j}(T)T^{-j}\sum_{k=j-1}^{\infty}\binom{k}{j-1}\alpha_i^k T^{-k}.\end{aligned}$$

Hence on the annulus $v_p(T) < \min_{i\in\{1,...,k\}}\{v_p(\alpha_i)\}$ the Laurent series with summands as defined above is convergent. □

Remark. Let α be a root of the polynomial $p(T)$. On the annulus $v_p(T) = v_p(\alpha)$ clearly there is no inverse to the polynomial $p(T)$. If this would be the case we could find an inverse of the 0, since $p(\alpha) \cdot p^{-1}(\alpha) = 1$ and $p(\alpha) = 0$.

So also for the case of the operator given by $\varphi \colon T \mapsto (1+T)^p - 1$ we have two different possibilities to express $\varphi(T^{-1})$ as an element of $\mathcal{L}_L$. We need to pick the right expression to make φ well-defined.

Proposition 4.5. *We can invert $\varphi(T)$ either on the annulus $v_p(T) < \frac{1}{p-1}$ or on the annulus $\frac{1}{p-1} < v_p(T) < \infty$.*

Proof. In Lemma 4.2 we have calculated the valuations of the non-zero roots of $\varphi(T)$ to be $1/(p-1)$. So with Proposition 4.4 this shows that on $v_p(T) < \frac{1}{p-1}$ we can find an inverse of $\varphi(T)$ as well as on the annulus $\infty > v_p(T) > \frac{1}{p-1}$, since the zero is also a root of $\varphi(T)$. $\square$

From now on define $\varphi(T)^{-1}$ to be the inverse defined on $v_p(T) < \frac{1}{p-1}$.

Proposition 4.6. *Let $x_+ \in L[\![T]\!]$ converge on $v_p(T) > r \in \mathbb{R}$ for $r < \frac{p}{p-1}$, then $\varphi(x_+)$ converges on $v_p(T) > r/p$.*

Proof. Clearly $\varphi(0) = 0$, so we can use Proposition 1.46 to calculate on which annulus the series $\varphi(x_+) = x_+ \circ \varphi(T)$ converges. So using this Proposition we see that $\varphi(x_+)$ converges on any $a \in C_p$ with $v_p(a) > r_{\varphi(T)} = -\infty$ and $v^{\{v_p(a)\}}(\varphi(T)) > \tilde{r}_{x^+} = 0$. So combined with Example 3.9

$$v^{\{v_p(a)\}}(\varphi(T)) = \begin{cases} v_p(a) + 1 & \text{if } v_p(a) \geq \frac{1}{(p-1)} \\ pv_p(a) & \text{if } v_p(a) < \frac{1}{(p-1)} \end{cases}$$

so since $r \leq \frac{p}{p-1}$ this means $v^{\{v_p(a)\}}(\varphi(T)) > r$ if $v_p(a) > r/p$ and hence $\varphi(x_+)$ converges on $v_p(T) > r/p$. $\square$

Lemma 4.7. *A power-series series $x = \sum_{i \in \mathbb{N}} a_i T^i \in$ converges on $v_p(T) = r$ if and only if $x \circ T^{-1} = \sum_{i \in \mathbb{N}} a_i T^{-i} \in$ converges on $v_p(T) = -r$.*

Proof. x converges on r if and only of $\lim_{i \to \infty} a_i + ir \to \infty$. This is the case if and only if $\lim_{i \to \infty} a_i - i(-r) \to \infty$ in other words if $x \circ T^{-1}$ converges on $-r$. $\square$

Proposition 4.8. *On $v_p(T) < \frac{1}{p-1}$ the inverse of $\varphi(T)$ is an element of $T^{-p}(1 + T^{-1}\mathfrak{m}_L[\![T^{-1}]\!])$.*

Proof. On $v_p(T) < \frac{1}{p-1}$ we can calculate

$$\begin{aligned}\varphi(T)^{-1} &= ((1+T)^p - 1)^{-1} = (T^p(pT^{-p+1} + ... + pT^{-1} + 1))^{-1} \\ &= T^{-p}(1 + a_1T^{-1} + ...) \in T^{-p}(1 + T^{-1}\mathfrak{m}_L[\![T^{-1}]\!])\end{aligned}$$

where the last step is not proven yet, but will be in Proposition 4.21. $\square$

Proposition 4.9. *Let $x_- \in L[\![T^{-1}]\!]$ converge on $v_p(T) < r$ with $r < \frac{1}{p-1}$. Then $\varphi(x_-)$ converges on $v_p(T) < r/p$.*

Proof. We want to use Proposition 1.46, which tells us something about the convergence of the composition of power series. x_- is not a power series, therefore we will restate the problem in a slightly different way. Since $r < \frac{1}{p-1}$ the series $\varphi(T)^{-1}$ is of the form

$$T^{-p}(1 + b_1T^{-1} + b_2T^{-2}...) \in T^{-p}(1 + T^{-1}\mathfrak{m}_L[\![T^{-1}]\!])$$

by Proposition 4.8. Let

$$\tilde{x}_- = x_- \circ T^{-1} = \sum_{i=1}^{\infty} a_iT^i$$

and

$$\tilde{\varphi}(T) = (T^{-1} \circ \varphi(T)) \circ T^{-1} = T^p(1 + b_1T^1 + b_2T^2...)$$

such that

$$\begin{aligned}x_- \circ \varphi(T) &= x_- \circ T^{-1} \circ T^{-1} \circ \varphi(T) \circ T^{-1} \circ T^{-1} \\ &= \tilde{x}_- \circ \tilde{\varphi}(T) \circ T^{-1}.\end{aligned}$$

So by Lemma 4.7 the series $x_- \circ \varphi(T)$ converges on $v_p(T) < r'$ if and only if $\tilde{x}_- \circ \tilde{\varphi}(T)$ converges on $v_p(T) > -r'$.

For calculating where $\tilde{x}_- \circ \tilde{\varphi}(T)$ converges we can use Proposition 1.46, since $\tilde{x}$ and $\tilde{\varphi}(T)$ are both power series and $\tilde{\varphi}(T) = 0$. By Lemma 4.7 the series $\tilde{x}_-$ converges on $-r < v_p(T)$, while the series $\tilde{\varphi}(T)$ converges for $v_p(T) > -\frac{1}{p-1}$. Hence $\tilde{x}_- \circ \tilde{\varphi}(T)$ converges for any $a \in \mathbb{C}_p$ with $v_p(a) > -\frac{1}{p-1}$ and $v^{\{v_p(a)\}}(\tilde{\varphi}(T)) > -r$. We have

$$\begin{aligned} 1 &= (T^{-1} \circ \varphi(T)) \cdot \varphi(T) \\ \iff 1 &= (T^{-1} \circ \varphi(T) \circ T^{-1}) \cdot (\varphi(T) \circ T^{-1}) \\ &= \tilde{\varphi}(T) \cdot (\varphi(T) \circ T^{-1})\end{aligned}$$

and since $v^{\{v_p(a)\}}$ is a valuation therefore

$$\begin{aligned}
v^{\{v_p(a)\}}(\tilde{\varphi}(T)) &= -v^{\{v_p(a)\}}(\varphi(T) \circ T^{-1}) \\
&= -v^{\{v_p(a)\}}((1+T^{-1})^p - 1) \\
&= -\inf\{1 - v_p(a), ..., 1-(p-1)v_p(a), -pv_p(a)\} = pv_p(a)
\end{aligned}$$

since $v_p(a) > -r > -\frac{1}{p-1}$.

$$v^{\{v_p(a)\}}(\tilde{\varphi}(T)) > -r \iff pv_p(a) > -r \iff v_p(a) > -r/p$$

Hence we can conclude that $\tilde{x}_- \circ \tilde{\varphi}$ converges on $v_p(T) > -r/p$ which implies that $\varphi(x_-) = x_- \circ \varphi = \tilde{x}_- \circ \tilde{\varphi} \circ T^{-1}$ converges on $v_p(T) < r/p$.

□

Proposition 4.10. *Let $0 < r \in \mathbb{R} < \frac{1}{p-1}$. If $x \in \mathcal{L}_L]0, r]$ then $\varphi(x) \in \mathcal{L}_L]0, r/p]$.*

Proof. Recall Proposition 1.46. To use this Proposition we write x as sum of two series, $x_+ \in L[\![T]\!]$ and $x_- \in TL[\![T^{-1}]\!]$, such that $x_+ + x_- = x$. By Proposition 1.42 then the series are convergent on $v_p(T) > r_1$, resp. $v_p(T) < r_2$. Since $\varphi(x) = \varphi(x_+ + x_-) = \varphi(x_+) + \varphi(x_-)$ we can calculate the interval, where the both series $\varphi(x_+)$ and $\varphi(x_-)$ converge separately and the sum $\varphi(x)$ will converge on the intersection of the two intervals. In Proposition 4.9 we have shown that $\varphi(x_-)$ converges on $v_p(T) < r/p$, while $\varphi(x_+)$ converges on $v_p(T) > 0$ by Proposition 4.6. Therefore $\varphi(x) = \varphi(x_+) + \varphi(x_-)$ converges on $0 < v_p(T) < r/p$. □

Remark. Whenever we used the notion x converges on $[r_1, r_2]$ we do not mean that x converges exclusively in this interval. It could be that x converges on a bigger interval as well.

Lemma 4.11. *$\gamma(T)$ is invertible on $0 < v_p(T) < \infty$.*

Proof.

$$\begin{aligned}
\gamma(T) &= (1+T)^{\chi(\gamma)} - 1 = \sum_{i=1}^{\infty} \binom{\chi(\gamma)}{i} T^i \\
&= T\chi(\gamma)(1 + \sum_{i=1}^{\infty} \chi(\gamma)^{-1} \binom{\chi(\gamma)}{i} T^{i-1}) \text{ with } v_p(\binom{\chi(\gamma)}{i}\chi(\gamma)^{-1}) \geq 0
\end{aligned}$$

So as we will see in Proposition 4.21 we can invert $\gamma(T)$ such that $\gamma(T)^{-1} = \chi(\gamma)^{-1}T^{-1}x^+$with $x^+ \in 1 + T\mathbb{Z}_p[\![T]\!]$ is convergent on $0 < v_p(T) < \infty$. □

From now we fix this inverse.

Proposition 4.12. *If $x = \sum_{i\in\mathbb{Z}} a_i T^i \in \mathcal{L}]0,r]$ then $\gamma(x) \in \mathcal{L}]0,r)$. Furthermore if x is bounded at 0, then so is $\gamma(x)$.*

Proof. We can decompose $x = x_+ + x_-$ for $x_+ = \sum_{i=0}^{\infty} a_i T^i$ and $x_- = \sum_{i=-\infty}^{-1} a_i T^i$. Then

$$\gamma(x) = x \circ \gamma(T) = x_+ \circ \gamma(T) + x_- \circ \gamma(T)$$

converges if both $x_+ \circ \gamma(T)$ and $x_- \circ \gamma(T)$ converge.

$$\begin{aligned} v^{\{r\}}(\gamma(x_+)) = v^{\{r\}}(\sum_{i=0}^{\infty} a_i \gamma(T)^i) &\geq \inf_{i\in\mathbb{N}}\{v^{\{r\}}(a_i\gamma(T)^i)\} \\ &= \inf_{i\in\mathbb{N}}\{v^{\{r\}}(a_iT^i(\sum_{j=1}^{\infty}\binom{\chi(\gamma)}{j}T^{j-1}))\} \\ &= \inf_{i\in\mathbb{N}}\{v_p(a_i) + ir + 0\} \\ &= v^{\{r\}}(x_+) \end{aligned}$$

So if x_+ converges on $v_p(T) > 0$ then also $\gamma(x_+)$ converges on $v_p(T) > 0$ and if x_+ is bounded at 0 then so is $\gamma(x_+)$.

For x_- note first that $\gamma(T)^{-1}$ by Proposition 4.11 is equal to $\chi(\gamma)^{-1}T^{-1}y^+$ for some $y^+ \in 1 + T\mathbb{Z}_p[\![T]\!]$ and hence converges on $0 < v_p(T) < \infty$.

$$\begin{aligned} v^{\{r\}}(\gamma(x_-)) = v^{\{r\}}(\sum_{i=1}^{\infty} a_{-i}\gamma(T)^{-i}) &\geq \inf_{i\in\mathbb{N}}\{v^{\{r\}}(a_{-i}(\gamma(T)^{-1})^i)\} \\ &= \inf_{i\in\mathbb{N}}\{v^{\{r\}}(a_{-i}T^{-i}(\chi(\gamma))^{-1}y^+)\} \\ &= \inf_{i\in\mathbb{N}}\{v_p(a_{-i}) - ir + 0\} \\ &= v^{\{r\}}(x_-). \end{aligned}$$

So if x_- converges on $v_p(T) \leq r$ then $\gamma(x_-)$ converges on $v_p(T) < r$ by proposition 3.8. $\square$

Definition 4.13 (See subsection I.2 of [Col10])**.** We define the operators $\varphi\colon \mathcal{R} \to \mathcal{R}$ and $\gamma\colon \mathcal{R} \to \mathcal{R}$ by

$$\begin{aligned} \varphi&\colon T \mapsto \varphi(T) = (1+T)^p - 1, \\ \gamma&\colon T \mapsto \gamma(T) = (1+T)^{\chi(\gamma)} - 1. \end{aligned}$$

Corollary 4.14. *These operators are well-defined.*

Proposition 4.15 (Statement in subsection I.2 of [Col10])**.** *$\varphi\colon \mathcal{R} \to \mathcal{R}$ and $\gamma\colon \mathcal{R} \to \mathcal{R}$ are continuous.*

Proof. Recall that a map is continuous for the Fréchet-topology on $\mathcal{R}$, if it is sequentially continuous for any $v^{\{r\}}$ with $0 < r < r'$ for some $r' > 0$ sufficiently small. So let $x_n = \sum_{i\in\mathbb{Z}} a_{n,i}T^i \in \mathcal{L}_L]0, r']$ be a series that converges to $x = \sum_{i\in\mathbb{Z}} a_iT^i \in \mathcal{L}_L]0, r']$ for all $v^{\{r\}}$ with $0 < r \leq r'$.

$$\begin{aligned}
v^{\{r\}}(\sum_{i\in\mathbb{Z}} a_{n,i}\varphi(T)^i - \sum_{i\in\mathbb{Z}} a_i\varphi(T)^i)& \\
&= v^{\{r\}}(\sum_{i\in\mathbb{Z}}(a_{n,i} - a_i)\varphi(T)^i) \\
&\geq \inf_{i\in\mathbb{Z}}\{v^{\{r\}}((a_{n,i} - a_i)\varphi(T)^i)\} \\
&= \inf_{i\in\mathbb{Z}}\{v_p(a_{n,i} - a_i) + v^{\{r\}}(T^i(p + ... + T^{p-1})^i)\} \\
&\geq \inf_{i\in\mathbb{Z}}\{v_p(a_{n,i} - a_i) + ir\} \\
&= v^{\{r\}}((a_{n,i} - a_i)T^i)
\end{aligned}$$

Therefore with $\lim_{n\to\infty} v^{\{r\}}(a_{n,i} - a_i)T^i = \infty$, also $\varphi(x_n)$ converges to $\varphi(x)$. For γ we can make the same kind of argument. □

Lemma 4.16 (See subsection I.2 of [Col10])**.** *For $\gamma \in \Gamma$ the induced operators $\gamma\colon \mathcal{R} \to \mathcal{R}$ define a group action of Γ on $\mathcal{R}$.*

Proof. Let $\gamma, \gamma' \in \Gamma$ and $\mathrm{id} \in \Gamma$ be the identity element.

$$\begin{aligned}
(\gamma \circ \gamma')(T) &= (1 + T)^{\chi(\gamma\circ\gamma')} - 1 = (1 + T)^{\chi(\gamma)\chi(\gamma')} - 1 \\
\gamma(\gamma'(T)) &= (((1 + T)^{\chi(\gamma)} - 1) + 1)^{\chi(\gamma')} - 1 \\
&= (1 + T)^{\chi(\gamma)\chi(\gamma')} - 1 \\
\mathrm{id}(T) &= (1 + T) - 1 = T
\end{aligned}$$

□

Lemma 4.17. *The action of Γ commutes with φ and $\varphi^n(T) = (1 + T)^{p^n} - 1$ for each $n \in \mathbb{N}$.*

Proof. Let $\gamma \in \Gamma$.

$$\begin{aligned}
\gamma(\varphi(T)) &= (((1 + T)^{\chi(\gamma)} - 1) + 1)^p - 1 \\
&= (1 + T)^{\chi(\gamma)p} - 1 \\
&= \varphi(\gamma(T)).
\end{aligned}$$

By the same calculation with p^{n-1} instead of $\chi(\gamma)$ inductively we get the second claim. □

4.2 The decomposition of elements of the Robba ring

The goal of this subsection is to find a unique decomposition of elements $x \in (\mathcal{E}^\dagger)^*$ into a product of a positive degree power series and a negative degree power series, both elements of $(\mathcal{E}^\dagger)^*$. More precisely we will show that any $x \in (\mathcal{E}^t)^*$ uniquely factorizes into $x = x^0 T^{k(x)} x^+ x^-$ with $x^0 \in L^*$, $k(x) \in \mathbb{Z}$, $x^+ \in 1 + T\mathcal{O}_L[\![T]\!]$ and $x^- \in 1 + \mathfrak{m}_L[\![T^{-1}]\!] \cap \mathcal{E}^t$. This is a result that is used by Colmez in [Col05] without reference. Similar statements can be found in [Laz62] and in [Ked04], there with proof. We will give a variation of the proof in [Ked04] where a more general statement about matrices with entries in $\mathcal{R}$ is made.

Before this we need to make some observations about invertible series in $\mathcal{R}$.

Recall definition 3.5 for $f \in \mathcal{L}_L$ and $r \in \mathrm{Conv}(f)$:

$$n(f,r) = \inf \left\{ i \in \mathbb{Z} \mid v^{\{r\}}(f) = v_p(a_i) + ir \right\}$$
$$N(f,r) = \sup \left\{ i \in \mathbb{Z} \mid v^{\{r\}}(f) = v_p(a_i) + ir \right\}$$

Lemma 4.18. *(Proposition 4 of [Laz62]) Let $-\infty < r_1 \leq r_2 \in \mathcal{R} \cup \{\infty\}$. An element $f \in \mathcal{L}_L[r_1, r_2]$ is invertible in $\mathcal{L}_L[r_1, r_2]$ if and only if $N(f, r_1) = n(f, r_2)$.*

Proposition 4.19. *An element $x^- \in 1 + \mathfrak{m}_L[\![T^{-1}]\!] \cap (\mathcal{E}^\dagger)^*$ has the property $v^{\{r\}}(x^- - 1) > 0$ for all r small enough.*

Proof. Assume for any $r > 0$ there is $0 < r' < r$ such that $v^{\{r'\}}(x^- - 1) \leq 0$. By Proposition 4.18 x^- is invertible in $\mathcal{L}_L]0, r]$ if and only if there is an $i \in \mathbb{Z}$ such that $v^{\{r''\}} = v_p(x^{-(i)}) + ir$ for all $r'' \in]0, r]$. Since $v_p(1) = 0$ therefore $v^{\{r\}}(x^- - 1) < 0$. Now we can let r go to 0 and get that $v_p(a_i) \leq 0$. This is a contradiction with $v_p(a_i) > 0$ because $x^- \in 1 + \mathfrak{m}_L[\![T^{-1}]\!]$. □

Proposition 4.20. *Any $x^+ \in 1 + T\mathcal{O}_L[\![T]\!]$ is an element of $\mathcal{E}^\dagger$.*

Proof. Let $x^+ = \sum_{i=0}^\infty a_i T^i$ with $a_0 = 1$ and $v_p(a_i) \geq 0$ for all $i \in \mathbb{N}$ so the coefficients are clearly bounded. Let $r > 0$.

$$\lim_{i \to \infty} v_p(a_i) + ir \geq \lim_{i \to \infty} ir = \infty$$

so x converges for all $r > 0$. □

Proposition 4.21. *The elements* $x^+ \in 1 + T\mathcal{O}_L[\![T]\!]$ *and* $x^- \in 1 + \mathfrak{m}_L[\![T^{-1}]\!] \cap (\mathcal{E}^\dagger)^*$ *as in Theorem 4.23 are are invertible in* $1+T\mathcal{O}_L[\![T]\!]$*, resp.* $1+\mathfrak{m}_L[\![T^{-1}]\!] \cap (\mathcal{E}^\dagger)^*$.

Proof. Let $x^+ = 1 + \sum_{i=1}^\infty a_i T^i$ with $v_p(a) \geq 0$. Define $(x^+)^{-1} = 1 + \sum_{i=1}^\infty b_i T^i$ recursively by setting $a_0, b_0 = 1$ and $b_i = -\sum_{k=1}^i a_k b_{i-k}$. Then

$$x^+ \cdot (x^+)^{-1} = \sum_{i=0}^\infty \sum_{k=0}^i a_k b_{i-k} T^i = a_0 b_0 + \sum_{i=1}^\infty (\sum_{k=1}^i a_k b_{i-k} + b_i) T^i = 1.$$

If we suppose that $v_p(b_l) \geq 0$ for all $l < i$, we have

$$\begin{aligned} v_p(b_i) = v_p(-\sum_{k=1}^i a_k b_{i-k}) &\geq \min_{1\leq k\leq i} \{v_p(a_k b_{i-k})\} \\ &\geq \min_{1\leq k\leq i}\{v_p(a_k) + v_p(b_{i-k})\} \geq 0 + 0 = 0 \end{aligned}$$

So inductively we get $v_p(b_i) \geq 0$ for all $i \in \mathbb{N}$.

Let $x^- = \sum_{i=0}^\infty a_i T^{-i}$ with $a_i \in \mathfrak{m}_L$ for all $i > 0$, in other words $v_p(a_i) > 0$, and $v_p(a_0) = 0$. Define again $(x^-)^{-1} = \sum_{i=0}^\infty b_i T^{-i}$ recursively by setting $b_0 = a_0^{-1}$ and again $b_i = -\sum_{k=1}^1 a_k b_{i-k}$ for $i > 0$. With the same calculation as above we get $x^- \cdot (x^-)^{-1} = 1$. From Proposition 4.19 we know for $r \in \mathbb{R}$ such that x^- converges on radius r we have $v_p(a_i) - ir > v^{\{r\}}(x_- - 1) > 0$ for all $i > 0$. Inductively

$$v_p(b_i) \geq \min_{1\leq k\leq i}\{v_p(a_k) + v_p(b_{i-k})\} > \min_{1\leq k\leq i}\{kr + (i-k)r\} = ir$$

so $v^{\{r\}}((x^-)^{-1}) = 0$ and therefore x^- converges on $v_p(T) < r$ by Proposition 3.8. $\square$

Proposition 4.22. *Let* $x \in \mathcal{L}]0, r']$ *such that for any* $r \in (0, r']$ *we can find* $c \in \mathbb{R}_{>0}$ *with* $v^{\{r\}}(x-1) \geq c > 0$*. Then there exist unique* $u = 1 + \sum_{i=1}^\infty u_i T^i$ *and* $v = 1 + \sum_{i=0}^\infty v_i T^{-i}$ *in* $\mathcal{R}$ *with* $v^{\{r\}}(u-1) > 0$ *and* $v^{\{r\}}(v-1) > 0$ *for all* $r \in (0, r']$*, such that* $x = u \cdot v$.

Proof. The proof is a variation of the proof of Lemma 6.4 of [Ked04].

In the following we want to define sequences $(b_j)_{j\in\mathbb{N}}$ resp. $(d_j)_{j\in\mathbb{N}}$ that converge to v resp. $u-1$ with respect to the Fréchet topology. In Proposition 1.14 we have shown that a sequence is convergent with respect to the Fréchet topology if and only if it is convergent with respect to any norm $|-|^{[1/n, r']}$ with $1/n < r'$. This is the case if and only if the function converges for any norm $|-|^{\{r\}}$ with $r \in (0, r']$. Let $r \in (0, r']$.

(I) The construction of the sequences $(b_j)_{j\in\mathbb{N}}$ and $(d_j)_{j\in\mathbb{N}}$: We set $b_0 = 1$ and define the sequences $(b_j)_{j\in\mathbb{N}}$, $(c_j)_{j\in\mathbb{N}}$ and $(d_j)_{j\in\mathbb{N}}$ recursively by writing xb_j^{-1} as $xb_j^{-1} = \sum_{i=-\infty}^{\infty} a_{j,i}T^i$ and setting

$$\begin{aligned} c_j &= \sum_{i\leq 0} a_{j,i}T^i \\ d_j &= \sum_{i>0} a_{j,i}T^i \\ b_{j+1} &= c_j \cdot b_j. \end{aligned}$$

We will see in (II) that b_j is actually invertible and hence these sequences are well-defined.

(II) $v^{\{r\}}(c_j - 1) \geq c$ and $v^{\{r\}}(d_j) \geq c$ for all $j \in \mathbb{N}$: By definition c_0 and d_0 each consists of some of the terms of x, hence $v^{\{r\}}(c_0 - 1) \geq v^{\{r\}}(x - 1) = c$ and $v^{\{r\}}(d_0) \geq v^{\{r\}}(x - 1) = c$. Also we have

$$\begin{aligned} v^{\{r\}}(xb_1^{-1} - 1) &= v^{\{r\}}(xb_0^{-1}c_0^{-1} - 1) = v^{\{r\}}(c_0^{-1}(x - 1 + 1 - c_0)) \\ &\geq -v^{\{r\}}(c_0) + \min(v^{\{r\}}(x - 1), v^{\{r\}}(c_0 - 1)) \\ &\geq 0 + c = c \end{aligned}$$

since $v^{\{r\}}(c_0 - 1) \geq c$ so $v^{\{r\}}(c_0) = v_p(1) = 0$. Supposing $v^{\{r\}}(xb_j^{-1} - 1) \geq c$, $v^{\{r\}}(c_{j-1} - 1) \geq c$ and $v^{\{r\}}(d_{j-1}) \geq c$ we get $v^{\{r\}}(c_j - 1) \geq v^{\{r\}}(xb_j^{-1} - 1) \geq c$ and $v^{\{r\}}(d_j) \geq v^{\{r\}}(xb_j^{-1} - 1) \geq c$ and with this

$$\begin{aligned} v^{\{r\}}(xb_{j+1}^{-1} - 1) &= v^{\{r\}}(xb_j^{-1}c_j^{-1} - 1) = v^{\{r\}}(c_j^{-1}(xb_j^{-1} - 1 + 1 - c_j)) \\ &\geq -v^{\{r\}}(c_j) + \min(v^{\{r\}}(xb_j^{-1} - 1), v^{\{r\}}(c_j - 1)) \\ &\geq 0 + c = c. \end{aligned}$$

So by induction this holds for all $j > 0$. $v^{\{r\}}(c_j - 1) > c > 0$, so c_j is invertible (see Proposition 4.21) and therefore inductively also $b_{j+1} = b_jc_j$ is invertible for all $j \geq 0$.

(III) $(c_j)_{j\in\mathbb{N}}$ converges towards 1: More specific we show that $v^{\{r\}}(c_j - 1) \geq (j + 1)c$, again by induction. We have just seen that this is true for $j = 0$. So suppose for $j \in \mathbb{N}$ we have $v^{\{r\}}(c_j - 1) \geq (j + 1)c$ given. In $c_{j+1} - 1$ we collect all terms with negative powers of T (and the constant term) from

$$xb_{j+1}^{-1} - 1 = xb_j^{-1}c_j^{-1} - 1 = c_j^{-1}(c_j + d_j) - 1 = d_jc_j^{-1} = d_j + d_j(c_j^{-1} - 1)$$

and since d_j only has terms with positive powers of T we get

$$\begin{aligned}
v^{\{r\}}(c_{j+1}-1) &\geq v^{\{r\}}(d_j(c_j^{-1}-1)) = v^{\{r\}}(d_j) + v^{\{r\}}(c_j^{-1}(1-c_j)) \\
&= v^{\{r\}}(d_j) - v^{\{r\}}(c_j) + v^{\{r\}}(1-c_j) \\
&\geq c - 0 + (j+1)c = (j+2)c.
\end{aligned}$$

Hence the sequence $(c_j)_{j\in\mathbb{N}}$ converges towards 1.

(IV)$(d_j)_{j\in\mathbb{N}}$ defines a Cauchy sequence, i.e. $v^{\{r\}}(d_{j+1}-d_j) \geq (j+2)c$ for $j \geq 0$:

$$\begin{aligned}
d_{j+1} - d_j &= xb_{j+1}^{-1} - c_{j+1} - d_j = xc_j^{-1}b_j^{-1} - c_{j+1} - d_j \\
&= c_j^{-1}(c_j + d_j) - c_{j+1} - d_j = 1 + d_jc_j^{-1} - c_{j+1} - d_j \\
&= d_j(c_j^{-1}-1) + 1 - c_{j+1}
\end{aligned}$$

and since d_{j+1} and d_j only consist of terms with positive powers of T, while c_{j+1} and 1 have no positive powers of T, we have

$$\begin{aligned}
v^{\{r\}}(d_{j+1}-d_j) &\geq v^{\{r\}}(d_j(c_j^{-1}-1)) = v^{\{r\}}(d_jc_j^{-1}(1-c_j)) \\
&= v^{\{r\}}(d_j) - v^{\{r\}}(c_j) + v^{\{r\}}(1-c_j) \\
&\geq c + 0 + (j+1)c = (j+2)c.
\end{aligned}$$

Hence $(d_j)_{j\in\mathbb{N}}$ defines a Cauchy sequence.

(V) Now we can define u and v and show that they have the desired properties. Since the above holds for any $r \in (0, r']$ the sequence $(c_j)_{j\in\mathbb{N}}$ converges to 1 with respect to the Fréchet topology. This means that the sequence $(b_j)_{j\in\mathbb{N}}$ is a Cauchy sequence with respect to the Fréchet topology. As we have seen also $(d_j)_{j\in\mathbb{N}}$ defines a Cauchy sequence with respect to the Fréchet topology. In Proposition 3.26 we have seen that the space $\mathcal{L}_L]0, r']$ is complete. Hence $(b_j)_{j\in\mathbb{N}}$ and $(d_j)_{j\in\mathbb{N}}$ are converging towards an element of $\mathcal{L}_L(0, r']$. We set $u := \lim_{j\to\infty}(d_j)+1$ and $v := \lim_{j\to\infty} b_j$.

By taking the limit, the equation $xb_j^{-1} = c_j + d_j$ converges towards $xv^{-1} = u$, hence $x = uv$. Furthermore $v^{\{r\}}(v-1) \geq c$ and $v^{\{r\}}(u-1) \geq c$.

(VI) It remains to show that the so found elements are indeed unique. So suppose there is another decomposition $x = u'v'$ with the required properties. Then u, v, u', and v' are all invertible in $\mathcal{R}$ as we have seen in Proposition 4.21 and $u'v' = uv$ implies $u'u^{-1} = vv'^{-1}$. But $u'u^{-1}-1$ has only terms with positive powers of T, while $vv'^{-1}-1$ has no terms with positive powers of T. This can only be true if $u'u^{-1}-1 = vv'^{-1}-1 = 0$, so if $u = u'$ and $v = v'$. □

Proposition 4.23 (Statement in the proof of Lemma 4.1 of [Col05]). *We can factorize any element $x \in (\mathcal{E}^t)^*$ uniquely into $x = x^0 T^{k(x)} x^+ x^-$ with $x^0 \in L^*$, $k(x) \in \mathbb{Z}$, $x^+ \in 1 + T\mathcal{O}_L [\![T]\!]$ and $x^- \in 1 + \mathfrak{m}_L[\![T^{-1}]\!] \cap \mathcal{E}^t$.*

Proof. First note that since x is invertible in $\mathcal{E}^\dagger$ we can find $r' \in \mathbb{R}$ such that $x, x^{-1} \in \mathcal{L}_L(0, r']$. We have $N(x, r_1) = n(x, r_2)$ for any $0 < r_1, r_2 \leq r'$ by Proposition 4.18. From this we can derive that there is one unique place n, where the infimum in $v^{\{r'\}}(x) = \inf_{n \in \mathbb{Z}}(v_p(x^{(n)}) + nr)$ is attained for any $0 < r \leq r'$. Let $k(x) = n(x, r')$ and $x^0 = x^{(k(x))}$, the $k(x)$-th coefficient of x. Take $\tilde{x} = xT^{-k(x)}x^{0^{-1}}$. Then we have $v_p(\tilde{x}^{(0)}) = v_p(1) = 0$, $n(\tilde{x}, r') = 0 = N(\tilde{x}, r')$ and therefore, since the coefficients of x are elements of L, and hence have discrete valuation, for all $0 < r' \leq r$ there is some $c \in \mathbb{R}_{>0}$ with $v^{\{r'\}}(\tilde{x} - 1) \geq c > 0$.

Now we can use Lemma 4.22 to find $u = 1 + \sum_{i=1}^{\infty} u_i T^i$ and $v = \sum_{i=0}^{\infty} 1 + v_i T^{-i}$ in $\mathcal{R}$, such that $v^{\{r\}}(u - 1) > 0$, $v^{\{r\}}(v - 1) > 0$ for all $r \in (0, r']$ and $\tilde{x} = u \cdot v$. Then for all $r \in (0, r']$

$$\begin{aligned} v^{\{r\}}(v-1) > 0 &\iff v_p(v_i) - ir > 0 \quad \text{for all } i \in \mathbb{N} \\ &\iff v_p(v_i) > ir \geq 0 \quad \text{for all } i \in \mathbb{N} \end{aligned}$$

hence $v \in 1 + \mathfrak{m}_L[\![T^{-1}]\!]$. Similarly for all $r \in (0, r']$

$$\begin{aligned} v^{\{r\}}(u-1) > 0 &\iff v_p(u_i) + ir > 0 \quad \text{for all } i \in \mathbb{N}_{>0} \\ &\iff v_p(u_i) > -ir \quad \text{for all } i \in \mathbb{N}_{>0} \end{aligned}$$

so by taking $r \to 0$ we get $v_p(u_i) \geq 0$, i.e. $u \in 1 + T\mathcal{O}_L [\![T]\!]$. Hence we define $x^+ = u$ and $x^- = v$.

Clearly x^+ and x^- are bounded at 0, since all the coefficients are elements of $\mathcal{O}_L$ and therefore bounded by 0. Hence $x^+, x^- \in \mathcal{E}^\dagger$.

The uniqueness of this decomposition follows similarly to the uniqueness of the decomposition in Proposition 4.22: Suppose we have two decompositions $x = x^0 T^k x^+ x^- = y^0 T^{k'} y^+ y^-$. Then x^- and y^+ are invertible in $1 + \mathfrak{m}_L[\![T^{-1}]\!] \cap \mathcal{E}^\dagger$ resp. $1 + T\mathcal{O}_L [\![T]\!]$. Then comparing the coefficients of

$$x^+(y^+)^{-1} = y^-(x^-)^{-1}(x^0)^{-1} y^0 T^{k'-k} = z^0 T^{k'-k} z^-$$

where $z^0 \in L*$, $z^- \in 1 + \mathfrak{m}_L[\![T^{-1}]\!] \cap \mathcal{E}^\dagger$. We find $z^0 = 1$, $k' - k = 0$ and $x^+(y^+)^{-1} = 1 = y^-(x^-)^{-1}$. Hence the decomposition is unique.

□

Lemma 4.24. *Let $x = x^0 T^k(x) x^+ x^-$ be a decomposition as in Proposition 4.23. Then we have* *(i)* $\varphi(x^0) = x^0 = \gamma(x^0)$

(ii) $\varphi(T^k) \subset T^{pk}(1 + \mathfrak{m}_L[\![T^{-1}]\!])$

(iii) $\gamma(T^k) \subset \chi(\gamma)^k T^k (1 + T\mathcal{O}_L[\![T]\!])$

(iv) $\varphi(x^+) \subset 1 + T\mathcal{O}_L[\![T]\!]$

(v) $\varphi(x^-) \subset 1 + \mathfrak{m}_L[\![T^{-1}]\!]$

(vi) $\gamma(x^+) \subset 1 + T\mathcal{O}_L[\![T]\!]$

(vii) $\gamma(x^-) \subset 1 + \mathfrak{m}_L[\![T^{-1}]\!]$

Proof. By definition φ and γ leave elements of L^* fixed. For the other identities we do some calculations.

$$\begin{aligned}
\varphi(T^k) = \varphi(T)^k &= ((1+T)^p - 1)^k \\
&= (pT + ... + pT^{p-1} + T^p)^k \\
&= (T^p(pT^{1-p} + ... + pT^{-1} + 1))^k \\
&\subset T^{pk}(1 + \mathfrak{m}_L[\![T^{-1}]\!]), \\
\gamma(T^k) = \gamma(T)^k &= ((1+T)^{\chi(\gamma)} - 1)^k \\
&= (\chi(\gamma)T + ... + \chi(\gamma)T^{\chi(\gamma)-1} + T^{\chi(\gamma)})^k \\
&= (\chi(\gamma)T(1 + ... + T^{\chi(\gamma)-2} + \chi(\gamma)^{-1}T^{\chi(\gamma)-1}))^k \\
&\subset \chi(\gamma)^k T^k (1 + T\mathcal{O}_L[\![T]\!]),
\end{aligned}$$

where we used that $\chi(\gamma) \in \mathbb{Z}_p^*$ and therefore $\chi(\gamma)^{-1} \in \mathbb{Z}_p^* \subset \mathcal{O}_L$.

$$\begin{aligned}
\varphi(1 + T\mathcal{O}_L[\![T]\!]) &= 1 + \varphi(T)\mathcal{O}_L[\![\varphi(T)]\!] \\
&= 1 + ((1+T)^p - 1)\mathcal{O}_L[\![((1+T)^p - 1)]\!] \\
&= 1 + T(p + ... + pT^{p-2} + T^{p-1})\mathcal{O}_L[\![((1+T)^p - 1)]\!] \\
&\subset 1 + T\mathcal{O}_L[\![T]\!], \\
\varphi(1 + \mathfrak{m}_L[\![T^{-1}]\!]) &= 1 + \mathfrak{m}_L[\![\varphi(T)^{-1}]\!] \\
&= 1 + \mathfrak{m}_L[\![((1+T)^p - 1)^{-1}]\!] \\
&= 1 + \mathfrak{m}_L[\![T^{-p}(pT^{1-p} + ... + pT^{-1} + 1)^{-1}]\!] \\
&\subset 1 + \mathfrak{m}_L[\![T^{-1}]\!],
\end{aligned}$$

since $(pT^{1-p} + ... + pT^{-1} + 1)$ is invertible in $\mathfrak{m}_L[\![T^{-1}]\!]$.

$$
\begin{aligned}
\gamma(1 + T\mathcal{O}_L[\![T]\!]) &= 1 + \gamma(T)\mathcal{O}_L[\![\gamma(T)]\!] \\
&= 1 + ((1+T)^{\chi(\gamma)} - 1)\mathcal{O}_L[\![((1+T)^{\chi(\gamma)} - 1)]\!] \\
&= 1 + T(\chi(\gamma) + ... + \chi(\gamma)T^{\chi(\gamma)-2} + T^{\chi(\gamma)-1})\mathcal{O}_L[\![((1+T)^{\chi(\gamma)} - 1)]\!] \\
&\subset 1 + T\mathcal{O}_L[\![T]\!], \\
\gamma(1 + \mathfrak{m}_L[\![T^{-1}]\!]) &= 1 + \mathfrak{m}_L[\![\gamma(T)^{-1}]\!] \\
&= 1 + \mathfrak{m}_L[\![((1+T)^{\chi(\gamma)} - 1)^{-1}]\!] \\
&= 1 + \mathfrak{m}_L[\![T^{-\chi(\gamma)}(\chi(\gamma)T^{1-\chi(\gamma)} + ... + \chi(\gamma)T^{-1} + 1)^{-1}]\!] \\
&\subset 1 + \mathfrak{m}_L[\![T^{-1}]\!].
\end{aligned}
$$

□

4.3 The logarithm function on x^+ and x^-

We want to extend the p-adic logarithm function log defined on $a \in C_p$ with $v_p(a) > 0$ to elements of the Robba ring of the form $x = x^0x^+x^-$ with $x^0 \in L$, $x^+ \in 1 + T\mathcal{O}_L[\![T]\!]$ and $x^- \in 1 + \mathfrak{m}_L[\![T^{-1}]\!] \cap \mathcal{E}^\dagger$. The notion of this logarithm can be found in Colmez' papers, for example in [Col10], but without definition or statement about where this logarithm is defined. Let $\overline{\log}(T) = \sum_{i=1}^{\infty} \frac{T^i}{i}$.

Definition 4.25. Let $x^+ \in 1 + T\mathcal{O}_L[\![T]\!]$ and $x^- \in 1 + \mathfrak{m}_L[\![T^{-1}]\!] \cap \mathcal{E}^\dagger$. We define $\log(x^+)$ to be the formal series given by

$$\log(x^+) \coloneqq \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x^+ - 1)^n}{n} = \overline{\log}(T) \circ (x^+ - 1)$$

and equivalently

$$\log(x^-) \coloneqq \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x^- - 1)^n}{n} = \overline{\log}(T) \circ (x^- - 1).$$

For the product we define $\log(x^0x^+x^-) = \log(x^0) + \log(x^+) + \log(x^-)$.

Proposition 4.26. *Let $x^+ \in 1 + T\mathcal{O}_L[\![T]\!]$. Then $\log(x^+)$ is an element of $\mathcal{R}$.*

Proof. We want to use Proposition 1.46 about the radius of convergence of the composition of two power series. Since $(x^+ - 1)(0) = 0$ we find that $\overline{\log}(T) \circ (x^+ - 1)$ is a well-defined power series that converges at any

$a \in C_p$ with $v_p(a) > 0$ (since x^+ converges there) and $v^{\{v_p(a)\}}(x^+ - 1) > 0$ (since $\overline{\log}(T)$ converges for $v_p(T) > 0$. But since all coefficients of x^+ have positive valuation we clearly have $v^{\{v_p(a)\}}(x^+ - 1) \geq v_p(a) > 0$ and hence the composition $\overline{\log}(T) \circ (x^+ - 1)$ converges for $v_p(T) > 0$. $\square$

Proposition 4.27. *Let* $x^- \in 1 + \mathfrak{m}_L[\![T^{-1}]\!] \cap \mathcal{E}^\dagger$ *converge on* $0 < v_p(T) \leq r$. *Then* $\log(x^-)$ *is an element of* $\mathcal{R}$.

Proof. We will use again Proposition 1.46 now combined with Lemma 4.7.

$$\log(x^-) = \overline{\log} \circ (x^- - 1) = (\overline{\log} \circ ((x^- - 1) \circ T^{-1})) \circ T^{-1}$$

so if $\overline{\log} \circ ((x^- - 1) \circ T^{-1})$ converges for $v_p(T) > r$ then $log(x^-)$ converges on $v_p(T) < r$. First note that $\tilde{x} = (x^- - 1) \circ T^{-1}$ defines a power series $\sum_{i=1}^{\infty} a_i T^i$ that converges on $v_p(T) > -r$. Furthermore $\tilde{x}(0) = 0$, so we can precompose this serie with $\overline{\log}(T)$ and this composition converges on any $a \in C_p$ with $v_p(a) > -r$ and $v^{\{v_p(a)\}}(\tilde{x}) > 0$. From Proposition 4.19 we know that $v^{\{r\}}(x^- - 1) > 0$, which is equivalent to $v_p(a_i) > ir$ (in Proposition 4.23 we named $x^- = v$). Hence

$$v^{\{v_p(a)\}}(\tilde{x}) = \inf_{i>1} \{v_p(a_i) + iv_p(a)\} > 0 \text{ if } v_p(a) \geq -r.$$

So $\overline{\log} \circ ((x^- - 1) \circ T^{-1})$ converges on $v_p(T) \geq -r$ and therefore $\log(x^-)$ converges on $v_p(T) \leq r$. $\square$

So by Proposition 4.26 $t := \log(1 + T)$ defines an element of $\mathcal{R}^+$.

Proposition 4.28 (Statement in subsection I.2 of [Col10])**.** *We have* $\varphi(t) = p \cdot t$ *and* $\gamma(t) = \chi(\gamma)t$ *for all* $\gamma \in \Gamma$.

Proof.

$$\begin{aligned}\varphi(t) &= \varphi(\log(1+T)) = \log(1 + \varphi(T)) = \log(1 + (T+1)^p - 1) \\ &= \log((T+1)^p) = p \cdot \log(1+T) = p \cdot t\end{aligned}$$

For γ equivalent. $\square$

For some elements we will later on also need the notion of the exponential function. Let $\exp(T) = \sum_{i=0}^{\infty} \frac{T^i}{i!}$.

Definition 4.29. For $x = \sum_{i \in \mathbb{Z}} a_i T^i \in \mathcal{R}$ define the formal *exponential* series

$$\exp(x) := \exp(a_0)(\exp \circ \sum_{i=-\infty}^{-1} a_i T^i) \cdot (\exp \circ \sum_{i=1}^{\infty} a_i T^i).$$

Remark. This is well-defined by Proposition 1.46 since x_+ and x_- have only positive resp. negative coefficients.

Lemma 4.30. *If on some annulus* $0 < v_p(a) < r$ *we have* $v_p(x(a)) > \frac{1}{p-1}$ *then* $\exp(x)$ *is an element of* $\mathcal{R}$*.*

Proof. In Proposition 1.36 it is stated that the radius of convergence of the p-adic exponential function equals $p^{-(p-1)}$, so it converges for $b \in C_p$ with $v_p(b) > \frac{1}{p-1}$. Since $v_p(x(a)) > \frac{1}{p-1}$ also $\exp(x)$ converges on a. □

4.4 The differential operator $\partial \colon \mathcal{R} \to \mathcal{R}$

We will now define the differential operator $\partial \colon \mathcal{R} \to \mathcal{R}$, show some of its properties and give examples that we are going to use in the proofs later on.

Definition 4.31 (See page 37 of [Col08])**.** Let $\partial \colon \mathcal{R} \to \mathcal{R}$ be the *differential operator*

$$\partial = \frac{d}{dt} = (1+T)\frac{d}{dT}.$$

Example 4.32.

$$\partial(T) = (1+T)\frac{dT}{dT} = 1+T$$
$$\partial(T^i) = (1+T)\frac{dT^i}{dT} = i(1+T)T^{i-1}$$

Proposition 4.33. *Let* $f, g \in \mathcal{R}$ *and* $f \circ g \in \mathcal{R}$*. Then*

(i) $\partial(f \cdot g) = f\partial g + g\partial f$*,*

(ii) $\partial(f \circ g) = (\frac{df}{dT} \circ g) \cdot \partial g$*.*

Proof. (i)

$$\begin{aligned}\partial(f \cdot g) &= (1+T)\frac{d(f \cdot g)}{dT} = (1+T)(f\frac{dg}{dT} + g\frac{df}{dT}) \\ &= f(1+T)\frac{dg}{dT} + g(1+T)\frac{df}{dT} = f\partial g + g\partial f\end{aligned}$$

(ii)

$$\begin{aligned}\partial(f \circ g) &= (1+T)\frac{d(f \circ g)}{dT} = (1+T)(\frac{df}{dT} \circ g) \cdot \frac{dg}{dT} \\ &= (\frac{df}{dT} \circ g) \cdot \partial g\end{aligned}$$

$\square$

Lemma 4.34 (Statement on page 37 of [Col08]). *We have*

$$\partial \circ \varphi = p(\varphi \circ \partial) \quad \text{and} \quad \partial \circ \gamma = \chi(\gamma)(\gamma \circ \partial).$$

Proof.

$$\begin{aligned}
\partial \circ \varphi(\sum a_k T^k) &= \partial(\sum a_k((T+1)^p - 1)^k) \\
&= (T+1)(\sum a_k k((T+1)^p - 1)^{k-1} p(T+1)^{p-1}) \\
&= p(T+1)^p(\sum a_k k((T+1)^p - 1)^{k-1}) \\
\varphi \circ \partial(\sum a_k T^k) &= \varphi(T+1)\varphi(\sum a_k k T^{k-1}) \\
&= (T+1)^p(\sum a_k k((T+1)^p - 1)^{k-1})
\end{aligned}$$

For γ equivalent. $\square$

Example 4.35.

$$\begin{aligned}
\partial\varphi(T^i) &= \partial(\varphi(T)^i) = i\varphi(T)^{i-1}\partial\varphi(T) \\
&= i\varphi(T)^{i-1}p\varphi(\partial(T)) = ip\varphi(T)^{i-1}\varphi(1+T) \\
&= ip\varphi(T)^{i-1}(1+\varphi(T)) \\
&= ip(\varphi(T)^{i-1} + \varphi(T)^i) \\
\partial\gamma(T^i) &= i\chi(\gamma)(\gamma(T)^{i-1} + \gamma(T)^i)
\end{aligned}$$

Example 4.36. Since by Lemma 4.24 $\frac{\varphi(T)}{T^p}$ is an element of $1+\mathfrak{m}_L[\![T^{-1}]\!]\cap \mathcal{E}^\dagger$, by Proposition 4.27 the series $\log(\frac{\varphi(T)}{T^p})$ is a well-defined element of $\mathcal{R}$. We want to calculate the Differential $\partial \log(\frac{\varphi(T)}{T^p})$ by using the previous Lemmas.

$$\begin{aligned}
\partial \log\left(\frac{\varphi(T)}{T^p}\right) &= \left(\frac{\varphi(T)}{T^p}\right)^{-1} \partial\left(\frac{\varphi(T)}{T^p}\right) = \frac{T^p}{\varphi(T)}\left(\frac{\partial\varphi(T)}{T^p} + \varphi(T)\partial\frac{1}{T^p}\right) \\
&= \frac{T^p}{\varphi(T)}\left(\frac{p\varphi(\partial T)}{T^p} + \varphi(T)\partial T^{-p}\right) \\
&= \frac{T^p}{\varphi(T)}\left(p\frac{\varphi(1+T)}{T^p} - p\frac{\varphi(T)(1+T)}{T^{p+1}}\right) \\
&= p\frac{1+\varphi(T)}{\varphi(T)} - p\frac{(1+T)}{T} \\
&= p\left(\frac{1}{\varphi(T)} + 1 - \frac{1}{T} - 1\right) = p\left(\frac{1}{\varphi(T)} - \frac{1}{T}\right)
\end{aligned}$$

Example 4.37. By Lemma 4.24 $\frac{\gamma(T)}{T}$ is an element of $\chi(\gamma)(1+T\mathcal{O}_L[\![T]\!])$. Furthermore $\chi(\gamma) \in \mathbb{Z}_p^*$, so the logarithms

$$\log\left(\chi(\gamma)\frac{\gamma(T)}{\chi(\gamma)T}\right) = \log(\chi(\gamma)) + \log\left(\frac{\gamma(T)}{\chi(\gamma)T}\right)$$

is well-defined. As above we can calculate

$$\partial \log\left(\frac{\gamma(T)}{T}\right) = \frac{\chi(\gamma)}{\gamma(T)} + \chi(\gamma) - \frac{1}{T} - 1.$$

4.5 The residue of an element of $\mathcal{R}$

Also the residue is a tool, that we will use in the proofs later on. We will again give examples that we will use later and show some properties.

Let $x = \sum_{k\in\mathbb{Z}} a_k T^k \in \mathcal{R}$. Let the residue of the differential form $\omega = xdT$ be defined as usual by $\operatorname{res}(\omega) = a_{-1}$.

Definition 4.38 (See page 37 of [Col08])**.** The *residue* $\operatorname{Res}(x)$ of an element $x \in \mathcal{R}$ is defined by $\operatorname{Res}(x) = \operatorname{res}(x\frac{dT}{1+T})$.

Example 4.39. The formal series $1 + T \in 1 + T\mathcal{O}_L[\![T]\!]$ is invertible in $1 + T\mathcal{O}_L[\![T]\!]$. The inverse equals $(1 + T)^{-1} = \sum_{i\in\mathbb{N}}(-1)^n T^n = 1 - T + T^2 - T^3 + ...$ Hence for all $i \geq 0$

$$\operatorname{Res}(T^i) = \operatorname{res}\left(\frac{T^i}{1+T}dT\right) = 0$$

and for $-i < 0$

$$\begin{aligned}\operatorname{Res}(T^{-i}) &= \operatorname{res}\left(\frac{T^{-i}}{1+T}dT\right)\\ &= \operatorname{res}(T^{-i} - T^{-i+1} + T^{-i+2} - ...dT) = (-1)^{i-1}.\end{aligned}$$

Lemma 4.40. *Let $x, y \in \mathcal{R}$. Then*

$$\operatorname{Res}(x + y) = \operatorname{Res}(x) + \operatorname{Res}(y).$$

Proof. We can use that res is additive:

$$\begin{aligned}\operatorname{Res}(x + y) &= \operatorname{res}((x + y)\frac{dT}{(1+T)}) = \operatorname{res}(x\frac{dT}{(1+T)} + y\frac{dT}{(1+T)})\\ &= \operatorname{res}(x\frac{dT}{(1+T)}) + \operatorname{res}(y\frac{dT}{(1+T)}) = \operatorname{Res}(x) + \operatorname{Res}(y)\end{aligned}$$

$\square$

Proposition 4.41 (Proposition A.6 in [Col08])**.** *$\ker \partial = L$ and an element $x \in \mathcal{R}$ is in the image of the differential operator ∂ if and only if* $\mathrm{Res}(x) = 0$.

Proof. For the first statement we see

$$x \in \ker \partial \iff (1+T)\frac{\partial x}{\partial T} = 0 \iff \frac{\partial x}{\partial T} = 0 \iff x \in L.$$

For the second statement suppose x is an element in the image of ∂, in other words there is a $\tilde{x} \in \mathcal{R}$ such that $\partial\tilde{x} = (1+T)\frac{d\tilde{x}}{dT} = x$. Then

$$\mathrm{Res}(x) = \mathrm{Res}((1+T)\frac{d\tilde{x}}{dT}) = \mathrm{res}(\frac{d\tilde{x}}{dT})$$

and since $\frac{d}{dT}(\sum_{i\in\mathbb{Z}} a_i T^i) = \sum_{i\in\mathbb{Z}} i a_i T^{i-1}$ we get $(\frac{d}{dT}\sum_{i\in\mathbb{Z}} a_i T^i)^{(-1)} = 0$, so the residue $\mathrm{res}(\frac{d\tilde{x}}{dT}dT)$ equals 0.

On the other hand, if $\mathrm{Res}(\frac{dx}{dT}) = 0$ then $\left(\frac{x}{1+T}\right)^{(-1)} = 0$. Hence we can find $\tilde{x}$ with $\frac{d\tilde{x}}{dT} = \frac{x}{1+T}$, which is equivalent with $x = (1+T)\frac{d\tilde{x}}{dT} = \partial\tilde{x}$ $\square$

Lemma 4.42 (Statement on page 37 of [Col08])**.**

$$\mathrm{Res}(\varphi(x)) = \mathrm{Res}(x), \quad \mathrm{Res}(\gamma(x)) = \mathrm{Res}(x)$$

Proof. In Lemma 4.40 we have seen that Res is additive. By Definition also φ and the action of Γ are additive. Therefore it is enough to show that the statement is true for $x = T^i$ for all $i \in \mathbb{Z}$. First consider $x = T^i$ with $i \geq 0$.

$$\mathrm{Res}(\varphi(T^i)) = \mathrm{res}((1+T)^{-1}\varphi(T)^i dT)$$

Since $(1+T)$ is invertible in $1+T\mathcal{O}_L[\![T]\!]$ and $\varphi(T) \in \mathcal{O}_L[\![T]\!]$, the series $(1+T)^{-1}\varphi(T)^i$ does not have any negative powers of T, hence for $i \geq 0$

$$\mathrm{Res}(\varphi(T^i)) = 0$$

and by Example 4.39 also $\mathrm{Res}(T^i) = 0$.

For $i = -1$ we use Example 4.36. In this example we have calculated that

$$\partial \log\left(\frac{\varphi(T)}{T^p}\right) = p\left(\frac{1}{\varphi(T)} - \frac{1}{T}\right).$$

By Proposition 4.41 this implies that

$$\mathrm{Res}\left(p(\frac{1}{\varphi(T)} - \frac{1}{T})\right) = 0$$

and therefore by additivity of Res we have

$$\operatorname{Res}(\varphi(T^{-1}) = \operatorname{Res}(T^{-1}).$$

Now assume that

$$\operatorname{Res}(\varphi(T^{-i}) = \operatorname{Res}(T^{-i})$$

for some $-i < 0$. So with Proposition 4.41 there is some $f \in \mathcal{R}$ such that $\partial(f) = \varphi(T^{-i}) - T^{-i}$. We have by example 4.35

$$\begin{aligned}\partial\bigg(\frac{1}{-ip}\varphi(T)^{-i} - f\bigg) &= \frac{1}{-ip}(-ip)(\varphi(T)^{-i-1} + \varphi(T^{-i})) - \varphi(T^{-i}) - T^{-i}\\ &= \varphi(T)^{-i-1} + T^{-i}\end{aligned}$$

and hence by Proposition 4.41

$$\operatorname{Res}(\varphi(T)^{-i-1}) = -\operatorname{Res}(T^{-i}) = \operatorname{Res}(T^{-i-1}).$$

By induction this holds for any $-i < 0$.

In the same way as before, we get the statement for γ and T^i with $i \geq 0$. For $i = -1$, we use Example 4.37 and Proposition 4.41 and get

$$\begin{aligned}0 = \operatorname{Res}\left(\partial \log\left(\frac{\gamma(T)}{T}\right)\right) &= \operatorname{Res}\left(\frac{\chi(\gamma)}{\gamma(T)} + \chi(\gamma) - \frac{1}{T} - 1\right)\\ &= \chi(\gamma)\operatorname{Res}\left(\frac{1}{\gamma(T)}\right) - \operatorname{Res}\left(\frac{1}{T}\right).\end{aligned}$$

And again assuming that the statement holds for T^{-i} with $-i < 0$ we have $g \in \mathcal{R}$ with $\partial(g) = \gamma(T)^{-i} - \chi(\gamma)^{-1}T^{-i}$ and we get

$$\begin{aligned}0 &= \operatorname{Res}\left(\partial\bigg(\frac{1}{-i\chi(\gamma)}\gamma(T)^{-i} - g\bigg)\right)\\ &= \operatorname{Res}(\gamma(T)^{-i-1} + \gamma(T)^{-i} - \gamma(T)^{-i} + \chi(\gamma)^{-1}T^{-i})\\ &= \operatorname{Res}(\gamma(T)^{-i-1}) + \chi(\gamma)^{-1}\operatorname{Res}(T^{-i})\\ &= \operatorname{Res}(\gamma(T)^{-i-1}) - \chi(\gamma)^{-1}\operatorname{Res}(T^{-i-1}).\end{aligned}$$

So inductively the statement holds for any $i \in \mathbb{Z}$. □

Lemma 4.43. *Let $x^+ \in 1 + T\mathcal{O}_L[\![T]\!]$ and $x^- \in 1 + \mathfrak{m}_L[\![T^{-1}]\!] \cap \mathcal{E}^\dagger$. Then*

(i) $\operatorname{Res}(\frac{\partial x^+}{x^+}) = 0$

(ii) $\operatorname{Res}(\frac{\partial x^-}{x^-}) = 0$

(iii) $\operatorname{Res}(\frac{\partial(x^+x^-)}{x^+x^-}) = 0$

Proof. (i)

$$\operatorname{Res}\left(\frac{\partial x^+}{x^+}\right) = \operatorname{Res}\left(\frac{(1+T)\frac{dx^+}{dT}}{x^+}\right) = \operatorname{res}\left(\frac{\frac{dx^+}{dT}}{x^+}dT\right),$$

and since x^+ is invertible in $1 + T\mathcal{O}_L[\![T]\!]$ neither $(x^+)^{-1}$ nor $\frac{dx^+}{dT}$ has any negative coefficients. Therefore the residue equals 0.
(ii)

$$\operatorname{Res}\left(\frac{\partial x^-}{x^-}\right) = \operatorname{res}\left(\frac{\frac{dx^-}{dT}}{x^-}dT\right).$$

$x^- = \sum_{n\leq 0} a_n T^n$ is invertible in $1 + \mathfrak{m}_L[\![T^{-1}]\!] \cap \mathcal{E}^\dagger$ and

$$\frac{dx^-}{dT} = -b_1 T^{-2} - 2b_2 T^{-3} - ... = T^{-2}(-b_1 - 2b_2 T^{-1} - ...).$$

The product $\frac{\frac{dx^-}{dT}}{x^-}$ therefore has only coefficients $\neq 0$ for T^n with $n < 1$.
(iii)

$$\begin{aligned}\operatorname{Res}\left(\frac{\partial(x^+x^-)}{x^+x^-}\right) &= \operatorname{res}\left(\frac{x^-\frac{d(x^+)}{dT} + x^+\frac{d(x^-)}{dT}}{x^+x^-}dT\right) = \operatorname{res}\left(\frac{\frac{d(x^+)}{dT}}{x^+} + \frac{\frac{d(x^-)}{dT}}{x^-}dT\right)\\ &= \operatorname{Res}\left(\frac{\partial(x^+)}{x^+}\right) + \operatorname{Res}\left(\frac{\partial(x^-)}{x^-}\right) = 0.\end{aligned}$$

□

Chapter 5

(φ, Γ)-modules over $\mathcal{R}$

In this final chapter we want to define (φ, Γ)-modules over the Robba ring. The goal was to proof that the (φ, Γ)-module over $\mathcal{R}$ of rank 1 can be characterized by the modules $\mathcal{R}(\delta)$ for $\delta\colon \mathbb{Q}_p^* \to L^*$ any continuous character. As a tool we define the cohomology of a certain chain complex of (φ, Γ)-module and we calculate the cohomology group $H^0(\mathcal{R}(\delta))$. A nice independent result is the isomorphism between the groups $\mathrm{Ext}^1(\mathcal{R}, M)$ and $H^1(M)$ for a (φ, Γ)-module M.

This chapter will mostly follow [Col08] and [Col10] but also [Col04] and [Col05]. The first two sources are published papers that have a similar content than the other two. As I know now the two unpublished papers are an abandoned preliminary version of the other papers and at least [Col05] has some mistakes in it. As mentioned in the introduction this leads to a mistake in the proof of proposition 5.26 and hence also in the proof of theorem 5.29.

We fix a finite extension L of $\mathbb{Q}_p$ and let $\mathcal{R}$ be the Robba ring $\mathcal{R}_L$ equipped with the Frobenius-endomorphism $\varphi_{\mathcal{R}} : T \mapsto (T+1)^p - 1$ and the action of $\Gamma = \mathrm{Gal}(L_\infty/L)$ (see section 1.7) on $\mathcal{R}$, given by $\gamma_{\mathcal{R}}(T) = (1+T)^{\chi(\gamma)} - 1$. In Lemma 4.17 we have shown that $\varphi_{\mathcal{R}}$ commutes with the action of Γ.

Definition 5.1 (See subsection 0.3 of [Col08])**.** A (φ, Γ)-*module over* $\mathcal{R}$ (resp. $\mathcal{E}^\dagger$) is a free $\mathcal{R}$-module (resp. $\mathcal{E}^\dagger$-module) M that is equipped with a $\varphi_{\mathcal{R}}$-semi-linear operation φ_M, such that if $\{e_1, ..., e_d\}$ is a basis of M, then so is $\{\varphi_M(e_1), ..., \varphi_M(e_d)\}$, and with a $\gamma_{\mathcal{R}}$-semi-linear operation γ_M for any $\gamma \in \Gamma$, such that they define an action of Γ on M, that commutes with φ.

Proposition 5.2. *The ring $\mathcal{R}$, seen as a $\mathcal{R}$-module, together with $\varphi_{\mathcal{R}}$*

and the action of Γ *on* $\mathcal{R}$ *defines a* (φ, Γ)*-module of rank 1.*

Proof. $\varphi_{\mathcal{R}}$ and $\gamma_{\mathcal{R}}$ clearly are semi-linear and commute as we have shown before. It remains to show that if e is a base of $\mathcal{R}$, so is $\varphi(e)$. Clearly if e is a base, there is an element $e^{-1} \in \mathcal{R}$ such that $e \cdot e^{-1} = 1 \in \mathcal{R}$. But then we also have $\varphi(e) \cdot \varphi(e^{-1}) = \varphi(e \cdot e^{-1}) = 1$ so $\varphi(e)$ generates the whole ring as well. $\square$

The goal of this chapter will be to characterize all (φ, Γ)-modules of rank 1 over $\mathcal{R}$.

5.1 Rank 1 (φ, Γ)-modules

For any rank one $\mathcal{R}$-module M there is a basis ν such that $M = \mathcal{R}.\nu$. Hence for φ and γ endomorphisms on M the elements $\varphi(\nu)$ and $\gamma(\nu)$ take the form $\varphi(\nu) = r_\varphi.\nu$ and $\gamma(\nu) = r_\gamma.\nu$ for some $r_\varphi, r_\gamma \in \mathcal{R}$.

Remark. Here we denote both $\varphi_{\mathcal{R}}$ and φ_M simply by φ, as it will be clear which one we are referring to (similarly for γ).

Proposition 5.3. *Let* M *be a rank one* $\mathcal{R}$*-module with such a basis* ν*. Then* M *is a* (φ, Γ)*-module if and only if*

$$\varphi(r_\gamma) r_\varphi = \gamma(r_\varphi) r_\gamma \text{ for all } \gamma \in \Gamma.$$

Proof. Since the action of Γ commutes with φ the following are equal for any $r \in \mathcal{R}$:

$$\varphi(\gamma(r.\nu)) = \varphi(\gamma(r) r_\gamma.\nu) = \varphi(\gamma(r))\varphi(r_\gamma).\varphi(\nu) = \varphi(\gamma(r))\varphi(r_\gamma) r_\varphi.\nu$$
$$\gamma(\varphi(r.\nu)) = \gamma(\varphi(r) r_\varphi.\nu) = \gamma(\varphi(r))\gamma(r_\varphi).\gamma(\nu) = \gamma(\varphi(r))\gamma(r_\varphi) r_\gamma.\nu$$

Since on $\mathcal{R}$ the action of φ and γ commute this is equivalent to

$$\varphi(r_\gamma) r_\varphi = \gamma(r_\varphi) r_\gamma.$$

$\square$

Let now $\delta\colon \mathbb{Q}_p^* \to L^*$ be a continuous character. Let $1\colon \mathbb{Q}_p^* \to L^*$ be the character sending any element of $\mathbb{Q}_p^*$ to $1 \in L^*$.

Definition 5.4 (See subsection 2.2 of [Col08])**.** We note $\mathcal{R}(\delta)$ the one dimensional (φ, Γ)-module that we get by twisting the actions of Γ and

φ on $\mathcal{R}$ by δ. So $\mathcal{R}(\delta)$ has a basis ν such that for all $x \in \mathcal{R}$ and $\gamma \in \Gamma$ we have

$$\varphi_\delta(x\nu) = \delta(p)\varphi(x)\nu \quad \text{and} \quad \gamma_\delta(x\nu) = \delta(\chi(\gamma))\gamma(x)\nu$$

where χ is the cyclotomic character.

Notation. If it is clear what is meant, we will sometimes just write φ or γ instead of φ_δ or γ_δ.

Proposition 5.5. *$\mathcal{R}(\delta)$ actually defines a (φ, Γ)-module.*

Proof. It is clear that the morphism φ_δ and the action of Γ are $\delta(p)\gamma$-resp. $\delta(\chi(\gamma))\gamma$-semi-linear. With $\varphi(\nu) = \delta(p)\nu = r_\varphi\nu$ and $\gamma(\nu) = \delta(\chi(\gamma))\nu = r_\gamma\nu$ we have $\gamma(r_\varphi) = r_\varphi$ and $\varphi(r_\gamma) = r_\gamma$, so these operations commute by Proposition 5.3. Furthermore if ν is a basis of $\mathcal{R}(\delta)$ then $\varphi(\nu) = \delta(p)\nu$ is also a basis since $\delta(p)$ is invertible in L^*. $\square$

In the following we want to proof that any rank one (φ, Γ)-module is of this form. We will first look into some cohomology of (φ, Γ)-module-complexes that we will use to proof this statement.

5.2 Cohomology for (φ, Γ)-modules over $\mathcal{R}$

Note that Γ is topologically cyclic, since $\mathbb{Z}_p^*$ is topologically cyclic. This is mentioned in chapter 2.1 of [Col08]. We will use this without a reference or proof. Let γ be a generator of Γ and let M be a (φ, Γ)-module over $\mathcal{R}$.

Definition 5.6 (See subsection 2.1 of [Col08])**.** We define $M^\bullet$ to be the sequence

$$0 \xrightarrow{d_0} M \xrightarrow{d_1} M \oplus M \xrightarrow{d_2} M \xrightarrow{d_3} 0$$

where $d_1(x) = ((\gamma - 1)x, (\varphi - 1)x)$ and $d_2(x, y) = (\varphi - 1)x - (\gamma - 1)y$. For any other $i \in \mathbb{N}$ we let d_i be the zero map.

Definition 5.7 (See subsection 2.1 of [Col08])**.** We define $B^i(M)$ to be the image of d_i and $Z^i(M)$ to be the kernel of $d_{i+1}(M)$. The quotient defines the cohomology group $H^i(M) = Z^i(M)/B^i(M)$. For $i > 2$ we get $H^i(M) = 0$.

Proposition 5.8. *$B^i(M) \subset Z^i(M)$ hence M is a chain complex and $H^i(M)$ is well-defined.*

Proof. The only interesting case is $i = 1$.

$$\begin{aligned} d_2 \circ d_1(x) &= d_2((\gamma - 1)x, (\varphi - 1)x) \\ &= (\varphi - 1)(\gamma - 1)x - (\gamma - 1)(\varphi - 1)x \\ &= 0 \end{aligned}$$

since φ and γ commute per definition of a (φ, Γ)-module. □

5.3 The isomorphism $\mathbf{Ext}^1(\mathcal{R}, M) \tilde{\to} H^1(M)$

In this chapter we want to construct an isomorphism between the group of extensions of the (φ, Γ)-module M by $\mathcal{R}$ and the first cohomology group $H^1(M)$. First we will define extensions of R-modules for a commutative ring R with unity. We take the definitions and propositions from [Wei94] and extend them to (φ, Γ)-module modules over R. Afterwards we set $R = \mathcal{R}$, explicitly construct the group homomorphism and show that it is bijective. The motivation for this section comes from chapter 2.1 of [Col08], where the isomorphism between these two groups is mentioned.

Fix a commutative ring commutative R with unity.

Definition 5.9 (See page 76 of [Wei94])**.** Let M, N be R-modules. An *extension of* N *by* M is a short exact sequence of R-modules

$$\xi\colon\ 0 \longrightarrow M \longrightarrow E \longrightarrow N \longrightarrow 0\ .$$

Two extensions

$$0 \longrightarrow M \longrightarrow E \longrightarrow N \longrightarrow 0$$

$$0 \longrightarrow M \longrightarrow E' \longrightarrow N \longrightarrow 0$$

are called equivalent if the diagram

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, "\|"] & E \arrow[r] \arrow[d, "\wr"] & N \arrow[r] \arrow[d, "\|"] & 0 \\ 0 \arrow[r] & M \arrow[r] & E' \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

commutes. An extension is *split* if it is equivalent to

$$0 \longrightarrow M \longrightarrow N \oplus M \longrightarrow N \longrightarrow 0\ .$$

Definition 5.10 (Definition 3.3.4 of [Wei94]). Let

$$\xi\colon\ 0 \longrightarrow M \xrightarrow{\alpha} E \xrightarrow{\beta} N \longrightarrow 0$$

$$\xi'\colon\ 0 \longrightarrow M \xrightarrow{\alpha'} E' \xrightarrow{\beta'} N \longrightarrow 0$$

be two extensions of N by M. Let E'' be the pullback $\{(e, e') \in E \times E' \mid \beta(e) = \beta'(e')\}$ and $X = E''/\{(\alpha(m), 0) - (0, \alpha'(m)) \mid m \in M\}$. Then we define the *Baer sum* of the two extensions ξ and ξ' to be the short exact sequence

$$\xi + \xi'\colon\ 0 \longrightarrow M \xrightarrow{\alpha''} X \xrightarrow{\beta''} N \longrightarrow 0$$

where $\alpha''\colon m \mapsto [(\alpha(m), 0)] = [(0, \alpha'(m)]$ and $\beta''\colon (e, e') \mapsto \beta(e) = \beta'(e')$.

Proposition 5.11. *The equivalence classes of extensions of N by M, together with the Baer sum form an abelian group that we will call $Ext^1(N, M)$.*

Proof. The proof can be found in [Wei94], Corollary 3.4.5. The zero element is given by the class of the split extensions. Note that usually $\mathrm{Ext}^1(N, M)$ is defined differently, but Theorem 3.4.3 and Corollary 3.4.5 in [Wei94] show that the two definitions are equivalent. □

We want to look at extensions of (φ, Γ)-modules. Hence from now if we say ξ is an extension of (φ, Γ)-modules we mean that it is an extension of R-modules such that any module is equipped with the additional structure of a (φ, Γ)-module and such that all the morphisms are morphisms of (φ, Γ)-modules.

Proposition 5.12. *If extensions ξ and ξ' in Definition 5.10 are extensions of (φ, Γ)-modules then also the Baer sum $\xi + \xi'$ is an extension of (φ, Γ)-modules.*

Proof. Let N, and M be (φ, Γ)-modules and the extensions ξ and ξ' be denoted as in Definition 5.10. We want to give X a (φ, Γ)-module structure. We use the structure on E and E'. So let $\varphi_X\colon X \to X$ be defined by $(e, e') \mapsto (\varphi(e), \varphi(e'))$ and $\gamma_X\colon X \to X$ be defined by $(e, e') \mapsto (\gamma(e), \gamma(e'))$. These maps are clearly module morphisms and semilinear, since φ and γ are semilinear endomorphisms for E (resp. E'). With the same reason γ and φ commmute. Also if $\{(e_i, e_i')\}_{i \in I} \subset X$ is a basis, then as well $\{\varphi(e_i, e_i')\}_{i \in I}$ is a basis of X.

It remains to show that φ and γ are well-defined and that α'' and β'' are morphisms of (φ, Γ)-modules. We have

$$\begin{aligned} \beta''(\varphi(e, e')) &= \beta''(\varphi(e), \varphi(e')) = \beta(\varphi(e)) \\ &= \varphi(\beta(e)) = \varphi(\beta''(e, e')) \end{aligned}$$

and

$$\begin{aligned} \alpha''(\varphi(m)) &= (\alpha(\varphi(m)), 0) = (\varphi(\alpha(m)), 0) \\ &= \varphi(\alpha(m), 0) = \varphi(\alpha''(m)), \end{aligned}$$

for γ equivalent. This shows that φ and γ are well-defined as well as that α'' and β'' commute with φ and γ. $\square$

Proposition 5.13. *If N and M are (φ, Γ)-modules, then $Ext^1(N, M)$, the set of extensions of (φ, Γ)-modules, together with the Baer-sum still defines a group.*

Proof. In the previous Proposition we have seen that $\mathrm{Ext}^1(N, M)$ is closed under addition. The neutral element is defined by the split sequence with φ and γ defined component wise on $N \oplus M$. All morphism commute with these morphisms φ and γ, so the split sequence is as well a sequence of (φ, Γ)-modules. Furthermore the inverse of an extension is just given by the same sequence but with $-\beta$ instead of β. This clearly as well is a sequence of (φ, Γ)-modules. $\square$

Proposition 5.14 (Statement in subsection 2.1 of [Col08])**.** *We can identify the elements of $Ext(\mathcal{R}, M)$ with the cohomology classes in $H^1(M)$.*

Proof. Let $0 \longrightarrow M \xrightarrow{\alpha} E \xrightarrow{\beta} \mathcal{R} \longrightarrow 0$ be an extension of $\mathcal{R}$ by M. We have the following commuting diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M \arrow[r] \arrow[d, "\alpha"] & M \oplus M \arrow[r] \arrow[d, "\alpha\oplus\alpha"] & M \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & E \arrow[r] \arrow[d, "\beta"] & E \oplus E \arrow[r] \arrow[d, "\beta\oplus\beta"] & E \arrow[r] \arrow[d, "\beta"] & 0 \\
0 \arrow[r] & \mathcal{R} \arrow[r] \arrow[d] & \mathcal{R} \oplus \mathcal{R} \arrow[r] \arrow[d] & \mathcal{R} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Then β is surjective, hence we can find $e \in E$, a preimage of $1 \in \mathcal{R}$. Since $(\gamma(1) - 1, \varphi(1) - 1) = (0, 0)$ we have $d_1(e) \in \ker(\beta \oplus \beta)$ hence there is an element $(m', m'') \in M \oplus M$ with $(\alpha \oplus \alpha)((m', m'')) = d_1(e)$. Furthermore clearly $d_2 \circ d_1(e) = 0$, so since α is injective $d_2((m', m'')) = 0$ so we have $(m', m'') \in Z^1(D)$. Hence we can identify e with an element of $Z^1(M)$. Now we want to show that this identification is independent of the choice of e. So let $e, e' \in E$ be two preimages of $1 \in \mathcal{R}$. We will show that they identify with the same element in the cohomology group, in other words that $e - e'$ identify with an element of $B^1(M)$. Since $\beta(e - e') = 1 - 1 = 0$ we can find a $m \in M$ with $\alpha(m) = e$. Hence the element of $M \oplus M$ that identifies with $e - e'$ lies in the image of d_{1M} and therefore in $B^1(M)$. So the cohomology class is independent of the choice of e and therefore this gives a well-defined morphism $Ext^1(\mathcal{R}, M) \to H^1(M)$.

It is clear that equivalent extensions are getting mapped to the same cohomology class, since if e' is a preimage of 1 in the extension

$$0 \longrightarrow M \longrightarrow E' \longrightarrow \mathcal{R} \longrightarrow 0$$

under the isomorphism $E' \to E$ it has to be mapped to a preimage $e \in E$ of 1. $\square$

Proposition 5.15 (Statement in subsection 2.1 of [Col08])**.** *This identification defines a group homomorphism.*

Proof. Let

$$\xi\colon\ 0 \longrightarrow M \xrightarrow{\alpha} E \xrightarrow{\beta} \mathcal{R} \longrightarrow 0$$

$$\xi'\colon\ 0 \longrightarrow M \xrightarrow{\alpha'} E' \xrightarrow{\beta'} \mathcal{R} \longrightarrow 0$$

be two extensions. We take a preimage of $1 \in \mathcal{R}$ in $\xi + \xi'$ by taking a pair of a preimage of the $1 \in \mathcal{R}$ in ξ and in ξ', so $\tilde{e} = (e, e')$ for $e \in E$, $e' \in E'$. Suppose that e corresponds to $(m_1, m_2) \in M \oplus M$ and e' corresponds to $(m_1', m_2') \in M \oplus M$. For the $(m_1'', m_2'') \in M \oplus M$ corresponding to $\tilde{e}$ the equation $d_1(e, e') = (\alpha''(m_1''), \alpha''(m_2''))$ holds. Furthermore

$$\begin{aligned} d_1(e, e') &= \big((\gamma - 1)(e, e'), (\varphi - 1)(e, e')\big) \\ &= \big(((\gamma(e), \gamma(e')) - (e, e')), ((\varphi(e), \varphi(e')) - (e, e'))\big) \\ &= \big((\gamma(e) - e, \gamma(e') - e'), (\varphi(e) - e, \varphi(e') - e')\big) \\ &= \big((\alpha(m_1), \alpha'(m_1)'), (\alpha(m_2), \alpha'(m_2)')\big) \\ &= \big((\alpha(m_1) + \alpha(m_1'), 0), (\alpha(m_2) + \alpha(m_2'), 0)\big) \end{aligned}$$

and on the other hand

$$(\alpha''(m_1''), \alpha''(m_2'')) = \big((\alpha(m_1''), 0), (\alpha(m_2''), 0)\big)$$

so $\alpha(m_1 + m_1') = \alpha(m_1'')$ and $\alpha(m_2 + m_2') = \alpha(m_2'')$. But since α is injective this already implies $m_1 + m_1' = m_1''$ and $m_2 + m_2' = m_2''$, so the sum $\xi + \xi'$ corresponds to the sum of the corresponding cohomology classes. $\square$

Proposition 5.16 (Statement in subsection 2.1 of [Col08])**.** *The identification as in Proposition 5.14 is an isomorphism $Ext^1(\mathcal{R}, M) \tilde{\to} H^1(M)$.*

Proof. Injectivity: The neutral element in the group $Ext^1(\mathcal{R}, M)$ is given by the extension with $E \simeq M \oplus \mathcal{R}$ and component wise structure of (φ, Γ)-modules. Suppose that the extension

$$0 \longrightarrow M \longrightarrow E \longrightarrow \mathcal{R} \longrightarrow 0$$

corresponds to the 0-class in $H^1(M)$. We want to show that then the extension splits. Fix a preimage $e \in E$ of the $1 \in \mathcal{R}$. This e corresponds then to an element in the image of d_{1M}. Hence there is a $m \in M$ with $d_{1E}\alpha(m) = d_{1E}(e)$, so

$$\begin{aligned}(\gamma - 1)e = (\gamma - 1)\alpha(m) &\iff \gamma(e - \alpha(m)) = e - \alpha(m)\\ (\varphi - 1)e = (\varphi - 1)\alpha(m) &\iff \varphi(e - \alpha(m)) = e - \alpha(m)\end{aligned}$$

So take $f \colon \mathcal{R} \to E$ with $1 \mapsto e - \alpha(m)$. This is a (φ, Γ)-module morphism since

$$\begin{aligned}\gamma(f(1)) &= \gamma(e - \alpha(m)) = e - \alpha(m) = f(\gamma(1))\\ \varphi(f(1)) &= \varphi(e - \alpha(m)) = e - \varphi(m) = f(\varphi(1)).\end{aligned}$$

Hence the sequence splits because clearly $\beta(f(1)) = \beta(e - \alpha(d)) = 1 - 0$.

Surjectivity: Let $(m_1, m_2) \in Z^1(M)$. We define the module $\mathcal{R} \oplus M$ together with the endomorphisms $\gamma(r, m) = (\gamma(r), \gamma(m) + \gamma(r).m_1)$ and $\varphi(r, m) = (\varphi(r), \varphi(m) + \varphi(r).m_2)$. This defines a (φ, Γ)-module because of the semi-linearity of γ_M and $\gamma_{\mathcal{R}}$. For $a \in \mathcal{R}$ we have:

$$\begin{aligned}\gamma(a.(r, m)) &= \gamma(ar, a.m) = (\gamma(ar), \gamma(a.m) + \gamma(ar).m_1)\\ &= (\gamma(a)\gamma(r), \gamma(a).\gamma(m) + \gamma(a)\gamma(r).m_1)\\ &= \gamma(a).(\gamma(r), \gamma(m) + \gamma(r).m_1) = \gamma(a).\gamma(r, m).\end{aligned}$$

for φ equivalently. Also φ and γ commute

$$\begin{aligned}\varphi(\gamma(r, m)) &= \varphi(\gamma(r), \gamma(m) + \gamma(r).m_1)\\ &= (\varphi(\gamma(r)), \varphi(\gamma(m) + \gamma(r).m_1) + \varphi(\gamma(r)).m_2)\\ &= (\varphi(\gamma(r)), \varphi(\gamma(m)) + \varphi(\gamma(r)).m_1 + \varphi(\gamma(r)).m_2)\end{aligned}$$

since φ and γ commute in M and $\mathcal{R}$. Let now

$$\xi\colon\ 0 \longrightarrow M \xrightarrow{\alpha} \mathcal{R} \oplus M \xrightarrow{\beta} \mathcal{R} \longrightarrow 0$$

be the extension with $\alpha\colon m \mapsto (0,m)$ and $\beta\colon (r,m) \mapsto r$. Clearly this sequence is exact as sequence of $\mathcal{R}$-modules. Furthermore α and β are morphisms of (φ,Γ)-modules, since

$$\begin{aligned}(\gamma\circ\alpha)(m) &= \gamma(0,m) = (0,\gamma(m)) = (\alpha\circ\gamma)(m)\\ (\beta\circ\gamma)(r,m) &= \beta(\gamma(r),\gamma(m)+\gamma(r).m_1) = \gamma(r) = (\gamma\circ\beta)(r,m).\end{aligned}$$

Hence ξ is an extension of (φ,Γ)-modules.

Now we want to show that this extension corresponds to the class of $(m_1,m_2) \in H^1(M)$. So we take a preimage of the $1 \in \mathcal{R}$, say $(1,0) \in \mathcal{R} \oplus M$. Then

$$\begin{aligned}&((\gamma-1)(1,0),(\varphi-1)(1,0))\\ &= ((\gamma(1)-1,\gamma(0)+\gamma(1).m_1-0),(\varphi(1)-1,\varphi(0)+\varphi(1).m_2-0))\\ &= ((0,m_1),(0,m_2)).\end{aligned}$$

Clearly $(\alpha\oplus\alpha)(m_1,m_2) = ((0,m_1),(0,m_2))$, so the class of (m_1,m_2) is corresponding to the extension ξ and hence for any class we can find an extension that corresponds to it. $\square$

5.4 The cohomology group $H^0(\mathcal{R}(\delta))$

We calculate now the cohomology group $H^0(\mathcal{R}(\delta))$. We will use this group to proof that $\mathcal{R}(\delta)$ and $\mathcal{R}(\delta')$ are isomorphic as (φ,Γ)-modules if and only if $\delta = \delta'$.

Let $\delta\colon \mathbb{Q}_p \to L^*$ again be a continuous character and $M = \mathcal{R}(\delta)$ be the one dimensional (φ,Γ)-module as defined in Definition 5.4.

Notation. We will sometimes write $H^i(\delta)$, $Z^i(\delta)$ or $B^i(\delta)$ for $H^i(\mathcal{R}(\delta))$, $Z^i(\mathcal{R}(\delta))$ or $B^i(\mathcal{R}(\delta))$ respectively.

Recall the expression $t \coloneqq \log(1+T)$ from Chapter 4.3.

Lemma 5.17 (Statement in the Proof of Lemma I.5 of [Col10])**.** *We can decompose* $\mathcal{R}^+ = \bigoplus_{i=0}^{k-1} L\cdot t^i \bigoplus T^k\mathcal{R}^+$*, where* $t = \log(1+T)$*.*

Proof. We can write

$$t = \log(1+T) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{T^k}{n} = T + \sum_{n=2}^{\infty}(-1)^{n+1}\frac{T^k}{n}.$$

Therefore t^i is of the form $T^i + \sum_{n=i+1}^{\infty} a_k T^n$. This gives us the desired decomposition. □

Lemma 5.18 (Lemma I.5 of [Col10]).

(i) If $\alpha \in L^*$ is not of the form p^{-i} with $i \in \mathbb{N}$, then $\alpha\varphi - 1\colon \mathcal{R}^+ \to \mathcal{R}^+$ is an isomorphism.

(ii) If $\alpha = p^{-i}$ with $i \in \mathbb{N}$, then the we have $\ker(\alpha\varphi - 1) = L \cdot t^i$ and $\sum_{k=0}^{\infty} a_k t^k$ is in the image of $\alpha\varphi - 1$ if and only if $a_i = 0$.

Proof. The idea behind the proof can be found in the proof of Lemma I.5 in [Col10]. Let $k > -v_p(\alpha)$. We will show that on $T^k\mathcal{R}^+$ the morphism $-\sum_{n=0}^{\infty}(\alpha\varphi)^n$ defines a inverse of $\alpha\varphi - 1$.

Let $r \in \mathbb{R}_{>0}$. We will begin with showing that there exists $N \in \mathbb{N}$ such that for all $n > N$ and some constant $C_r \in \mathbb{R}$ we have $v^{\{r\}}(\varphi^n(T)) = n + C_r > 0$.

With the same kind of argument as in Proposition 3.9 we see that

$$\begin{aligned} v^{\{r\}}(\varphi^n(T)) &= v^{\{r\}}((1+T)^{p^n} - 1) \\ &= v^{\{r\}}(p^n T + \binom{p^n}{2} T^2 + ... + T^{p^n}) \\ &= \inf_{1<i<p^n} \{v_p(\binom{p^n}{i} + ir)\} \\ &= \min(v_p(p^n) + r, v_p(1) + p^n r) = \min(n + r, p^n r) \end{aligned}$$

So for n big enough it is $v^{\{r\}}(\varphi^n(T)) = n + r$. Define $C_r = r$.

Let now $x = \sum_{i=k}^{\infty} a_i T^i \in T^k\mathcal{R}$. First note that $(\alpha\varphi)^n(x) = \alpha^n(x \circ \varphi^n(T))$ is a well-defined element of $\mathcal{R}^+$, since x converges on the whole $v_p(T) > 0$ and hence so does $\varphi^n(x)$. We want to show that the series $-\sum_{n=0}^{\infty}(\alpha\varphi)^n(x)$ as well is an element of $\mathcal{R}^+$. This is the case if for any $r > 0$ the sequence $((\alpha\varphi)^n(x))_{n\in\mathbb{N}}$ is a zero sequence with respect to $v^{\{r\}}$, since then the series $\sum_{n=0}^{\infty}(\alpha\varphi)^n(x)$ converges in $\mathcal{R}$ by Proposition 1.7.

Let now $n > N$ such that $v^{\{r\}}(\varphi^n(T)) = n + C_r$.

$$\begin{aligned}
v^{\{r\}}(\alpha^n \varphi^n(x)) &= v^{\{r\}}(\alpha^n) + v^{\{r\}}(\sum_{i=k}^{\infty} a_i \varphi^n(T)^i) \\
&= n v_p(\alpha) + v^{\{r\}}(\sum_{i=k}^{\infty} a_i T^i \cdot \varphi^n(T)^i \cdot T^{-i}) \\
&\geq n v_p(\alpha) + \inf_{i>k} \{v^{\{r\}}(a_i T^i \cdot \varphi^n(T)^i \cdot T^{-i})\} \\
&= n v_p(\alpha) + \inf_{i>k} \{v^{\{r\}}(a_i T^i) + v^{\{r\}}(\varphi^n(T)^i) + v^{\{r\}}(T^{-i})\} \\
&\geq n v_p(\alpha) + \inf_{i>k} \{v^{\{r\}}(x) + i v^{\{r\}}(\varphi^n(T)) - i v^{\{r\}}(T)\} \\
&= n v_p(\alpha) + v^{\{r\}}(x) + \inf_{i>k} \{i(n + C_r) - ir\} \\
&= n v_p(\alpha) + v^{\{r\}}(x) + k(n + C_r - r) \\
&= n(v_p(\alpha) + k) + v^{\{r\}}(x) + k(C_r - r)
\end{aligned}$$

where we used that $n + C_r - r > 0$ for n sufficiently large. So if we let $n \to \infty$, since $v_p(\alpha) + k > 0$, also $v^{\{r\}}(\alpha^n \varphi^n(x)) \to \infty$ and therefore $\sum_{n=0}^{\infty} (\alpha\varphi)^n(x)$ converges in $\mathcal{R}$.

So $-\sum_{n=0}^{\infty} (\alpha\varphi)^n$ is well-defined on $T^k\mathcal{R}$ and clearly is the inverse of $\alpha\varphi - 1$. Therefore $\alpha\varphi - 1$ defines an isomorphism on $T^k\mathcal{R}$.

So with the decomposition $\mathcal{R}^+ = \oplus_{i=0}^{k-1} L \cdot t^i \oplus T^k \mathcal{R}^+$ it remains to check how $\alpha\varphi - 1$ is is operating on t^i for $0 \leq i \leq k-1$. We have

$$(\alpha\varphi - 1)(t^i) = \alpha p^i t^i - t^i = (\alpha p^i - 1) t^i = 0 \iff \alpha = p^{-i}$$

Therefore if α is not of the form p^{-i} the kernel of $\alpha\varphi - 1$ is trivial, while for $\alpha = p^{-i}$ the kernel is equal to $L \cdot t^i$. In the last case with the isomorphism theorem we get the image consists of all elements with $a_i \neq 0$. □

Remark. In [Col10] the calculation of the valuation $v^{\{r\}}(\varphi^n(T))$ is done slightly different. There it is shown as well that for n big enough the valuation equals $n + C_r$. But while in my calculation C_r equals r in [Col10] it equals $v^{\{r\}}(\log(1+T))$. Since those two values are not equal, there must be a mistake in either of the calculations, probably in mine, that I can not find. For the rest of the proof the value of C_s does not matter.

Lemma 5.19. *(Proposition 2.1 of [Col08])*

i If δ is not of the form x^{-i} with $i \in \mathbb{N}$ then $H^0(\delta) = 0$.

ii If $i \in \mathbb{N}$ then $H^0(x^{-i}) = L \cdot t^i$.

Proof. Let ν be a basis of $\mathcal{R}(\delta)$ as in Chapter 5.1 and $x \in \mathcal{R}$. We have $\varphi_\delta(x\nu) = \delta(p)\varphi(x).\nu$. The image of d_0 is trivial, therefore $B^0(\mathcal{R}(\delta)) = 0$. To calculate $H^0(\mathcal{R}(\delta)) = Z^0(\mathcal{R}(\delta))$ we need to calcualte the kernel of the map $d_1 = (\gamma_\delta - 1, \varphi_\delta - 1) = (\delta(\chi(\gamma))\gamma - 1, \delta(p)\varphi - 1)$. In Lemma 5.18 we have seen that the kernel for the map $\delta(p)\varphi - 1$ is given by $L \cdot t^i$ in the case that $\delta(p) = p^{-i}$ and it is trivial otherwise. Therefore $H^0(x^{-i}) \subset L \cdot t^i$ and $H^0(\delta) = 0$ for any δ not of this form. Furthermore with $\gamma(t) = \chi(\gamma)t$ by Proposition 4.28

$$(\gamma_{x^{-i}} - 1)(t^i) = (\chi(\gamma))^{-i}\gamma(t^i) - t^i = \chi(\gamma)^{-i}\chi(\gamma)^i t^i - t^i = 0.$$

and hence $\ker(\gamma_{x^{-i}} - 1) \subset L \cdot t^i$. Therefore $H^0(x^{-i}) = L \cdot t^i$. □

Lemma 5.20 (Corollary 2.2 of [Col08])**.** *$\mathcal{R}(\delta)$ and $\mathcal{R}(\delta')$ are isomorphic as (φ, Γ)-modules if and only if $\delta = \delta'$.*

Proof. We first proof that an isomorphism $f : \mathcal{R}(\delta) \to \mathcal{R}(\delta')$ induces an isomorphism between $\mathcal{R}(\delta\delta'^{-1})$ and $\mathcal{R}$, where δ'^{-1} describes the character $\mathbb{Q}_p^* \ni a \mapsto 1/\delta'(a)$.

So let $f : \mathcal{R}(\delta) \to \mathcal{R}(\delta')$ be an isomorphism with $f(\nu) = x'.\nu'$, where ν and ν' are bases as in 5.1.

$$\begin{aligned} f(\varphi_\delta(\nu)) &= f(\delta(p).\nu) = \delta(p).f(\nu) = \delta(p)x'.\nu' \\ \varphi_{\delta'}(f(\nu)) &= \varphi_{\delta'}(x'.\nu') = \delta'(p)\varphi(x').\nu' \end{aligned}$$

and since f as a morphism of (φ, Γ)-modules and therefore commutes with φ,

$$\delta(p)x'.\nu' = \delta'(p)\varphi(x').\nu'.$$

Hence we have $\delta(p)(\delta'(p))^{-1}x' = \varphi(x')$.

Now if $\tilde{\nu}$ is a basis as in 5.1 of $\mathcal{R}(\delta\delta'^{-1})$ let $g : \mathcal{R}(\delta\delta'^{-1}) \to \mathcal{R}$ be the morphism given by $\tilde{\nu} \mapsto x'$. Let $x \in \mathcal{R}$. The φ-structure induced by g is then given by

$$\begin{aligned} \varphi_{ind}(xx') &= \varphi_{ind}(g(x\tilde{\nu})) = g(\varphi_{\delta\delta'^{-1}}(x\tilde{\nu})) = g((\delta(\delta')^{-1}(p))\varphi(x)\tilde{\nu}) \\ &= (\delta(\delta')^{-1})(p)\varphi(x)x' = \varphi(x)\varphi(x') = \varphi(xx') \end{aligned}$$

where we used that $\delta(p)(\delta'(p))^{-1}x' = \varphi(x')$. So g induces indeed the "normal" φ structure on $\mathcal{R}$. On the same way we observe that the action of Γ on $\mathcal{R}$ induces by g (and therefore by f) is just the "normal" action. Therefore $\mathcal{R}(\delta\delta'^{-1})$ and $\mathcal{R}$ are isomorphic as (φ, Γ)-modules.

Suppose now $\mathcal{R}(\delta)$ is isomorphic to $\mathcal{R} = \mathcal{R}(1)$. We then also have $H^0(\delta) = H^0(\mathcal{R}) = L$. But Lemma 5.19 shows that $H^0(\delta) = L$ if and only if $\delta = 1$. Hence $\mathcal{R}(\delta\delta'^{-1})$ is isomorphic to $\mathcal{R}$ if and only if $\delta\delta'^{-1} = 1$. So $\mathcal{R}(\delta)$ and $\mathcal{R}(\delta')$ are isomorphic if and only if $\delta = \delta'$. □

5.5 The characterization of all (φ, Γ)-modules of rank 1 over $\mathcal{R}$

In this section I wanted to give a proof that any rank 1 (φ, Γ)-module over $\mathcal{R}$ can be given by $\mathcal{R}(\delta)$ for some continuous character $\delta\colon \mathbb{Q}_p^* \to L^*$. As source I was using Lemma 4.1 and Proposition 4.2 in [Col05]. I wanted to work out the details such that with the theory developed before one could read and understand the proof of this theorem. There was always one piece missing at the very end of the proof, that I thought I just could not yet understand because of the lack of some theory, so I left it open in the hope to figure it out after reading some more in other papers. This one step turned out to be wrong, which makes this chapter kind of pointless. Maybe one could fix the proof by adding some theory, maybe this is just not the way to go. I will leave this part in the thesis, because still every lemma should be correct on its own, just not relevant to prove the theorem. I marked the point where the mistake is happening.

Recall that in Proposition 4.23 we have shown that any element $x \in \mathcal{E}^t$ can be written uniquely in the form $x = x^0 T^{k(x)} x^+ x^-$, where $x^0 \in L^*$, $x^+ \in 1 + T\mathcal{O}_L [\![T]\!]$ and $x^- \in 1 + \mathfrak{m}_L[\![T^{-1}]\!] \cap \mathcal{E}^t$.

Lemma 5.21 (Statement in Lemma 4.1 of [Col05])**.** *Let $a, b \in (\mathcal{E}^t)^*$ with $a^{-1}\varphi(a) = b^{-1}\gamma(b)$ and $a^0, b^0 \in L^*$ as in Proposition 4.23. Then we can find $s \in \mathbb{Z}$ such that $\tilde{a} = (a^0)^{-1} T^s \gamma(T^{-s}) a$ and $\tilde{b} = (b^0)^{-1} T^s \varphi(T^{-s}) b$ satisfy the following properties:*

(i) There are $\tilde{a}^+, \tilde{b}^+ \in 1 + T\mathcal{O}_L [\![T]\!]$ and $\tilde{a}^-, \tilde{b}^- \in 1 + \mathfrak{m}_L[\![T^{-1}]\!] \cap \mathcal{E}^t$ with $\tilde{a} = \tilde{a}^+\tilde{a}^-$ and $\tilde{b} = \tilde{b}^+\tilde{b}^-$.

(ii) $\tilde{a}^{-1}\varphi(\tilde{a}) = \tilde{b}^{-1}\gamma(\tilde{b})$.

Proof. Since φ and γ operate trivial on elements of L^* we can multiply a and b by $(a^0)^{-1}$ and $(b^0)^{-1}$ and the equation still holds.

$$((a^0)^{-1}a)^{-1}\varphi((a^0)^{-1}a) = a^{-1}\varphi(a) = b^{-1}\gamma(b) = ((b^0)^{-1}b)^{-1}\gamma((b^0)^{-1}b)$$

Also if we take $a' = a \cdot T^{-1}\gamma(T)$ and $b' = b \cdot T^{-1}\varphi(T)$ the equation $a'^{-1}\varphi(a') = b'^{-1}\gamma(b')$ still holds true:

$$\begin{aligned} b'^{-1}\gamma(b') &= b^{-1}T\varphi(T)^{-1}\gamma(b)\gamma(T^{-1})\gamma(\varphi(T)) \\ &= b^{-1}\gamma(b)T\gamma(T^{-1})\varphi(T^{-1}\gamma(T)) \\ a'^{-1}\varphi(a') &= a^{-1}\varphi(a)T\gamma(T^{-1})\varphi(T^{-1}\gamma(T)) \end{aligned}$$

where we used that the action of Γ commutes with φ. So $a^{-1}\varphi(a) = b^{-1}\gamma(b)$ holds if and only if $b'^{-1}\gamma(b') = a'^{-1}\varphi(a')$.

Now suppose that $a^0 = b^0 = 1$. Using $\varphi(x) = \varphi(x^0)\varphi(T^{k(x)})\varphi(x^+)\varphi(x^-)$ with Lemma 4.24 we have $k(\varphi(x)) = pk(x)$ and $k(\gamma(x)) = k(x)$. With $k(a^{-1}\varphi(a)) = k(b^{-1}\gamma(b))$ this means

$$-k(a) + pk(a) = k(a)(p-1) = -k(b) + k(b) = 0$$

and therefore $k(a) = 0$. Hence a is of the form $a = a^+a^-$.

Now we look at $a^{-1}\varphi(a) = b^{-1}\gamma(b)$ modulo the maximal ideal $\mathfrak{m}_L$. Then a^- and b^- equal 1. Moreover we have seen that a^+, b^+ and $\varphi(a^+)$, $\gamma(b^+)$ are all elements of $1 + T\mathcal{O}_L[\![T]\!]$. Also a^+ and b^+ are invertible in $1 + T\mathcal{O}_L[\![T]\!]$. Furthermore by Lemma 4.24 $(T^{k(b)})^{-1}\gamma(T^{k(b)}) \subset \chi(\gamma)^{k(b)}(1 + T\mathcal{O}_L[\![T]\!])$. So comparing the constant coefficients $(b^{-1}\gamma(b))^{(0)}$ and $(a^{-1}\varphi(a))^{(0)}$ we see that $\chi(\gamma)^{k(b)} \equiv 1 \pmod{\mathfrak{m}_L}$. This means, since $\chi(\gamma) \in \mathbb{Z}_p^*$, that $\chi(\gamma)^{k(b)} \equiv 1 \pmod{p}$. We have chosen γ to be a generator of Γ, therefore this is only the case if $p-1|k(b)$ in other words $k(b) = s\cdot(p-1)$ for some integer $s \in \mathbb{Z}$.

So let $\tilde{b} = (b^0)^{-1}T^s\varphi(T^{-s})b$ and $\tilde{a} = (a^0)^{-1}T^s\gamma(T^{-s})a$, then $k(\tilde{a}) = 0$, $\tilde{a}^0 = 1 = \tilde{b}^0$ and

$$k(\tilde{b}) = k(b) + k(T^s) + k(\varphi(T^{-s})) = k(b) + s - sp = k(b) - s(p-1) = 0.$$

□

Lemma 5.22 (Statement in Lemma 4.1 of [Col05])**.** *For $a = a^+a^-$ and $b = b^+b^-$ with $a^{-1}\varphi(a) = b^{-1}\gamma(b)$ we can find $c_0 \in \mathcal{R}$ such that*

$$\left(\frac{\partial a}{a}, \frac{\partial b}{b}\right) = ((\chi(\gamma)\gamma - 1)c_0, (p\varphi - 1)c_0).$$

Proof. In Definition 4.25 we have defined a logarithm on the elements a^+a^- and b^+b^-. In Proposition 4.26 and 4.27 we have seen that these logarithms are well-defined elements of $\mathcal{R}$. So $(\log(a), \log(b))$ is a well-defined element of $\mathcal{R} \oplus \mathcal{R}$.

$$\begin{aligned}(\varphi - 1)\log a - (\gamma - 1)\log b &= \log\varphi(a) - \log a - \log\gamma(b) + \log b \\ &= \log\left(\frac{\varphi(a)}{a}\frac{b}{\gamma(b)}\right) \\ &= \log 1 = 0.\end{aligned}$$

Hence it is an element of the cocycles $Z^1(1)$ as defined in Chapter 5.4. Now we use Proposition 3.8 (ii) in [Col05], which states that

$\partial\colon H^1(1) \to H^1(x)$ is the zero map. Therefore $\partial(\log a, \log b) \equiv (0,0)$ modulo $B^1(x)$. This means there exists $c_0 \in \mathcal{R}$ such that

$$\begin{aligned}\partial(\log a, \log b) = \left(\frac{\partial a}{a}, \frac{\partial b}{b}\right) &= (0,0) + ((\gamma_{\delta=x} - 1)c_0, (\varphi_{\delta=x} - 1)c_0) \\ &= ((\chi(\gamma)\gamma - 1)c_0, (p\varphi - 1)c_0).\end{aligned}$$

□

Suppose for $c_0 \in \mathcal{R}$ as in the previous lemma there is some $c_1 \in \mathcal{R}$ such that $\partial c_1 = c_0$. Let $\frac{1}{p-1} + 1 > r > 0$ such that c_1 converges on $0 < v_p(T) \leq r$. Take $n \in \mathbb{N}$ such that on the annulus $p^{-2}r \leq v_p(a) \leq r$ we have $n + v_p(c_1(a)) > 1$ and let $c_2 = \exp(p^n c_1)$.

Lemma 5.23 (Statement in Lemma 4.1 of [Col05])**.** *Let $0 < r < 1+\frac{1}{p-1}$ and $x = \sum_{i\in\mathbb{Z}} a_i T^i$ such that*

(i) x converges for $p^{-2}r \leq v_p(T) \leq r$

(ii) there is $b \in (\mathcal{E}^\dagger)^*$ with $x = b\varphi(x)$.

Then x converges already on $0 < v_p(T) \leq r$.

Proof. We know that x converges on $p^{-2}r \leq v_p(T) \leq r$, so by Proposition 1.42 we can split $x = x_+ + x_-$ with $x_+ \in L[\![T]\!]$ and $x_- \in T^{-1}L[\![T^{-1}]\!]$ such that x_+ converges on $v_p(T) > p^{-2}r$ and x_- converges on $v_p(T) < r$. By Propositions 4.6 and 4.9 then $\varphi(x) = \varphi(x_+) + \varphi(x_-)$ converges on $p^{-3}r \leq v_p(T) \leq p^{-1}r$. Hence both x and $\varphi(x)$ converge on $p^{-2}r \leq v_p(T) \leq p^{-1}r$. With $x = b\varphi(x)$ then also b converges on this interval. Therefore $b \in \mathcal{L}_L(0, p^{-1}r]$. Hence with $x = b\varphi(x)$ the elements x must already converge on the interval $p^{-3}r \leq v_p(T) \leq r$. By induction we then get that x converges on $0 < v_p(T) \leq r$. □

Lemma 5.24 (Statement in Lemma 4.1 of [Col05])**.** *c_2 as defined above fulfills the equation $c_2 = b^{-p^n}\varphi(c_2)$ and is an element of $(\mathcal{E}^\dagger)^*$.*

Proof. The p-adic exponential function converges for elements with valuation bigger than $1/(p-1)$. Since $v_p(p^n c_1) = n + v_p(c_1) > 1$ on the annulus $p^{-2}r \leq v_p(T) \leq r$ we have that c_2 converges there. To show that c_2 fulfills the equation $c_2 = b^{-p^n}\varphi(c_2)$ we start with the equation

$$\partial(\log b) = (p\varphi - 1)c_0 = (p\varphi - 1)\partial c_1 = \partial(\varphi(c_1) - c_1).$$

Taking the antiderivative and multiplying by p^n we get

$$p^n \log b = \log b^{p^n} = p^n(\varphi - 1)c_1 + k \text{ for some } k \in L.$$

Comparing the constant coefficients on both sides shows that $k = 0$, since $\log(b) = \log(b^+) + \log(b^-)$ does not have a constant coefficient and $\varphi(c_1)^{(0)} = c_1^{(0)}$. Finally we take the exponent

$$\begin{aligned} b^{p^n} &= exp(p^n\varphi(c_1))/exp(p^n c_1) \\ &= \varphi(exp(p^n c_1))/exp(p^n c_1) = \varphi(c_2)/c_2. \end{aligned}$$

With Lemma 5.23 this gives us that c_2 is analytic on $0 < v_p(T) \le r$. To show that c_2 is bounded at 0 note that inductively for $m \in \mathbb{N}$ we have

$$\begin{aligned} c_2 = b^{-p^n}\varphi(c_2) = b^{-p^n}\varphi(b^{-p^n}\varphi(c_2)) &= \big(\prod_{i=0}^{m}\varphi^i(b^{-p^n})\big)\varphi^m(c_2) \\ &= \big(\prod_{i=0}^{m}\varphi^i(b^{-p^n})\big)\sum_{j\in\mathbb{Z}} c_2^{(j)}((1+T)^{p^m}-1)^j \\ &= \big(\prod_{i=0}^{m}\varphi^i(b^{-p^n})\big)\sum_{j\in\mathbb{Z}} c_2^{(j)}(p^mT + ... + p^mT^{p^m-1} + T^{p^m})^j. \end{aligned}$$

Since b^+ and b^- are invertible in $1 + T\mathcal{O}_L[\![T]\!]$ resp. $1 + \mathfrak{m}_L[\![T^{-1}]\!] \cap \mathcal{E}^\dagger$ we get $v_p(b^{(j)}) \ge 0$ for all $j \in \mathbb{Z}$ and hence also $v_p(\varphi^i(b)^{(j)}) \ge 0$ for all $i \in \mathbb{N}, j \in \mathbb{Z}$. The coefficients of $\sum_{j\in\mathbb{Z}} c_2^{(j)}(p^mT + ... + p^mT^{p^m-1} + T^{p^m})^j$ are converging towards 0 (their valuation tends to ∞) if m goes to ∞. Hence c_2 is bounded at 0 and therefore $c_2 \in (\mathcal{E}^t)^*$. □

Lemma 5.25 (Statement in Lemma 4.1 of [Col05]). *For $a, b, c \in (\mathcal{E}^\dagger)^*$ such that*

(i) $b = c^{-1}\varphi(c)$,

(ii) $a^{-1}\varphi(a) = b^{-1}\gamma(b)$,

there is a $\tilde{l} \in L^*$ with $a = \tilde{l}c^{-1}\gamma(c)$.

Proof. We substitute $b = c^{-1}\varphi(c)$:

$$\begin{aligned} a^{-1}\varphi(a) &= b^{-1}\gamma(b) \\ &= (c^{-1}\varphi(c))^{-1}\gamma(c^{-1}\varphi(c)) \\ &= c\gamma(c)^{-1}\varphi(c^{-1}\gamma(c)). \end{aligned}$$

Therefore by rearranging the equation

$$\varphi(c\gamma(c)^{-1}a) = c\gamma(c)^{-1}a.$$

The only nonzero elements that are fixed by φ are elements of L^*, hence

$$c\gamma(c)^{-1}a = \tilde{l} \in L^*$$

in other words

$$a = \tilde{l}c^{-1}\gamma(c).$$

$\square$

At the end of the following proof there is the mistake that we have already mentioned.

Lemma 5.26. *(Lemma 4.1 in [Col05]) For $a, b \in (\mathcal{E}^\dagger)^*$ with $a^{-1}\varphi(a) = b^{-1}\gamma(b)$ there exist $\alpha, \beta \in L^*$, $c \in (\mathcal{E}^\dagger)^*$ with $a = \alpha c^{-1}\gamma(c)$ and $b = \beta c^{-1}\varphi(c)$.*

Proof. We follow the proof of Lemma 4.1 of [Col05]. In Lemma 5.21 we have seen that we can find $s \in \mathbb{Z}$ such that $\tilde{a} = (a^0)^{-1}T^s\gamma(T^{-s})a$ and $\tilde{b} = (b^0)^{-1}T^s\varphi(T^{-s})$ are of the form $\tilde{a} = \tilde{a}^+\tilde{a}^-$ and $\tilde{b} = \tilde{b}^+\tilde{b}^-$ and fulfill the equation $\tilde{a}^{-1}\varphi(\tilde{a}) = \tilde{b}^{-1}\gamma(\tilde{b})$. So it suffices to find $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{c}$ for $\tilde{b} = \tilde{b}^+\tilde{b}^-$ and $\tilde{a} = \tilde{a}^+\tilde{a}^-$ with $\tilde{a} = \tilde{\alpha}\tilde{c}^{-1}\gamma(\tilde{c})$ and $\tilde{b} = \tilde{\beta}\tilde{c}^{-1}\varphi(\tilde{c})$ because then we can take $\alpha = a^0\tilde{\alpha}$, $\beta = b^0\tilde{\beta}$ and $c = T^s\tilde{c}$ and get the desired equations, since

$$\tilde{a} = \tilde{\alpha}\tilde{c}^{-1}\gamma(\tilde{c}) = (a^0)^{-1}T^s\gamma(T^{-s})a$$
$$\tilde{b} = \tilde{\beta}\tilde{c}^{-1}\varphi(\tilde{c}) = (b^0)^{-1}T^s\varphi(T^{-s})b$$

and hence by rearranging

$$a = a^0\tilde{\alpha}(T^s\tilde{c})^{-1}\gamma(T^s\tilde{c}) = \alpha c^{-1}\gamma(c)$$
$$b = b^0\tilde{\beta}(T^s\tilde{c})^{-1}\varphi(T^s\tilde{c}) = \beta c^{-1}\varphi(c).$$

So from now let $a = a^+a^-$ and $b = b^+b^-$. With Lemma 5.22 we find $c_0 \in \mathcal{R}$ with $(\frac{\partial a}{a}, \frac{\partial b}{b}) = ((\chi(\gamma)\gamma - 1)c_0, (p\varphi - 1)c_0)$.

In Lemma 4.43 we have shown that $\mathrm{Res}(\frac{\partial b}{b}) = 0$, which implies that $\mathrm{Res}(c_0) = 0$, since with Lemma 4.42 and Lemma 4.40

$$\mathrm{Res}(\frac{\partial b}{b}) = \mathrm{Res}(p\varphi(c_0) - c_0) = p\,\mathrm{Res}(\varphi(c_0)) - \mathrm{Res}(c_0) = (p-1)\,\mathrm{Res}(c_0).$$

Therefore by Proposition 4.41 there is some $c_1 \in \mathcal{R}$ such that $\partial c_1 = c_0$.

So let $r > 0$ such that c_1 converges on $0 < v_p(T) \leq r$. Take $n \in \mathbb{N}$ such that on the annulus $p^{-2}r \leq v_p(T) \leq r$ we have $n + v_p(c_1) > 1$. Let $c_2 = \exp(p^n c_1)$. Lemma 5.24 tells us that $c_2 \in (\mathcal{E}^\dagger)^*$ and $c_2 = b^{-p^n}\varphi(c_2)$.

Here is where the argumentation in [Col05] is going wrong. Colmez states that for $b \in (\mathcal{E}^t)^*$ there exists some $c \in \mathcal{B}^t$ such that $\varphi(c) = bc$. But as Laurent Berger pointed out for me on mathoverflow.net that is not true for $b = p$.

If this was true we could proceed as follows:

$$\varphi(c^{-p^n}c_2) = \varphi(c)^{-p^n}\varphi(c_2)) = b^{-p^n}c^{-p^n}c_2 b^{p^n} = c^{-p^n}c_2.$$

The only elements of the Robba ring that are fixed by φ are elements of L. So since c_2 and c are non zero, $c_2 = lc^{p^n}$ for some scalar $l \in L^*$ and $c^{p^n} \in \mathcal{E}^t$. The extension B^t over $\mathcal{E}^t$ is unramified (this is something which is still not clear to me whether it is true or not), therefore c was already in $\mathcal{E}^t$.

With Lemma 5.25 there is a $\tilde{l} \in L^*$ such that $a = \tilde{l}c^{-1}\gamma(c)$. □

Remark. Proposition 3.1 of [Col10] shows the statement with an alternative proof. Here the equivalence between the category of étale (φ, Γ)-module and the category of L-representations of $G_{\mathbb{Q}_p}$ is used.

Proposition 5.27. *Let $x, y \in \mathcal{R}$, $z \in (\mathcal{E}^\dagger)^*$ with $x \cdot y = z$. Then $x, y \in (\mathcal{E}^\dagger)^*$.*

Proof. I got a hint from Laurent Berger on mathoverflow.net to proof this with the theory of Newton polygons. Because of lack of time for now this Proposition remains unproven. □

Lemma 5.28. *Let $e \in \mathcal{R}$ be a basis of the rank one φ module $\mathcal{R}$. Then e is an element of $(\mathcal{E}^\dagger)^*$.*

Proof. Clearly e needs to be invertible to be a basis. So there is $e^{-1} \in \mathcal{R}$ with $e^{-1} \cdot e = 1$. Moreover by Proposition 5.27 we have for $a \in \mathcal{R}$ that $a \cdot e \in \mathcal{E}^\dagger$ if and only if $a, e \in \mathcal{E}^\dagger$. Hence e is element of $\mathcal{E}^\dagger$. □

Theorem 5.29 (Proposition 4.2 of [Col05]). *If M is a (φ, Γ)-module of rank one over $\mathcal{R}$, then there is a continuous character $\delta \colon \mathbb{Q}_p^* \to L^*$ such that M is isomorphic to $\mathcal{R}(\delta)$.*

Proof. We follow the proof of Lemma 4.2 in [Col05]. Let ν be a basis of the module M over $\mathcal{R}$. Then by definition $\varphi(\nu) = r_\varphi.\nu$ is a basis as well. Hence there must exist an element $r \in \mathcal{R}$ such that $r_\varphi \cdot r = 1$. This can only be the case if $r_\varphi \in (\mathcal{E}^\dagger)^*$. Therefore $\mathcal{E}^\dagger e$ is stabel under φ. With Chapter 1.1 and 1.2, especially Proposition 1.4 in [Col08] (or Proposition 2.4 in [Col05]), since M is of rank one, $\mathcal{E}^\dagger e$ is as well stable under the action of Γ, i.e. there exists $r_\gamma \in (\mathcal{E}^\dagger)^*$ such that $\gamma(\nu) = r_\gamma \nu$. We have seen that in 5.1 that the property that φ and γ commute translates to $\varphi(r_\gamma) \cdot r_\gamma^{-1} = \gamma(r_\varphi) \cdot r_\varphi^{-1}$. So with Lemma 5.26 there are $\alpha_\gamma, \beta \in L^*$ and $c \in (\mathcal{E}^t)^*$ with $r_\gamma = \alpha_\gamma c^{-1}\gamma(c)$ and $r_\varphi = \beta c^{-1}\varphi(c)$.

Switching the basis ν to $e = c^{-1}\nu$ we get

$$\varphi(e) = \varphi(c^{-1}\nu) = \varphi(c^{-1})r_\varphi\nu = \varphi(c^{-1})\beta c^{-1}\varphi(c)\nu = \beta e$$
$$\gamma(e) = \gamma(c^{-1}\nu) = \gamma(c^{-1})r_\gamma\nu = \gamma(c^{-1})\alpha_\gamma c^{-1}\gamma(c)\nu = \alpha_\gamma e$$

Define

$$\delta\colon \mathbb{Z}_p^* \to L^*, \quad p \mapsto \beta, \quad \chi(\gamma) \mapsto \alpha_\gamma.$$

We have

$$\alpha_{\gamma\circ\gamma'} = \alpha_\gamma \cdot \alpha_{\gamma'}$$

because

$$\gamma \circ \gamma'(e) = \gamma(\gamma'(e)) = \gamma(\alpha_{\gamma'}e) = \gamma(\alpha_{\gamma'})\gamma(e) = \alpha_\gamma \cdot \alpha_{\gamma'}e.$$

Hence δ defines a character that we can extend to $\delta\colon \mathbb{Q}_p^* \to L^*$. $\square$

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