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Periods and Algebraic deRham Cohomology

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Abstract

It is known that the algebraic deRham cohomology group $H^i_{dR}(X_0/\mathbb{Q})$ of a nonsingular variety $X_0/\mathbb{Q}$ has the same rank as the rational singular cohomology group $H^i_{sing}(X^{an};\mathbb{Q})$ of the complex manifold X^{an} associated to the base change $X_0 \times_\mathbb{Q} \mathbb{C}$. However, we do not have a natural isomorphism $H^i_{dR}(X_0/\mathbb{Q}) \cong H^i_{sing}(X^{an};\mathbb{Q})$. Any choice of such an isomorphism produces certain integrals, so called periods, which reveal valuable information about X_0. The aim of this thesis is to explain these classical facts in detail. Based on an approach of Kontsevich [K, pp. 62–64], different definitions of a period are compared and their properties discussed. Finally, the theory is applied to some examples. These examples include a representation of $\zeta(2)$ as a period and a variation of mixed Hodge structures used by Goncharov [G1].

Contents

1   Introduction

The prehistory of Algebraic Topology dates back to Euler, Riemann and Betti, who started the idea of attaching various invariants to a topological space. With his simplicial (co)homology theory, Poincaré was the first to give an instance of what in modern terms we would call a contravariant functor $H^\bullet$ from the category of (sufficiently nice) topological spaces to the category of cyclic complexes of abelian groups.

Many of such functors have been found so far; the most common examples are the standard cohomology theories (i.e. those satisfying the Eilenberg-Steenrod axioms), which measure quite different phenomena relating to diverse branches of mathematics. It is a beautiful basic fact that all these standard cohomology theories agree (when restricted to an appropriate subcategory).

This does not imply that we cannot hope for more. If the topological space in question enjoys additional structure, one defines more elaborate invariants which take values in an abelian category of higher complexity. For example, Hodge theory gives us a functor from the category of compact Kähler manifolds to the category of cyclic complexes of pure Hodge structures.

In this thesis, we will concentrate on spaces originating from Algebraic Geometry; these may be regarded as spaces carrying an algebraic structure.

Generalizing the concept of deRham theory to "nice" schemes over $\mathbb{Q}$ gives us algebraic deRham cohomology groups where each is nothing but a full $\mathbb{Q}$-lattice inside the $\mathbb{C}$-vector space of the corresponding classical deRham cohomology group. So, after tensoring with $\mathbb{C}$, algebraic deRham cohomology agrees with all the standard cohomology theories with complex coefficients. However, a natural isomorphism between the original $\mathbb{Q}$-vector space and a standard cohomology group with rational coefficients cannot exist.

We will illustrate this phenomenon in the following example (see Example 4.12.1 for details). Let $X^{\mathrm{an}} := \mathbb{C}^\times$ be the complex plane with the point 0 deleted and let t be the standard coordinate on X^{an}. Then the first singular cohomology group of X^{an} is generated by the dual σ^* of the unit circle $\sigma := S^1$

$$\mathrm{H}^1_{\mathrm{sing}}(X^{\mathrm{an}}; \mathbb{Q}) = \mathbb{Q}\,\sigma^* \quad \text{and} \quad \mathrm{H}^1_{\mathrm{sing}}(X^{\mathrm{an}}; \mathbb{C}) = \mathbb{C}\,\sigma^*;$$

while for the first classical deRham cohomology group, we have

$$\mathrm{H}^1_{\mathrm{dR}}(X^{\mathrm{an}}; \mathbb{C}) = \mathbb{C}\,\frac{dt}{t}.$$

Under the comparison isomorphism

$$\mathrm{H}^1_{\mathrm{sing}}(X^{\mathrm{an}}; \mathbb{C}) \cong \mathrm{H}^1_{\mathrm{dR}}(X^{\mathrm{an}}; \mathbb{C})$$

the generator σ^* of $\mathrm{H}^1_{\mathrm{sing}}(X^{\mathrm{an}}; \mathbb{C})$ is mapped to

$$\left(\int_{S^1} \frac{dt}{t} \right)^{-1} \frac{dt}{t} = \frac{1}{2\pi i}\frac{dt}{t}.$$

If we view X^{an} as the complex manifold associated to the base change to $\mathbb{C}$ of the algebraic variety $X_0 := \operatorname{Spec} \mathbb{Q}[t, t^{-1}]$ over $\mathbb{Q}$, we can also compute the algebraic de Rham cohomology group $\mathrm{H}^1_{\mathrm{dR}}(X_0/\mathbb{Q})$ of X_0 and embed it into $\mathrm{H}^1_{\mathrm{dR}}(X^{\mathrm{an}}; \mathbb{C})$

$$\mathrm{H}^1_{\mathrm{dR}}(X_0/\mathbb{Q}) = \mathbb{Q}\frac{dt}{t} \subset \mathrm{H}^1_{\mathrm{dR}}(X^{\mathrm{an}}; \mathbb{C}) = \mathbb{C}\frac{dt}{t}.$$

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Thus we get two $\mathbb{Q}$-lattices inside $\mathrm{H}^1_{\mathrm{dR}}(X^{\mathrm{an}}; \mathbb{C})$, $\mathrm{H}^1_{\mathrm{sing}}(X^{\mathrm{an}}; \mathbb{Q})$ and $\mathrm{H}^1_{\mathrm{dR}}(X_0/\mathbb{Q})$, which do not coincide. In fact, they differ by the factor $2\pi i$ — our first example of what we will call a period. Other examples will produce period numbers like π, $\ln 2$, elliptic integrals, or $\zeta(2)$, which are interesting also from a number theoretical point of view (cf. page 47).

There is some ambiguity about the precise definition of a period; actually we will give four definitions in total:

(i) pairing periods (cf. Definition 5.1.1 on page 43)

(ii) abstract periods (cf. Definition 5.2.1 on page 45)

(iii) naïve periods (cf. Definition 5.3.2 on page 46)

(iv) effective periods (cf. Definition 7.3.1 on page 63)

For X_0 a nonsingular variety over $\mathbb{Q}$, we have a natural pairing between the ith algebraic deRham cohomology of X_0 and the ith singular homology group of the complex manifold X^{an} associated to the base change $X_0 \times_{\mathbb{Q}} \mathbb{C}$

$$\mathrm{H}^{\mathrm{sing}}_i(X^{\mathrm{an}}; \mathbb{Q}) \times \mathrm{H}^i_{\mathrm{dR}}(X_0/\mathbb{Q}) \longrightarrow \mathbb{C}.$$

The numbers which can appear in the image of this pairing (or its version for relative cohomology) are called *pairing periods*; this is the most traditional way to define a period.

In [K, p. 62], Kontsevich gives the alternative definition of *effective periods* which does not need algebraic deRham cohomology and, at least conjecturally, gives the set of all periods some extra algebraic structure. We present his ideas in Subsection 7.3. *Abstract periods* describe just a variant of Kontsevich's definition. In fact, we have a surjection from the set of effective periods to the set of abstract ones (cf. page 65), which is conjectured to be an isomorphism.

Naïve periods are defined in an elementary way and are used to provide a connection between pairing periods and abstract periods.

In Kontsevich's paper [K, p. 63], it is used that the notion of pairing and abstract period coincide. The aim of this thesis is to show that the following implications hold true (cf. Theorem 7.1.1)

$$\text{abstract period} \Leftrightarrow \text{naïve period} \Rightarrow \text{pairing period}.$$

The thesis is organized as follows. The discussion of the various definitions of a period makes up the principal part of the work filling sections five to seven.

Section two gives an introduction to complex analytic spaces. Additionally, we provide the connection to Algebraic Geometry by defining the associated complex analytic space of a variety.

In **Section three**, we define algebraic deRham cohomology for pairs consisting of a variety and a divisor on it. We also give some working tools for this cohomology.

The aim of **Section four** is to give a comparison theorem (Theorem 4.10.1) which states that algebraic deRham and singular cohomology agree.

In **Section five**, we present the definition of pairing, abstract, and naïve periods and prove some of their properties.

Section six provides some facts about the triangulation of algebraic varieties.

Section seven contains the main result (Theorem 7.1.1) about the implications between the various definitions of a period mentioned above. Furthermore, we give the definition of effective periods which motivated the definition of abstract periods. In the last section, **Section eight**, we consider five examples to give an application of the general theory. Among them is a representation of $\zeta(2)$ as an abstract period and the famous double logarithm variation of mixed Hodge structures used by Goncharov [G1] whose geometric origin is emphasized.

Conventions. By a *variety*, we will always mean a reduced, quasi-projective scheme. We will often deal with a variety X_0 defined over some algebraic extension of $\mathbb{Q}$. As a rule, skipping the subscript 0 will always mean base change to $\mathbb{C}$

$$X := \text{``}X_0 \times_{\mathbb{Q}} \mathbb{C}\text{''} = X_0 \times_{\operatorname{Spec} \mathbb{Q}} \operatorname{Spec} \mathbb{C}.$$

(An exception is section three, where arbitrary base fields are used.) The complex analytic space associated to X will be denoted by X^{an} (cf. Subsection 2.1).
The sign conventions used throughout this thesis are listed in the appendix.

2 The Associated Complex Analytic Space

Let X be a variety over $\mathbb{C}$. The set $|X|$ of closed points of X inherits the Zariski topology. However, we can also equip this set with the standard topology: For smooth X this gives a complex manifold; in general we get a complex analytic space X^{an}. The main reference for this section is [Ha, B.1].

2.1 The Definition of the Associated Complex Analytic Space

We consider an example before giving the general definition of a complex analytic space.

Example 2.1.1. Let $D^n \subset \mathbb{C}^n$ be the polycylinder

$$D^n := \{ \underline{z} \in \mathbb{C}^n \, | \, |z_i| < 1, i = 1, \ldots, n \}$$

and $\mathcal{O}_{D^n}$ the sheaf of holomorphic functions on D^n. For a set of holomorphic functions $f_1, \ldots, f_m \in \Gamma(D^n, \mathcal{O}_{D^n})$ we define

$$\begin{aligned}
\mathcal{X}_{D^n} &:= \{ \underline{z} \in D^n \, | \, f_1(\underline{z}) = \ldots = f_m(\underline{z}) = 0 \} \\
\mathcal{O}_{\mathcal{X}_{D^n}} &:= \mathcal{O}_{D^n}/(f_1, \ldots, f_m).
\end{aligned} \tag{1}$$

The locally ringed space $(\mathcal{X}_{D^n}, \mathcal{O}_{\mathcal{X}_{D^n}})$ from this example is a complex analytic space. In general, complex analytic spaces are obtained by glueing spaces of the form (1).

Definition 2.1.2 (Complex analytic space, [Ha, B.1, p. 438]). *A locally ringed space $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is called* complex analytic *if it is locally (as a locally ringed space) isomorphic to one of the form (1). A morphism of complex analytic spaces is a morphism of locally ringed spaces.*

For any scheme $(X, \mathcal{O}_X)$ of finite type over $\mathbb{C}$ we have an *associated complex analytic space* $(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}})$.

Definition 2.1.3 (Associated complex analytic space, [Ha, B.1, p. 439]). *Assume first that X is affine. We fix an isomorphism*

$$X \cong \operatorname{Spec} \mathbb{C}[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$$

and then consider the f_i as holomorphic functions on $\mathbb{C}^n$ in order to set

$$\begin{aligned}
X^{\mathrm{an}} &:= \{ \underline{z} \in \mathbb{C}^n \, | \, f_1(\underline{z}) = \ldots = f_m(\underline{z}) = 0 \} \\
\mathcal{O}_{X^{\mathrm{an}}} &:= \mathcal{O}_{\mathbb{C}^n}/(f_1, \ldots, f_m),
\end{aligned}$$

where $\mathcal{O}_{\mathbb{C}^n}$ denotes the sheaf of holomorphic functions on $\mathbb{C}^n$.
For an arbitrary scheme X of finite type over $\mathbb{C}$, we take a covering of X by open affine subsets U_i. The scheme X is obtained by glueing the open sets U_i, so we can use the same glueing data to glue the complex analytic space $(U_i)^{\mathrm{an}}$ into an analytic space X^{an}. This is the associated complex analytic space of X.

This construction is natural and we obtain a functor **an** from the category of schemes of finite type over $\mathbb{C}$ to the category of complex analytic spaces. Note that its restriction to the subcategory of smooth schemes maps into the category of complex manifolds as a consequence of the inverse function theorem (cf. [Gun, Thm. 6, p. 20]).

Example 2.1.4. The complex analytic space associated to complex projective space is again complex projective space, but considered as a complex manifold. To avoid confusion in the subsequent sections, the notation $\mathbb{C}P^n$ will be reserved for *complex projective space in the category of schemes*, whereas we write $\mathbb{C}P^n_{\mathrm{an}}$ for *complex projective space in the category of complex analytic spaces*.

For any scheme X of finite type over $\mathbb{C}$, we have a natural map of locally ringed spaces

$$\phi : X^{\mathrm{an}} \to X \tag{2}$$

which induces the identity on the set of closed points $|X|$ of X. Note that $\phi^* \mathcal{O}_X = \mathcal{O}_{X^{\mathrm{an}}}$.

2.2 Algebraic and Analytic Coherent Sheaves

Let us consider sheaves of $\mathcal{O}_X$-modules. The equality of functors

$$\Gamma(X, ?) = \Gamma(X^{\mathrm{an}}, \phi^{-1}?)$$

gives an equality of their right derived functors

$$\mathrm{H}^i(X; ?) = \mathrm{R}^i\Gamma(X; ?) = \mathrm{R}^i\Gamma(X^{\mathrm{an}}; \phi^{-1}?).$$

Since ϕ^{-1} is an exact functor, the spectral sequence for the composition of the functors ϕ^{-1} and $\Gamma(X^{\mathrm{an}}; ?)$ degenerates and we obtain

$$\mathrm{H}^i(X^{\mathrm{an}}; \phi^{-1}?) = \mathrm{R}^i\Gamma(X^{\mathrm{an}}; ?) \circ \phi^{-1} = \mathrm{R}^i\Gamma(X^{\mathrm{an}}; \phi^{-1}?).$$

Thus the natural map for $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules

$$\phi^{-1}\mathcal{F} \to \phi^*\mathcal{F}$$

gives a natural map of cohomology groups

$$\mathrm{H}^i(X; \mathcal{F}) = \mathrm{H}^i(X^{\mathrm{an}}; \phi^{-1}\mathcal{F}) \to \mathrm{H}^i(X^{\mathrm{an}}; \phi^*\mathcal{F}). \tag{3}$$

Sheaf cohomology behaves particularly nice for coherent sheaves, this notion being defined as follows.

Definition 2.2.1 (Coherent sheaf). *We define a* coherent sheaf $\mathcal{F}$ *on* X *(resp.* X^{an}*) to be a sheaf of* $\mathcal{O}_X$*-modules (resp.* $\mathcal{O}_{X^{\mathrm{an}}}$*-modules) that is Zariski-locally (resp. locally in the standard topology) isomorphic to the cokernel of a morphism of free* $\mathcal{O}_X$*-modules (resp.* $\mathcal{O}_{X^{\mathrm{an}}}$*-modules) of finite rank*

$$\mathcal{O}_U^r \longrightarrow \mathcal{O}_U^s \longrightarrow \mathcal{F}_{|U} \longrightarrow 0, \quad for \quad U \subseteq X \ (resp. \ U \subseteq X^{\mathrm{an}}) \quad open, \tag{4}$$

where $r, s \in \mathbb{N}$.

For sheaves on X this agrees with the definition given in [Ha, II.5, p. 111]. For this alternate definition, we need some notation: If $U = \operatorname{Spec} A$ is an affine variety and M an A-module, we denote by $\widetilde{M}$ the sheaf on U associated to M (i.e. the sheaf associated to the presheaf $V \mapsto \Gamma(V; \mathcal{O}_V) \otimes_A M$ *for* $V \subseteq U$ open, see [Ha, II.5, p. 110]).

Lemma 2.2.2 (cf. [Ha, II.5 Exercise 5.4, p. 124]). *A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is coherent if and only if X can be covered by open affine subsets $U_i = \operatorname{Spec} A_i$ such that $F_{|U_i} \cong \widetilde{M_i}$ for some finitely generated A_i-modules M_i.*

Proof. "if": The A_i's are Noetherian rings. Therefore any finitely generated A_i-module M_i will be finitely presented

$$A_i^r \longrightarrow A_i^s \longrightarrow M_i \longrightarrow 0.$$

Since localization is an exact functor, we get

$$\mathcal{O}_{U_i}^r \longrightarrow \mathcal{O}_{U_i}^s \longrightarrow \widetilde{M_i} \longrightarrow 0,$$

which proves the "if"-part. ∎

"only if": W.l.o.g. we may assume that the open subsets $U \subseteq X$ in (4) are affine $U = \operatorname{Spec} A$. Then $\Gamma(U; \mathcal{O}_U) = A$ and $\mathcal{O}_U = \widetilde{A}$. Now the A-module

$$
\begin{aligned}
M := & \operatorname{coker}\big(\Gamma(U; \mathcal{O}_U^r) \longrightarrow \Gamma(U; \mathcal{O}_U^s)\big) \\
= & \operatorname{coker}(A^r \longrightarrow A^s)
\end{aligned}
$$

is clearly finitely generated. Since

$$A^r \longrightarrow A^s \longrightarrow M \longrightarrow 0$$

gives

$$\widetilde{A}^r \longrightarrow \widetilde{A}^s \longrightarrow \widetilde{M} \longrightarrow 0,$$

we conclude

$$\mathcal{F}_{|U} = \operatorname{coker}(\mathcal{O}_U^r \longrightarrow \mathcal{O}_U^s) = \operatorname{coker}(\widetilde{A}^r \longrightarrow \widetilde{A}^s) = \widetilde{M}.$$

□

As an immediate consequence of the definition of a coherent sheaf $\mathcal{F}$ on X, we see that the sheaf

$$\mathcal{F}_{\mathrm{an}} := \phi^* \mathcal{F}$$

will be coherent as well: If

$$\mathcal{O}_U^r \longrightarrow \mathcal{O}_U^s \longrightarrow \mathcal{F}_{|U} \longrightarrow 0$$

is exact, so is

$$\mathcal{O}_{U^{\mathrm{an}}}^r \longrightarrow \mathcal{O}_{U^{\mathrm{an}}}^s \longrightarrow \phi^* \mathcal{F}_{|U^{\mathrm{an}}} \longrightarrow 0,$$

since ϕ^{-1} is exact and tensoring is a right exact functor.

There is a famous theorem by Serre usually referred to as GAGA, since it is contained in his paper "*Géométrie algébrique et géométrie analytique*" [Se:gaga].

Theorem 2.2.3 (Serre, [Ha, B.2.1, p. 440]). *Let X be a **projective** scheme over $\mathbb{C}$. Then the map*

$$\mathcal{F} \mapsto \mathcal{F}_{\mathrm{an}}$$

induces an equivalence between the category of coherent sheaves on X and the category of coherent sheaves on X^{an}. Furthermore, the natural map (3)

$$\mathrm{H}^i(X; \mathcal{F}) \xrightarrow{\sim} \mathrm{H}^i(X^{\mathrm{an}}; \mathcal{F}_{\mathrm{an}})$$

is an isomorphism for all i.

Let us state a corollary of Theorem 2.2.3, which is not included in [Ha].

Corollary 2.2.4. *In the situation of Theorem 2.2.3, we also have a natural isomorphism for hypercohomology for all i*

$$\mathbb{H}^i(X; \mathcal{F}^\bullet) \cong \mathbb{H}^i(X^{\mathrm{an}}; \mathcal{F}^\bullet_{\mathrm{an}}),$$

where $\mathcal{F}^\bullet$ is a bounded complex of coherent sheaves on X.

Here we only require the boundary morphisms of $\mathcal{F}^\bullet$ to be morphisms of sheaves of abelian groups. They do not need to be $\mathcal{O}_X$-linear.

Before we begin proving Corollary 2.2.4, we need some homological algebra.

Lemma 2.2.5. *Let $\mathfrak{A}$ be an abelian category and*

$$\mathrm{F}^\bullet[0] \longrightarrow \mathrm{G}^{\bullet,\bullet}, \tag{5}$$

a morphism of double complexes of $\mathfrak{A}$-objects (cf. the appendix), where

- $\mathrm{F}^\bullet[0]$ *is a double complex concentrated in the zeroth row with $\mathrm{F}^\bullet$ being a complex vanishing below degree zero, i.e. $\mathrm{F}^n = 0$ for $n < 0$, and*

- $\mathrm{G}^{\bullet,\bullet}$ *is a double complex living only in non-negative degrees.*

If for all $q \in \mathbb{Z}$

$$0 \to \mathrm{F}^q \to \mathrm{G}^{0,q} \to \mathrm{G}^{1,q} \to \cdots$$

is a resolution of F^q, then the map of total complexes induced by (5)

$$\mathrm{F}^\bullet \xrightarrow{\sim} \operatorname{tot} \mathrm{G}^{\bullet,\bullet}$$

is a quasi-isomorphism.

Proof. From (5) we obtain a morphism of spectral sequences

$$\mathrm{h}_{\mathrm{II}}^p \, \mathrm{h}_{\mathrm{I}}^q \, \mathrm{F}^\bullet[0] \Rightarrow \mathrm{h}^n \, \mathrm{F}^\bullet$$
$$\downarrow \qquad\quad \downarrow$$
$$\mathrm{h}_{\mathrm{II}}^p \, \mathrm{h}_{\mathrm{I}}^q \, \mathrm{G}^{\bullet,\bullet} \Rightarrow \mathrm{h}^n \, \operatorname{tot} \mathrm{G}^{\bullet,\bullet}.$$

Both spectral sequences degenerate because of

$$\mathrm{h}_{\mathrm{II}}^p \, \mathrm{h}_{\mathrm{I}}^q \, \mathrm{F}^\bullet[0] = \begin{cases} \mathrm{h}^p \, \mathrm{F}^\bullet & \text{if } q = 0, \\ 0 & \text{else} \end{cases}$$

and

$$\mathrm{h}_{\mathrm{II}}^p \, \mathrm{h}_{\mathrm{I}}^q \, \mathrm{G}^{\bullet,\bullet} = \begin{cases} \mathrm{h}^p \, \mathrm{F}^\bullet & \text{if } q = 0, \\ 0 & \text{else} \end{cases}$$

and so we get an isomorphism on the initial terms. Hence we also have an isomorphism on the limit terms and our assertion follows. $\square$

Remark 2.2.6. A similar statement holds with $\mathrm{F}^\bullet[0]$ considered as a double complex concentrated in the zeroth column.

Godement resolutions. As we also need Γ-acyclic resolutions that behave functorial in the proof of Corollary 2.2.4, we now describe the concept of Godement resolutions. For any sheaf $\mathcal{F}$ on X (or X^{an}) define

$$g(\mathcal{F}) := \prod_{x \in |X|} i_* \mathcal{F}_x,$$

where $i : x \hookrightarrow X$ denotes the closed immersion of the point x. The sheaf $g(\mathcal{F})$ is flabby and we have a natural inclusion

$$\mathcal{F} \rightarrowtail g(\mathcal{F}).$$

Setting $\mathcal{G}^{i+1} := g\left(\mathrm{coker}(\mathcal{G}^{i-1} \to \mathcal{G}^i)\right)$ with $\mathcal{G}^{-1} := \mathcal{F}$ and $\mathcal{G}^0 := g(\mathcal{F})$ gives an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G}^0 \longrightarrow \mathcal{G}^1 \longrightarrow \dots.$$

We define the *Godement resolution* of $\mathcal{F}$ to be

$$\mathcal{G}_{\mathcal{F}}^\bullet := 0 \longrightarrow \mathcal{G}^0 \longrightarrow \mathcal{G}^1 \longrightarrow \dots.$$

It is Γ-acyclic and functorial in $\mathcal{F}$. Extending this definition to a bounded complex $\mathcal{F}^\bullet$

$$
\begin{aligned}
\mathcal{G}_{\mathcal{F}^\bullet}^{\bullet;q} &:= \mathcal{G}_{\mathcal{F}^q}^\bullet \\
\mathcal{G}_{\mathcal{F}^\bullet}^\bullet &:= \mathrm{tot}\, \mathcal{G}_{\mathcal{F}^\bullet}^{\bullet;\bullet}
\end{aligned}
\tag{6}
$$

yields a map of double complexes

$$\mathcal{F}^\bullet[0] \longrightarrow \mathcal{G}^{\bullet,\bullet}$$

and a quasi-isomorphism by Lemma 2.2.5

$$\mathcal{F}^\bullet \xrightarrow{\ \sim\ } \mathcal{G}_{\mathcal{F}^\bullet}^\bullet.$$

Let $x \in |X|$ be a closed point of X. The map ϕ from (2) gives a commutative square

$$
\begin{array}{ccc}
\{x\}^{\mathrm{an}} & \overset{i}{\hookrightarrow} & X^{\mathrm{an}} \\
\phi \downarrow & & \downarrow \phi \\
\{x\} & \overset{i}{\hookrightarrow} & X
\end{array}
$$

and a natural morphism (cf. [Ha, II.5, p. 110])

$$\mathcal{F}_x \longrightarrow \phi_* \phi^* \mathcal{F}_x,$$

where we consider the stalk $\mathcal{F}_x$ as a sheaf on $\{x\}$. This map induces another map

$$i_* \mathcal{F}_x \longrightarrow i_* \phi_* \phi^* \mathcal{F}_x = (i \circ \phi)_* \phi^* \mathcal{F}_x = (\phi \circ i)_* \phi^* \mathcal{F}_x = \phi_* i_* \phi^* \mathcal{F}_x,$$

and yet a third map (cf. loc. cit.)

$$\varepsilon : \phi^* i_* \mathcal{F}_x \longrightarrow i_* \phi^* \mathcal{F}_x, \tag{7}$$

which is an isomorphism, as can be seen on the stalks

$$\varepsilon_x : \phi^* \mathcal{F}_x \xrightarrow{\sim} \phi^* \mathcal{F}_x,$$
$$\varepsilon_y : \quad 0 \longrightarrow 0, \qquad \text{for } y \neq x.$$

Consequently,

$$\phi^* g(\mathcal{F}) = \phi^* \prod_x i_* \mathcal{F}_x = \prod_x \phi^* i_* \mathcal{F}_x \overset{(7)}{=} \prod_x i_* \phi^* \mathcal{F}_x = \prod_x i_* (\phi^* \mathcal{F})_x = g(\phi^* \mathcal{F}_x).$$

Thus we get first

$$\phi^* \mathcal{G}^\bullet_{\mathcal{F}^p} = \mathcal{G}^\bullet_{\phi^* \mathcal{F}^p},$$

then

$$\phi^* \mathcal{G}^\bullet_{\mathcal{F}^\bullet} = \mathcal{G}^\bullet_{\phi^* \mathcal{F}^\bullet}.$$

This gives us a natural map of hypercohomology groups

$$\mathbb{H}^i(X; \mathcal{F}^\bullet) = \mathrm{h}^i \Gamma(X; \mathcal{G}^\bullet_{\mathcal{F}^\bullet}) = \mathrm{h}^i \Gamma(X^{\mathrm{an}}; \phi^{-1} \mathcal{G}^\bullet_{\mathcal{F}^\bullet})$$
$$\longrightarrow$$
$$\mathrm{h}^i \Gamma(X^{\mathrm{an}}; \phi^* \mathcal{G}^\bullet_{\mathcal{F}^\bullet}) = \mathrm{h}^i \Gamma(X^{\mathrm{an}}; \mathcal{G}^\bullet_{\phi^* \mathcal{F}^\bullet}) = \mathrm{h}^i \Gamma(X^{\mathrm{an}}; \mathcal{G}^\bullet_{\mathcal{F}^\bullet_{\mathrm{an}}}) = \mathbb{H}^i(X^{\mathrm{an}}; \mathcal{F}^\bullet_{\mathrm{an}}). \quad (8)$$

Proof of Corollary 2.2.4. We claim that the natural map (8) is an isomorphism for all i if X is projective.

If $\mathcal{F}^\bullet$ has has length one, Theorem 2.2.3 tells us that this is indeed true. So let us assume that (8) is an isomorphism for all complexes of length $\leq n$ and let $\mathcal{F}^\bullet$ be a complex of coherent sheaves on X of length $n + 1$. W.l.o.g. $\mathcal{F}^{n+1} \neq 0$ but $\mathcal{F}^p = 0$ for $p > n + 1$. We write $\sigma_{\leq n} \mathcal{F}^\bullet$ for the complex $\mathcal{F}^\bullet$ cut off above degree n. The short exact sequence

$$0 \longrightarrow \mathcal{F}^{n+1}[-n-1] \longrightarrow \mathcal{F}^\bullet \longrightarrow \sigma_{\leq n} \mathcal{F}^\bullet \longrightarrow 0$$

remains exact if we take the inverse image along ϕ

$$0 \longrightarrow \mathcal{F}^{n+1}_{\mathrm{an}}[-n-1] \longrightarrow \mathcal{F}^\bullet_{\mathrm{an}} \longrightarrow \sigma_{\leq n} \mathcal{F}^\bullet_{\mathrm{an}} \longrightarrow 0.$$

Using the naturality of (8), we obtain the following "ladder" with commuting squares

$$\begin{array}{ccccccc}
\cdots \to & \mathrm{H}^{-n-1+i}(X; \mathcal{F}^{n+1}) & \longrightarrow & \mathbb{H}^i(X; \mathcal{F}^\bullet) & \longrightarrow & \mathbb{H}^i(X; \sigma_{\leq n} \mathcal{F}^\bullet) & \to \cdots \\
& \downarrow \wr & & \downarrow & & \downarrow \wr & \\
\cdots \to & \mathrm{H}^{-n-1+i}(X^{\mathrm{an}}; \mathcal{F}^{n+1}_{\mathrm{an}}) & \longrightarrow & \mathbb{H}^i(X^{\mathrm{an}}; \mathcal{F}^\bullet_{\mathrm{an}}) & \longrightarrow & \mathbb{H}^i(X^{\mathrm{an}}; \sigma_{\leq n} \mathcal{F}^\bullet_{\mathrm{an}}) & \to \cdots
\end{array}$$

and the induction step follows from the 5-lemma. □

3 Algebraic deRham Theory

In this section, we define the algebraic deRham cohomology $\mathrm{H}^\bullet_{\mathrm{dR}}(X, D/k)$ of a smooth variety X over a field k and a normal-crossings-divisor D on X (cf. definitions 3.2.3, 3.2.4, and 3.2.6). We also give some working tools for this cohomology: a base change theorem (Proposition 3.5.1) and two spectral sequences (Corollary 3.6.3 and Proposition 3.6.4).

3.1 Classical deRham Cohomology

This subsection only serves as a motivation for the following giving an overview of classical deRham theory. For a complex manifold, analytic deRham, complex, and singular cohomology are defined and shown to be equal to classical deRham cohomology. All material presented here will be generalized to a relative setup later on.

Let M be a complex manifold. We recall two standard exact sequences,

(i) the *analytic deRham complex* of holomorphic differential forms on M

$$0 \longrightarrow \mathbb{C}_M \longrightarrow \Omega^0_M \xrightarrow{\partial} \Omega^1_M \xrightarrow{\partial} \dots, \text{ and}$$

(ii) the *classical deRham complex* of smooth $\mathbb{C}$-valued differential forms on M

$$0 \longrightarrow \mathbb{C}_M \longrightarrow \mathcal{E}^0_M \xrightarrow{d} \mathcal{E}^1_M \xrightarrow{d} \dots,$$

where $\mathbb{C}_M$ is the constant sheaf with fibre $\mathbb{C}$ on M.

In both cases exactness is a consequence of the respective Poincaré lemmas [W, 4.18, p. 155] and [GH, p. 25].

Now consider the commutative diagram

$$\begin{array}{ccc} \mathbb{C}_M[0] & = & \mathbb{C}_M[0] \\ \downarrow \wr & & \downarrow \wr \\ \Omega^\bullet_M & \hookrightarrow & \mathcal{E}^\bullet_M. \end{array}$$

We can rephrase the exactness of the sequences (i) and (ii) by saying that the vertical maps are quasi-isomorphisms. We indicate quasi-isomorphisms by a tilde. Hence the natural inclusion

$$\Omega^\bullet_M \hookrightarrow \mathcal{E}^\bullet_M$$

is a quasi-isomorphism as well. Therefore the hypercohomology of the two complexes coincides

$$\mathbb{H}^\bullet(M; \Omega^\bullet_M) = \mathbb{H}^\bullet(M; \mathcal{E}^\bullet_M).$$

The sheaves $\mathcal{E}^p_M$ are fine, since they admit a partition of unity. In particular they are acyclic for the global section functor $\Gamma(M, ?)$ and we obtain

$$\mathbb{H}^\bullet(M; \mathcal{E}^\bullet_M) = \mathrm{h}^\bullet\Gamma(M; \mathcal{E}^\bullet_M).$$

The right-hand-side is usually called the *classical deRham cohomology* of M, denoted

$$\mathrm{H}^\bullet_{\mathrm{dR}}(M; \mathbb{C}).$$

The equalities above give

$$\mathrm{H}_{\mathrm{dR}}^\bullet(M;\mathbb{C}) = \mathbb{H}^\bullet(M;\Omega_M^\bullet).$$

We refer to the right-hand-side $\mathbb{H}^\bullet(M;\Omega_M^\bullet)$ as *analytic deRham cohomology*, for which we want to use the same symbol $\mathrm{H}_{\mathrm{dR}}^\bullet(M;\mathbb{C})$. The hypercohomology $\mathbb{H}^\bullet(M;\Omega_M^\bullet)$ turns out to be a good candidate for generalizing deRham theory to algebraic varieties. Both variants of deRham cohomology agree with *complex cohomology*

$$\mathrm{H}^\bullet(M;\mathbb{C}) := \mathrm{H}^\bullet(M;\mathbb{C}_M) = \mathbb{H}^\bullet(M;\mathbb{C}_M[0]).$$

We have yet a third resolution of the constant sheaf $\mathbb{C}_M$ given by the *complex of singular cochains*: For any open set $U \subseteq M$, we write $\mathrm{C}_{\mathrm{sing}}^p(U;\mathbb{C})$ for the vector space of singular p-cochains on U with coefficients in $\mathbb{C}$. The sheaf of singular p-cochains is now defined as

$$\mathcal{C}_{\mathrm{sing}}^p(M;\mathbb{C}) : U \mapsto \mathrm{C}_{\mathrm{sing}}^p(U;\mathbb{C}) \quad \text{for} \quad U \subseteq M \quad \text{open}$$

with the obvious restriction maps. The sheaves $\mathcal{C}_{\mathrm{sing}}^p(M;\mathbb{C})$ are flabby: The restriction maps are the duals of injections between vector spaces of singular p-chains and the functor $\mathrm{Hom}_\mathbb{C}(?,\mathbb{C})$ is exact. In particular these sheaves are acyclic for the global section functor $\Gamma(M;?)$ and we obtain

$$\mathbb{H}^\bullet(M;\mathcal{C}_{\mathrm{sing}}^\bullet(M;\mathbb{C})) = \mathrm{h}^\bullet\Gamma(M;\mathcal{C}_{\mathrm{sing}}^\bullet(M;\mathbb{C})) = \mathrm{h}^\bullet\mathrm{C}_{\mathrm{sing}}^\bullet(M;\mathbb{C}) = \mathrm{H}_{\mathrm{sing}}^\bullet(M;\mathbb{C}).$$

By the following lemma, we conclude $\mathrm{H}^\bullet(M;\mathbb{C}) = \mathrm{H}_{\mathrm{sing}}^\bullet(M;\mathbb{C})$.

Lemma 3.1.1. *For any locally contractible, locally path-connected topological space M the sequence*

$$0 \longrightarrow \mathbb{C}_M \longrightarrow \mathcal{C}_{\mathrm{sing}}^0(M;\mathbb{C}) \longrightarrow \mathcal{C}_{\mathrm{sing}}^1(M;\mathbb{C}) \longrightarrow \dots .$$

is exact.

Proof. Note first that $\mathbb{C}_M = \mathrm{h}^0\,\mathcal{C}_{\mathrm{sing}}^\bullet(M;\mathbb{C})$, since M is locally path-connected. For the higher cohomology sheaves $\mathrm{h}^p\,\mathcal{C}_{\mathrm{sing}}^\bullet(M;\mathbb{C})$, $p > 0$, we observe that any element s_x of the stalk $\mathrm{h}^p\,\mathcal{C}_{\mathrm{sing}}^\bullet(M;\mathbb{C})_x$ at $x \in M$ not only lifts to a section s of $\mathrm{C}_{\mathrm{sing}}^p(U;\mathbb{C})$ for some contractible open subset $x \in U \subset M$, but that we can assume s to be a cocycle by eventually shrinking U. Now this s is also a coboundary because of

$$\mathrm{h}^p\mathrm{C}_{\mathrm{sing}}^\bullet(U;\mathbb{C}) = \mathrm{H}_{\mathrm{sing}}^p(U;\mathbb{C}) = 0 \quad \text{for all} \quad p > 0.$$

$\square$

We summarize this subsection in the following proposition.

Proposition 3.1.2. *Let M be a complex manifold. Then we have a chain of natural isomorphisms between the various cohomology groups defined in this subsection*

$$\mathrm{H}_{\mathrm{dR}}^\bullet(M;\mathbb{C}) \cong \mathbb{H}^\bullet(M;\Omega_M^\bullet) \cong \mathrm{H}^\bullet(M;\mathbb{C}) \cong \mathrm{H}_{\mathrm{sing}}^\bullet(M;\mathbb{C}).$$

3.2 Algebraic deRham Cohomology

Let X be a smooth variety defined over a field k and D a divisor with normal crossings on X; where having normal crossings means, that locally D looks like a collection of coordinate hypersurfaces, or more precisely:

Definition 3.2.1 (Divisor with normal crossings, [Ha, p. 391]). *A divisor $D \subset X$ is said to have normal crossings, if each irreducible component of D is nonsingular and whenever s irreducible components $D_1, \ldots, D_s$ meet at a closed point P, then the local equations $f_1, \ldots, f_s$ of the D_i form part of a regular system of parameters $f_1, \ldots, f_d$ at P.*

It is proved in [Ma, 12, p. 78] that in this case the $f_1, \ldots, f_d$ are linearly independent modulo m_P^2, where m_P is the maximal ideal of the local ring $\mathcal{O}_{X,P}$ at P. By the inverse function theorem for holomorphic functions [Gun, Thm. 6, p. 20], we find in a neighbourhood of any $P \in X^{\mathrm{an}}$ a holomorphic chart $z_1, \ldots, z_d$ such that D^{an} is given as the zero-set $\{z_1 \cdot \ldots \cdot z_s = 0\}$.

We are now going to define algebraic deRham cohomology groups

$$H_{\mathrm{dR}}^\bullet(X/k), \quad H_{\mathrm{dR}}^\bullet(D/k) \quad \text{and} \quad H_{\mathrm{dR}}^\bullet(X, D/k).$$

Remark 3.2.2. In [Ha:dR], algebraic deRham cohomology is defined for varieties with arbitrary singularities. However, the relative version of algebraic deRham cohomology discussed in [Ha:dR] deals with morphisms of varieties, not pairs of them.

3.2.1 The Smooth Case

Definition 3.2.3 (Algebraic deRham cohomology for a smooth variety). *We set*

$$H_{\mathrm{dR}}^\bullet(X/k) := \mathbb{H}^\bullet(X; \Omega_{X/k}^\bullet),$$

where $\Omega_{X/k}^\bullet$ is the complex of algebraic differential forms on the smooth variety X over k (cf. [Ha, II.8, p. 175]).

3.2.2 The Case of a Divisor with Normal Crossings

We write $D = \sum_{i=1}^r D_i$ as a sum of its irreducible components and use the short-hand notation (cf. Figure 1)

$$D_I := D_{i_0 \cdots i_p} := \bigcap_{k=0}^p D_{i_k} \quad \text{for} \quad I = \{i_0, \ldots, i_k\}.$$

Associated to the decomposition $D = \sum D_i$ is a simplicial scheme

$$D^\bullet := \coprod D_a \Longleftarrow \coprod D_{ab} \Longleftarrow \coprod D_{abc} \cdots,$$
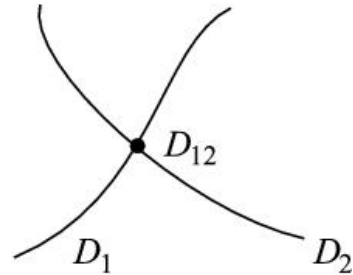
this notion being defined as follows.

Figure 1: Divisor with normal crossings

Simplicial sets and simplicial schemes. We consider a category $\mathfrak{Simplex}$ with objects $[m] := \{0, \ldots, m\}$, $m \in \mathbb{N}_0$, and $<$-order-preserving maps as morphisms. (Thus the existence of a map $f : [m] \to [n]$ implies $m \leq n$.)

This category can be thought of as the prototype of a simplicial complex: Let K be a simplicial complex and assign to $[m]$ the set of m-simplices of K. Any morphism $f : [m] \to [n]$ maps each n-simplex to its m-dimensional "f-face"

$$f^*(e_0, \ldots, e_n) = (e_{f(0)}, \ldots, e_{f(m)}).$$

Thus K can be described by a contravariant functor to the category $\mathfrak{Sets}$ of sets

$$\mathfrak{Simplex} \longrightarrow \mathfrak{Sets}.$$

A contravariant functor from $\mathfrak{Simplex}$ to the category of sets, schemes, ... is called a *simplicial set*, *simplicial scheme* and so forth [GM, Ch. 1, 2.2, p. 9]. (There, as a slight generalization, the maps $f : [m] \to [n]$ are only required to be non-decreasing instead of being $<$-order-preserving, thus allowing for "degenerate simplices".)

In the above situation we define a simplicial scheme $D^\bullet : \mathfrak{Simplex} \to \mathfrak{Schemes}$ by the assignment

$$[m] \mapsto \coprod_{1 \leq i_0 < \cdots < i_m \leq r} D_{i_0 \cdots i_m}$$

$$(f : [m] \to [n]) \mapsto \left(D^\bullet(f) : \coprod D_{i_0 \cdots i_n} \to \coprod D_{i_0 \cdots i_m} \right);$$

here $D^\bullet(f)$ is the sum of the natural inclusions

$$D_{i_0 \cdots i_n} \hookrightarrow D_{i_{f(0)} \cdots i_{f(m)}}.$$

If $\delta_l^m : [m] \to [m+1]$ denotes the unique $<$-order-preserving map, whose image does not contain l, we can represent $D^\bullet$ by a diagram

$$\coprod_{a=1}^{r} D_a \quad \begin{array}{c} \xleftarrow{D^\bullet(\delta_0^0)} \\ \xleftarrow[D^\bullet(\delta_1^0)]{} \end{array} \quad \coprod_{1 \leq a < b \leq r} D_{ab} \quad \begin{array}{c} \xleftarrow{D^\bullet(\delta_0^1)} \\ \xleftarrow{D^\bullet(\delta_1^1)} \\ \xleftarrow[D^\bullet(\delta_2^1)]{} \end{array} \quad \coprod_{1 \leq a < b < c \leq r} D_{abc} \cdots . \qquad (D^\bullet)$$

Now the differentials come into play again. We consider the contravariant functor $i_*\Omega_{?/k}^\bullet$ defined for subschemes Z of $\mathrm{Supp}(D)$

$$i_*\Omega_{?/k}^\bullet : Z \mapsto i_*\Omega_{Z/k}^\bullet.$$

where i stands for the natural inclusion $Z \overset{i}{\hookrightarrow} \mathrm{Supp}(D)$. We enlarge the scope of $i_*\Omega_{?/k}^\bullet$ to schemes of the form $\coprod_a Z_a$, $Z_a \subseteq D$ by making the convention

$$\left(i_*\Omega_{?/k}^\bullet \right) \left(\coprod_a Z_a \right) := \bigoplus_a i_*\Omega_{Z_a/k}^\bullet.$$

Composing $D^\bullet$ with the contravariant functor $i_*\Omega_{?/k}^\bullet$ yields a diagram

$$\bigoplus_{a=1}^{r} i_*\Omega_{D_a/k}^\bullet \quad \begin{array}{c} \xrightarrow{d_0^0} \\ \xrightarrow[d_1^0]{} \end{array} \quad \bigoplus_{1 \leq a < b \leq r} i_*\Omega_{D_{ab}/k}^\bullet \quad \begin{array}{c} \xrightarrow{d_0^1} \\ \xrightarrow{d_1^1} \\ \xrightarrow[d_2^1]{} \end{array} \quad \bigoplus_{1 \leq a < b < c \leq r} i_*\Omega_{D_{abc}/k}^\bullet \cdots ,$$

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where we have written d_l^m for $(i_*\Omega_{?/k}^\bullet \circ D^\bullet)(\delta_l^m)$. Summing up these maps d_l^m with alternating signs

$$d^m := \sum_{l=0}^{m}(-1)^l d_l^m$$

gives a new diagram

$$\bigoplus_{a=1}^{r} i_*\Omega_{D_a/k}^\bullet \xrightarrow{\ d^0\ } \bigoplus_{1\le a<b\le r} i_*\Omega_{D_{ab}/k}^\bullet \xrightarrow{\ d^1\ } \bigoplus_{1\le a<b<c\le r} i_*\Omega_{D_{abc}/k}^\bullet \cdots,$$

which turns out to be a double complex (cf. the appendix) as a little calculation shows

$$
\begin{aligned}
d^{m+1}\circ d^m &= \sum_{l=0}^{m+1}\sum_{k=0}^{m}(-1)^{l+k}\, d_l^{m+1}\circ d_k^m\\
&= \sum_{0\le k<l\le m+1}(-1)^{l+k}\, d_l^{m+1}\circ d_k^m + \underbrace{\sum_{0\le l\le k\le m}(-1)^{l+k}\, d_l^{m+1}\circ d_k^m}_{\displaystyle \sum_{0\le l\le k\le m}(-1)^{l+k}\, d_{k+1}^{m+1}\circ d_l^m}\\
&= \sum_{0\le l\le k\le m}(-1)^{l+k}\, d_{k+1}^{m+1}\circ d_l^m\\
&= \sum_{0\le k'<l'\le m+1}(-1)^{k'+l'-1}\, d_{l'}^{m+1}\circ d_{k'}^m\\
&= 0.
\end{aligned}
$$

We denote this double complex by $\Omega_{D^\bullet/k}^{\bullet;\bullet}$ and its total complex (cf. the appendix) by

$$\widetilde{\Omega}_{D/k}^\bullet := \operatorname{tot}\Omega_{D^\bullet/k}^{\bullet;\bullet}.$$

Definition 3.2.4 (Algebraic deRham cohomology for a divisor). *If D is a divisor with normal crossings on a smooth variety X over k, we define its* algebraic deRham cohomology *by*

$$\mathrm{H}_{\mathrm{dR}}^\bullet(D/k) := \mathbb{H}^\bullet(D;\widetilde{\Omega}_{D/k}^\bullet).$$

Remark 3.2.5. Actually, we do not need that $D\subseteq X$ has codimension 1 for this definition; but D having normal crossings is essential.

3.2.3 The Relative Case

The natural restriction maps

$$\Omega_{X/k}^\bullet \to (D_i\hookrightarrow X)_*\Omega_{D_i/k}^\bullet$$

sum up to a natural map of double complexes

$$\Omega_{X/k}^\bullet[0] \to (\operatorname{Supp}(D)\hookrightarrow X)_*\,\Omega_{D^\bullet/k}^{\bullet;\bullet}, \tag{9}$$

where we view $\Omega^\bullet_{X/k}[0]$ as a double complex concentrated in the zeroth column. Somewhat sloppy, we will often write $i_*\mathcal{F}$ instead of $(\mathrm{Supp}(D) \hookrightarrow X)_* \mathcal{F}$. Taking the total complex in (9) yields a natural map

$$f : \Omega^\bullet_{X/k} \to i_*\widetilde{\Omega}^\bullet_{D/k},$$

for whose mapping cone (cf. the appendix)

$$M_f = i_*\widetilde{\Omega}^\bullet_{D/k}[-1] \oplus \Omega^\bullet_{X/k} \text{ with differential } \begin{pmatrix} -d_D & f \\ 0 & d_X \end{pmatrix}$$

we write $\widetilde{\Omega}^\bullet_{X,D/k}$.

Definition 3.2.6 (Relative algebraic deRham cohomology). *For X a smooth variety over k and D a divisor with normal crossings on X, we define the* relative algebraic deRham cohomology *of the pair (X, D) by*

$$\mathrm{H}^\bullet_{\mathrm{dR}}(X, D/k) := \mathbb{H}^\bullet(X; \widetilde{\Omega}^\bullet_{X,D/k}).$$

These definitions may seem a bit technical at first glance. However, they will hopefully become more transparent in the sequel, when the existence of a long exact sequence in algebraic deRham cohomology and various comparison isomorphisms are proved.

3.3 Basic Lemmas

In this subsection we gather two basic lemmas for the ease of reference.

Lemma 3.3.1. *If $i : Z \hookrightarrow X$ is a closed immersion and $\mathcal{I}$ an injective sheaf of abelian groups on Z, then $i_*\mathcal{I}$ is also injective.*

Proof. Let $\mathcal{A} \rightarrowtail \mathcal{B}$ be an injective morphism of sheaves of abelian groups on X. Since i^{-1} is an exact functor, the induced map

$$i^{-1}\mathcal{A} \rightarrowtail i^{-1}\mathcal{B}$$

will be injective as well. Now the adjoint property of i^{-1} provides us with the following commutative square

$$\mathrm{Hom}_X(\mathcal{B}, i_*\mathcal{I}) \cong \mathrm{Hom}_Z(i^{-1}\mathcal{B}, \mathcal{I})$$
$$\downarrow \qquad\qquad \downarrow$$
$$\mathrm{Hom}_X(\mathcal{A}, i_*\mathcal{I}) \cong \mathrm{Hom}_Z(i^{-1}\mathcal{A}, \mathcal{I}).$$

Because $\mathcal{I}$ is injective, the map on the right-hand-side is surjective. Hence so is the left-hand map, which in turn implies the injectivity of $i_*\mathcal{I}$, since $\mathcal{A}$ and $\mathcal{B}$ were arbitrary. $\square$

Lemma 3.3.2. *If $i : Z \hookrightarrow X$ is a closed immersion and $\mathcal{F}^\bullet$ a complex of sheaves on Z, which is bounded below, then there is a natural isomorphism*

$$\mathbb{H}^\bullet(X; i_*\mathcal{F}^\bullet) \cong \mathbb{H}^\bullet(Z; \mathcal{F}^\bullet).$$

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Proof. The point is that i_* is an exact functor for closed immersions, which can be easily checked on the stalks. Thus if $\mathcal{F}^\bullet \rightarrowtail \mathcal{I}^\bullet$ is a quasi-isomorphism between $\mathcal{F}^\bullet$ and a complex $\mathcal{I}^\bullet$ consisting of injective sheaves, the induced map

$$i_*\mathcal{F}^\bullet \rightarrowtail i_*\mathcal{I}^\bullet$$

will be a quasi-isomorphism as well. But the sheaves $i_*\mathcal{I}^p$ are injective by Lemma 3.3.1, hence we can use $i_*\mathcal{I}^\bullet$ to compute the hypercohomology of $i_*\mathcal{F}^\bullet$

$$\mathbb{H}^\bullet(X; i_*\mathcal{F}^\bullet) = \mathrm{h}^\bullet\Gamma(X; i_*\mathcal{I}^\bullet).$$

Now the claim follows from the identity of functors $\Gamma(X, i_*?) = \Gamma(Z; ?)$

$$\mathbb{H}^\bullet(X; i_*\mathcal{F}^\bullet) = \mathrm{h}^\bullet\Gamma(X; i_*\mathcal{I}^\bullet) = \mathrm{h}^\bullet\Gamma(Z; \mathcal{I}^\bullet) = \mathbb{H}^\bullet(Z; \mathcal{F}^\bullet).$$

$\square$

3.4 The Long Exact Sequence in Algebraic deRham Cohomology

Proposition 3.4.1. *We have a natural long exact sequence in algebraic deRham cohomology (as defined in Definition 3.2.3, 3.2.4 and 3.2.6)*

$$\cdots \to \mathrm{H}_{\mathrm{dR}}^{p-1}(D/k) \to$$
$$\mathrm{H}_{\mathrm{dR}}^p(X, D/k) \to \mathrm{H}_{\mathrm{dR}}^p(X/k) \to \mathrm{H}_{\mathrm{dR}}^p(D/k) \to$$
$$\mathrm{H}_{\mathrm{dR}}^{p+1}(X, D/k) \to \cdots,$$

where X is a smooth variety over k and D a normal-crossings-divisor on X.

Proof. The short exact sequence of the mapping cone

$$0 \to i_*\widetilde{\Omega}_{D/k}^\bullet[-1] \to \widetilde{\Omega}_{X,D/k}^\bullet \to \Omega_{X/k}^\bullet \to 0$$

gives us a long exact sequence in hypercohomology

$$\cdots \to \mathbb{H}^p(X; i_*\widetilde{\Omega}_{D/k}^\bullet[-1]) \to$$
$$\mathbb{H}^p(X; \widetilde{\Omega}_{X,D/k}^\bullet) \to \mathbb{H}^p(X; \Omega_{X/k}^\bullet) \to \mathbb{H}^{p+1}(X; i_*\widetilde{\Omega}_{D/k}^\bullet[-1]) \to$$
$$\mathbb{H}^{p+1}(X; \widetilde{\Omega}_{X,D/k}^\bullet) \to \cdots,$$

which we can rewrite as

$$\cdots \to \mathbb{H}^{p-1}(X; i_*\widetilde{\Omega}_{D/k}^\bullet) \to$$
$$\mathrm{H}_{\mathrm{dR}}^p(X, D/k) \to \mathrm{H}_{\mathrm{dR}}^p(X/k) \to \mathbb{H}^p(X; i_*\widetilde{\Omega}_{D/k}^\bullet) \to$$
$$\mathrm{H}_{\mathrm{dR}}^{p+1}(X, D/k) \to \cdots.$$

By Lemma 3.3.2, we see

$$\mathbb{H}^\bullet(X; i_*\widetilde{\Omega}_{D/k}^\bullet) = \mathbb{H}^\bullet(D; \widetilde{\Omega}_{D/k}^\bullet) = \mathrm{H}_{\mathrm{dR}}^\bullet(D/k)$$

and our assertion follows. $\square$

3.5 Behaviour Under Base Change

Let X_0 be a smooth variety defined over a field k_0 and D_0 a divisor with normal crossings on X_0. For the base change to an extension field k of k_0 we write

$$X := X_0 \times_{k_0} k \text{ and } D := D_0 \times_{k_0} k.$$

Then we have the following proposition.

Proposition 3.5.1. *In the above situation, there are natural isomorphisms between algebraic deRham cohomology groups (as defined in definitions 3.2.3, 3.2.4, 3.2.6)*

$$\mathrm{H}^\bullet_{\mathrm{dR}}(X/k) \cong \mathrm{H}^\bullet_{\mathrm{dR}}(X_0/k_0) \otimes_{k_0} k,$$
$$\mathrm{H}^\bullet_{\mathrm{dR}}(D/k) \cong \mathrm{H}^\bullet_{\mathrm{dR}}(D_0/k_0) \otimes_{k_0} k \quad \text{and}$$
$$\mathrm{H}^\bullet_{\mathrm{dR}}(X, D/k) \cong \mathrm{H}^\bullet_{\mathrm{dR}}(X_0, D_0/k_0) \otimes_{k_0} k.$$

Proof. Denote by $\pi : X \to X_0$ the natural projection. By [Ha, Prop. II.8.10, p. 175], we have

$$\Omega^\bullet_{X/k} = \pi^* \Omega^\bullet_{X_0/k_0}.$$

Now the natural map

$$\pi^{-1} \Omega^\bullet_{X_0/k_0} \to \pi^* \Omega^\bullet_{X_0/k_0}$$

factors through

$$\pi^{-1} \Omega^\bullet_{X_0/k_0} \to \pi^{-1} \Omega^\bullet_{X_0/k_0} \otimes_{k_0} k$$

yielding a natural map

$$\pi^{-1} \Omega^\bullet_{X_0/k_0} \otimes_{k_0} k \to \pi^* \Omega^\bullet_{X_0/k_0} = \Omega^\bullet_{X/k},$$

which provides us with a morphism of spectral sequences

$$\begin{array}{ccc}
\mathrm{h}^p \, \mathrm{H}^q(X; \pi^{-1} \Omega^\bullet_{X_0/k_0} \otimes_{k_0} k) & \Rightarrow & \mathbb{H}^n(X; \pi^{-1} \Omega^\bullet_{X_0/k_0} \otimes_{k_0} k) \\
\downarrow & & \downarrow \\
\mathrm{h}^p \, \mathrm{H}^q(X; \pi^* \Omega^\bullet_{X_0/k_0}) & \Rightarrow & \mathbb{H}^n(X; \pi^* \Omega^\bullet_{X_0/k_0}).
\end{array}$$

We can rewrite the first initial term using the exactness of the functor $? \otimes_{k_0} k$

$$\mathrm{H}^q(X; \pi^{-1} \Omega^p_{X_0/k_0} \otimes_{k_0} k) = \mathrm{H}^q(X; \pi^{-1} \Omega^p_{X_0/k_0}) \otimes_{k_0} k$$
$$= \mathrm{H}^q(X_0; \Omega^p_{X_0/k_0}) \otimes_{k_0} k.$$

Since "*cohomology commutes with flat base extension for quasi-coherent sheaves*" [Ha, Prop. III.9.3, p. 255], the map

$$\mathrm{H}^q(X_0; \Omega^p_{X_0/k_0}) \otimes_{k_0} k \longrightarrow \mathrm{H}^q(X; \pi^* \Omega^p_{X_0/k_0})$$

is an isomorphism. Hence we also have an isomorphism on the limit terms, for which we obtain

$$\mathbb{H}^n(X; \pi^{-1} \Omega^\bullet_{X_0/k_0} \otimes_{k_0} k) = \mathbb{H}^n(X; \pi^{-1} \Omega^\bullet_{X_0/k_0}) \otimes_{k_0} k$$
$$= \mathbb{H}^n(X_0; \Omega^\bullet_{X_0/k_0}) \otimes_{k_0} k$$
$$= \mathrm{H}^n_{\mathrm{dR}}(X_0/k_0) \otimes_{k_0} k$$

and

$$\mathbb{H}^n(X; \pi^*\Omega^\bullet_{X_0/k_0}) = \mathbb{H}^n(X; \Omega^\bullet_{X/k})$$
$$= \mathrm{H}^n_{\mathrm{dR}}(X/k).$$

The other two statements can be proved analogously. (Observe that the sheaves $\widetilde{\Omega}^p_{D_0/k_0}$ and $\widetilde{\Omega}^p_{X_0,D_0/k_0}$ are quasi-coherent by [Ha, Prop. II.5.8(c), p. 115].) $\qquad\square$

3.6 Some Spectral Sequences

Before we proceed in the discussion of algebraic deRham cohomology, we have to make a digression and provide some spectral sequences which we will need later on.

3.6.1 Čech Cohomology

We adopt the notation from [Ha, III.4]: Let X be a topological space, $\mathfrak{U} = (U_j)_{j \in J}$ an open covering of X and $\mathcal{F}$ a sheaf of abelian groups on X. We assume J to be well-ordered. As usual, we use the short-hand notation

$$U_I := U_{i_0 \cdots i_m} := \bigcap_{k=0}^m U_{i_k} \quad \text{for} \quad I = \{i_0, \ldots, i_m\}.$$

Recall that in this situation we have a *Čech functor* defined on the category $\mathfrak{Ab}(X)$ of sheaves of abelian groups on X (cf. [Ha, III.4, p. 220])

$$\check{\mathcal{C}}^p(\mathfrak{U}, ?) : \mathfrak{Ab}(X) \to \mathfrak{Ab}(X)$$
$$\mathcal{F} \mapsto \bigoplus_{\substack{|I|=p+1 \\ I \subseteq J}} i_* \mathcal{F}_{|U_I},$$

where i stands for the respective inclusions $U_I \hookrightarrow X$, and a differential

$$d : \check{\mathcal{C}}^p(\mathfrak{U}; \mathcal{F}) \to \check{\mathcal{C}}^{p+1}(\mathfrak{U}; \mathcal{F}),$$

which makes $\check{\mathcal{C}}^\bullet(\mathfrak{U}, \mathcal{F})$ into a complex. We explain this d below. We also need *Čech groups*

$$\check{\mathrm{C}}^p(\mathfrak{U}; \mathcal{F}) := \Gamma(X; \check{\mathcal{C}}^p(\mathfrak{U}; \mathcal{F}))$$

and of course *Čech cohomology*

$$\check{\mathrm{H}}^p(\mathfrak{U}; \mathcal{F}) := \mathrm{h}^p \check{\mathrm{C}}^\bullet(\mathfrak{U}; \mathcal{F}).$$

Clearly, we can consider the Čech functor also for closed coverings; but then Čech cohomology will not behave particularly well.

We can give the following fancy definition of the Čech complex $\check{\mathcal{C}}^\bullet(\mathfrak{U}; \mathcal{F})$. The assignment

$$\mathfrak{U}^\bullet : \mathfrak{Simplex} \to \mathfrak{Schemes}$$
$$[m] \mapsto \coprod_{i_0 < \cdots < i_m} U_{i_0 \cdots i_m}$$
$$(f : [m] \to [n]) \mapsto \left(\mathfrak{U}^\bullet(f) : \coprod U_{i_0 \cdots i_n} \to \coprod U_{i_0 \cdots i_m} \right)$$

provides us with a simplicial scheme $\mathfrak{U}^\bullet$ (cf. page 13), and thus we have a diagram

$$\coprod_a U_a \underset{\mathfrak{U}^\bullet(\delta_1^0)}{\overset{\mathfrak{U}^\bullet(\delta_0^0)}{\leftleftarrows}} \coprod_{a<b} U_{ab} \underset{\mathfrak{U}^\bullet(\delta_2^1)}{\overset{\mathfrak{U}^\bullet(\delta_0^1)}{\underset{\mathfrak{U}^\bullet(\delta_1^1)}{\leftleftarrows}}} \coprod_{a<b<c} U_{abc} \cdots .$$

Denote by

$$\varepsilon : \coprod_{|I|=m+1} U_I \to X$$

the natural morphism induced by $U_I \hookrightarrow X$. Then

$$\varepsilon_* \varepsilon^{-1} \mathcal{F} = \bigoplus_{|I|=m+1} i_* \mathcal{F}_{|U_I},$$

where $i : U_I \hookrightarrow X$. Composing $\mathfrak{U}^\bullet$ with the contravariant functor

$$\varepsilon_* \varepsilon^{-1} \mathcal{F} : \coprod_{|I|=m+1} U_I \mapsto \bigoplus_{|I|=m+1} i_* \mathcal{F}_{|U_I}$$

yields a diagram

$$\bigoplus_a i_* \mathcal{F}_{xU_a} \underset{d_1^0}{\overset{d_0^0}{\rightrightarrows}} \bigoplus_{a<b} i_* \mathcal{F}_{|U_{ab}} \underset{d_2^1}{\overset{d_0^1}{\underset{d_1^1}{\rightrightarrows}}} \bigoplus_{a<b<c} i_* \mathcal{F}_{|U_{abc}} \cdots$$

with maps

$$d_l^m := \left(\varepsilon_* \varepsilon^{-1} \mathcal{F} \circ \mathfrak{U}^\bullet \right) (\delta_l^m).$$

Summing up these maps with alternating signs

$$d^m := \sum_{l=0}^m (-1)^l \, d_l^m$$

gives us the Čech complex $\check{\mathcal{C}}^\bullet(\mathfrak{U}; \mathcal{F})$.

We have the following lemma.

Lemma 3.6.1 ([Ha, III.4.2, p. 220]). *Let X be a topological space, $\mathcal{F}$ a sheaf of abelian groups on X, and $\mathfrak{U} = (U_j)_{j \in J}$ an open covering of X. Then the Čech complex $\check{\mathcal{C}}^\bullet(\mathfrak{U}; \mathcal{F})$, as defined above, is a resolution for $\mathcal{F}$, i.e. the sequence*

$$0 \to \mathcal{F} \rightarrowtail \check{\mathcal{C}}^0(\mathfrak{U}; \mathcal{F}) \xrightarrow{d^0} \check{\mathcal{C}}^1(\mathfrak{U}; \mathcal{F}) \xrightarrow{d^1} \cdots$$

is exact. In particular, we have a natural isomorphism

$$\mathrm{H}^\bullet(X; \mathcal{F}) \cong \mathbb{H}^\bullet(X; \check{\mathcal{C}}^\bullet(\mathfrak{U}; \mathcal{F})).$$

3.6.2 A Spectral Sequence for Hypercohomology of an Open Covering

We generalize the well-known spectral sequence of an open covering $\mathfrak{U}$ of a space X (where $\mathcal{H}^q(\mathcal{F})$ is the presheaf $V \mapsto \mathrm{H}^q(V; \mathcal{F}_{|V})$)

$$\check{\mathrm{H}}^p(\mathfrak{U}; \mathcal{H}^q(\mathcal{F})) \Rightarrow \mathrm{H}^n(X; \mathcal{F})$$

to the case of a complex $\mathcal{F}^\bullet$ of abelian sheaves and its hypercohomology.

Let X be a topological space, $\mathfrak{U} = (U_j)_{j \in J}$ an open covering of X and $\mathcal{F}^\bullet$ a complex of sheaves of abelian groups on X, which is bounded below. We define a presheaf

$$\boldsymbol{\mathcal{H}}^p(\mathcal{F}^\bullet) : V \mapsto \mathbb{H}^p(V; \mathcal{F}^\bullet_{|V}) \quad \text{for} \quad V \subseteq X \quad \text{open}$$

as a "sheafified" version of hypercohomology.

Proposition 3.6.2. *In the situation above, the following spectral sequence converges*

$$E_2^{p,q} := \check{\mathrm{H}}^p(\mathfrak{U}; \boldsymbol{\mathcal{H}}^q(\mathcal{F}^\bullet)) \Rightarrow E_\infty^n := \mathbb{H}^n(X; \mathcal{F}^\bullet).$$

Proof. We choose a quasi-isomorphism $\mathcal{F}^\bullet \xrightarrow{\sim} \mathcal{I}^\bullet$ with $\mathcal{I}^\bullet$ being a complex of flabby sheaves. Note that the double complex $\check{\mathcal{C}}^\bullet \mathcal{I}^\bullet := \check{\mathcal{C}}^\bullet(\mathfrak{U}; \mathcal{I}^\bullet)$ consists of flabby sheaves, as well.

Now we consider the two spectral sequences of a double complex (cf. [GM, Ch. 1, 3.5, p. 20]) for the double complex $\check{\mathrm{C}}^\bullet \mathcal{I}^\bullet := \check{\mathrm{C}}^\bullet(\mathfrak{U}; \mathcal{I}^\bullet) = \Gamma(X; \check{\mathcal{C}}^\bullet \mathcal{I}^\bullet)$

$$\mathrm{h}_{\mathrm{I}}^p \mathrm{h}_{\mathrm{II}}^q \check{\mathrm{C}}^\bullet \mathcal{I}^\bullet \Rightarrow \mathrm{h}^n \operatorname{tot} \check{\mathrm{C}}^\bullet \mathcal{I}^\bullet \quad \text{and} \tag{10}$$

$$\mathrm{h}_{\mathrm{II}}^p \mathrm{h}_{\mathrm{I}}^q \check{\mathrm{C}}^\bullet \mathcal{I}^\bullet \Rightarrow \mathrm{h}^n \operatorname{tot} \check{\mathrm{C}}^\bullet \mathcal{I}^\bullet. \tag{11}$$

The first spectral sequence (10). Since $\check{\mathcal{C}}^\bullet \mathcal{I}^p$ is a flabby resolution of $\mathcal{I}^p$ (by Lemma 3.6.1), we have

$$\mathrm{h}_{\mathrm{II}}^q \check{\mathrm{C}}^\bullet \mathcal{I}^p = \mathrm{h}^q \Gamma(X; \check{\mathcal{C}}^\bullet \mathcal{I}^p) = \mathrm{H}^q(X; \mathcal{I}^p) = \begin{cases} \Gamma(X; \mathcal{I}^p) & \text{if } q = 0, \\ 0 & \text{else.} \end{cases}$$

Therefore

$$\mathrm{h}_{\mathrm{I}}^p \mathrm{h}_{\mathrm{II}}^q \check{\mathrm{C}}^\bullet \mathcal{I}^\bullet = \begin{cases} \mathrm{h}^p \Gamma(X; \mathcal{I}^\bullet) & \text{if } q = 0, \\ 0 & \text{else} \end{cases}$$

$$= \begin{cases} \mathbb{H}^p(X; \mathcal{F}^\bullet) & \text{if } q = 0, \\ 0 & \text{else.} \end{cases}$$

Thus the first spectral sequence degenerates and we obtain a natural isomorphism

$$\mathbb{H}^\bullet(X; \mathcal{F}^\bullet) \cong \mathrm{h}^\bullet \operatorname{tot} \check{\mathrm{C}}^\bullet \mathcal{I}^\bullet.$$

The second spectral sequence (11). From

$$\check{\mathrm{C}}^p \mathcal{I}^q = \Gamma\Big(X; \bigoplus_{|I| = p+1} i_* \mathcal{I}^q_{|U_I}\Big)$$

$$= \bigoplus_{|I| = p+1} \Gamma\big(X; i_* \mathcal{I}^q_{|U_I}\big)$$

$$= \bigoplus_{|I| = p+1} \Gamma\big(U_I; \mathcal{I}^q_{|U_I}\big),$$

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we see

$$
\begin{aligned}
\mathrm{h}_{\mathrm{I}}^{q}\check{\mathrm{C}}^{p}\mathcal{I}^{\bullet} &= \bigoplus_{|I|=p+1} \mathrm{h}^{q}\Gamma\big(U_{I}; \mathcal{I}_{|U_{I}}^{\bullet}\big) \\
&= \bigoplus_{|I|=p+1} \mathbb{H}^{q}\big(U_{I}; \mathcal{F}_{|U_{I}}^{\bullet}\big) \\
&= \bigoplus_{|I|=p+1} \Gamma\big(U_{I}; \boldsymbol{\mathcal{H}}^{q}(\mathcal{F}^{\bullet})_{|U_{I}}\big) \\
&= \bigoplus_{|I|=p+1} \Gamma\big(X; i_{*}\boldsymbol{\mathcal{H}}^{q}(\mathcal{F}^{\bullet})_{|U_{I}}\big) \\
&= \Gamma\big(X; \bigoplus_{|I|=p+1} i_{*}\boldsymbol{\mathcal{H}}^{q}(\mathcal{F}^{\bullet})_{|U_{I}}\big) \\
&= \Gamma\big(X; \check{\mathcal{C}}^{p}\big(\mathfrak{U}; \boldsymbol{\mathcal{H}}^{q}(\mathcal{F}^{\bullet})\big)\big) \\
&= \check{\mathrm{C}}^{p}\big(\mathfrak{U}; \boldsymbol{\mathcal{H}}^{q}(\mathcal{F}^{\bullet})\big).
\end{aligned}
$$

This gives

$$
\mathrm{h}_{\mathrm{II}}^{p}\,\mathrm{h}_{\mathrm{I}}^{q}\,\check{\mathrm{C}}^{\bullet}\mathcal{I}^{\bullet} = \mathrm{h}^{p}\,\check{\mathrm{C}}^{\bullet}\big(\mathfrak{U}; \boldsymbol{\mathcal{H}}^{q}(\mathcal{F}^{\bullet})\big) = \check{\mathrm{H}}^{p}\big(\mathfrak{U}; \boldsymbol{\mathcal{H}}^{q}(\mathcal{F}^{\bullet})\big)
$$

and concludes the proof. $\qquad\square$

3.6.3 A Spectral Sequence for Algebraic deRham Cohomology of a Smooth Variety

We are especially interested in Proposition 3.6.2 for $\mathcal{F}^{\bullet} = \Omega_{X/k}^{\bullet}$.

Corollary 3.6.3. *Let X be a smooth variety over a field k and $\mathfrak{U}$ an open covering of X. Then we have a convergent spectral sequence for algebraic deRham cohomology of smooth varieties (as defined in Definition 3.2.3)*

$$
E_{2}^{p,q} := \mathrm{h}^{p}\bigoplus_{|I|=\bullet+1} \mathrm{H}_{\mathrm{dR}}^{q}(U_{I}/k) \Rightarrow E_{\infty}^{n} := \mathrm{H}_{\mathrm{dR}}^{n}(X/k).
$$

Proof. Let us compute the initial terms of the spectral sequence

$$
\check{\mathrm{H}}(\mathfrak{U}; \boldsymbol{\mathcal{H}}^{q}(\mathcal{F}^{\bullet})) \Rightarrow \mathbb{H}^{n}(X; \mathcal{F}^{\bullet})
$$

of Proposition 3.6.2 for $\mathcal{F}^{\bullet} = \Omega_{X/k}^{\bullet}$

$$
\begin{aligned}
\check{\mathrm{H}}^{p}(\mathfrak{U}; \boldsymbol{\mathcal{H}}^{q}(\Omega_{X/k}^{\bullet})) &= \mathrm{h}^{p}\Gamma\big(X; \check{\mathcal{C}}^{\bullet}\big(\mathfrak{U}; \boldsymbol{\mathcal{H}}^{q}(\Omega_{X/k}^{\bullet})\big)\big) \\
&= \mathrm{h}^{p}\Gamma\Big(X; \bigoplus_{|I|=\bullet+1} i_{*}\boldsymbol{\mathcal{H}}^{q}(\Omega_{X/k}^{\bullet})_{|U_{I}}\Big) \\
&= \mathrm{h}^{p}\bigoplus_{|I|=\bullet+1} \Gamma\big(X; i_{*}\boldsymbol{\mathcal{H}}^{q}(\Omega_{X/k}^{\bullet})_{|U_{I}}\big) \\
&= \mathrm{h}^{p}\bigoplus_{|I|=\bullet+1} \Gamma\big(U_{I}; \boldsymbol{\mathcal{H}}^{q}(\Omega_{X/k}^{\bullet})_{|U_{I}}\big) \\
&= \mathrm{h}^{p}\bigoplus_{|I|=\bullet+1} \mathbb{H}^{q}\big(U_{I}; \Omega_{X/k|U_{I}}^{\bullet}\big)
\end{aligned}
$$

$$= \mathrm{h}^p \bigoplus_{|I|=\bullet+1} \mathbb{H}^q(U_I; \Omega^\bullet_{U_I/k})$$

$$= \mathrm{h}^p \bigoplus_{|I|=\bullet+1} \mathrm{H}^q_{\mathrm{dR}}(U_I/k).$$

For the limit term we get immediately

$$\mathbb{H}^\bullet(X; \Omega^\bullet_{X/k}) = \mathrm{H}^\bullet_{\mathrm{dR}}(X/k)$$

and the corollary is proved. $\qquad\square$

3.6.4 A Spectral Sequence for Algebraic deRham Cohomology of a Divisor with Normal Crossings

For a divisor D with normal crossings, we have a spectral sequence expressing $\mathrm{H}^\bullet_{\mathrm{dR}}(D/k)$ in terms of $\mathrm{H}^\bullet_{\mathrm{dR}}(D_{i_0\cdots i_m}/k)$.
Recall that (cf. page 14)

$$\widetilde{\Omega}^\bullet_{D/k} = \mathrm{tot}\, \Omega^{\bullet;\bullet}_{D\bullet/k} = \mathrm{tot} \bigoplus_{|I|} i_* \Omega^\bullet_{D_I/k}.$$

Since we are interested in hypercohomology, we replace the $\Omega^\bullet_*$ by their Godement resolutions (cf. page 8), where we shall use the following abbreviations

- $\mathcal{G}^p_{D_I/k} := \mathcal{G}^p_{\Omega^\bullet_{D_I/k}}$,

- $\mathcal{G}^{p,q}_{D\bullet/k} := \mathcal{G}^p_{\Omega^{\bullet,q}_{D\bullet/k}}$
  $$= \bigoplus_{|I|=q+1} \mathcal{G}^p_{i_*\Omega^\bullet_{D_I/k}}$$
  $$\overset{(*)}{=} \bigoplus_{|I|=q+1} i_* \mathcal{G}^p_{D_I/k},$$

  (we have equality at $(*)$ since i_* is exact for closed immersions i)

and

- $\mathcal{G}^\bullet_{D/k} := \mathcal{G}^\bullet_{\widetilde{\Omega}^\bullet_{D/k}}$
  $$= \mathcal{G}^\bullet_{\mathrm{tot}\, \Omega^{\bullet;\bullet}_{D\bullet/k}}$$
  $$= \mathrm{tot}\, \mathcal{G}^{\bullet;\bullet}_{D\bullet/k}.$$

Furthermore, we write

$$\mathrm{G}^p_{D_I/k} := \Gamma\big(D_I; \mathcal{G}^p_{D_I/k}\big),\ \mathrm{G}^{p,q}_{D\bullet/k} := \Gamma\big(D; \mathcal{G}^{p,q}_{D\bullet/k}\big)\ \text{and}\ \mathrm{G}^p_{D/k} := \Gamma\big(D; \mathcal{G}^p_{D/k}\big)$$

for the groups of global sections of these sheaves.
We consider the second spectral sequence for the double complex $\mathrm{G}^{p,q}_{D\bullet/k}$

$$\mathrm{h}^p_{\mathrm{II}}\, \mathrm{h}^q_{\mathrm{I}}\, \mathrm{G}^{\bullet;\bullet}_{D\bullet/k} \Rightarrow \mathrm{h}^n\, \mathrm{tot}\, \mathrm{G}^{\bullet;\bullet}_{D\bullet/k}.$$

For the limit terms, we obtain

$$\mathrm{h}^n\, \mathrm{tot}\, \mathrm{G}^{\bullet;\bullet}_{D\bullet/k} = \mathrm{h}^n \mathrm{G}^\bullet_{D/k} = \mathrm{h}^n \Gamma(D; \mathcal{G}^\bullet_{D/k}) = \mathbb{H}^n(D; \widetilde{\Omega}^\bullet_{D/k}) = \mathrm{H}^n_{\mathrm{dR}}(D/k).$$

Let us compute the initial terms

$$\mathrm{h}^q \mathrm{G}_{D^\bullet/k}^{\bullet,p} = \mathrm{h}^q \Gamma(D; \bigoplus_{|I|=p+1} i_* \mathcal{G}_{D_I/k}^\bullet)$$

$$= \mathrm{h}^q \bigoplus_{|I|=p+1} \Gamma(D; i_* \mathcal{G}_{D_I/k}^\bullet)$$

$$= \bigoplus_{|I|=p+1} \mathrm{h}^q \Gamma(D; i_* \mathcal{G}_{D_I/k}^\bullet)$$

$$= \bigoplus_{|I|=p+1} \mathrm{h}^q \Gamma(D_I; \mathcal{G}_{D_I/k}^\bullet)$$

$$= \bigoplus_{|I|=p+1} \mathbb{H}^q(D_I; \Omega_{D_I/k}^\bullet)$$

$$= \bigoplus_{|I|=p+1} \mathrm{H}_{\mathrm{dR}}^q(D_I/k).$$

Thus we have proved the following proposition.

Proposition 3.6.4. *Let X be a smooth variety over a field k and D a divisor with normal crossings on X. Then we have a convergent spectral sequence for algebraic de Rham cohomology of a divisor (as defined in Definition 3.2.4)*

$$E_2^{p,q} := \mathrm{h}^p \bigoplus_{|I|=\bullet+1} \mathrm{H}_{\mathrm{dR}}^q(D_I/k) \Rightarrow E_\infty^n := \mathrm{H}_{\mathrm{dR}}^n(D/k).$$

4 Comparison Isomorphisms

If we consider a smooth variety X defined over $\mathbb{C}$ and a divisor D with normal crossings on X, then their algebraic deRham cohomology as defined in Subsection 3.2 turns out to be naturally isomorphic to the singular cohomology of the associated complex analytic spaces X^{an} and D^{an}.

Our proof proceeds in three steps: (1) First we mimic the definition of algebraic deRham cohomology in the complex analytic setting using holomorphic differential forms instead of algebraic ones, thus defining analytic deRham cohomology groups

$$\mathrm{H}^\bullet_{\mathrm{dR}}(X^{\mathrm{an}}; \mathbb{C}), \quad \mathrm{H}^\bullet_{\mathrm{dR}}(D^{\mathrm{an}}; \mathbb{C}) \quad \text{and} \quad \mathrm{H}^\bullet_{\mathrm{dR}}(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{C}).$$

These are then shown to be isomorphic to their algebraic counterparts using an application of Serre's "GAGA-type" results by Grothendieck.

(2) In the second step, analytic deRham cohomology is proved to coincide with complex cohomology, (3) which in turn is isomorphic to singular cohomology, as we show in the third step. (In the smooth case this has already been shown in Proposition 3.1.2.)

4.1 Situation

Throughout this section, X will be a smooth variety over $\mathbb{C}$ and D a divisor with normal crossings on X. As usual, we denote the complex analytic spaces associated to X and D by X^{an} and D^{an}, respectively (cf. Definition 2.1.3).

4.2 Analytic deRham Cohomology

If we replace in Subsection 3.2 every occurrence of the complex of algebraic differential forms $\Omega^\bullet_{Y/\mathbb{C}}$ on a complex variety Y by the complex of holomorphic differential forms $\Omega^\bullet_{Y^{\mathrm{an}}}$ on the associated complex analytic space Y^{an}, we obtain complexes

$$\Omega^\bullet_{X^{\mathrm{an}}}, \quad \Omega^{\bullet;\bullet}_{D^{\mathrm{an}\bullet}}, \quad \widetilde{\Omega}^\bullet_{D^{\mathrm{an}}} \quad \text{and} \quad \widetilde{\Omega}^\bullet_{X^{\mathrm{an}}, D^{\mathrm{an}}}$$

and thus are able to define:

Definition 4.2.1 (Analytic deRham cohomology). *For X^{an} and D^{an} as in Subsection 4.1, we define* analytic deRham cohomology *by*

$$\mathrm{H}^\bullet_{\mathrm{dR}}(X^{\mathrm{an}}; \mathbb{C}) := \mathbb{H}^\bullet(X^{\mathrm{an}}; \Omega^\bullet_{X^{\mathrm{an}}}), \qquad (*)$$

$$\mathrm{H}^\bullet_{\mathrm{dR}}(D^{\mathrm{an}}; \mathbb{C}) := \mathbb{H}^\bullet(D^{\mathrm{an}}; \widetilde{\Omega}^\bullet_{D^{\mathrm{an}}}) \quad and$$

$$\mathrm{H}^\bullet_{\mathrm{dR}}(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{C}) := \mathbb{H}^\bullet(X^{\mathrm{an}}; \widetilde{\Omega}^\bullet_{X^{\mathrm{an}}, D^{\mathrm{an}}}).$$

Here $(*)$ is the definition of analytic deRham cohomology for complex manifolds already considered on page 11, which we listed again for completeness.

Note that $(\Omega^\bullet_{Y/\mathbb{C}})_{\mathrm{an}} = \Omega^\bullet_{Y^{\mathrm{an}}}$, hence

$$\begin{aligned}
(\Omega^\bullet_{X/\mathbb{C}})_{\mathrm{an}} &= \Omega^\bullet_{X^{\mathrm{an}}}, \\
(\Omega^{\bullet;\bullet}_{D^\bullet/\mathbb{C}})_{\mathrm{an}} &= \Omega^{\bullet;\bullet}_{D^{\mathrm{an}\bullet}}, \\
(\widetilde{\Omega}^\bullet_{D/\mathbb{C}})_{\mathrm{an}} &= \widetilde{\Omega}^\bullet_{D^{\mathrm{an}}} \quad \text{and} \\
(\widetilde{\Omega}^\bullet_{X,D/\mathbb{C}})_{\mathrm{an}} &= \widetilde{\Omega}^\bullet_{X^{\mathrm{an}}, D^{\mathrm{an}}}.
\end{aligned} \qquad (12)$$

Remark 4.2.2. We could easily generalize Definition 4.2.1 to arbitrary complex manifolds (and normal-crossings-divisors on them) not necessarily associated to something algebraic, but we will not need this.

4.3 The Long Exact Sequence in Analytic deRham Cohomology

The results of Subsection 3.4 carry over one-to-one to the complex analytic case.

Proposition 4.3.1. *We have naturally a long exact sequence in analytic deRham cohomology (as defined in Definition 4.2.1)*

$$\cdots \to \mathrm{H}_{\mathrm{dR}}^{p-1}(D^{\mathrm{an}};\mathbb{C}) \to$$
$$\mathrm{H}_{\mathrm{dR}}^{p}(X^{\mathrm{an}}, D^{\mathrm{an}};\mathbb{C}) \to \mathrm{H}_{\mathrm{dR}}^{p}(X^{\mathrm{an}};\mathbb{C}) \to \mathrm{H}_{\mathrm{dR}}^{p}(D^{\mathrm{an}};\mathbb{C}) \to$$
$$\mathrm{H}_{\mathrm{dR}}^{p+1}(X^{\mathrm{an}}, D^{\mathrm{an}};\mathbb{C}) \to \cdots,$$

where X^{an} and D^{an} are as in Subsection 4.1.

Moreover, by Equation (8) on page 9 and Equation (12), we have a natural morphism from the long exact sequence in algebraic deRham cohomology to the one in analytic deRham cohomology

$$
\begin{array}{ccccccc}
\cdots \to & \mathrm{H}_{\mathrm{dR}}^{p}(X, D/\mathbb{C}) & \longrightarrow & \mathrm{H}_{\mathrm{dR}}^{p}(X/\mathbb{C}) & \longrightarrow & \mathrm{H}_{\mathrm{dR}}^{p}(D/\mathbb{C}) & \to \cdots \\
& \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & \mathrm{H}_{\mathrm{dR}}^{p}(X^{\mathrm{an}}, D^{\mathrm{an}};\mathbb{C}) & \longrightarrow & \mathrm{H}_{\mathrm{dR}}^{p}(X^{\mathrm{an}};\mathbb{C}) & \longrightarrow & \mathrm{H}_{\mathrm{dR}}^{p}(D^{\mathrm{an}};\mathbb{C}) & \to \cdots.
\end{array}
\tag{13}
$$

We want to show that all the vertical maps are isomorphisms.

4.4 Some Spectral Sequences for Analytic deRham Cohomology

The proof of Proposition 3.6.3 applies in the analytic case as well and we obtain

Proposition 4.4.1. *We have a convergent spectral sequence for analytic deRham cohomology (as defined in Definition 4.2.1)*

$$E_2^{p,q} := \mathrm{h}^p \bigoplus_{|I|=\bullet+1} \mathrm{H}_{\mathrm{dR}}^q(U_I^{\mathrm{an}};\mathbb{C}) \Rightarrow E_\infty^n := \mathrm{H}_{\mathrm{dR}}^n(X^{\mathrm{an}};\mathbb{C}),$$

where X^{an} is associated to a smooth variety X over $\mathbb{C}$ and $\mathfrak{U} = (U_j)_{j\in J}$ is an open covering of X.

Furthermore, the natural map of hypercohomology groups (cf. (8) and (12))

$$\mathbb{H}^\bullet(U; \Omega_{U/\mathbb{C}}^\bullet) \to \mathbb{H}^\bullet(U^{\mathrm{an}}; \Omega_{U^{\mathrm{an}}}^\bullet) \quad \text{for} \quad U \subseteq X \text{ open,}$$

provides us with a natural map of spectral sequences

$$
\begin{array}{ccc}
\check{\mathrm{H}}^p(\mathfrak{U}; \boldsymbol{\mathcal{H}}^q(\Omega_{X/\mathbb{C}}^\bullet)) & \Rightarrow & \mathbb{H}^n(X; \Omega_{X/\mathbb{C}}^\bullet) \\
\downarrow & & \downarrow \\
\check{\mathrm{H}}^p(\mathfrak{U}^{\mathrm{an}}; \boldsymbol{\mathcal{H}}^q(\Omega_{X^{\mathrm{an}}}^\bullet)) & \Rightarrow & \mathbb{H}^n(X^{\mathrm{an}}; \Omega_{X^{\mathrm{an}}}^\bullet),
\end{array}
$$

which we can rewrite using Corollary 3.6.3 and Proposition 4.4.1

$$\mathrm{h}^p \bigoplus_{|I|=\bullet+1} \mathrm{H}_{\mathrm{dR}}^q(U_I/\mathbb{C}) \Rightarrow \mathrm{H}_{\mathrm{dR}}^n(X/\mathbb{C})$$
$$\downarrow \qquad\qquad\qquad \downarrow \qquad\qquad (14)$$
$$\mathrm{h}^p \bigoplus_{|I|=\bullet+1} \mathrm{H}_{\mathrm{dR}}^q(U_I^{\mathrm{an}};\mathbb{C}) \Rightarrow \mathrm{H}_{\mathrm{dR}}^n(X^{\mathrm{an}};\mathbb{C}).$$

Likewise, we can transfer Proposition 3.6.4 to the analytic case.

Proposition 4.4.2. *We have a convergent spectral sequence*

$$E_2^{p,q} := \mathrm{h}^p \bigoplus_{|I|=\bullet+1} \mathrm{H}_{\mathrm{dR}}^q(D_I^{\mathrm{an}};\mathbb{C}) \Rightarrow E_\infty^n := \mathrm{H}_{\mathrm{dR}}^n(D^{\mathrm{an}};\mathbb{C}),$$

where D^{an} is associated to a normal-crossings-divisor D on a smooth variety X defined over $\mathbb{C}$ as in Subsection 4.1.

Again, we have a natural map of spectral sequences. We write (cf. page 22)

$$\mathcal{G}_{D^{\mathrm{an}}\bullet}^{p,q} := \left(\mathcal{G}_{D^\bullet/\mathbb{C}}^{p,q}\right)_{\mathrm{an}}$$

and

$$\mathrm{G}_{D^{\mathrm{an}}\bullet}^{p,q} := \Gamma(D^{\mathrm{an}};\mathcal{G}_{D^{\mathrm{an}}\bullet}^{p,q}).$$

Now the natural map of zeroth homology (cf. Equation (3) for $i=0$)

$$\mathrm{G}_{D^\bullet/\mathbb{C}}^{\bullet,\bullet} \to \mathrm{G}_{D^{\mathrm{an}}\bullet}^{\bullet,\bullet}$$

gives us a natural map of spectral sequences

$$\mathrm{h}_{\mathrm{I}}^p \mathrm{h}_{\mathrm{II}}^q \mathrm{G}_{D^\bullet/\mathbb{C}}^{\bullet,\bullet} \Rightarrow \mathrm{h}^n \operatorname{tot} \mathrm{G}_{D^\bullet/\mathbb{C}}^{\bullet,\bullet}$$
$$\downarrow \qquad\qquad \downarrow$$
$$\mathrm{h}_{\mathrm{I}}^p \mathrm{h}_{\mathrm{II}}^q \mathrm{G}_{D^{\mathrm{an}}\bullet}^{\bullet,\bullet} \Rightarrow \mathrm{h}^n \operatorname{tot} \mathrm{G}_{D^{\mathrm{an}}\bullet}^{\bullet,\bullet}.$$

With propositions 3.6.4 and 4.4.2, this map takes the form

$$\mathrm{h}^p \bigoplus_{|I|=\bullet+1} \mathrm{H}_{\mathrm{dR}}^q(D_I/\mathbb{C}) \Rightarrow \mathrm{H}_{\mathrm{dR}}^n(D/\mathbb{C})$$
$$\downarrow \qquad\qquad\qquad \downarrow \qquad\qquad (15)$$
$$\mathrm{h}^p \bigoplus_{|I|=\bullet+1} \mathrm{H}_{\mathrm{dR}}^q(D_I^{\mathrm{an}};\mathbb{C}) \Rightarrow \mathrm{H}_{\mathrm{dR}}^n(D^{\mathrm{an}};\mathbb{C}).$$

4.5 Comparison of Algebraic and Analytic deRham Cohomology

The proof of $\mathrm{H}_{\mathrm{dR}}^\bullet(X/\mathbb{C}) \cong \mathrm{H}_{\mathrm{dR}}^\bullet(X^{\mathrm{an}};\mathbb{C})$, i.e. $\mathbb{H}^\bullet(X;\Omega_{X/\mathbb{C}}^\bullet) \cong \mathbb{H}^\bullet(X^{\mathrm{an}};\Omega_{X^{\mathrm{an}}}^\bullet)$ is easy in the **projective case**, because of (12) and Corollary 2.2.4.
The **affine case** is contained in a theorem of Grothendieck.

Theorem 4.5.1 (Grothendieck, [Gro, Thm. 1, p. 95]). *Let X be a smooth affine variety over $\mathbb{C}$. Then the natural map between algebraic and analytic deRham cohomology (as defined in definitions 3.2.3 and 4.2.1)*

$$\mathrm{H}_{\mathrm{dR}}^\bullet(X/\mathbb{C}) \xrightarrow{\sim} \mathrm{H}_{\mathrm{dR}}^\bullet(X^{\mathrm{an}};\mathbb{C})$$

is an isomorphism.

We give only some comments on the proof. Let $j : X \hookrightarrow \overline{X}$ be the projective closure of X and let $Z := \overline{X} \setminus X$. Furthermore, we set

$$\Omega^\bullet_{\overline{X}/\mathbb{C}}(*Z) := \varinjlim_n \Omega^\bullet_{\overline{X}/\mathbb{C}}(nZ) \quad \text{and}$$

$$\Omega^\bullet_{\overline{X}^{\mathrm{an}}}(*Z^{\mathrm{an}}) := \varinjlim_n \Omega^\bullet_{\overline{X}^{\mathrm{an}}}(nZ^{\mathrm{an}}),$$

where $\mathcal{F}(nZ')$ denotes the sheaf $\mathcal{F}$ twisted by the n-fold of the divisor Z'. In his paper [Gro], Grothendieck considers the following chain of isomorphisms:

$$\mathrm{H}^\bullet_{\mathrm{dR}}(X/\mathbb{C}) = \mathbb{H}^\bullet(X; \Omega^\bullet_{X/\mathbb{C}})$$

$$\uparrow \wr \quad \Omega^p_{X/\mathbb{C}} \text{ are } \Gamma(X;?)\text{-acyclic [Ha, Thm. III.3.5, p. 215]}$$

$$\mathrm{h}^\bullet\Gamma(X; \Omega^\bullet_{X/\mathbb{C}})$$

$$\|$$

$$\mathrm{h}^\bullet\Gamma(\overline{X}; j_*\Omega^\bullet_{X/\mathbb{C}})$$

$$\|$$

$$\mathrm{h}^\bullet\Gamma(\overline{X}; \Omega^\bullet_{\overline{X}/\mathbb{C}}(*Z))$$

$$\downarrow \wr \quad \text{[Ha, Prop. III.2.9, p. 209]}$$

$$\varinjlim_n \mathrm{h}^\bullet\Gamma(\overline{X}; \Omega^\bullet_{\overline{X}/\mathbb{C}}(nZ))$$

$$\downarrow \wr \quad \text{Thm. 2.2.3 (GAGA)}$$

$$\varinjlim_n \mathrm{h}^\bullet\Gamma(\overline{X}^{\mathrm{an}}; \Omega^\bullet_{\overline{X}^{\mathrm{an}}}(nZ^{\mathrm{an}}))$$

$$\downarrow \wr \quad \text{[AH, Lemma 6]}$$

$$\mathrm{h}^\bullet\Gamma(\overline{X}^{\mathrm{an}}; \Omega^\bullet_{\overline{X}^{\mathrm{an}}}(*Z^{\mathrm{an}}))$$

$$\downarrow \wr \quad \textbf{main step}$$

$$\mathrm{h}^\bullet\Gamma(\overline{X}^{\mathrm{an}}; j_*\Omega^\bullet_{X^{\mathrm{an}}})$$

$$\|$$

$$\mathrm{h}^\bullet\Gamma(X^{\mathrm{an}}; \Omega^\bullet_{X^{\mathrm{an}}})$$

$$\downarrow \wr \quad \begin{array}{l} \Omega^p_{X^{\mathrm{an}}} \text{ are } \Gamma(X^{\mathrm{an}};?)\text{-acyclic since } X^{\mathrm{an}} \text{ is Stein} \\ \text{[GR, IV.1.1, Def. 1, p. 103; V.1, Bem., p. 130]} \end{array}$$

$$\mathrm{H}^\bullet_{\mathrm{dR}}(X^{\mathrm{an}}; \mathbb{C}) = \mathbb{H}^\bullet(X^{\mathrm{an}}; \Omega^\bullet_{X^{\mathrm{an}}}).$$

In the case that Z has normal crossings, the main step is due to Atiyah and Hodge [AH, Lemma 17] and is proved by explicit calculations. Grothendieck reduces the general case to this one by using the resolution of singularities according to Hironaka [Hi1].

The smooth case. Choose an open affine covering $\mathfrak{U} = (U_j)_{j \in J}$ of X. Since X is separated, all the U_I are affine. Hence by Theorem 4.5.1 we get an isomorphism on the initial terms in (14), and therefore also on the limit terms

$$\mathrm{H}^\bullet_{\mathrm{dR}}(X/\mathbb{C}) \cong \mathrm{H}^\bullet_{\mathrm{dR}}(X^{\mathrm{an}}; \mathbb{C}).$$

The case of a normal-crossings-divisor. Since D is a divisor with normal crossings, all the D_I are smooth. Thus we see from the discussion of the smooth case, that we have an isomorphism on the initial terms in (15), hence also on the limit terms

$$\mathrm{H}^\bullet_{\mathrm{dR}}(D/\mathbb{C}) \cong \mathrm{H}^\bullet_{\mathrm{dR}}(D^{\mathrm{an}}; \mathbb{C}).$$

The relative case. By the 5-lemma, this case follows immediately from the two cases considered above using diagram (13).

We summarize our results so far.

Proposition 4.5.2. *For X a smooth variety over $\mathbb{C}$ and D a normal-crossings-divisor on X, we have a natural isomorphism between algebraic deRham cohomology of X and D (see definitions 3.2.3, 3.2.4, 3.2.6) and analytic deRham cohomology (see Definition 4.2.1) of the associated complex analytic spaces X^{an} and D^{an}*

$$
\begin{array}{ccccccc}
\cdots \to & \mathrm{H}^p_{\mathrm{dR}}(X, D/\mathbb{C}) & \longrightarrow & \mathrm{H}^p_{\mathrm{dR}}(X/\mathbb{C}) & \longrightarrow & \mathrm{H}^p_{\mathrm{dR}}(D/\mathbb{C}) & \to \cdots \\
& \downarrow \wr & & \downarrow \wr & & \downarrow \wr & \\
\cdots \to & \mathrm{H}^p_{\mathrm{dR}}(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{C}) & \longrightarrow & \mathrm{H}^p_{\mathrm{dR}}(X^{\mathrm{an}}; \mathbb{C}) & \longrightarrow & \mathrm{H}^p_{\mathrm{dR}}(D^{\mathrm{an}}; \mathbb{C}) & \to \cdots .
\end{array}
$$

4.6 Complex Cohomology

Definition 4.6.1 (Complex cohomology). *Let X^{an} and D^{an} be as in Subsection 4.1. In the absolute case,* complex cohomology *is simply sheaf cohomology of the constant sheaf with fibre $\mathbb{C}$*

$$\mathrm{H}^\bullet(X^{\mathrm{an}}; \mathbb{C}) := \mathrm{H}^\bullet(X^{\mathrm{an}}; \mathbb{C}_{X^{\mathrm{an}}}) \quad and \quad \mathrm{H}^\bullet(D^{\mathrm{an}}; \mathbb{C}) := \mathrm{H}^\bullet(D^{\mathrm{an}}; \mathbb{C}_{D^{\mathrm{an}}}).$$

We define a relative version by

$$\mathrm{H}^\bullet(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{C}) := \mathrm{H}^\bullet(X^{\mathrm{an}}; j_! \mathbb{C}_{U^{\mathrm{an}}}),$$

where $U := X \setminus D$ and $j : U \hookrightarrow X$.

Remark 4.6.2. Taking $\mathbb{Q}$ instead of $\mathbb{C}$ in the above definition would give us *rational cohomology groups* $\mathrm{H}^\bullet(X^{\mathrm{an}}; \mathbb{Q})$, $\mathrm{H}^\bullet(D^{\mathrm{an}}; \mathbb{Q})$ and $\mathrm{H}^\bullet(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{Q})$.

The short exact sequence

$$0 \to j_! \mathbb{C}_{U^{\mathrm{an}}} \xrightarrow{\alpha} \mathbb{C}_{X^{\mathrm{an}}} \to i_* \mathbb{C}_{D^{\mathrm{an}}} \to 0 \tag{16}$$

with $j : U^{\mathrm{an}} \hookrightarrow X^{\mathrm{an}}$ and $i : D^{\mathrm{an}} \hookrightarrow X^{\mathrm{an}}$, yields a long exact sequence in complex cohomology

$$
\cdots \to \mathrm{H}^{p-1}(D^{\mathrm{an}}; \mathbb{C}) \to
$$
$$
\mathrm{H}^p(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{C}) \to \mathrm{H}^p(X^{\mathrm{an}}; \mathbb{C}) \to \mathrm{H}^p(D^{\mathrm{an}}; \mathbb{C}) \to
$$
$$
\mathrm{H}^{p+1}(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{C}) \to \cdots .
$$

The mapping cone $M_{\mathbb{C}}$ (cf. the appendix) of the natural restriction map $\mathbb{C}_{X^{\mathrm{an}}} \to i_* \mathbb{C}_{D^{\mathrm{an}}}$ gives us another short exact sequence

$$0 \to i_* \mathbb{C}_{D^{\mathrm{an}}}[-1] \to M_{\mathbb{C}} \to \mathbb{C}_{X^{\mathrm{an}}}[0] \to 0. \tag{17}$$

Consider the map

$$\gamma := (0, \alpha) : j_! \mathbb{C}_{U^{\mathrm{an}}}[0] \to M_{\mathbb{C}} = i_* \mathbb{C}_{D^{\mathrm{an}}}[-1] \oplus \mathbb{C}_{X^{\mathrm{an}}}[0],$$

where $\alpha : j_! \mathbb{C}_{U^{\mathrm{an}}}[0] \to \mathbb{C}_{X^{\mathrm{an}}}[0]$ was defined in (16) and $0 : j_! \mathbb{C}_{U^{\mathrm{an}}} \to i_* \mathbb{C}_{D^{\mathrm{an}}}[-1]$ denotes the zero map. In order to show that α is a morphism of complexes, we only have to check the compatibility of the differentials, but this follows from the fact that (16) is a complex. The exactness of the sequence in (16) translates into γ being a quasi-isomorphism. This gives us the following diagram

$$
\begin{array}{ccccccc}
\mathrm{H}^{p-1}(D^{\mathrm{an}}; \mathbb{C}) & \to \mathrm{H}^p(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{C}) \to & \mathrm{H}^p(X^{\mathrm{an}}; \mathbb{C}) & \to & \mathrm{H}^p(D^{\mathrm{an}}; \mathbb{C}) \\
\| & \| & \| & \| \\
\mathbb{H}^p(X^{\mathrm{an}}; i_* \mathbb{C}_{D^{\mathrm{an}}}[-1]) \to & \mathbb{H}^p(X^{\mathrm{an}}; M_{\mathbb{C}}) & \to \mathbb{H}^p(X^{\mathrm{an}}; \mathbb{C}_{X^{\mathrm{an}}}[0]) \to \mathbb{H}^{p+1}(X^{\mathrm{an}}; i_* \mathbb{C}_{D^{\mathrm{an}}}[-1]).
\end{array}
$$

Unfortunately, this diagram does not commute (there is a sign mismatch), so we consider the natural transformation

$$\varepsilon^p = (-1)^p \, \mathrm{id} \quad \text{for} \quad p \in \mathbb{Z}$$

and write down a new diagram

$$
\begin{array}{ccccccc}
\mathrm{H}^{p-1}(D^{\mathrm{an}}; \mathbb{C}) & \to \mathrm{H}^p(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{C}) \to & \mathrm{H}^p(X^{\mathrm{an}}; \mathbb{C}) & \to & \mathrm{H}^p(D^{\mathrm{an}}; \mathbb{C}) \\
\varepsilon^{p-1} \downarrow \wr & \varepsilon^p \downarrow \wr & \varepsilon^p \downarrow \wr & \varepsilon^p \downarrow \wr \\
\mathbb{H}^p(X^{\mathrm{an}}; i_* \mathbb{C}_{D^{\mathrm{an}}}[-1]) \to & \mathbb{H}^p(X^{\mathrm{an}}; M_{\mathbb{C}}) & \to \mathbb{H}^p(X^{\mathrm{an}}; \mathbb{C}_{X^{\mathrm{an}}}[0]) \to \mathbb{H}^{p+1}(X^{\mathrm{an}}; i_* \mathbb{C}_{D^{\mathrm{an}}}[-1]).
\end{array}
$$

This diagram commutes as we see by replacing the sheaves involved in (16) by injective resolutions $\mathcal{A}^\bullet$, $\mathcal{B}^\bullet$ and $\mathcal{C}^\bullet$ and applying Lemma 4.6.3 below to the complexes $\mathrm{A}^\bullet$, $\mathrm{B}^\bullet$ and $\mathrm{C}^\bullet$ of their global sections.

Lemma 4.6.3. *Let*

$$0 \longrightarrow \mathrm{A}^\bullet \overset{\alpha}{\longrightarrow} \mathrm{B}^\bullet \overset{\beta}{\longrightarrow} \mathrm{C}^\bullet \longrightarrow 0$$

be a short exact sequence of complexes of abelian groups and

$$0 \longrightarrow \mathrm{C}^\bullet[-1] \overset{\iota}{\longrightarrow} \mathrm{M}^\bullet \overset{\pi}{\longrightarrow} \mathrm{B}^\bullet \longrightarrow 0$$

the mapping cone of $\beta : \mathrm{B}^\bullet \to \mathrm{C}^\bullet$. Then we have a quasi-isomorphism

$$\gamma := (0, \alpha) : \mathrm{A}^\bullet \overset{\sim}{\longrightarrow} \mathrm{M}^\bullet = \mathrm{C}^\bullet[-1] \oplus \mathrm{B}^\bullet$$

and a commutative diagram

$$
\begin{array}{ccccccc}
\mathrm{h}^{p-1}\mathrm{C}^\bullet & \overset{\delta}{\longrightarrow} & \mathrm{h}^p\mathrm{A}^\bullet & \overset{\alpha_*}{\longrightarrow} & \mathrm{h}^p\mathrm{B}^\bullet & \overset{\beta_*}{\longrightarrow} & \mathrm{h}^p\mathrm{C}^\bullet \\
\varepsilon^{p-1} \downarrow \wr & \boxed{1} \; \varepsilon^p \circ \gamma_* \downarrow \wr & \circlearrowleft & \varepsilon^p \downarrow \wr & \boxed{2} & \varepsilon^p \downarrow \wr \\
\mathrm{h}^{p-1}\mathrm{C}^\bullet & \overset{\iota_*}{\longrightarrow} & \mathrm{h}^p\mathrm{M}^\bullet & \overset{\pi_*}{\longrightarrow} & \mathrm{h}^p\mathrm{B}^\bullet & \overset{\delta'}{\longrightarrow} & \mathrm{h}^p\mathrm{C}^\bullet.
\end{array}
$$

Proof. The assertion about the quasi-isomorphism $\gamma : \mathrm{A}^\bullet \to \mathrm{M}^\bullet$ follows from the exactness of $0 \to \mathrm{A}^\bullet \to \mathrm{B}^\bullet \to \mathrm{C}^\bullet \to 0$. The middle square commutes because of $\pi \circ \gamma = \alpha$. So we are left to show the commutativity of the squares $\boxed{1}$ and $\boxed{2}$.

Ad 2 **.)** From the diagram

$$
\begin{array}{ccc}
\mathrm{C}^p & \overset{\iota}{\longrightarrow} & \mathrm{M}^{p+1} \\
& & \uparrow d_{\mathrm{M}} \\
& \mathrm{M}^p & \overset{\pi}{\longrightarrow} & \mathrm{B}^p
\end{array}
$$

and the definition of the connecting morphism δ' we see for $b \in \mathrm{B}^p$

$$
\delta'[b] = [\beta(b)] = \beta_*[b],
$$

hence

$$
(\delta' \circ \varepsilon^p)[b] = (\varepsilon^p \circ \beta_*)[b].
$$

Ad 1 **.)** Pick an element $c \in \mathrm{C}^{p-1}$ and choose a preimage $b \in \mathrm{B}^{p-1}$. With

$$
\begin{array}{ccc}
\mathrm{A}^p & \overset{\alpha}{\longrightarrow} & \mathrm{B}^p \\
& & \uparrow d_{\mathrm{B}} \\
& \mathrm{B}^{p-1} & \overset{\beta}{\longrightarrow} & \mathrm{C}^{p-1}
\end{array}
$$

we get

$$
\delta[c] = [\alpha^{-1} \circ d_{\mathrm{B}} b] \in \mathrm{h}^p \mathrm{A}^\bullet.
$$

Applying γ_* gives

$$
(\gamma_* \circ \delta)[c] = (0, \alpha)_*[\alpha^{-1} \circ d_{\mathrm{B}} b] = [0, d_{\mathrm{B}} b] \in \mathrm{h}^p \mathrm{M}^\bullet = \mathrm{h}^p(\mathrm{C}^\bullet[-1] \oplus \mathrm{B}^\bullet)
$$

or after subtracting the coboundary $d_{\mathrm{M}}(0, b) = (\beta(b), d_{\mathrm{B}} b) = (c, d_{\mathrm{B}} b)$

$$
(\gamma_* \circ \delta)[c] = [-c, 0] = -\iota_*[c],
$$

hence

$$
(\varepsilon^p \circ \gamma_* \circ \delta)[c] = (\iota_* \circ \varepsilon^{p-1})[c].
$$

$\square$

4.7 Comparison of Analytic deRham Cohomology and Complex Cohomology

We apply the Čech functor (cf. Subsection 3.6.1) for the closed covering $\mathfrak{D} := (D_i^{\mathrm{an}})_{i=1}^r$ to the constant sheaf $\mathbb{C}_{D^{\mathrm{an}}}$, and thus get $\check{\mathcal{C}}^q(\mathfrak{D}; \mathbb{C}) := \check{\mathcal{C}}^q(\mathfrak{D}; \mathbb{C}_{D^{\mathrm{an}}})$. Now the natural inclusions $\mathbb{C}_{D_I^{\mathrm{an}}} \hookrightarrow \Omega_{D_I^{\mathrm{an}}}^0$ sum up to a natural morphism of sheaves on D^{an}

$$
\begin{array}{c}
\check{\mathcal{C}}^q(\mathfrak{D}; \mathbb{C}) = \displaystyle\bigoplus_{|I|=q+1} i_* \mathbb{C}_{D_I^{\mathrm{an}}} \\
\downarrow \\
\Omega_{D^{\mathrm{an}}\bullet}^{0,q} = \displaystyle\bigoplus_{|I|=q+1} i_* \Omega_{D_I^{\mathrm{an}}}^0,
\end{array} \tag{18}
$$

where $i : D_I^{\mathrm{an}} \hookrightarrow D^{\mathrm{an}}$. Since the complexes $\Omega_{D_I^{\mathrm{an}}}^\bullet$ are resolutions of the sheaves $\mathbb{C}_{D_I^{\mathrm{an}}}$ and i_* and $\oplus$ are exact functors, this shows that $\Omega_{D^{\mathrm{an}}\bullet}^{\bullet;q}$ is a resolution of $\check{\mathcal{C}}^q(\mathfrak{D}; \mathbb{C})$.

If we consider $\check{\mathcal{C}}^\bullet(\mathfrak{D};\mathbb{C})[0]$ as a double complex (cf. the appendix) concentrated in the zeroth row, we can rewrite (18) as

$$\check{\mathcal{C}}^\bullet(\mathfrak{D};\mathbb{C})[0] \to \Omega^{\bullet,\bullet}_{D^{\mathrm{an}}\bullet}.$$

or, after taking the total complex, as

$$\check{\mathcal{C}}^\bullet(\mathfrak{D};\mathbb{C}) \to \widetilde{\Omega}^\bullet_{D^{\mathrm{an}}}.$$

This morphism is a quasi-isomorphism by Lemma 2.2.5 (and Remark 2.2.6) and fits into the following commutative diagram

$$
\begin{array}{ccc}
\mathbb{C}_{X^{\mathrm{an}}}[0] & \to & i_*\mathbb{C}_{D^{\mathrm{an}}}[0] \\
\| & & \downarrow \alpha \\
\mathbb{C}_{X^{\mathrm{an}}}[0] & \to & i_*\check{\mathcal{C}}^\bullet(\mathfrak{D};\mathbb{C}) \\
\downarrow \wr & & \downarrow \wr \\
\Omega^\bullet_{X^{\mathrm{an}}} & \to & i_*\widetilde{\Omega}^\bullet_{D^{\mathrm{an}}} .
\end{array}
$$

The map α is a quasi-isomorphism as a consequence of the following proposition.

Proposition 4.7.1. *We have an exact sequence*

$$0 \to \mathbb{C}_{D^{\mathrm{an}}} \to \check{\mathcal{C}}^0(\mathfrak{D};\mathbb{C}) \to \check{\mathcal{C}}^1(\mathfrak{D};\mathbb{C}) \to \cdots .$$

Proof (cf. [Ha, Lemma III.4.2, p. 220]). This can be checked on the stalks and is therefore a purely combinatorial problem. Let $x \in D^{\mathrm{an}}$ and assume w.l.o.g. that the irreducible components $D^{\mathrm{an}}_1,\ldots,D^{\mathrm{an}}_s$ but no other component of D^{an} are passing through x. For the stalks at x, we write

$$\check{\mathrm{C}}^p := \check{\mathcal{C}}^p(\mathfrak{D};\mathbb{C})_x = \bigoplus_{1 \le i_0 < \cdots < i_p \le s} \mathbb{C}_{D^{\mathrm{an}}_{i_0 \cdots i_p},x}.$$

Observe that exactness at the zeroth and first step is obvious. Using the "coordinate functions" for a set of indices $j_0,\ldots,j_p$

$$\check{\mathrm{C}}^p = \bigoplus_{1 \le i_0 < \cdots < i_p \le s} \mathbb{C}_{D^{\mathrm{an}}_{i_0 \cdots i_p},x} \twoheadrightarrow \mathbb{C}_{D^{\mathrm{an}}_{j_0 \cdots j_p},x}$$

$$\oplus\, \alpha_{i_0 \cdots i_p} \mapsto \alpha_{j_0 \cdots j_p}$$

we define a homotopy operator

$$k^p : \check{\mathrm{C}}^p \to \check{\mathrm{C}}^{p-1}$$

such that for any $\alpha \in \check{\mathrm{C}}^p$ the "coordinates" of its image $k^p(\alpha)$ satisfy

$$k^p(\alpha)_{j_0 \cdots j_{p-1}} = \begin{cases} 0 & \text{if } j_0 = 1 \\ \alpha_{1\,j_0 \cdots j_{p-1}} & \text{else.} \end{cases}$$

Now an elementary calculation shows that for $p \ge 1$

$$d^{p-1} \circ k^p + k^{p+1} \circ d^p = \mathrm{id}_{\check{\mathrm{C}}^p}.$$

Hence k is a homotopy between the identity map and the zero map on $\check{\mathrm{C}}^\bullet$, and we conclude that

$$\mathrm{h}^p\,\check{\mathrm{C}}^\bullet = 0 \quad \text{for} \quad p \ge 1.$$

$\square$

Thus we have a commutative diagram

$$
\begin{array}{ccc}
\mathbb{C}_{X^{\mathrm{an}}}[0] & \longrightarrow & i_*\mathbb{C}_{D^{\mathrm{an}}}[0] \\
\downarrow \wr & & \downarrow \wr \\
\Omega^\bullet_{X^{\mathrm{an}}} & \longrightarrow & i_*\widetilde{\Omega}^\bullet_{D^{\mathrm{an}}},
\end{array}
$$

where the vertical maps are quasi-isomorphisms. As a consequence, we also have a quasi-isomorphism between the respective mapping cones

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & i_*\mathbb{C}_{D^{\mathrm{an}}}[-1] & \longrightarrow & M_{\mathbb{C}} & \longrightarrow & \mathbb{C}_{X^{\mathrm{an}}}[0] & \longrightarrow & 0 \\
& & \downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \\
0 & \longrightarrow & i_*\widetilde{\Omega}^\bullet_{D^{\mathrm{an}}}[-1] & \longrightarrow & \widetilde{\Omega}^\bullet_{X^{\mathrm{an}},D^{\mathrm{an}}} & \longrightarrow & \Omega^\bullet_{X^{\mathrm{an}}} & \longrightarrow & 0 \, .
\end{array}
\tag{19}
$$

Taking hypercohomology proves the following proposition.

Proposition 4.7.2. *We have a natural isomorphism between the long exact sequence in analytic deRham cohomology (cf. Definition 4.2.1) and the sequence in complex cohomology (cf. Definition 4.6.1)*

$$
\begin{array}{ccccccc}
\cdots \to \mathrm{H}^{p-1}_{\mathrm{dR}}(X^{\mathrm{an}};\mathbb{C}) & \longrightarrow & \mathrm{H}^{p-1}_{\mathrm{dR}}(D^{\mathrm{an}};\mathbb{C}) & \longrightarrow & \mathrm{H}^{p}_{\mathrm{dR}}(X^{\mathrm{an}},D^{\mathrm{an}};\mathbb{C}) & \to \cdots \\
\uparrow \wr & & \uparrow \wr & & \uparrow \wr & \\
\cdots \to \mathrm{H}^{p-1}(X^{\mathrm{an}};\mathbb{C}) & \longrightarrow & \mathrm{H}^{p-1}(D^{\mathrm{an}};\mathbb{C}) & \longrightarrow & \mathrm{H}^{p}(X^{\mathrm{an}},D^{\mathrm{an}};\mathbb{C}) & \to \cdots \, ,
\end{array}
$$

where X^{an} and D^{an} are as usual (see Subsection 4.1).

4.8 Singular Cohomology

We extend the definition of singular cohomology given on page 11

$$
\begin{aligned}
\mathrm{H}^\bullet_{\mathrm{sing}}(X^{\mathrm{an}};\mathbb{C}) &:= \mathrm{h}^\bullet\Gamma\big(X^{\mathrm{an}};\mathcal{C}^\bullet_{\mathrm{sing}}(X^{\mathrm{an}};\mathbb{C})\big) \quad \text{and} \\
\mathrm{H}^\bullet_{\mathrm{sing}}(D^{\mathrm{an}};\mathbb{C}) &:= \mathrm{h}^\bullet\Gamma\big(D^{\mathrm{an}};\mathcal{C}^\bullet_{\mathrm{sing}}(D^{\mathrm{an}};\mathbb{C})\big)
\end{aligned}
$$

by a relative version.

Definition 4.8.1 (Relative singular cohomology). *As usual, let X^{an} and D^{an} be as in Subsection 4.1. We set*

$$
\mathcal{C}^\bullet_{\mathrm{sing}}(X^{\mathrm{an}},D^{\mathrm{an}};\mathbb{C}) := \ker(\mathcal{C}^\bullet_{\mathrm{sing}}(X^{\mathrm{an}};\mathbb{C}) \to i_*\mathcal{C}^\bullet_{\mathrm{sing}}(D^{\mathrm{an}};\mathbb{C}))
$$

and define

$$
\mathrm{H}^\bullet_{\mathrm{sing}}(X^{\mathrm{an}},D^{\mathrm{an}};\mathbb{C}) := \mathrm{h}^\bullet\Gamma(X^{\mathrm{an}};\mathcal{C}^\bullet_{\mathrm{sing}}(X^{\mathrm{an}},D^{\mathrm{an}};\mathbb{C})).
$$

Remark 4.8.2. Similarly, one defines singular cohomology $\mathrm{H}^\bullet_{\mathrm{sing}}(X^{\mathrm{an}},D^{\mathrm{an}};k)$ with coefficients in a field k different from $\mathbb{C}$. We will need $\mathrm{H}^\bullet_{\mathrm{sing}}(X^{\mathrm{an}},D^{\mathrm{an}};\mathbb{Q})$ occasionly. Since $? \otimes_{\mathbb{Q}} \mathbb{C}$ is an exact functor, we have

$$
\mathrm{H}^\bullet_{\mathrm{sing}}(X^{\mathrm{an}},D^{\mathrm{an}};\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} = \mathrm{H}^\bullet_{\mathrm{sing}}(X^{\mathrm{an}},D^{\mathrm{an}};\mathbb{C}).
$$

The sheaf $\mathcal{C}_{\text{sing}}^{\bullet}(X^{\text{an}}, D^{\text{an}}; \mathbb{C})$ is $\Gamma(X^{\text{an}}; ?)$-acyclic because of the surjectivity of

$$\mathrm{C}_{\text{sing}}^{\bullet}(X^{\text{an}}; \mathbb{C}) \twoheadrightarrow \mathrm{C}_{\text{sing}}^{\bullet}(D^{\text{an}}; \mathbb{C}) = \Gamma(X^{\text{an}}; i_* \mathcal{C}_{\text{sing}}^{\bullet}(D^{\text{an}}; \mathbb{C})).$$

Thus the short exact sequence

$$0 \to \mathcal{C}_{\text{sing}}^{\bullet}(X^{\text{an}}, D^{\text{an}}; \mathbb{C}) \to \mathcal{C}_{\text{sing}}^{\bullet}(X^{\text{an}}; \mathbb{C}) \to i_* \mathcal{C}_{\text{sing}}^{\bullet}(D^{\text{an}}; \mathbb{C}) \to 0$$

gives rise to a long exact sequence in singular cohomology

$$\cdots \to \mathrm{H}_{\text{sing}}^{p-1}(D^{\text{an}}; \mathbb{C}) \to$$
$$\mathrm{H}_{\text{sing}}^{p}(X^{\text{an}}, D^{\text{an}}; \mathbb{C}) \to \mathrm{H}_{\text{sing}}^{p}(X^{\text{an}}; \mathbb{C}) \to \mathrm{H}_{\text{sing}}^{p}(D^{\text{an}}; \mathbb{C}) \to$$
$$\mathrm{H}_{\text{sing}}^{p+1}(X^{\text{an}}, D^{\text{an}}; \mathbb{C}) \to \cdots .$$

4.9 Comparison of Complex Cohomology and Singular Cohomology

We have a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & j_! \mathbb{C}_{U^{\text{an}}}[0] & \longrightarrow & \mathbb{C}_{X^{\text{an}}}[0] & \longrightarrow & i_* \mathbb{C}_{D^{\text{an}}}[0] & \longrightarrow & 0 \\
& & \downarrow{\alpha} & & \downarrow{\beta} & & \downarrow{\gamma} & & \\
0 & \longrightarrow & \mathcal{C}_{\text{sing}}^{\bullet}(X^{\text{an}}, D^{\text{an}}; \mathbb{C}) & \longrightarrow & \mathcal{C}_{\text{sing}}^{\bullet}(X^{\text{an}}; \mathbb{C}) & \longrightarrow & i_* \mathcal{C}_{\text{sing}}^{\bullet}(D^{\text{an}}; \mathbb{C}) & \longrightarrow & 0,
\end{array}
$$
$$(20)$$

where β and γ are quasi-isomorphisms, since the singular cochain complex is a resolution of the constant sheaf (see Proposition 3.1.1) and i_* is exact for closed immersions. Therefore α is also a quasi-isomorphism.

Taking hypercohomology gives a natural isomorphism between the long exact sequence in complex cohomology and the sequence in singular cohomology. Thus we have finally proved the following theorem.

4.10 Comparison Theorem

Theorem 4.10.1 (Comparison theorem). *Let X be a smooth variety defined over $\mathbb{C}$ and D a divisor on X with normal crossings (cf. Definition 3.2.1). The associated complex analytic spaces (cf. Subsection 2.1) are denoted by X^{an} and D^{an}, respectively. Then we have natural isomorphisms between the long exact sequences in*

- *algebraic deRham cohomology (cf. definitions 3.2.3, 3.2.4, 3.2.6),*

- *analytic deRham cohomology (cf. Definition 4.2.1),*

- *complex cohomology (cf. Definition 4.6.1), and*

- *singular cohomology (cf. Definition 4.8.1)*

as shown in the commutative diagram below

$$\cdots \longrightarrow \; \mathrm{H}^p_{\mathrm{dR}}(X, D/\mathbb{C}) \; \longrightarrow \; \mathrm{H}^p_{\mathrm{dR}}(X/\mathbb{C}) \; \longrightarrow \; \mathrm{H}^p_{\mathrm{dR}}(D/\mathbb{C}) \; \longrightarrow \cdots$$
$$\text{algebraic deRham cohomology}$$

$$\Big\downarrow{\wr} \qquad\qquad \Big\downarrow{\wr} \qquad\qquad \Big\downarrow{\wr}$$

$$\cdots \longrightarrow \mathrm{H}^p_{\mathrm{dR}}(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{C}) \; \longrightarrow \; \mathrm{H}^p_{\mathrm{dR}}(X^{\mathrm{an}}; \mathbb{C}) \; \longrightarrow \; \mathrm{H}^p_{\mathrm{dR}}(D^{\mathrm{an}}; \mathbb{C}) \; \longrightarrow \cdots$$
$$\text{analytic deRham cohomology}$$

$$\Big\uparrow{\wr} \qquad\qquad \Big\uparrow{\wr} \qquad\qquad \Big\uparrow{\wr}$$

$$\cdots \longrightarrow \; \mathrm{H}^p(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{C}) \; \longrightarrow \; \mathrm{H}^p(X^{\mathrm{an}}; \mathbb{C}) \; \longrightarrow \; \mathrm{H}^p(D^{\mathrm{an}}; \mathbb{C}) \; \longrightarrow \cdots$$
$$\text{complex cohomology}$$

$$\Big\downarrow{\wr} \qquad\qquad \Big\downarrow{\wr} \qquad\qquad \Big\downarrow{\wr}$$

$$\cdots \longrightarrow \mathrm{H}^p_{\mathrm{sing}}(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{C}) \; \longrightarrow \; \mathrm{H}^p_{\mathrm{sing}}(X^{\mathrm{an}}; \mathbb{C}) \; \longrightarrow \; \mathrm{H}^p_{\mathrm{sing}}(D^{\mathrm{an}}; \mathbb{C}) \longrightarrow \cdots .$$
$$\text{singular cohomology}$$

Proof. Combine propositions 4.5.2 and 4.7.2 with the result from Subsection 4.9. $\square$

4.11 An Alternative Description of the Comparison Isomorphism

For computational purposes, it will be useful to describe the isomorphism from the Comparison Theorem 4.10.1

$$\mathrm{H}^\bullet_{\mathrm{dR}}(X^{\mathrm{an}}; \mathbb{C}) \cong \mathrm{H}^\bullet_{\mathrm{sing}}(X^{\mathrm{an}}; \mathbb{C})$$

more explicitely. We will formulate our result in Proposition 4.11.5 and explain its significance in Subsection 4.11.5. We start by defining yet another cohomology theory.

4.11.1 Smooth Singular Cohomology

Let M be a complex manifold. We write $\triangle^{\mathrm{std}}_p$ for the standard p-simplex spanned by the basis $\{(0, \ldots, 0, \underset{i}{1}, 0, \ldots, 0) \,|\, i = 1, \ldots, p+1\}$ of $\mathbb{R}^{p+1}$. A singular p-simplex $\sigma : \triangle^{\mathrm{std}}_p \to M$ is said to be *smooth*, if σ extends to a smooth map defined on a neighbourhood of the standard p-simplex $\triangle^{\mathrm{std}}_p$ in its p-plane. We denote by $\mathrm{C}^\infty_p(M; \mathbb{C})$ the $\mathbb{C}$-vector space generated by all smooth p-simplices of M, thus getting a subcomplex $\mathrm{C}^\infty_\bullet(M; \mathbb{C})$ of the complex $\mathrm{C}^{\mathrm{sing}}_\bullet(M; \mathbb{C})$ of all singular simplices on M

$$\mathrm{C}^\infty_\bullet(M; \mathbb{C}) \subset \mathrm{C}^{\mathrm{sing}}_\bullet(M; \mathbb{C}).$$

We can dualize this subcomplex

$$\mathrm{C}^\bullet_\infty(M; \mathbb{C}) := \mathrm{C}^\infty_\bullet(M; \mathbb{C})^\vee,$$

and define a complex of sheaves

$$\mathcal{C}^\bullet_\infty(M; \mathbb{C}) : V \mapsto \mathrm{C}^\bullet_\infty(V; \mathbb{C}) \quad \text{for} \quad V \subseteq M \quad \text{open.}$$

The sheaves $\mathcal{C}^p_\infty(M; \mathbb{Q})$ are flabby and define smooth singular cohomology of M.

Definition 4.11.1 (Smooth singular cohomology of a manifold). *Let M be a complex manifold. We define* smooth singular cohomology *groups*

$$\mathrm{H}^\bullet_\infty(M; \mathbb{C}) := \mathrm{h}^\bullet \, \mathrm{C}^\bullet_\infty(M; \mathbb{C}) = \mathbb{H}^\bullet\big(M; \mathcal{C}^\bullet_\infty(M; \mathbb{C})\big).$$

Since smooth singular cohomology coincides with singular cohomology for smooth manifolds (cf. [Br, p. 291]), we can show that $\mathcal{C}_\infty^\bullet(M;\mathbb{C})$ is a resolution of the constant sheaf $\mathbb{C}_M$ exactly as we did for $\mathcal{C}_{\mathrm{sing}}^\bullet(M;\mathbb{C})$ in Lemma 3.1.1. Thus we have the following lemma.

Lemma 4.11.2. *For M a complex manifold, the following sequence is exact*

$$0 \longrightarrow \mathbb{C}_M \longrightarrow \mathcal{C}_\infty^0(M;\mathbb{C}) \longrightarrow \mathcal{C}_\infty^1(M;\mathbb{C}) \longrightarrow \dots.$$

We are now going to extend the definition of smooth singular cohomology to the case of a divisor and to the relative case. Let X^{an} and D^{an} be as in Subsection 4.1.
First, recall how $\widetilde{\Omega}_{D^{\mathrm{an}}}^\bullet$ was defined (cf. pages 14 and 24): We composed the simplicial scheme $D^\bullet$ with the functor $i_*\Omega_{?^{\mathrm{an}}}^\bullet$, then obtained a double complex $\Omega_{D^{\mathrm{an}}\bullet}^{\bullet;\bullet}$ by summing up certain maps and finally denoted the total complex of $\Omega_{D^{\mathrm{an}}\bullet}^{\bullet;\bullet}$ by $\widetilde{\Omega}_{D^{\mathrm{an}}}^\bullet$.
Similarly, we can use the functor $i_*\mathcal{C}_\infty^\bullet(?^{\mathrm{an}};\mathbb{C})$, compose it to the simplicial scheme $D^\bullet$ and denote the resulting double complex by $\mathcal{C}_\infty^{\bullet;\bullet}(D^{\mathrm{an}};\mathbb{C})$. For the corresponding total complex we write $\mathcal{C}_\infty^\bullet(D^{\mathrm{an}};\mathbb{C})$.
We define $\mathcal{C}_\infty^\bullet(X^{\mathrm{an}},D^{\mathrm{an}};\mathbb{C})$ similar to $\widetilde{\Omega}_{X^{\mathrm{an}},D^{\mathrm{an}}}^\bullet$ (cf. pages 14 and 24): The natural restriction maps

$$\mathcal{C}_\infty^\bullet(X^{\mathrm{an}};\mathbb{C}) \to i_*\mathcal{C}_\infty^\bullet(D_j^{\mathrm{an}};\mathbb{C})$$

sum up to a natural map of double complexes

$$\mathcal{C}_\infty^\bullet(X^{\mathrm{an}};\mathbb{C})[0] \to i_*\mathcal{C}_\infty^{\bullet;\bullet}(D^{\mathrm{an}};\mathbb{C}), \tag{21}$$

where we view $\mathcal{C}_\infty^\bullet(X^{\mathrm{an}};\mathbb{C})[0]$ as a double complex concentrated in the zeroth column. Taking the total complex in (21) yields a natural map

$$\mathcal{C}_\infty^\bullet(X^{\mathrm{an}};\mathbb{C}) \longrightarrow i_*\mathcal{C}_\infty^\bullet(D^{\mathrm{an}};\mathbb{C}),$$

for whose mapping cone (cf. the appendix) we write $\mathcal{C}_\infty^\bullet(X^{\mathrm{an}},D^{\mathrm{an}};\mathbb{C})$.
Thus we are able to formulate:

Definition 4.11.3 (Smooth singular cohomology for the case of a divisor and for the relative case). *We define* smooth singular cohomology *groups*

$$\mathrm{H}_\infty^\bullet(D^{\mathrm{an}};\mathbb{C}) := \mathrm{h}^\bullet\,\mathrm{C}_\infty^\bullet(D^{\mathrm{an}};\mathbb{C}) = \mathbb{H}^\bullet(D^{\mathrm{an}};\mathcal{C}_\infty^\bullet(D^{\mathrm{an}};\mathbb{C})) \quad and$$
$$\mathrm{H}_\infty^\bullet(X^{\mathrm{an}},D^{\mathrm{an}};\mathbb{C}) := \mathrm{h}^\bullet\,\mathrm{C}_\infty^\bullet(X^{\mathrm{an}},D^{\mathrm{an}};\mathbb{C}) = \mathbb{H}^\bullet(X^{\mathrm{an}};\mathcal{C}_\infty^\bullet(X^{\mathrm{an}},D^{\mathrm{an}};\mathbb{C})),$$

where X^{an} and D^{an} are as in Subsection 4.1.

Remark 4.11.4. Similarly one defines smooth singular cohomology with coefficients in $\mathbb{Q}$. All statements made below about smooth singular cohomology remain valid for rational coefficients except those involving holomorphic differentials.

As usual, we have a long exact sequence in smooth singular cohomology

$$\cdots \to \mathrm{H}_\infty^p(X^{\mathrm{an}},D^{\mathrm{an}};\mathbb{C}) \to \mathrm{H}_\infty^p(X^{\mathrm{an}};\mathbb{C}) \to \mathrm{H}_\infty^p(D^{\mathrm{an}};\mathbb{C}) \to \cdots.$$

4.11.2 Comparison with Analytic deRham Cohomology

Our motivation for considering smooth singular cohomology instead of ordinary singular cohomology comes from the following fact: We can integrate differential-p-forms over smooth p-simplices γ and thus have a natural morphism

$$\Omega^p_{X^{\mathrm{an}}} \to \mathcal{C}^p_\infty(X^{\mathrm{an}}; \mathbb{C})$$

$$\omega \mapsto \begin{pmatrix} \mathcal{C}^\infty_p(X^{\mathrm{an}}; \mathbb{C}) \to \mathbb{C} \\ \gamma \mapsto \int_\gamma \omega \end{pmatrix}.$$

As a consequence of Stoke's theorem

$$\int_{\partial\gamma} \omega = \int_\gamma d\omega,$$

we get a well-defined map of complexes

$$\Omega^\bullet_{X^{\mathrm{an}}} \to \mathcal{C}^\bullet_\infty(X^{\mathrm{an}}; \mathbb{C}).$$

This map is functorial, hence gives a natural transformation of functors

$$\Omega^\bullet_{?^{\mathrm{an}}} \longrightarrow \mathcal{C}^\bullet_\infty(?^{\mathrm{an}}; \mathbb{C}).$$

With the definitions of the complexes

$$\widetilde{\Omega}^\bullet_{D^{\mathrm{an}}}, \mathcal{C}^\bullet_\infty(D^{\mathrm{an}}; \mathbb{C}) \quad \text{and} \quad \widetilde{\Omega}^\bullet_{X^{\mathrm{an}}, D^{\mathrm{an}}}, \mathcal{C}^\bullet_\infty(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{C})$$

(cf. pages 24 and 35), we get a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & i_*\widetilde{\Omega}^\bullet_{D^{\mathrm{an}}}[-1] & \longrightarrow & \widetilde{\Omega}^\bullet_{X^{\mathrm{an}}, D^{\mathrm{an}}} & \longrightarrow & \Omega^\bullet_{X^{\mathrm{an}}} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & i_*\mathcal{C}^\bullet_\infty(D^{\mathrm{an}}; \mathbb{C})[-1] & \longrightarrow & \mathcal{C}^\bullet_\infty(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{C}) & \longrightarrow & \mathcal{C}^\bullet_\infty(X^{\mathrm{an}}; \mathbb{C}) & \longrightarrow & 0.
\end{array}
\tag{22}
$$

Recall the short exact sequence (17) of the mapping cone $M_\mathbb{C}$ from Subsection 4.6.1

$$0 \to i_*\mathbb{C}_{D^{\mathrm{an}}}[-1] \to M_\mathbb{C} \to \mathbb{C}_{X^{\mathrm{an}}}[0] \to 0$$

and Diagram (19) relating this short exact sequence with the first line of (22). Combining (19) and (22) gives a big diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & i_*\mathbb{C}_{D^{\mathrm{an}}}[-1] & \longrightarrow & M_\mathbb{C} & \longrightarrow & \mathbb{C}_{X^{\mathrm{an}}}[0] & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & i_*\widetilde{\Omega}^\bullet_{D^{\mathrm{an}}}[-1] & \longrightarrow & \widetilde{\Omega}^\bullet_{X^{\mathrm{an}}, D^{\mathrm{an}}} & \longrightarrow & \Omega^\bullet_{X^{\mathrm{an}}} & \longrightarrow & 0 \\
& & \downarrow \quad {}^\alpha & & \downarrow \quad {}^\beta & & \downarrow \quad {}^\gamma & & \\
0 & \longrightarrow & i_*\mathcal{C}^\bullet_\infty(D^{\mathrm{an}}; \mathbb{C})[-1] & \longrightarrow & \mathcal{C}^\bullet_\infty(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{C}) & \longrightarrow & \mathcal{C}^\bullet_\infty(X^{\mathrm{an}}; \mathbb{C}) & \longrightarrow & 0
\end{array}
\tag{23}
$$

with α, β, and γ being just the composition of the two vertical maps. We want to show that α, β, and γ are quasi-isomorphisms. We already know that γ is a quasi-isomorphism by Lemma 4.11.2. If α is also a quasi-isomorphism, then it will follow

from the 5-lemma that β is a quasi-isomorphism as well. We split α as a composition of a quasi-isomorphism (cf. Proposition 4.7.1) and a map $\widetilde{\alpha}$

$$\alpha : \mathbb{C}_{D^{\mathrm{an}}}[0] \xrightarrow{\sim} \check{\mathcal{C}}^\bullet(\mathfrak{D};\mathbb{C}) \xrightarrow{\widetilde{\alpha}} \mathcal{C}^\bullet_\infty(D^{\mathrm{an}};\mathbb{C}).$$

The map $\widetilde{\alpha}$ is a map of total complexes induced by the morphism of double complexes

$$\check{\mathcal{C}}^\bullet(\mathfrak{D};\mathbb{C})[0] \longrightarrow \mathcal{C}^{\bullet,\bullet}_\infty(D^{\mathrm{an}};\mathbb{C}),$$

where we consider $\check{\mathcal{C}}^\bullet(\mathfrak{D};\mathbb{C})[0]$ as a double complex concentrated in the zeroth row. By Lemma 4.11.2 the sequence

$$0 \longrightarrow \mathbb{C}_{D^{\mathrm{an}}_I} \longrightarrow \mathcal{C}^0_\infty(D^{\mathrm{an}}_I;\mathbb{C}) \longrightarrow \mathcal{C}^1_\infty(D^{\mathrm{an}}_I;\mathbb{C}) \longrightarrow \dots$$

is exact. Hence the sequence

$$0 \longrightarrow \check{\mathcal{C}}^q(\mathfrak{D};\mathbb{C}) \longrightarrow \mathcal{C}^{0,q}_\infty(D^{\mathrm{an}};\mathbb{C}) \longrightarrow \mathcal{C}^{1,q}_\infty(D^{\mathrm{an}};\mathbb{C}) \longrightarrow \dots$$

is also exact, since $\oplus$ and i_* are exact functors. Applying Lemma 2.2.5 shows that $\widetilde{\alpha}$ is a quasi-isomorphism. Thus all vertical maps in diagram (23) are indeed quasi-isomorphisms. Taking hypercohomology provides us with isomorphisms between long exact cohomology sequences generalizing the Comparison Theorem 4.10.1

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & \mathrm{H}^p(X^{\mathrm{an}};M_{\mathbb{C}}) & \longrightarrow & \mathrm{H}^p(X^{\mathrm{an}};\mathbb{C}) & \longrightarrow & \mathrm{H}^p(D^{\mathrm{an}};\mathbb{C}) & \longrightarrow & \cdots \\
& & \downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \\
\cdots & \longrightarrow & \mathrm{H}^p_{\mathrm{dR}}(X^{\mathrm{an}},D^{\mathrm{an}};\mathbb{C}) & \longrightarrow & \mathrm{H}^p_{\mathrm{dR}}(X^{\mathrm{an}};\mathbb{C}) & \longrightarrow & \mathrm{H}^p_{\mathrm{dR}}(D^{\mathrm{an}};\mathbb{C}) & \longrightarrow & \cdots \\
& & \downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \\
\cdots & \longrightarrow & \mathrm{H}^p_\infty(X^{\mathrm{an}},D^{\mathrm{an}};\mathbb{C}) & \longrightarrow & \mathrm{H}^p_\infty(X^{\mathrm{an}};\mathbb{C}) & \longrightarrow & \mathrm{H}^p_\infty(D^{\mathrm{an}};\mathbb{C}) & \longrightarrow & \cdots \, .
\end{array}
$$
$$\tag{24}$$

4.11.3 Comparison with Singular Cohomology

We will compare smooth singular cohomology with ordinary singular cohomology. First we need morphisms between the defining complexes. We already have a natural map

$$\mathcal{C}^\bullet_{\mathrm{sing}}(X^{\mathrm{an}};\mathbb{C}) \xrightarrow{\sim} \mathcal{C}^\bullet_\infty(X^{\mathrm{an}};\mathbb{C})$$

induced by the duals of the standard inclusions $\mathrm{C}^\infty_\bullet(V;\mathbb{C}) \subset \mathrm{C}^{\mathrm{sing}}_\bullet(V;\mathbb{C})$ for $V \subseteq X^{\mathrm{an}}$ open.

Next we discuss the case of a divisor: By restricting singular cochains from D^{an} to an irreducible component D^{an}_j of D^{an}, we get a map

$$\mathcal{C}^\bullet_{\mathrm{sing}}(D^{\mathrm{an}};\mathbb{C}) \to i_*\mathcal{C}^\bullet_{\mathrm{sing}}(D^{\mathrm{an}}_j;\mathbb{C}),$$

where $i : D^{\mathrm{an}}_j \hookrightarrow D^{\mathrm{an}}$ is the natural inclusion. By summing up the composition maps

$$\mathcal{C}^\bullet_{\mathrm{sing}}(D^{\mathrm{an}};\mathbb{C}) \to i_*\mathcal{C}^\bullet_{\mathrm{sing}}(D^{\mathrm{an}}_j;\mathbb{C}) \to i_*\mathcal{C}^\bullet_\infty(D^{\mathrm{an}}_j;\mathbb{C}),$$

we obtain a morphism of complexes

$$\mathcal{C}^\bullet_{\mathrm{sing}}(D^{\mathrm{an}};\mathbb{C}) \to \mathcal{C}^{\bullet,0}_\infty(D^{\mathrm{an}};\mathbb{C}),$$

which gives a morphism of double complexes

$$\mathcal{C}^\bullet_{\mathrm{sing}}(D^{\mathrm{an}}; \mathbb{C})[0] \to \mathcal{C}^{\bullet,\bullet}_\infty(D^{\mathrm{an}}; \mathbb{C}),$$

where we consider $\mathcal{C}^\bullet_{\mathrm{sing}}(D^{\mathrm{an}}; \mathbb{C})[0]$ as a double complex concentrated in the zeroth column.

Now we turn to the relative case of a pair $(X^{\mathrm{an}}; D^{\mathrm{an}})$: Let $M^\bullet_{\mathrm{sing}}$ denote the mapping cone of

$$\mathcal{C}^\bullet_{\mathrm{sing}}(X^{\mathrm{an}}; \mathbb{C}) \longrightarrow i_* \mathcal{C}^\bullet_{\mathrm{sing}}(D^{\mathrm{an}}; \mathbb{C}).$$

By the functoriality of the mapping cone construction, we get a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & i_* \mathcal{C}^\bullet_{\mathrm{sing}}(D^{\mathrm{an}}; \mathbb{C})[-1] & \longrightarrow & M^\bullet_{\mathrm{sing}} & \longrightarrow & \mathcal{C}^\bullet_{\mathrm{sing}}(X^{\mathrm{an}}; \mathbb{C}) & \longrightarrow & 0 \\
 & & {\scriptstyle \alpha'}\downarrow & & {\scriptstyle \beta'}\downarrow & & {\scriptstyle \gamma'}\downarrow & & \\
0 & \longrightarrow & i_* \mathcal{C}^\bullet_\infty(D^{\mathrm{an}}; \mathbb{C})[-1] & \longrightarrow & \mathcal{C}^\bullet_\infty(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{C}) & \longrightarrow & \mathcal{C}^\bullet_\infty(X^{\mathrm{an}}; \mathbb{C}) & \longrightarrow & 0.
\end{array}
$$

It is not hard to show directly that α', β', γ' are quasi-isomorphisms. However, we can use a shorter indirect argument: We have the following commutative diagram (cf. diagrams (20), (23) and the one above)

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & i_* \mathbb{C}_{D^{\mathrm{an}}}[-1] & \longrightarrow & M_\mathbb{C} & \longrightarrow & \mathbb{C}_{X^{\mathrm{an}}}[0] & \longrightarrow & 0 \\
 & & {\scriptstyle \alpha}\downarrow{\scriptstyle\wr} & & {\scriptstyle \beta}\downarrow{\scriptstyle\wr} & & {\scriptstyle \gamma}\downarrow{\scriptstyle\wr} & & \\
0 & \longrightarrow & i_* \mathcal{C}^\bullet_\infty(D^{\mathrm{an}}; \mathbb{C})[-1] & \longrightarrow & \mathcal{C}^\bullet_\infty(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{C}) & \longrightarrow & \mathcal{C}^\bullet_\infty(X^{\mathrm{an}}; \mathbb{C}) & \longrightarrow & 0 \\
 & & {\scriptstyle \alpha'}\uparrow & & {\scriptstyle \beta'}\uparrow & & {\scriptstyle \gamma'}\uparrow & & \\
0 & \longrightarrow & i_* \mathcal{C}^\bullet_{\mathrm{sing}}(D^{\mathrm{an}}; \mathbb{C})[-1] & \longrightarrow & M^\bullet_{\mathrm{sing}} & \longrightarrow & \mathcal{C}^\bullet_{\mathrm{sing}}(X^{\mathrm{an}}; \mathbb{C}) & \longrightarrow & 0
\end{array}
$$

with diagonal maps α'', β'', γ''.

In Subsection 4.11.2, we have already seen that α, β, γ are quasi-isomorphisms. Furthermore α'' and γ'' are quasi-isomorphisms (cf. Subsection 4.9), hence so is β'' (by the 5-lemma). Therefore α', β', γ' have to be quasi-isomorphisms as well. Taking hypercohomology in this diagram gives us

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & \mathrm{H}^p(X^{\mathrm{an}}; M_\mathbb{C}) & \longrightarrow & \mathrm{H}^p(X^{\mathrm{an}}; \mathbb{C}) & \longrightarrow & \mathrm{H}^p(D^{\mathrm{an}}; \mathbb{C}) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle\wr} & & \downarrow{\scriptstyle\wr} & & \downarrow{\scriptstyle\wr} & & \\
\cdots & \longrightarrow & \mathrm{H}^p_\infty(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{C}) & \longrightarrow & \mathrm{H}^p_\infty(X^{\mathrm{an}}; \mathbb{C}) & \longrightarrow & \mathrm{H}^p_\infty(D^{\mathrm{an}}; \mathbb{C}) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle\wr}\ {\scriptstyle\beta''_*} & & \downarrow{\scriptstyle\wr}\ {\scriptstyle\gamma''_*} & & \downarrow{\scriptstyle\wr}\ {\scriptstyle\alpha''_*} & & \\
\cdots & \longrightarrow & \mathrm{H}^p(X^{\mathrm{an}}; M^\bullet_{\mathrm{sing}}) & \longrightarrow & \mathrm{H}^p_{\mathrm{sing}}(X^{\mathrm{an}}; \mathbb{C}) & \longrightarrow & \mathrm{H}^p_{\mathrm{sing}}(D^{\mathrm{an}}; \mathbb{C}) & \longrightarrow & \cdots.
\end{array}
$$
(25)

In order to get rid of the mapping cones $M_\mathbb{C}$ and $M^\bullet_{\mathrm{sing}}$, we apply Lemma 4.6.3 two each of the two short exact sequences of complexes below (where $\mathcal{G}^\bullet_?$ denotes the Godement resolution — see page 8)

$$
\begin{array}{ccccccc}
0 \to & \Gamma(X^{\mathrm{an}}; \mathcal{G}^\bullet_{j_! \mathbb{C}_{U^{\mathrm{an}}}}) & \to & \Gamma(X^{\mathrm{an}}; \mathcal{G}^\bullet_{\mathbb{C}_{X^{\mathrm{an}}}}) & \to & \Gamma(X^{\mathrm{an}}; \mathcal{G}^\bullet_{i_* \mathbb{C}_{D^{\mathrm{an}}}}) & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to \Gamma(X^{\mathrm{an}}; \mathcal{G}^\bullet_{\mathcal{C}^\bullet_{\mathrm{sing}}(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{C})}) & \to & \Gamma(X^{\mathrm{an}}; \mathcal{G}^\bullet_{\mathcal{C}^\bullet_{\mathrm{sing}}(X^{\mathrm{an}}; \mathbb{C})}) & \to & \Gamma(X^{\mathrm{an}}; \mathcal{G}^\bullet_{i_* \mathcal{C}^\bullet_{\mathrm{sing}}(D^{\mathrm{an}}; \mathbb{C})}) & \to 0
\end{array}
$$

and obtain a commutative diagram (cf. definitions 4.6.1 and 4.8.1)

$$
\begin{array}{c}
\text{(commutative diagram with nodes } H^p(D^{\mathrm{an}};\mathbb{C}),\ H^p(D^{\mathrm{an}};\mathbb{C}),\ H^p_{\mathrm{sing}}(D^{\mathrm{an}};\mathbb{C}),\ H^p_{\mathrm{sing}}(D^{\mathrm{an}};\mathbb{C}),\\
H^p(X^{\mathrm{an}};\mathbb{C}),\ H^p(X^{\mathrm{an}};\mathbb{C}),\ H^p_{\mathrm{sing}}(X^{\mathrm{an}};\mathbb{C}),\ H^p_{\mathrm{sing}}(X^{\mathrm{an}};\mathbb{C}),\\
H^p(X^{\mathrm{an}};M_{\mathbb{C}}),\ H^p(X^{\mathrm{an}};M^{\bullet}_{\mathrm{sing}}),\\
H^p(X^{\mathrm{an}},D^{\mathrm{an}};\mathbb{C}),\ H^p_{\mathrm{sing}}(X^{\mathrm{an}},D^{\mathrm{an}};\mathbb{C})\\
\text{with maps } \varepsilon_p,\ \alpha''_*,\ \gamma''_*,\ \beta''_* \text{)}
\end{array}
\tag{26}
$$

Thus we have the following supplement to the Comparison Theorem 4.10.1.

4.11.4 Supplement to the Comparison Theorem 4.10.1

Proposition 4.11.5. *With the notation of Theorem 4.10.1 and definitions 4.11.1, 4.11.3, we have the following commutative diagram, all whose vertical maps are isomorphisms*

$$
\begin{array}{ccccccc}
\cdots \longrightarrow & H^p(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{C}) & \longrightarrow & H^p(X^{\mathrm{an}}; \mathbb{C}) & \longrightarrow & H^p(D^{\mathrm{an}}; \mathbb{C}) & \longrightarrow \cdots \\
& \downarrow\wr & & \downarrow\wr & & \downarrow\wr & \\
\cdots \longrightarrow & H^p_{\mathrm{dR}}(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{C}) & \longrightarrow & H^p_{\mathrm{dR}}(X^{\mathrm{an}}; \mathbb{C}) & \longrightarrow & H^p_{\mathrm{dR}}(D^{\mathrm{an}}; \mathbb{C}) & \longrightarrow \cdots \\
& \downarrow\wr & & \downarrow\wr & & \downarrow\wr & \\
\cdots \longrightarrow & H^p_{\infty}(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{C}) & \longrightarrow & H^p_{\infty}(X^{\mathrm{an}}; \mathbb{C}) & \longrightarrow & H^p_{\infty}(D^{\mathrm{an}}; \mathbb{C}) & \longrightarrow \cdots \\
& \uparrow\wr & & \uparrow\wr & & \uparrow\wr & \\
\cdots \longrightarrow & H^p_{\mathrm{sing}}(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{C}) & \longrightarrow & H^p_{\mathrm{sing}}(X^{\mathrm{an}}; \mathbb{C}) & \longrightarrow & H^p_{\mathrm{sing}}(D^{\mathrm{an}}; \mathbb{C}) & \longrightarrow \cdots .
\end{array}
$$

Proof. Combine diagrams (24), (25) and (26). □

4.11.5 Application of the Supplement to the Comparison Theorem

With Proposition 4.11.5, we can compute the isomorphism from the Comparison Theorem 4.10.1

$$
\begin{array}{ccccc}
\text{analytic} & \xleftarrow{\ \sim\ } & \text{complex} & \xrightarrow{\ \sim\ } & \text{singular} \\
\text{deRham cohomology} & & \text{cohomology} & & \text{cohomology}
\end{array}
$$

alternatively as

$$
\begin{array}{ccccc}
\text{analytic} & \xrightarrow[\alpha]{\ \sim\ } & \text{smooth singular} & \xleftarrow[\beta]{\ \sim\ } & \text{singular} \\
\text{deRham cohomology} & & \text{cohomology} & & \text{cohomology} .
\end{array}
$$

Why should this be useful? We show that we can easily find the image of a deRham cohomology class under the second isomorphism $\beta^{-1} \circ \alpha$ by computing a few integrals. First, we define *smooth singular homology* $H^{\infty}_{\bullet}$ as the dual of $H^{\bullet}_{\infty}$ (cf. definitions 4.11.1, 4.11.3), or equivalently:

Definition 4.11.6 (Smooth singular homology). *For X^{an}, D^{an} as usual (see Subsection 4.1), we define* smooth singular homology groups

$H^{\infty}_{\bullet}(X^{\mathrm{an}}; \mathbb{C}) := h^{\bullet} C^{\infty}_{\bullet}(X^{\mathrm{an}}; \mathbb{C})$

$H^{\infty}_{\bullet}(D^{\mathrm{an}}; \mathbb{C}) := h^{\bullet} \operatorname{tot} \oplus_{|I|=\bullet+1} i_* C^{\infty}_{\bullet}(D^{\mathrm{an}}_I; \mathbb{C})$

$H^{\infty}_{\bullet}(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{C}) := h^{\bullet} \operatorname{mapping\ cone} \left(\operatorname{tot} \oplus_{|I|=\bullet+1} i_* C^{\infty}_{\bullet}(D^{\mathrm{an}}_I; \mathbb{C}) \to C^{\infty}_{\bullet}(X^{\mathrm{an}}; \mathbb{C}) \right) .$

This gives immediately an isomorphism between singular homology and its smooth version.

Now if $[\omega]$ is a deRham cohomology class in the image of

$$
h^{\bullet} \Gamma(X^{\mathrm{an}}; \Omega^{\bullet}_{X^{\mathrm{an}}}) \longrightarrow H^{\bullet}_{\mathrm{dR}}(X^{\mathrm{an}}; \mathbb{C}),
$$

represented by a differential form $\omega \in \Gamma(X^{\mathrm{an}}; \Omega^{\bullet}_{X^{\mathrm{an}}})$, we can find the image $(\beta^{-1} \circ \alpha)[\omega]$ of $[\omega]$ in $H^{\bullet}_{\mathrm{sing}}(X^{\mathrm{an}}; \mathbb{C})$ as follows: Let $\{\gamma_1, \ldots, \gamma_t\}$ be a basis of $H^{\infty}_{\bullet}(X^{\mathrm{an}}; \mathbb{C})$ and

$\{\gamma_1^*, \ldots, \gamma_t^*\}$ its dual. Then $\{\beta^{-1}\gamma_1^*, \ldots, \beta^{-1}\gamma_t^*\}$ is a basis of $\mathrm{H}_{\mathrm{sing}}^\bullet(X^{\mathrm{an}}; \mathbb{C})$ and we can express $(\beta^{-1} \circ \alpha)[\omega]$ in terms of this basis

$$(\beta^{-1} \circ \alpha)[\omega] = \sum_{i=1}^{t} \langle \beta\gamma_i, (\beta^{-1} \circ \alpha)[\omega] \rangle \cdot \beta^{-1}\gamma_i^*$$

$$= \sum_{i=1}^{t} \langle \gamma_i, \alpha[\omega] \rangle \cdot \beta^{-1}\gamma_i^*$$

$$= \sum_{i=1}^{t} \left(\int_{\gamma_i} \omega \right) \cdot \beta^{-1}\gamma_i^*,$$

where $\langle \cdot, \cdot \rangle$ denotes the homology-cohomology-pairing for singular and smooth singular (co)homology.

Thus, if $[\omega] \in \mathrm{H}_{\mathrm{dR}}^\bullet(X^{\mathrm{an}}; \mathbb{C})$ is a cohomology class represented by a differential form $\omega \in \Gamma(X^{\mathrm{an}}; \Omega_{X^{\mathrm{an}}}^\bullet)$, we can find its image $(\beta^{-1} \circ \alpha)[\omega]$ in $\mathrm{H}_{\mathrm{sing}}^\bullet(X^{\mathrm{an}}; \mathbb{C})$ by computing the integrals $\int_{\gamma_i} \omega$ for $i = 1, \ldots, t$. For an application, see Example 4.12.1.

The case of a divisor and the relative case can be dealt with analogously. The only difference is that a cohomology class $[\omega]$ in the image of

$$\mathrm{h}^\bullet\Gamma(D^{\mathrm{an}}; \widetilde{\Omega}_{D^{\mathrm{an}}}^\bullet) \longrightarrow \mathrm{H}_{\mathrm{dR}}^\bullet(D^{\mathrm{an}}; \mathbb{C})$$

or

$$\mathrm{h}^\bullet\Gamma(X^{\mathrm{an}}; \widetilde{\Omega}_{X^{\mathrm{an}}; D^{\mathrm{an}}}^\bullet) \longrightarrow \mathrm{H}_{\mathrm{dR}}^\bullet(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{C})$$

is represented by a formal sum $\oplus_I \omega_I$ of differential forms ω_I living on either X^{an} or one of the various D_I^{an}. Similarly, a homology class $\gamma \in \mathrm{H}_\bullet^\infty(D^{\mathrm{an}}; \mathbb{C})$ or $\gamma \in \mathrm{H}_\bullet^\infty(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{C})$ is represented by a formal sum $\oplus_I \Gamma_I$ of smooth simplicial chains Γ_I living on X^{an} and the D_I^{an}'s. In computing the pairing $\langle \gamma, [\omega] \rangle$, a summand $\langle \gamma_J, [\omega_I] \rangle$ can only give a non-zero contribution if the domains of definition coincide $D_I^{\mathrm{an}} = D_J^{\mathrm{an}}$. (Here ω_I lives on D_I^{an} with the convention that $\omega_\varnothing$ corresponds to X^{an}.)

4.12 A Motivation for the Theory of Periods

Assume we are dealing with a smooth variety X_0 defined over $\mathbb{Q}$ and a normal-crossings-divisor D_0 on X_0. We denote the base change to $\mathbb{C}$ by X and D, respectively. It seems a natural question to ask whether the isomorphism

$$\mathrm{H}_{\mathrm{dR}}^\bullet(X_0, D_0/\mathbb{Q}) \otimes_\mathbb{Q} \mathbb{C} = \mathrm{H}_{\mathrm{dR}}^\bullet(X, D/\mathbb{C})$$
$$\downarrow \wr$$
$$\mathrm{H}_{\mathrm{sing}}^\bullet(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{Q}) \otimes_\mathbb{Q} \mathbb{C} = \mathrm{H}_{\mathrm{sing}}^\bullet(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{C})$$

is induced by a natural isomorphism of the form

$$\mathrm{H}_{\mathrm{dR}}^\bullet(X_0, D_0/\mathbb{Q}) \to \mathrm{H}_{\mathrm{sing}}^\bullet(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{Q}).$$

The following example shows that the answer to this question is **no** and this negative result may be regarded as the starting point of the theory of periods. (This example uses absolute (co)homology, but taking $D_0 := \varnothing$ we can reformulate it for relative (co)homology.)

Example 4.12.1. Let

$$X_0 := \operatorname{Spec} \mathbb{Q}[t, t^{-1}] = \mathbb{A}_{\mathbb{Q}}^1 \setminus \{0\}$$

be the affine line with point 0 deleted.

Then the singular homology group $\mathrm{H}_1^{\mathrm{sing}}(X^{\mathrm{an}}; \mathbb{Q})$ of $X^{\mathrm{an}} = \mathbb{C}^{\times}$ is generated by the unit circle $\sigma := S^1$. Hence $\mathrm{H}_{\mathrm{sing}}^1(X^{\mathrm{an}}; \mathbb{Q})$ is generated by its dual σ^*.

For the algebraic deRham cohomology group $\mathrm{H}_{\mathrm{dR}}^1(X_0/\mathbb{Q})$ of X_0 we get

$$
\begin{aligned}
\mathrm{H}_{\mathrm{dR}}^1(X_0/\mathbb{Q}) &= \mathbb{H}^1(X_0; \Omega_{X_0/\mathbb{Q}}^{\bullet}) \\
&\overset{(*)}{=} \mathrm{h}^1\Gamma(X_0; \Omega_{X_0/\mathbb{Q}}^{\bullet}) \\
&= \operatorname{coker}(d : \mathbb{Q}[t, t^{-1}] \to \mathbb{Q}[t, t^{-1}]dt) \\
&= \mathbb{Q}\frac{dt}{t}.
\end{aligned}
$$

Here we used at $(*)$ that the sheaves $\Omega_{X_0/\mathbb{Q}}^p$ are quasi-coherent, hence acyclic for the global section functor $\Gamma(X_0; ?)$ by [Ha, Thm. III.3.5, p. 215], since X_0 is affine.

Under the isomorphism $\mathrm{H}_{\mathrm{dR}}^1(X/\mathbb{C}) \overset{\sim}{\longrightarrow} \mathrm{H}_{\mathrm{sing}}^1(X; \mathbb{C})$ the generator $\frac{dt}{t}$ is mapped to $2\pi i \, \sigma^*$ because of $\int_{\sigma} \frac{dt}{t} = 2\pi i$. But $2\pi i$ is not a rational number, hence the isomorphism

$$\mathrm{H}_{\mathrm{dR}}^1(X/\mathbb{C}) \to \mathrm{H}_{\mathrm{sing}}^1(X; \mathbb{C})$$

is **not** induced by a map

$$\mathrm{H}_{\mathrm{dR}}^1(X_0/\mathbb{Q}) \to \mathrm{H}_{\mathrm{sing}}^1(X; \mathbb{Q}).$$

The complex number $2\pi i$ is our first example of a period.

5 Definitions of Periods

A period is to be thought of as an integral that occurs in a geometric context. In their papers [K] and [KZ], Kontsevich and Zagier list various ways of how to define a period.

It is stated in their papers without reference, that all these variants give the same definition. We give a partial proof of this statement in the Period Theorem 7.1.1.

5.1 First Definition of a Period: Pairing Periods

Let X_0 be a smooth variety defined over $\mathbb{Q}$ and D_0 a divisor with normal crossings on X_0. We denote by X^{an} and D^{an} the complex analytic spaces (cf. Subsection 2.1) associated to the base change to $\mathbb{C}$ of X_0 and D_0.

From the discussion of deRham cohomology in the previous sections (especially Proposition 3.5.1 and Theorem 4.10.1), we see that inside the $\mathbb{C}$-vector space $\mathrm{H}_{\mathrm{dR}}^p(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{C})$ there sits a $\mathbb{Q}$-lattice $\mathrm{H}_{\mathrm{dR}}^p(X_0, D_0/\mathbb{Q})$ of full rank. Since

$$\mathrm{H}_{\mathrm{dR}}^\bullet(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{C}) \overset{(4.10.1)}{\cong} \mathrm{H}_{\mathrm{sing}}^\bullet(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{C}) = \mathrm{H}_{\mathrm{sing}}^\bullet(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C},$$

the classical perfect pairing

$$\langle \cdot, \cdot \rangle : \mathrm{H}_\bullet^{\mathrm{sing}}(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{Q}) \times \mathrm{H}_{\mathrm{sing}}^\bullet(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{Q}) \to \mathbb{Q}$$

gives us a new non-degenerated pairing

$$\langle \cdot, \cdot \rangle : \mathrm{H}_\bullet^{\mathrm{sing}}(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{Q}) \times \mathrm{H}_{\mathrm{dR}}^\bullet(X_0, D_0/\mathbb{Q}) \to \mathbb{C}, \tag{27}$$

which is natural in (X_0, D_0).

Definition 5.1.1 (Pairing period, short: p-period). *The complex numbers*

$$\langle \gamma, \omega_0 \rangle \in \mathbb{C} \quad for \quad \omega_0 \in \mathrm{H}_{\mathrm{dR}}^\bullet(X_0, D_0/\mathbb{Q}),\, \gamma \in \mathrm{H}_\bullet^{\mathrm{sing}}(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{Q}),$$

which appear in the image of the natural pairing (27), are called p-periods. *The set of all p-periods for all pairs (X_0, D_0) will be denoted by $\mathbb{P}_p$.*
We call a p-period $\langle \gamma, \omega_0 \rangle$ special or a p'-period, if ω_0 is contained in the image of

$$\mathrm{h}^\bullet \Gamma(X_0; \widetilde{\Omega}_{X_0, D_0/\mathbb{Q}}^\bullet) \to \mathrm{H}_{\mathrm{dR}}^\bullet(X_0, D_0/\mathbb{Q})$$

and write $\mathbb{P}_p'$ for the set of all special pairing periods for all $X_0, D_0, \omega_0, \gamma$.
Running through bases

$$\{\omega_1, \ldots, \omega_t\} \text{ of } \mathrm{H}_{\mathrm{dR}}^\bullet(X_0, D_0/\mathbb{Q}) \text{ and } \{\gamma_1, \ldots, \gamma_t\} \text{ of } \mathrm{H}_\bullet^{\mathrm{sing}}(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{Q})$$

yields the so-called period matrix $P := \left(\langle \gamma_i, \omega_j \rangle \right)_{i,j=1}^t$ *of (X_0, D_0).*

Remark 5.1.2. As a consequence of the Comparison Theorem 4.10.1, the period matrix must be a square matrix (cf. [K, p. 63]). In loc. cit. also a statement about its determinant is made, being a square root of a rational number times a power of $2\pi i$

$$\text{determinant} = \pm\sqrt{a} \cdot (2\pi i)^n \quad \text{for} \quad a \in \mathbb{Q}^\times,\, n \in \mathbb{N}_0.$$

An indication of a proof is not given there.

5.2 Second Definition of a Period: Abstract Periods

There exists another definition of a period, which also emphasizes their geometric origin but does not involve algebraic deRham cohomology.
For this definition the following data is needed,

- X_0 a smooth algebraic variety defined over $\mathbb{Q}$ of dimension d,

- D_0 a divisor on X_0 with normal crossings,

- $\omega_0 \in \Gamma(X_0; \Omega^d_{X_0/\mathbb{Q}})$ an algebraic differential form of top degree,

- $\gamma \in \mathrm{H}^{\mathrm{sing}}_d(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{Q})$ a homology class of singular chains on the complex manifold X^{an} with boundary on the divisor D^{an}.

As usual, we denote by X^{an} and D^{an} the complex analytic spaces (cf. Subsection 2.1) associated to the base change to $\mathbb{C}$ of X_0 and D_0. Thus, we are dealing with objects X^{an}, D^{an} and γ of real dimension $2d$, $2d-2$ and d, respectively.
The differential form ω_0 on X_0 gives rise to a differential form

$$\omega := \pi^* \omega_0$$

on the base change $X = X_0 \times_\mathbb{Q} \mathbb{C}$, where $\pi : X \to X_0$ is the natural projection. Now ω_0 is closed for dimension reasons, hence ω is closed as well.
We choose a representative $\Gamma \in \alpha^{-1}\gamma$ of the preimage $\alpha^{-1}\gamma$ of γ under the isomorphism (cf. Proposition 4.11.5)

$$\alpha : \mathrm{H}^\infty_d(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^{\mathrm{sing}}_d(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{Q})$$

and define

$$\int_\gamma \omega := \int_{\Gamma_d} \omega,$$

where the chain Γ_d consists of the d-simplices of Γ (that is we ignore those $d-q-1$-simplices of Γ that live on one of the D^{an}_I for $|I| = q+1$, $q \geq 0$ — see page 12 for the definition of D_I).

Observe that this integral $\int_\gamma \omega$ is well-defined. The restriction of ω to some irreducible component D^{an}_j of D^{an} is a holomorphic d-form on a complex manifold of dimension $d-1$, hence zero. Therefore the integral $\int_\triangle \omega$ evaluates to zero for smooth singular simplices $\triangle$ that are supported on D^{an}. Now if Γ', Γ'' are two representatives for $\alpha^{-1}\gamma$, we have

$$\Gamma'_d - \Gamma''_d \sim \partial(\Gamma_{d+1})$$

modulo simplices living on some D^{an}_I for a smooth singular chain Γ of dimension $d+1$

$$\Gamma \in \mathrm{C}^\infty_{d+1}(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{Q}).$$

Using Stoke's theorem we get

$$\int_{\Gamma'_d} \omega - \int_{\Gamma''_d} \omega = \int_{\partial(\Gamma_{d+1})} \omega = \int_{\Gamma_{d+1}} d\omega = 0,$$

since ω is closed. $\qquad\square$

Definition 5.2.1 (Abstract period, short: a-period). *We will call the complex number $\int_\gamma \omega$ the a-period of the quadruple $(X_0, D_0, \omega_0, \gamma)$ and denote the set of all a-periods for all $(X_0, D_0, \omega_0, \gamma)$ by $\mathbb{P}_a$.*

Remark 5.2.2. This definition was motivated by Kontsevich's discussion of effective periods [K, def. 20, p. 62], which we will partly quote in subsection 7.3. Modulo Conjecture 7.3.4, abstract and effective periods are essentially the same.

We can ask whether a different definition of the tuples $(X_0, D_0, \omega_0, \gamma)$ yields more period values. A partial answer is given by the following remarks.

Remark 5.2.3. (cf. [K, p. 62]) For example, we could have considered algebraic varieties defined over $\overline{\mathbb{Q}}$. However, doing so does not give us more period values.

Proof.

- Any variety X_0 defined over $\overline{\mathbb{Q}}$ is already defined over a finite extension $\mathbb{Q}'$ of $\mathbb{Q}$, i.e. there exists a variety X_0' such that

$$X_0 = X_0' \times_{\mathbb{Q}'} \overline{\mathbb{Q}}.$$

  Via

$$X_0' \longrightarrow \operatorname{Spec} \mathbb{Q}' \longrightarrow \operatorname{Spec} \mathbb{Q}$$

  we may consider X_0' as a variety defined over $\mathbb{Q}$.

- An analogous argument applies to a normal-crossings-divisor D_0 on X_0. By possibly replacing $\mathbb{Q}'$ by a finite extension field, we can therefore assume w.l.o.g. that D_0 is the base extension of a normal-crossings-divisor D_0' on X_0'

$$D_0 = D_0' \times_{\mathbb{Q}} \overline{\mathbb{Q}}.$$

- Also any differential d-form $\omega_0 \in \Gamma(X_0; \Omega^d_{X_0/\overline{\mathbb{Q}}})$ can be considered as a differential form ω_0' on X_0' by eventually replacing $\mathbb{Q}'$ by a finite extension.

Finally, we discuss singular homology classes.

- First observe that

$$
\begin{aligned}
|\operatorname{Spec} \mathbb{C} \otimes_{\mathbb{Q}} \mathbb{Q}'| &= \operatorname{Hom}_{\mathbb{C}}(\mathbb{C} \otimes_{\mathbb{Q}} \mathbb{Q}', \mathbb{C}) \\
&= \operatorname{Hom}_{\mathbb{Q}}(\mathbb{Q}', \mathbb{C})
\end{aligned}
$$

  is a finite discrete set of points. Thus X' is a finite union of disjoint copies of X

$$
\begin{aligned}
X' &= X_0' \times_{\mathbb{Q}} \mathbb{C} \\
&= X_0' \times_{\mathbb{Q}'} \mathbb{Q}' \times_{\mathbb{Q}} \mathbb{C} \times_{\mathbb{C}} \mathbb{C} \\
&= (X_0' \times_{\mathbb{Q}'} \mathbb{C}) \times_{\mathbb{C}} (\mathbb{Q}' \times_{\mathbb{Q}} \mathbb{C})) \\
&= X \times \operatorname{Hom}_{\mathbb{Q}}(\mathbb{Q}', \mathbb{C}) \\
&= \coprod_{\sigma \,:\, \mathbb{Q}' \to \mathbb{C}} X.
\end{aligned}
$$

Similarly, $D' = \coprod_\sigma D$. Therefore we have

$$\mathrm{H}_d^{\mathrm{sing}}(X'^{\mathrm{an}}, D'^{\mathrm{an}}; \mathbb{Q}) = \bigoplus_\sigma \mathrm{H}_d^{\mathrm{sing}}(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{Q}).$$

If $\gamma \in \mathrm{H}_d^{\mathrm{sing}}(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{Q})$ is some singular homology class, we can pick any $\sigma' : \mathbb{Q}' \to \mathbb{C}$ and find

$$\gamma' := \{\sigma'\} \times \gamma \in \mathrm{H}_d^{\mathrm{sing}}(X'^{\mathrm{an}}, D'^{\mathrm{an}}; \mathbb{Q}) = \oplus_\sigma \mathrm{H}_d^{\mathrm{sing}}(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{Q}).$$

Now

$$\int_{\gamma'} \omega' = \int_\gamma \omega$$

which finishes the proof. $\qquad\square$

Remark 5.2.4. (cf. [K, p. 62]) The requirement of X to be smooth was also made only for convenience and is not a real restriction.

We postpone the proof of Remark 5.2.4 until the Period Theorem 7.1.1 is proved.

5.3 Third Definition of a Period: Naïve Periods

For the last definition of a period, we need the notion of semi-algebraic sets.
Let $\mathbb{K}$ be any field contained in $\mathbb{R}$.

Definition 5.3.1 ($\mathbb{K}$-semi-algebraic sets, [Hi2, Def. 1.1, p. 166]). *A subset of* $\mathbb{R}^n$ *is said to be $\mathbb{K}$-semi-algebraic, if it is of the form*

$$\{\underline{x} \in \mathbb{R}^n | f(\underline{x}) \geq 0\}$$

for some polynomial $f \in \mathbb{K}[x_1, \ldots, x_n]$ *or can be obtained from sets of this form in a finite number of steps, where each step consists of one of the following basic operations:*

(i) complementary set,

(ii) finite intersection,

(iii) finite union.

Denote the integral closure of $\mathbb{Q}$ in $\mathbb{R}$ by $\widetilde{\mathbb{Q}}$. Note that $\widetilde{\mathbb{Q}}$ is a field. Now we can define

Definition 5.3.2 (Naïve periods, short: n-periods). *Let*

- $G \subseteq \mathbb{R}^n$ *be an oriented compact $\widetilde{\mathbb{Q}}$-semi-algebraic set which is equidimensional of dimension d, and*

- ω_0 *a rational differential d-form on $\mathbb{R}^n$ having coefficients in $\overline{\mathbb{Q}}$, which does not have poles on G.*

Then we call the complex number $\int_G \omega_0$ a n-period and denote the set of all n-periods for all G and ω_0 by $\mathbb{P}_n$.

This set $\mathbb{P}_n$ enjoys additional structure.

Proposition 5.3.3. *The set $\mathbb{P}_n$ is a $\overline{\mathbb{Q}}$-algebra.*

Proof. Additive structure: Let $\int_{G_1} \omega_1$ and $\int_{G_2} \omega_2 \in \mathbb{P}_n$ be periods with domains of integration $G_1 \subseteq \mathbb{R}^{n_1}$ and $G_2 \subseteq \mathbb{R}^{n_2}$. Using the inclusions

$$i_1 : \mathbb{R}^{n_1} \cong \mathbb{R}^{n_1} \times \{1/2\} \times \{\underline{0}\} \subset \mathbb{R}^{n_1} \times \mathbb{R} \times \mathbb{R}^{n_2} \quad \text{and}$$
$$i_2 : \mathbb{R}^{n_2} \cong \{\underline{0}\} \times \{-1/2\} \times \mathbb{R}^{n_2} \subset \mathbb{R}^{n_1} \times \mathbb{R} \times \mathbb{R}^{n_2},$$

we can write $i_1(G_1) \cup i_2(G_2)$ for the disjoint union of G_1 and G_2. With the projections $p_j : \mathbb{R}^{n_1} \times \mathbb{R} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_j}$ for $j = 1, 2$, we can lift ω_j on $\mathbb{R}^{n_j}$ to $p_j^* \omega_j$ on $\mathbb{R}^{n_1} \times \mathbb{R} \times \mathbb{R}^{n_2}$. For $q_1, q_2 \in \overline{Q}$ we get

$$q_1 \int_{G_1} \omega_1 + q_2 \int_{G_2} \omega_2 = \int_{i_1(G_1) \cup i_2(G_2)} q_1 \cdot (1/2 + t) \cdot p_1^* \omega_1 + q_2 \cdot (1/2 - t) \cdot p_2^* \omega_2 \in \mathbb{P}_n,$$

where t is the coordinate of the "middle" factor $\mathbb{R}$ of $\mathbb{R}^{n_1} \times \mathbb{R} \times \mathbb{R}^{n_2}$. This shows that $\mathbb{P}_n$ is a $\overline{\mathbb{Q}}$-vector space.

Multiplicative structure: In order to show that $\mathbb{P}_n$ is closed under multiplication, we write

$$p_i : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \longrightarrow \mathbb{R}^{n_i}, \quad i = 1, 2$$

for the natural projections and obtain

$$\left(\int_{G_1} \omega_1 \right) \cdot \left(\int_{G_2} \omega_2 \right) = \int_{G_1 \times G_2} p_1^* \omega_1 \wedge p_2^* \omega_2 \in \mathbb{P}_n$$

by the Fubini formula. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 5.3.4. The Definition 5.3.2 was inspired by the one given in [KZ, p. 772]

> **Definition.** *A [naïve] period is a complex number whose real and imaginary part are values of absolutely convergent integrals of rational functions with rational coefficients, over domains in $\mathbb{R}^n$ given by polynomial inequalities with rational coefficients.*

We will not work with this definition and use the modified version 5.3.2 instead, since this gives us more flexibility in proofs.

From the statements made in [KZ, p. 773] it would follow that both definitions of a naïve period agree.

Examples of naïve periods are

- $\int_1^2 \dfrac{dt}{t} = \ln 2$,

- $\int_0^1 \dfrac{1}{t^2 + 1} dt = \pi$ and

- $\int_G \dfrac{dt}{s} = \int_1^2 \dfrac{dt}{\sqrt{t^3 + 1}} = $ elliptic integrals,

  for $G := \{(t, s) \in \mathbb{R}^2 \mid 1 \leq t \leq 2, \, 0 \leq s, \, s^2 = t^3 + 1\}$.

As a problematic example, we consider the following identity.

Proposition 5.3.5 (cf. [K, p. 62]). *We have*

$$\int_{0 \le t_1 \le t_2 \le 1} \frac{dt_1 \wedge dt_2}{(1 - t_1)\, t_2} = \zeta(2). \tag{28}$$

Proof. This equality follows by a simple power series manipulation: For $0 \le t_2 < 1$, we have

$$\int_0^{t_2} \frac{dt_1}{1 - t_1} = -\log(1 - t_2) = \sum_{n=1}^{\infty} \frac{t_2^n}{n}.$$

Let $\varepsilon > 0$. The power series $\sum_{n=1}^{\infty} \frac{t_2^{n-1}}{n}$ converges uniformly for $0 \le t_2 \le 1 - \varepsilon$ and we get

$$\int_{0 \le t_1 \le t_2 \le 1 - \varepsilon} \frac{dt_1\, dt_2}{(1 - t_1)\, t_2} = \int_0^{1-\varepsilon} \sum_{n=1}^{\infty} \frac{t_2^{n-1}}{n}\, dt_2 = \sum_{n=1}^{\infty} \frac{(1 - \varepsilon)^n}{n^2}.$$

Applying Abel's Theorem [Fi, vol. 2, XII, 438, 6°, p. 411] at $(*)$, using $\sum_{n=1}^{\infty} \frac{1}{n^3} < \infty$ gives us

$$\int_{0 \le t_1 \le t_2 \le 1} \frac{dt_1\, dt_2}{(1 - t_1)\, t_2} = \lim_{\varepsilon \to 0} \sum_{n=1}^{\infty} \frac{(1 - \varepsilon)^n}{n^2} \overset{(*)}{=} \sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2).$$

$\square$

Equation (28) is not a valid representation of $\zeta(2)$ as an integral for a naïve period in our sense, because the pole locus $\{t_1 = 1\} \cup \{t_2 = 0\}$ of $\frac{t_1 \wedge t_2}{(1-t_1)\, t_2}$ is not disjoint with the domain of integration $\{0 \le t_1 \le t_2 \le 1\}$. But (28) gives a valid period integral according to the original definition of [KZ, p. 772] — see Remark 5.3.4. We will show in Example 8.4 how to circumvent this difficulty.

We will prove in the Period Theorem 7.1.1, that special pairing, abstract and naïve periods are essentially the same, i.e.

$$\mathbb{P}_p' = \mathbb{P}_a = \mathbb{P}_n,$$

but have to make a digression on the triangulation of varieties first.
Note that in Subsection 7.3 we will give a fourth definition of a period.

6 Triangulation of Algebraic Varieties

If X_0 is a variety defined over $\mathbb{Q}$ we may ask whether any singular homology class $\gamma \in \mathrm{H}_\bullet^{\mathrm{sing}}(X^{\mathrm{an}}; \mathbb{Q})$ can be represented by an object described by polynomials. This is indeed the case: For a precise statement we need several definitions. The result will be formulated in Proposition 6.2.2.

6.1 Semi-algebraic Sets

We already defined $\widetilde{\mathbb{Q}}$-semi-algebraic sets in Definition 5.3.1.

Definition 6.1.1 ($\widetilde{\mathbb{Q}}$-semi-algebraic map [Hi2, p. 168]). *A continuous map f between $\widetilde{\mathbb{Q}}$-semi-algebraic sets $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ is said to be $\widetilde{\mathbb{Q}}$-semi-algebraic if its graph*

$$\Gamma_f := \left\{ \big(a, f(a)\big) \,\middle|\, a \in A \right\} \subseteq A \times B \subseteq \mathbb{R}^{n+m}$$

is $\widetilde{\mathbb{Q}}$-semi-algebraic.

Example 6.1.2. Any polynomial map

$$f : A \longrightarrow B$$
$$(a_1, \dots, a_n) \mapsto (f_1(a_1, \dots, a_n), \dots, f_m(a_1, \dots, a_n))$$

between $\widetilde{\mathbb{Q}}$-semi-algebraic sets $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ with $f_i \in \widetilde{\mathbb{Q}}[x_1, \dots, x_n]$ for $i = 1, \dots, m$ is $\widetilde{\mathbb{Q}}$-semi-algebraic, since it is continuous and its graph $\Gamma_f \subseteq \mathbb{R}^{n+m}$ is cut out from $A \times B$ by the polynomials

$$y_i - f_i(x_1, \dots, x_n) \in \widetilde{\mathbb{Q}}[x_1, \dots, x_n, y_1, \dots, y_m] \quad \text{for} \quad i = 1, \dots, m. \tag{29}$$

We can even allow f to be a rational map with rational component functions

$$f_i \in \widetilde{\mathbb{Q}}(x_1, \dots, x_n), \quad i = 1, \dots, m$$

as long as none of the denominators of the f_i vanish at a point of A. The argument remains the same except that the expression (29) has to be multiplied by the denominator of f_i.

Fact 6.1.3 ([Hi2, Prop. II, p. 167], [Sb, Thm. 3, p. 370]). *By a result of Seidenberg-Tarski, the image (respectively preimage) of a $\widetilde{\mathbb{Q}}$-semi-algebraic set under a $\widetilde{\mathbb{Q}}$-semi-algebraic map is again $\widetilde{\mathbb{Q}}$-semi-algebraic.*

As the name suggests any algebraic set should be in particular $\widetilde{\mathbb{Q}}$-semi-algebraic.

Lemma 6.1.4. *Let X_0 be an algebraic variety defined over $\widetilde{\mathbb{Q}}$. Then we can regard the complex analytic space X^{an}(cf. Subsection 2.1) associated to the base change $X = X_0 \times_{\widetilde{\mathbb{Q}}} \mathbb{C}$ as a bounded $\widetilde{\mathbb{Q}}$-semi-algebraic subset*

$$X^{\mathrm{an}} \subseteq \mathbb{R}^N \tag{30}$$

for some N. Moreover, if $f_0 : X_0 \to Y_0$ is a morphism of varieties defined over $\widetilde{\mathbb{Q}}$, we can consider $f_{\mathrm{an}} : X^{\mathrm{an}} \to Y^{\mathrm{an}}$ as a $\widetilde{\mathbb{Q}}$-semi-algebraic map.

Proof. First step $X_0 = \mathbb{Q}P^n$: Consider

- $\mathbb{C}P^n_{\mathrm{an}}$ with homogenous coordinates $x_0, \ldots, x_n$, which we split as $x_m = a_m + ib_m$ with $a_m, b_m \in \mathbb{R}$ in real and imaginary part, and

- $\mathbb{R}^N$, $N = 2(n+1)^2$, with coordinates $\{y_{kl}, z_{kl}\}_{k,l=0,\ldots,n}$.

We define a map

$$\psi: \begin{array}{c} \mathbb{C}P^n_{\mathrm{an}} \\ [x_0:\ldots:x_n] \end{array} \longrightarrow \begin{array}{c} \mathbb{R}^N \\ (y_{00}, z_{00}, \ldots, y_{nn}, z_{nn}) \end{array}$$

$$[x_0 : \ldots : x_n] \mapsto \left(\ldots, \underbrace{\frac{\operatorname{Re} x_k \overline{x}_l}{\sum_{m=0}^n |x_m|^2}}_{y_{kl}}, \underbrace{\frac{\operatorname{Im} x_k \overline{x}_l}{\sum_{m=0}^n |x_m|^2}}_{z_{kl}}, \ldots \right)$$

$$[a_0 + ib_0 : \ldots : a_n + ib_n] \mapsto \left(\ldots, \underbrace{\frac{a_k a_l + b_k b_l}{\sum_{m=0}^n a_m^2 + b_m^2}}_{y_{kl}}, \underbrace{\frac{b_k a_l - a_k b_l}{\sum_{m=0}^n a_m^2 + b_m^2}}_{z_{kl}}, \ldots \right).$$

Rewriting the last line (with the convention $0 \cdot \cos(\genfrac{}{}{0pt}{}{\text{indeterminate}}{\text{angle}}) = 0$) as

$$[r_0 e^{i\phi_0} : \ldots : r_n e^{i\phi_n}] \mapsto \left(\ldots, \frac{r_k r_l \cos(\phi_k - \phi_l)}{\sum_{m=0}^n r_m^2}, \frac{r_k r_l \sin(\phi_k - \phi_l)}{\sum_{m=0}^n r_m^2}, \ldots \right) \qquad (31)$$

shows that ψ is injective: Assume

$$\psi\left([r_0 e^{i\phi_0} : \ldots : r_n e^{i\phi_n}]\right) = (y_{00}, z_{00}, \ldots, y_{nn}, z_{nn})$$

where $r_k \neq 0$, or equivalently $y_{kk} \neq 0$, for a fixed k. We find

$$\frac{r_l}{r_k} = \frac{\sqrt{y_{kl}^2 + z_{kl}^2}}{y_{kk}}, \quad \text{and}$$

$$\phi_k - \phi_l = \begin{cases} \arctan(z_{kl}/y_{kl}) & \text{if } y_{kl} \neq 0, \\ \pi/2 & \text{if } y_{kl} = 0, z_{kl} > 0, \\ \text{indeterminate} & \text{if } y_{kl} = z_{kl} = 0, \\ -\pi/2 & \text{if } y_{kl} = 0, z_{kl} < 0; \end{cases}$$

that is the preimage of $(y_{00}, z_{00}, \ldots, y_{nn}, z_{nn})$ is uniquely determined.

Therefore we can consider $\mathbb{C}P^n_{\mathrm{an}}$ via ψ as a subset of $\mathbb{R}^N$. It is bounded since it is contained in the unit sphere $S^{N-1} \subset \mathbb{R}^N$. We claim that $\psi(\mathbb{C}P^n_{\mathrm{an}})$ is also $\widetilde{\mathbb{Q}}$-semi-algebraic. The composition of the projection

$$\pi : \mathbb{R}^{2(n+1)} \setminus \{(0, \ldots, 0)\} \longrightarrow \mathbb{C}P^n_{\mathrm{an}}$$
$$(a_0, b_0, \ldots, a_n, b_n) \mapsto [a_0 + ib_0 : \ldots : a_n + ib_n]$$

with the map ψ is a polynomial map, hence $\widetilde{\mathbb{Q}}$-semi-algebraic by Example 6.1.2. Thus

$$\operatorname{im} \psi \circ \pi = \operatorname{im} \psi \subseteq \mathbb{R}^N$$

is $\widetilde{\mathbb{Q}}$-semi-algebraic by Fact 6.1.3.

Second step (zero set of a polynomial): We use the notation

$$V(g) := \{\underline{x} \in \mathbb{C}P_{\mathrm{an}}^n \mid g(\underline{x}) = 0\} \quad \text{for} \quad g \in \mathbb{C}[x_0, \ldots, x_n] \text{ homogenous}, \quad \text{and}$$
$$W(h) := \{\underline{t} \in \mathbb{R}^N \mid h(\underline{t}) = 0\} \quad \text{for} \quad h \in \mathbb{C}[y_{00}, \ldots, z_{nn}].$$

Let $X^{\mathrm{an}} = V(g)$ for some homogenous $g \in \widetilde{\mathbb{Q}}[x_0, \ldots, x_n]$. Then $\psi(X^{\mathrm{an}}) \subseteq \mathbb{R}^N$ is a $\widetilde{\mathbb{Q}}$-semi-algebraic subset, as a little calculation shows. Setting for $k = 0, \ldots, n$

$$g_k := \text{``}g(\underline{x}\,\overline{x}_k)\text{''}$$
$$= g(x_0\overline{x}_k, \ldots, x_n\overline{x}_k)$$
$$= g\big((a_0 a_k + b_0 b_k) + i(b_0 a_k - a_0 b_k), \ldots, (a_n a_k + b_n b_k) + i(b_n a_k - a_n b_k)\big),$$

where $x_j = a_j + ib_j$ for $j = 0, \ldots, n$, and

$$h_k := g(y_{0k} + iz_{0k}, \ldots, y_{nk} + iz_{nk}),$$

we obtain

$$\psi(X^{\mathrm{an}}) = \psi(V(g))$$
$$= \bigcap_{k=0}^n \psi(V(g_k))$$
$$= \bigcap_{k=0}^n \psi(\mathbb{C}P_{\mathrm{an}}^n) \cap W(h_k)$$
$$= \bigcap_{k=0}^n \psi(\mathbb{C}P_{\mathrm{an}}^n) \cap W(\operatorname{Re} h_k) \cap W(\operatorname{Im} h_k).$$

Final step: We can choose an embedding

$$X_0 \subseteq \widetilde{\mathbb{Q}}P^n,$$

thus getting

$$X^{\mathrm{an}} \subseteq \mathbb{C}P_{\mathrm{an}}^n.$$

Since X_0 is a locally closed subvariety of $\widetilde{\mathbb{Q}}P^n$, X^{an} can be expressed in terms of subvarieties of the form $V(g)$ with $g \in \widetilde{\mathbb{Q}}[x_0, \ldots, x_n]$, using only the following basic operations

(i) complementary set,

(ii) finite intersection,

(iii) finite union.

Now $\widetilde{\mathbb{Q}}$-semi-algebraic sets are stable under these operations as well and the first assertion is proved. ∎

Second assertion: The first part of the lemma provides us with $\widetilde{\mathbb{Q}}$-semi-algebraic inclusions

$$\psi : X^{\mathrm{an}} \subseteq \underset{\underline{x}=[x_0:\ldots:x_n]}{\mathbb{C}P_{\mathrm{an}}^n} \subseteq \underset{(y_{00}, z_{00}, \ldots, y_{nn}, z_{nn})}{\mathbb{R}^N},$$

$$\phi : Y^{\mathrm{an}} \subseteq \underset{\underline{u}=[u_0:\ldots:u_m]}{\mathbb{C}P_{\mathrm{an}}^m} \subseteq \underset{(v_{00}, w_{00}, \ldots, v_{mm}, w_{mm})}{\mathbb{R}^M},$$

and a choice of coordinates as indicated. We use the notation

$$V(g) := \{(\underline{x}, \underline{u}) \in \mathbb{C}P^n_{\mathrm{an}} \times \mathbb{C}P^m_{\mathrm{an}} \mid g(\underline{x}, \underline{u}) = 0\},$$

$$\text{for } g \in \mathbb{C}[x_0, \ldots, x_n, u_0, \ldots, u_m] \text{ homogenous in both } \underline{x} \text{ and } \underline{u}, \quad \text{and}$$

$$W(h) := \{\underline{t} \in \mathbb{R}^{N+M} \mid h(\underline{t}) = 0\}, \quad \text{for } h \in \mathbb{C}[y_{00}, \ldots, z_{nn}, v_{00}, \ldots, w_{mm}].$$

Let $\{U_i\}$ be a finite open affine covering of X_0 such that $f_0(U_i)$ satisfies

- $f_0(U_i)$ does not meet the hyperplane $\{u_j = 0\} \subset \widetilde{\mathbb{Q}}P^m$ for some j, and

- $f_0(U_i)$ is contained in an open affine subset V_i of Y_0.

This is always possible, since we can start with the open covering $Y_0 \cap \{u_j \neq 0\}$ of Y_0, take a subordinated open affine covering $\{V_{i'}\}$, and then choose a finite open affine covering $\{U_i\}$ subordinated to $\{f^{-1}(V_{i'})\}$. Now each of the maps

$$f_i := f_{\mathrm{an}|U_i} : U_i^{\mathrm{an}} \longrightarrow Y^{\mathrm{an}}$$

has image contained in V_i^{an} and does not meet the hyperplane $\{\underline{u} \in \mathbb{C}P^m_{\mathrm{an}} \mid u_j = 0\}$ for an appropriate j

$$f_i : U_i^{\mathrm{an}} \longrightarrow V_i^{\mathrm{an}}.$$

Being associated to an algebraic map between affine varieties, this map is rational

$$f_i : \underline{x} \mapsto \left[\frac{g'_0(\underline{x})}{g''_0(\underline{x})} : \ldots : \frac{g'_{j-1}(\underline{x})}{g''_{j-1}(\underline{x})} : \underset{j}{1} : \frac{g'_{j+1}(\underline{x})}{g''_{j+1}(\underline{x})} : \ldots : \frac{g'_m(\underline{x})}{g''_m(\underline{x})} \right],$$

with $g'_k, g''_k \in \widetilde{\mathbb{Q}}[x_0, \ldots, x_n]$, $k = 0, \ldots, \widehat{j}, \ldots, m$. Since the graph $\Gamma_{f_{\mathrm{an}}}$ of f_{an} is the finite union of the graphs Γ_{f_i} of the f_i, it is sufficient to prove that $(\psi \times \phi)(\Gamma_{f_i})$ is a $\widetilde{\mathbb{Q}}$-semi-algebraic subset of $\mathbb{R}^{N+M}$. Now

$$\Gamma_{f_i} = (U_i^{\mathrm{an}} \times V_i^{\mathrm{an}}) \cap \bigcap_{\substack{k=0 \\ k \neq j}}^{n} V\left(\frac{y_k}{y_j} - \frac{g'_k(\underline{x})}{g''_k(\underline{x})}\right) = (U_i^{\mathrm{an}} \times V_i^{\mathrm{an}}) \cap \bigcap_{\substack{k=0 \\ k \neq j}}^{n} V(y_k g''_k(\underline{x}) - y_j g'_k(\underline{x})),$$

so all we have to deal with is

$$V(y_k g''_k(\underline{x}) - y_j g'_k(\underline{x})).$$

Again a little calculation is necessary. Setting

$$\begin{aligned}
g_{pq} &:= \text{``} u_k \overline{u}_q g''_k(\underline{x}\, \overline{x}_p) - u_j \overline{u}_q g'_k(\underline{x}\, \overline{x}_p) \text{''} \\
&= u_k \overline{u}_q g''_k(x_0 \overline{x}_p, \ldots, x_n \overline{x}_p) - u_j \overline{u}_q g'_k(x_0 \overline{x}_p, \ldots, x_n \overline{x}_p) \\
&= \left((c_k c_q + d_k d_q) + i(d_k c_q - c_k d_q)\right) \\
&\qquad g''_k\left((a_0 a_p + b_0 b_p) + i(b_0 a_p - a_0 b_p), \ldots, (a_n a_p + b_n b_p) + i(b_n a_p - a_n b_p)\right) \\
&\quad - \left((c_j c_q + d_j d_q) + i(d_j c_q - c_j d_q)\right) \\
&\qquad g'_k\left((a_0 a_p + b_0 b_p) + i(b_0 a_p - a_0 b_p), \ldots, (a_n a_p + b_n b_p) + i(b_n a_p - a_n b_p)\right),
\end{aligned}$$

where $x_l = a_l + i b_l$ for $l = 0, \ldots, n$, $u_l = c_l + i d_l$ for $l = 0, \ldots, m$, and

$$h_{pq} := (v_{kq} + i w_{kq}) g''_k(y_{0p} + i z_{0p}, \ldots, y_{np} + i z_{np}) - (v_{jq} + i w_{jq}) g'_k(y_{0p} + i z_{0p}, \ldots, y_{np} + i z_{np}),$$

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we obtain

$$(\psi \times \phi)\left(V\big(y_k g_k''(\underline{x}) - y_j g_k'(\underline{x})\big)\right) =$$

$$= \bigcap_{p=0}^{n} \bigcap_{q=0}^{m} (\psi \times \phi)(V(g_{pq}))$$

$$= \bigcap_{p=0}^{n} \bigcap_{q=0}^{m} (\psi \times \phi)(U_i^{\mathrm{an}} \times V_j^{\mathrm{an}}) \cap W(h_{pq})$$

$$= \bigcap_{p=0}^{n} \bigcap_{q=0}^{m} (\psi \times \phi)(U_i^{\mathrm{an}} \times V_j^{\mathrm{an}}) \cap W(\operatorname{Re} h_{pq}) \cap W(\operatorname{Im} h_{pq}).$$

$\square$

6.2 Semi-algebraic Singular Chains

We need further prerequisites in order to state the announced Proposition 6.2.2.

Definition 6.2.1 (Open simplex, [Hi2, p. 168]). *By an open simplex $\triangle^{\circ}$ we mean the interior of a simplex (= the convex hull of $r+1$ points in $\mathbb{R}^n$ which span a r-dimensional subspace). For convenience, a point is considered as an open simplex as well.*
The notation $\triangle_d^{\mathrm{std}}$ will be reserved for the closed standard simplex spanned by the standard basis $\{(0,\ldots,0,\underset{i}{1},0,\ldots,0) \,|\, i = 1,\ldots,d+1\}$ of $\mathbb{R}^{d+1}$.

Consider the following data $(*)$:

- X_0 a variety defined over $\widetilde{\mathbb{Q}}$,

- D_0 a divisor on X_0 with normal crossings,

- and finally $\gamma \in \mathrm{H}_p^{\mathrm{sing}}(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{Q})$, $p \in \mathbb{N}_0$.

As usual, we have denoted by X^{an} (resp. D^{an}) the complex analytic space associated to the base change $X = X_0 \times_{\widetilde{\mathbb{Q}}} \mathbb{C}$ (resp. $D = D_0 \times_{\widetilde{\mathbb{Q}}} \mathbb{C}$).

By Lemma 6.1.4, we may consider both X^{an} and D^{an} as bounded $\widetilde{\mathbb{Q}}$-semi-algebraic subsets of $\mathbb{R}^N$.

We are now able to formulate our proposition.

Proposition 6.2.2. *With data $(*)$ as above, we can find a representative of γ that is a rational linear combination of singular simplices each of which is $\widetilde{\mathbb{Q}}$-semi-algebraic.*

The proof of this proposition relies on the following proposition due to Lojasiewicz which has been written down by Hironaka.

Proposition 6.2.3 (Triangulation of $\widetilde{\mathbb{Q}}$-semi-algebraic sets, [Hi2, p. 170]).
For $\{X_i\}$ a finite system of bounded $\widetilde{\mathbb{Q}}$-semi-algebraic sets in $\mathbb{R}^n$, there exists a simplicial decomposition

$$\mathbb{R}^n = \coprod_j \triangle_j^{\circ}$$

by open simplices $\triangle_j^{\circ}$ and a $\widetilde{\mathbb{Q}}$-semi-algebraic automorphism

$$\kappa : \mathbb{R}^n \to \mathbb{R}^n$$

such that each X_i is a finite union of some of the $\kappa(\triangle_j^{\circ})$.

Note 6.2.4. Although Hironaka considers $\mathbb{R}$-semi-algebraic sets, it has been checked by the author, that we can safely replace $\mathbb{R}$ by $\widetilde{\mathbb{Q}}$ in his article (including the fact he cites from [Sb]). The only problem that could possibly arise concerns a "good direction lemma":

Lemma 6.2.5 (Good direction lemma for $\mathbb{R}$, [Hi2, p. 172], [KB, Thm. 5.I, p. 242]). *Let Z be a $\mathbb{R}$-semi-algebraic subset of $\mathbb{R}^n$, which is nowhere dense. A direction $v \in |\mathbb{R}P^{n-1}|$ is called* good, *if any line l in $\mathbb{R}^n$ parallel to v meets Z in a discrete (maybe empty) set of points; otherwise v is called* bad. *Then the set $B(Z)$ of bad directions is a Baire category set in $|\mathbb{R}P^{n-1}|$.*

This gives immediately good directions $v \in |\mathbb{R}P^{n-1}| \setminus B(Z)$, but not necessarily $v \in |\widetilde{\mathbb{Q}}P^{n-1}| \setminus B(Z)$. However, in Remark 2.1 of [Hi2], which follows directly after the lemma, the following statement is made: If Z is compact, then $B(Z)$ is closed in $|\mathbb{R}P^{n-1}|$. In particular $|\widetilde{\mathbb{Q}}P^{n-1}| \setminus B(Z)$ will be non-empty. Since we only consider **bounded** $\widetilde{\mathbb{Q}}$-semi-algebraic sets Z', we may take $Z := \overline{Z'}$ (which is compact by Heine-Borel), and thus find a good direction $v \in |\widetilde{\mathbb{Q}}P^{n-1}| \setminus B(Z')$ using $B(Z') \subseteq B(Z)$.

Lemma 6.2.6 (Good direction lemma for $\widetilde{\mathbb{Q}}$). *Let Z' be a bounded $\widetilde{\mathbb{Q}}$-semi-algebraic subset of $\mathbb{R}^n$, which is nowhere dense. Then the set $|\widetilde{\mathbb{Q}}P^{n-1}| \setminus B(Z)$ of good directions is non-empty.*

Proof of Proposition 6.2.2. Applying Proposition 6.2.3 to the two-element system of $\widetilde{\mathbb{Q}}$-semi-algebraic sets $X^{\mathrm{an}}, D^{\mathrm{an}} \subseteq \mathbb{R}^N$, we obtain a $\widetilde{\mathbb{Q}}$-semi-algebraic decomposition

$$\mathbb{R}^N = \coprod_j \triangle_j^\circ$$

of $\mathbb{R}^N$ by open simplices $\triangle_j^\circ$ and a $\widetilde{\mathbb{Q}}$-semi-algebraic automorphism

$$\kappa : \mathbb{R}^N \to \mathbb{R}^N.$$

We write $\triangle_j$ for the closure of $\triangle_j^\circ$. The sets

$$K := \{\triangle_j^\circ \,|\, \kappa(\triangle_j^\circ) \subseteq X^{\mathrm{an}}\} \quad \text{and} \quad L := \{\triangle_j^\circ \,|\, \kappa(\triangle_j^\circ) \subseteq D^{\mathrm{an}}\}$$

can be thought of as finite simplicial complexes, but built out of open simplices instead of closed ones. We define their *geometric realizations*

$$|K| := \bigcup_{\triangle_j^\circ \in K} \triangle_j^\circ \quad \text{and} \quad |L| := \bigcup_{\triangle_j^\circ \in L} \triangle_j^\circ.$$

Then Proposition 6.2.3 states that κ maps the pair of topological spaces $(|K|, |L|)$ homeomorphically to $(X^{\mathrm{an}}, D^{\mathrm{an}})$.
Easy case: If X_0 is complete, so is X (by [Ha, Cor. II.4.8(c), p. 102]), hence X^{an} and D^{an} will be compact [Ha, B.1, p. 439]. In this situation

$$\overline{K} := \{\triangle_j \,|\, \kappa(\triangle_j) \subseteq X^{\mathrm{an}}\} \quad \text{and} \quad \overline{L} := \{\triangle_j \,|\, \kappa(\triangle_j) \subseteq D^{\mathrm{an}}\}$$

are (ordinary) simplicial complexes, whose geometric realizations coincide with those of K and L, respectively. In particular

$$\begin{aligned}
\mathrm{H}_\bullet^{\mathrm{simpl}}(\overline{K}, \overline{L}; \mathbb{Q}) &\cong \mathrm{H}_\bullet^{\mathrm{sing}}(|\overline{K}|, |\overline{L}|; \mathbb{Q}) \\
&\cong \mathrm{H}_\bullet^{\mathrm{sing}}(|K|, |L|; \mathbb{Q}) \\
&\cong \mathrm{H}_\bullet^{\mathrm{sing}}(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{Q}).
\end{aligned} \tag{32}$$

Here $H_\bullet^{simpl}(\overline{K}, \overline{L}; \mathbb{Q})$ denotes simplicial homology of course.

We write $\gamma_{simpl} \in H_p^{simpl}(\overline{K}, \overline{L}; \mathbb{Q})$ and $\gamma_{sing} \in H_p^{sing}(|\overline{K}|, |\overline{L}|; \mathbb{Q})$ for the image of γ under this isomorphism. Any representative Γ_{simpl} of γ_{simpl} is a rational linear combination

$$\Gamma_{simpl} = \sum_j a_j \triangle_j, \quad a_j \in \mathbb{Q}$$

of oriented closed simplices $\triangle_j \in \overline{K}$. We can choose orientation-preserving affine-linear maps of the standard simplex $\triangle_p^{std}$ to $\triangle_j$

$$\sigma_j : \triangle_p^{std} \longrightarrow \triangle_j \quad \text{for} \quad \triangle_j \in \Gamma_{simpl}.$$

These maps yield a representative

$$\Gamma_{sing} := \sum_j a_j \, \sigma_j$$

of γ_{sing}. Composing with κ yields $\Gamma := \kappa_* \Gamma_{sing} \in \gamma$, where Γ has the desired properties. In the **general case**, we perform a barycentric subdivision $\mathcal{B}$ on K twice (once is not enough) and define $|K|$ and $|L|$ not as the "closure" of K and L, but as follows (see Figure 2)

$$\begin{aligned}
\overline{K} &:= \{ \triangle \, | \, \triangle^\circ \in \mathcal{B}^2(K) \text{ and } \triangle \subseteq |K| \}, \\
\overline{L} &:= \{ \triangle \, | \, \triangle^\circ \in \mathcal{B}^2(K) \text{ and } \triangle \subseteq |L| \}.
\end{aligned} \tag{33}$$

$$\kappa^{-1}(X^{an}) \cap \triangle_j \qquad\qquad K \cap \triangle_j \qquad\qquad |\overline{K}| \cap \triangle_j$$

Intersection of $\kappa^{-1}(X^{an})$ with a closed 2-simplex $\triangle_j$, where we assume that part of the boundary $\partial \triangle_j$ does not belong to $\kappa^{-1}(X^{an})$

Open simplices of K contained in $\triangle_j$

Intersection of $|\overline{K}|$ with $\triangle_j$ (the dashed lines show the barycentric subdivision)
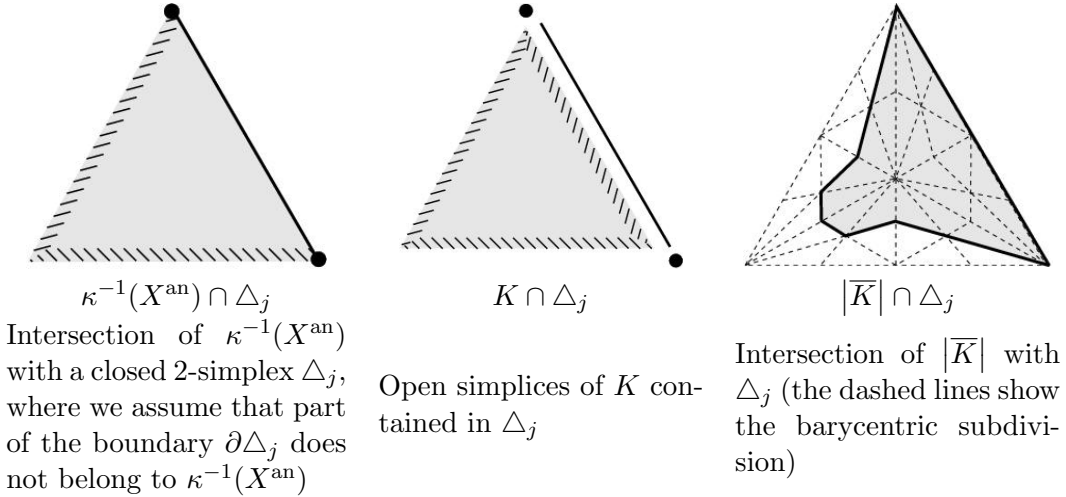
Figure 2: Definition of $\overline{K}$

The point is that the pair of topological spaces $(|\overline{K}|, |\overline{L}|)$ is a strong deformation retract of $(|K|, |L|)$. Assuming this we see that in the general case with $\overline{K}$, $\overline{L}$ defined as in (33), the isomorphism (32) still holds and we can proceed as in the easy case to prove the proposition.

We define the retraction map

$$\rho : (|K| \times [0,1], |L| \times [0,1]) \to (|\overline{K}|, |\overline{L}|)$$

as follows: Let $\triangle_j^\circ \in K$ be an open simplex which is not contained in the boundary of any other simplex of K and set

$$inner := \triangle_j \cap \overline{K}, \qquad outer := \triangle_j \setminus \overline{K}.$$

Note that *inner* is closed. For any point $p \in outer$ the ray $\overrightarrow{cp}$ from the center c of $\triangle_j^\circ$ through p "leaves" the set *inner* at a point q_p, i.e. $\overrightarrow{cp} \cap inner$ equals the line segment $c\,q_p$; see Figure 3. The map

$$\rho_j : \triangle_j \times [0,1] \to \triangle_j$$

$$(p,t) \mapsto \begin{cases} p & \text{if } p \in inner, \\ q_p + t \cdot (p - q_p) & \text{if } p \in outer \end{cases}$$

retracts $\triangle_j$ onto *inner*.

Now these maps ρ_j glue together to give the desired homotopy ρ. $\qquad\square$

Figure 3: Definition of q_p

We want to state one of the intermediate results of this proof explicitly.

Corollary 6.2.7. *Let X_0 and D_0 be as above. Then the pair of topological spaces $(X^{\mathrm{an}}, D^{\mathrm{an}})$ is homotopic equivalent to a pair of (realizations of) simplicial complexes $(|X^{\mathrm{simpl}}|, |D^{\mathrm{simpl}}|)$.*

7 Periods Revisited

7.1 Comparison of Definitions of Periods

We are now ready to prove the theorem announced in the introduction.

Main Theorem 7.1.1 (Period theorem). *It holds*

$$\mathbb{P}'_p = \mathbb{P}_a = \mathbb{P}_n,$$

that is the following three definitions of a period given in Section 5 coincide

- *special pairing periods (cf. Definition 5.1.1),*
- *abstract periods (cf. Definition 5.2.1), and*
- *naïve periods (cf. Definition 5.3.2).*

Note that we will give a fourth definition of a period in Subsection 7.3.

Proof. $\mathbb{P}_a \subseteq \mathbb{P}'_p$: Let $\int_\gamma \omega \in \mathbb{P}_a$ be an abstract period coming from $(X_0, D_0, \omega_0, \gamma)$. We want to understand ω_0 as an element of $\Gamma(X_0; \widetilde{\Omega}^d_{X_0, D_0/\mathbb{Q}}) = \Gamma(X_0; \Omega^d_{X_0/\mathbb{Q}}) \oplus \Gamma(D_0; \widetilde{\Omega}^{d-1}_{D_0/\mathbb{Q}})$ via $\Gamma(X_0; \Omega^d_{X_0/\mathbb{Q}}) \rightarrowtail \Gamma(X_0; \widetilde{\Omega}^d_{X_0, D_0/\mathbb{Q}})$, where $d := \dim X_0$. Then $\omega_0 \in \Gamma(X_0; \widetilde{\Omega}^\bullet_{X_0, D_0/\mathbb{Q}})$ is a cocycle, since ω_0 is closed (for dimension reasons) and its restriction to each of the irreducible components D_j of D_0 is zero (also for dimension reasons). Thus we can consider $[\omega_0]$ via

$$\mathrm{h}^\bullet \Gamma(X_0; \widetilde{\Omega}^\bullet_{X_0, D_0/\mathbb{Q}}) \to \mathrm{H}^\bullet_{\mathrm{dR}}(X_0, D_0/\mathbb{Q})$$

as an algebraic deRham cohomology class and find (cf. Subsection 4.11.5)

$$\int_\gamma \omega = \langle \gamma, [\omega_0] \rangle \in \mathbb{P}'_p.$$

∎

$\mathbb{P}'_p \subseteq \mathbb{P}_n$: As explained in Subsection 4.11.5, a special pairing period

$$\langle \gamma, \omega_0 \rangle \in \mathbb{P}'_p$$

is a sum of integrals

$$\sum_I \langle \gamma_I, \omega_I \rangle = \sum_I \int_{\gamma_I} \omega_I,$$

where $\omega_I \in \mathrm{H}^\bullet_{\mathrm{dR}}(D_I/\mathbb{Q})$ and $\gamma_I \in \mathrm{H}^\bullet_\infty(D_I; \mathbb{Q})$. Since $\mathbb{P}_n$ is closed under addition by Proposition 5.3.3, it suffices to prove $\int_{\gamma_I} \omega_I \in \mathbb{P}_n$ for all index sets I.
We choose an embedding

$$D_I \subseteq \underset{(x_0:\ldots:x_n)}{\mathbb{Q}P^n}$$

and equip $\mathbb{Q}P^n$ with coordinates as indicated. Lemma 6.1.4 provides us with a map

$$\psi : \mathbb{C}P^n_{\mathrm{an}} \hookrightarrow \mathbb{R}^N$$

such that D^{an}_I and $\mathbb{C}P^n_{\mathrm{an}}$ become $\widetilde{\mathbb{Q}}$-semi-algebraic subsets of $\mathbb{R}^N$. Then, by Proposition 6.2.2, $\psi_* \gamma_I$ has a representative which is a rational linear combination of singular simplices Γ_i, each of which is $\widetilde{\mathbb{Q}}$-semi-algebraic.

We set $G_i := \operatorname{im} \Gamma_i$ and are left to prove

$$\int_{\psi^{-1}(G_i)} \omega_I \in \mathbb{P}_n.$$

This will be clear as soon as we find a rational differential form ω_I' on $\mathbb{R}^N$ such that $\psi^* \omega_I' = \omega_I$, since then

$$\int_{\psi^{-1}(G_i)} \omega_I = \int_{\psi^{-1}(G_i)} \psi^* \omega_I' = \int_{G_i} \omega_I' \in \mathbb{P}_n.$$

After eventually applying a barycentric subdivision to Γ_i, we may assume w.l.o.g. that there exists a hyperplane in $\mathbb{C}P_{\mathrm{an}}^n$, say $\{x_0 = 0\}$, which does not meet $\psi^{-1}(G_i)$. Furthermore, we may assume that $\psi^{-1}(G_i)$ lies entirely in U^{an} for U_0 an open affine subset of $D_I \cap \{x_0 \neq 0\}$. (As usual, U^{an} denotes the complex analytic space associated to the base change to $\mathbb{C}$ of U.) The restriction of ω_I to the open affine subset can be represented in the form (cf. [Ha, II.8.4A, II.8.2.1, II.8.2A])

$$\sum_{|J|=d} f_J(x_0, \ldots, x_n) \, d\left(\frac{x_{j_1}}{x_0}\right) \wedge \ldots \wedge d\left(\frac{x_{j_d}}{x_0}\right)$$

with $f_J(x_1, \cdots, x_n) \in \mathbb{Q}(x_0, \cdots, x_n)$ being homogenous of degree zero. This expression defines a rational differential form on all of $\mathbb{Q}P^n$ with coefficients in $\mathbb{Q}$ and it does not have poles on $\psi^{-1}(G_i)$.

We construct the rational differential form ω_I' on $\mathbb{R}^N$ with coefficients in $\mathbb{Q}(i)$ as follows

$$\omega_I' := \sum_{|J|=d} f_J\left(1, \frac{y_{10} + i z_{10}}{y_{00} + i z_{00}}, \cdots, \frac{y_{n0} + i z_{n0}}{y_{00} + i z_{00}}\right) d\left(\frac{y_{j_1 0} + i z_{j_1 0}}{y_{00} + i z_{00}}\right) \wedge \ldots \wedge d\left(\frac{y_{j_d 0} + i z_{j_d 0}}{y_{00} + i z_{00}}\right),$$

where we have used the notation from the proof of Lemma 6.1.4. Using the explicit form of ψ given in this proof, we obtain

$$\psi^* f_J\left(1, \frac{y_{10} + i z_{10}}{y_{00} + i z_{00}}, \cdots, \frac{y_{n0} + i z_{n0}}{y_{00} + i z_{00}}\right) = f_J\left(\frac{x_0 \overline{x}_0}{|x_0|^2}, \frac{x_1 \overline{x}_0}{|x_0|^2}, \ldots, \frac{x_n \overline{x}_0}{|x_0|^2}\right)$$

$$= f_J(x_0, x_1, \ldots, x_n)$$

and

$$\psi^* d\left(\frac{y_{j0} + i z_{j0}}{y_{00} + i z_{00}}\right) = d\left(\frac{x_j \overline{x}_0}{|x_0|^2}\right) = d\left(\frac{x_j}{x_0}\right).$$

This shows that $\psi^* \omega_I' = \omega_I$ and we are done. $\blacksquare$

$\mathbb{P}_n \subseteq \mathbb{P}_a$: In this part of the proof, we will use objects over various base fields: We will use subscripts to indicate which base field is used:

subscript	base field
0	$\widetilde{\mathbb{Q}}$
1	$\overline{\mathbb{Q}}$
$\mathbb{R}$	$\mathbb{R}$
none	$\mathbb{C}$

Furthermore, we fix an embedding $\overline{\mathbb{Q}} \subset \mathbb{C}$.

Let $\int_G \omega_{\mathbb{R}} \in \mathbb{P}_n$ be a naïve period with

- $G \subset \mathbb{R}^n$ an oriented $\widetilde{\mathbb{Q}}$-semi-algebraic set, equidimensional of dimension d, and

- $\omega_{\mathbb{R}}$ a rational differential d-form on $\mathbb{R}^n$ with coefficients in $\overline{\mathbb{Q}}$, which does not have poles on G.

The $\widetilde{\mathbb{Q}}$-semi-algebraic set $G \subset \mathbb{R}^n$ is given by polynomial inequalities and equalities. By omitting the inequalities but keeping the equalities in the definition of G, we see that G is supported on (the set of $\mathbb{R}$-valued points of) a variety $Y_{\mathbb{R}} \subseteq \mathbb{A}_{\mathbb{R}}^n$ of same dimension d. This variety $Y_{\mathbb{R}}$ is already defined over $\widetilde{\mathbb{Q}}$

$$Y_{\mathbb{R}} = Y_0 \times_{\widetilde{\mathbb{Q}}} \mathbb{R}$$

for a variety $Y_0 \subseteq \mathbb{A}_{\widetilde{\mathbb{Q}}}^n$ over $\widetilde{\mathbb{Q}}$. Similarly the boundary ∂G of G is supported on a variety $E_{\mathbb{R}}$, likewise defined over $\widetilde{\mathbb{Q}}$

$$E_{\mathbb{R}} = E_0 \times_{\widetilde{\mathbb{Q}}} \mathbb{R}.$$

Note that E_0 is a divisor on Y_0. By eventually enlarging E_0, we may assume w.l.o.g. that E_0 contains the singular locus of Y_0. In order to obtain an abstract period, we need smooth varieties. The resolution of singularities according to Hironaka [Hi1] provides us with a Cartesian square

$$
\begin{array}{ccc}
\widetilde{E}_0 & \subseteq & \widetilde{Y}_0 \\
\downarrow & & \downarrow \pi_0 \\
E_0 & \subseteq & Y_0
\end{array}
\tag{34}
$$

where

- $\widetilde{Y}_0$ is smooth and quasi-projective,

- π_0 is proper, surjective and birational, and

- $\widetilde{E}_0$ is a divisor with normal crossings.

In fact, π_0 is an isomorphism away from $\widetilde{E}_0$ since the singular locus of Y_0 is contained in E_0

$$\pi_{0|\widetilde{U}_0} : \widetilde{U}_0 \xrightarrow{\sim} U_0 \tag{35}$$

with $\widetilde{U}_0 := \widetilde{Y}_0 \setminus \widetilde{E}_0$ and $U_0 := Y_0 \setminus E_0$.

We apply the functor **an** (associated complex analytic space — cf. Subsection 2.1) to the base change to $\mathbb{C}$ of the map $\pi_0 : \widetilde{Y}_0 \to Y_0$ and obtain a projection

$$\pi_{\mathrm{an}} : \widetilde{Y}^{\mathrm{an}} \to Y^{\mathrm{an}}.$$

We want to show that the "strict transform" of G

$$\widetilde{G} := \overline{\pi_{\mathrm{an}}^{-1}(G \setminus E^{\mathrm{an}})} \subseteq \widetilde{Y}^{\mathrm{an}}$$

can be triangulated. Since $\mathbb{C}P^n_{\mathrm{an}}$ is the projective closure of $\mathbb{C}^n$, we have $\mathbb{C}^n \subset \mathbb{C}P^n_{\mathrm{an}}$ and thus get an embedding

$$Y^{\mathrm{an}} \subseteq \mathbb{C}^n \subset \mathbb{C}P^n_{\mathrm{an}}.$$

We also choose an embedding

$$\widetilde{Y}^{\mathrm{an}} \subseteq \mathbb{C}P^m_{\mathrm{an}}$$

for some $m \in \mathbb{N}$. Using Lemma 6.1.4, we may consider both Y^{an} and $\widetilde{Y}^{\mathrm{an}}$ as $\widetilde{\mathbb{Q}}$-semi-algebraic sets via some maps

$$\psi : Y^{\mathrm{an}} \subset \mathbb{C}P^n_{\mathrm{an}} \hookrightarrow \mathbb{R}^N, \quad \text{and}$$
$$\widetilde{\psi} : \widetilde{Y}^{\mathrm{an}} \subseteq \mathbb{C}P^m_{\mathrm{an}} \hookrightarrow \mathbb{R}^M.$$

In this setting, the induced projection

$$\pi_{\mathrm{an}} : \widetilde{Y}^{\mathrm{an}} \longrightarrow Y^{\mathrm{an}}$$

becomes a $\widetilde{\mathbb{Q}}$-semi-algebraic map. The composition of ψ with the inclusion $G \subseteq Y^{\mathrm{an}}$ is a $\widetilde{\mathbb{Q}}$-semi-algebraic map; hence $G \subset \mathbb{R}^N$ is $\widetilde{\mathbb{Q}}$-semi-algebraic by Fact 6.1.3. Since E^{an} is also $\widetilde{\mathbb{Q}}$-semi-algebraic via ψ, we find that $G \setminus E^{\mathrm{an}}$ is $\widetilde{\mathbb{Q}}$-semi-algebraic. Again by Fact 6.1.3, $\pi_{\mathrm{an}}^{-1}(G \setminus E^{\mathrm{an}}) \subset \mathbb{R}^M$ is $\widetilde{\mathbb{Q}}$-semi-algebraic. Thus $\widetilde{G} \subset \mathbb{R}^M$, being the closure of a $\widetilde{\mathbb{Q}}$-semi-algebraic set, is $\widetilde{\mathbb{Q}}$-semi-algebraic. From Proposition 6.2.2, we see that $\widetilde{G}$ can be triangulated

$$\widetilde{G} = \cup_j \triangle_j, \tag{36}$$

where the $\triangle_j$ are (homeomorphic images of) d-dimensional simplices.

Our next aim is to define an algebraic differential form $\widetilde{\omega}_1$ replacing $\omega_{\mathbb{R}}$. We first make a base change in (34) from $\widetilde{\mathbb{Q}}$ to $\overline{\mathbb{Q}}$ and obtain

$$\begin{array}{c} \widetilde{E}_1 \subseteq \widetilde{Y}_1 \\ \downarrow \quad \downarrow \pi_1 \\ E_1 \subseteq Y_1 \,. \end{array}$$

The differential d-form $\omega_{\mathbb{R}}$ can be written as

$$\omega_{\mathbb{R}} = \sum_{|J|=d} f_J(x_1, \ldots, x_n)\, dx_{j_1} \wedge \ldots \wedge dx_{j_d}, \tag{37}$$

where $x_1, \ldots, x_n$ are coordinates of $\mathbb{R}^n$ and $f_J \in \overline{\mathbb{Q}}(x_1, \ldots, x_n)$. We can use equation (37) to define a differential form ω_1 on $\mathbb{A}^n_{\overline{\mathbb{Q}}}$

$$\omega_{\mathbb{R}} = \sum_{|J|=d} f_J(x_1, \ldots, x_n)\, dx_{j_1} \wedge \ldots \wedge dx_{j_d},$$

where now $x_1, \ldots, x_n$ denote coordinates of $\mathbb{A}^n_{\overline{\mathbb{Q}}}$. The pole locus of ω_1 gives us a variety $Z_1 \subset \mathbb{A}^n_{\overline{\mathbb{Q}}}$. We set

$$X_1 := Y_1 \setminus Z_1, \quad D_1 := E_1 \setminus Z_1, \quad \text{and}$$
$$\widetilde{X}_1 := \pi_1^{-1}(X_1), \quad \widetilde{D}_1 := \pi_1^{-1}(D_1).$$

The restriction $\omega_{1|X_1}$ of ω_1 to X_1 is a (regular) algebraic differential form on X_1; the pullback

$$\widetilde{\omega}_1 := \pi_1^*(\omega_{1|X_1})$$

is an algebraic differential form on $\widetilde{X}_1$.

We consider the complex analytic spaces $\widetilde{X}^{\mathrm{an}}$, $\widetilde{D}^{\mathrm{an}}$, Z^{an} associated to the base change to $\mathbb{C}$ of $\widetilde{X}_1$, $\widetilde{D}_1$, Z_1. Since $\omega_{\mathbb{R}}$ has no poles on G, we have $G \cap Z^{\mathrm{an}} = \varnothing$; hence $\widetilde{G} \cap \pi_{\mathrm{an}}^{-1}(Z^{\mathrm{an}}) = \varnothing$. This shows $\widetilde{G} \subseteq \widetilde{X} = \widetilde{Y} \setminus \pi_{\mathrm{an}}^{-1}(Z^{\mathrm{an}})$.

Since G is oriented, so is $\pi_{\mathrm{an}}^{-1}(G \setminus E^{\mathrm{an}})$, because π_{an} is an isomorphism away from E^{an}. Every d-simplex $\triangle_j$ in (36) intersects $\pi_{\mathrm{an}}^{-1}(G \setminus E^{\mathrm{an}})$ in a dense open subset, hence inherits an orientation. As in the proof of Proposition 6.2.2, we choose orientation-preserving homeomorphisms from the standard d-simplex $\triangle_d^{\mathrm{std}}$ to $\triangle_j$

$$\sigma_j : \triangle_d^{\mathrm{std}} \longrightarrow \triangle_j.$$

These maps sum up to a singular chain

$$\widetilde{\Gamma} = \oplus_j \, \sigma_j \in \mathrm{C}_d^{\mathrm{sing}}(\widetilde{X}^{\mathrm{an}}; \mathbb{Q}).$$

It might happen that the boundary of the singular chain $\widetilde{\Gamma}$ is not supported on $\partial \widetilde{G}$. Nevertheless, it will always be supported on $\widetilde{D}^{\mathrm{an}}$: The set $\pi_{\mathrm{an}}^{-1}(G \setminus E^{\mathrm{an}})$ is oriented and therefore the boundary components of $\partial \triangle_j$ that do not belong to $\partial \widetilde{G}$ cancel if they have non-zero intersection with $\pi_{\mathrm{an}}^{-1}(G \setminus E^{\mathrm{an}})$. Thus $\widetilde{\Gamma}$ gives rise to a singular homology class

$$\widetilde{\gamma} \in \mathrm{H}_d^{\mathrm{sing}}(\widetilde{X}^{\mathrm{an}}, \widetilde{D}^{\mathrm{an}}; \mathbb{Q}).$$

We denote the base change to $\mathbb{C}$ of ω_1 and $\widetilde{\omega}_1$ by ω and $\widetilde{\omega}$, respectively. Now

$$\int_G \omega_{\mathbb{R}} = \int_G \omega = \int_{G \cap U^{\mathrm{an}}} \omega$$

$$\overset{(35)}{=} \int_{\pi^{-1}(G \cap U^{\mathrm{an}})} \pi^* \omega = \int_{\widetilde{G} \cap \widetilde{U}^{\mathrm{an}}} \widetilde{\omega}$$

$$= \int_{\widetilde{G}} \widetilde{\omega} = \int_{\widetilde{\Gamma}} \widetilde{\omega} = \int_{\widetilde{\gamma}} \widetilde{\omega} \in \mathbb{P}_a$$

is an abstract period for the quadruple $(\widetilde{X}_1, \widetilde{D}_1, \widetilde{\omega}_1, \widetilde{\gamma})$ by Remark 5.2.3. $\qquad \square$

Now Remark 5.2.4 is an easy corollary of the Period Theorem 7.1.1.

Proof of Remark 5.2.4. Let X_0 be a possibly non-regular variety defined over $\mathbb{Q}$ of dimension d and D_0 a Cartier divisor on X_0. Furthermore, let $\omega_0 \in \Omega_{X_0}^d$ and $\gamma \in \mathrm{H}_d^{\mathrm{sing}}(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{Q})$.

Exactly the same argumentation used for $\int_{D_I} \omega_I$ in proving $\mathbb{P}_p' \subseteq \mathbb{P}_n$ shows that $\int_\gamma \omega \in \mathbb{P}_n$. Because of $\mathbb{P}_n = \mathbb{P}_a$, there must exists a "smooth" quadruple $(\widetilde{X}_0, \widetilde{D}_0, \omega_0, \gamma)$ as described in the definition of an abstract period (cf. Definition 5.2.1), which gives the same period value. $\qquad \square$

Remark 7.1.2. It might be expected that also $\mathbb{P}_p = \mathbb{P}_p'$ holds. The following ideas make this conjecture probable. Let $\langle \gamma, \omega_0 \rangle$ be a pairing period for (X_0, D_0).

- If X_0 is affine, then the complex $\widetilde{\Omega}_{X_0, D_0/\mathbb{Q}}^\bullet$ consists of quasi-coherent sheaves by [Ha, Thm. III.3.5, p. 215]. Hence we have a surjection

$$\mathrm{H}^\bullet \Gamma(X_0; \widetilde{\Omega}_{X_0, D_0/\mathbb{Q}}^\bullet) \twoheadrightarrow \mathrm{H}_{\mathrm{dR}}^\bullet(X_0, D_0/\mathbb{Q}),$$

  that is, only special periods appear.

- If X_0 is not affine, one might try to break up γ into subchains γ_j with $\widetilde{\mathbb{Q}}$-semi-algebraic representatives Γ_j that are already supported on U_j^{an} for open affine subsets $i : U_j \hookrightarrow X_0$. Finding a divisor D_j on U_j such that $\partial \Gamma_j \subset D_j^{\mathrm{an}}$ would give us a singular chain

$$[i^{-1}\Gamma_j] \in \mathrm{H}_\bullet^{\mathrm{sing}}(U_j^{\mathrm{an}}, D_j^{\mathrm{an}}; \mathbb{Q})$$

and so would allow us to reduce the general case to the affine one

$$\langle \gamma_j, \omega_0 \rangle = \langle [i^{-1}\Gamma_j], i^*\omega_0 \rangle.$$

- If γ and ω_0 are top-dimensional of real dimension $2d$, the procedure described above will definitely not work: A boundary $\partial \Gamma_j$ has real codimension one, but D^{an} has real codimension two for D_0 a divisor on X_0. On the other hand, the top-dimensional case is simpler, since the relative algebraic deRham cohomology group involved is in fact an absolute one

$$\mathrm{H}_{\mathrm{dR}}^{2d}(X_0, D_0/\mathbb{Q}) = \mathrm{H}_{\mathrm{dR}}^{2d}(X_0/\mathbb{Q}),$$

because of the long exact sequence in algebraic deRham cohomology and the vanishing of $\mathrm{H}_{\mathrm{dR}}^p(D_0/\mathbb{Q})$ for $p \geq \dim_{\mathbb{R}}(D^{\mathrm{an}}) = 2d - 2$.

Now, generalizing a result of Huisman [Hui, Thm 5.1, p. 7] to varieties over $\mathbb{Q}(i)$, we find that every algebraic deRham cohomology class in $\mathrm{H}_{\mathrm{dR}}^\bullet\big(X_0 \times_{\mathbb{Q}} \mathbb{Q}(i)/\mathbb{Q}(i)\big)$ can be realized by a rational differential form on $\pi_{\mathbb{Q}(i)/\mathbb{Q}}X_0$ with coefficients in $\mathbb{Q}$, where $\pi_{\mathbb{Q}(i)/\mathbb{Q}}X_0$ is the restriction of scalars according to Weil with respect to the field extension $\mathbb{Q}(i)/\mathbb{Q}$ of X_0.

The set $\mathbb{P} := \mathbb{P}_a = \mathbb{P}'_p = \mathbb{P}_n$ is not only a $\overline{\mathbb{Q}}$-algebra, but, at least conjecturally, comes with a triple coproduct imposing on $\operatorname{Spec} \mathbb{P}[\frac{1}{2\pi i}]$ the structure of a torsor, this notion being defined as follows.

7.2 Torsors

Heuristically, a torsor is a group that has forgotten its identity, more formally:

Definition 7.2.1 (Torsor, [K, p. 61]). *A torsor is a non-empty set X together with a map*

$$(\cdot, \cdot, \cdot) : X \times X \times X \to X$$

satisfying

(i) $(a, a, c) = c$

(ii) $(a, b, b) = a$

(iii) $((a, b, c), d, e) = (a, (b, c, d), e) = (a, b, (c, d, e)).$

Example 7.2.2. A group G becomes a torsor by setting

$$(\cdot, \cdot, \cdot) : G \times G \times G \to G$$
$$(a, b, c) \mapsto a \circ b^{-1} \circ c.$$

Conversely, a torsor X becomes a group by choosing a distinguished element $e \in X$ ('the identity') and setting

$$a \circ b := (a, e, b)$$
$$a^{-1} := (e, a, e).$$

Example 7.2.3. The fibre X of a principal G-bundle is a torsor: For any $a, b, c \in X$ we can work out the 'difference' of a and b, which is an element $g \in G$, and let g act on c in order to define (a, b, c).

Example 7.2.4. Let $\mathcal{C}$ be a category and E, F two isomorphic objects of $\mathcal{C}$. The set of isomorphisms $\mathrm{Iso}(E, F)$ comes naturally with a right action of the automorphism group $\mathrm{Aut}(E)$ and a left action of $\mathrm{Aut}(F)$; both actions being simply transitive. This can be encoded in a single map, thus turning $\mathrm{Iso}(E, F)$ into a torsor:

$$\mathrm{Iso}(E, F) \times \mathrm{Iso}(E, F) \times \mathrm{Iso}(E, F) \to \mathrm{Iso}(E, F)$$
$$(a, b, c) \mapsto a \circ b^{-1} \circ c.$$

These examples are generic in the sense that we could have defined a torsor alternatively as a principal homogenous space over a group (Example 7.2.3), or as a category containing just two isomorphic objects (Example 7.2.4).

7.3 Fourth Definition of a Period: Effective Periods

We pick up our discussion of periods and show how periods could give us a torsor.

The usual tools for proving identities between integrals are the *change of variables* and the *Stoke's formula*. These rules are formalized by the following definition.

Definition 7.3.1 (Effective Periods, [K, def. 20, p. 62]). *The $\mathbb{Q}$-vector space $\mathcal{P}_+$ of effective periods has as generators the quadruples $(X_0, D_0, \omega_0, \gamma)$, as considered in the definition of an abstract period (cf. Definition 5.2.1), modulo the following relations*

- *(linearity in both ω_0 and γ)*

  $(X_0, D_0, q_1\omega_{0,1} + q_2\omega_{0,2}, \gamma) \sim q_1(X_0, D_0, \omega_{0,1}, \gamma) + q_2(X_0, D_0, \omega_{0,2}, \gamma)$ *for $q_1, q_2 \in \mathbb{Q}$,*
  $(X_0, D_0, \omega_0, q_1\gamma_1 + q_2\gamma_2) \sim q_1(X_0, D_0, \omega_0, \gamma_1) + q_2(X_0, D_0, \omega_0, \gamma_2)$ *for $q_1, q_2 \in \mathbb{Q}$.*

- *(change of variables)*

  If $f : (X_1, D_1) \to (X_2, D_2)$ is a morphism of pairs defined over $\mathbb{Q}$, $\gamma_1 \in \mathrm{H}_d^{\mathrm{sing}}(X_1^{\mathrm{an}}, D_1^{\mathrm{an}}; \mathbb{Q})$ and $\omega_2 \in \Omega_{X_2}^d$ then

  $$(X_1, D_1, f^*\omega_2, \gamma_1) \sim (X_2, D_2, \omega_2, f_*\gamma_1).$$

- *(Stoke's formula)*

  Write $D_0 = \bigcup_{i=1}^n D_i$ as union of its prime divisors, then

  $$(X_0, D_0, d\omega_0, \gamma) \sim \text{``}(D_0, \varnothing, \omega_0, \partial\gamma)\text{''} = \sum_{i=1}^n (D_i, \bigcup_{j \neq i} D_j \cap D_i, \omega_0, \partial\gamma \cap D_i^{\mathrm{an}}),$$

  $$(38)$$

  where $\partial : \mathrm{H}_d^{\mathrm{sing}}(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{Q}) \to \mathrm{H}_{d-1}^{\mathrm{sing}}(D^{\mathrm{an}}; \mathbb{Q})$ is the boundary operator.

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Here the term $\partial\gamma \cap D_i^{\mathrm{an}}$ in the Stoke's formula is defined in an ad-hoc fashion: We consider D^{an} as a $\widetilde{\mathbb{Q}}$-semi-algebraic set (by Lemma 6.1.4)

$$D^{\mathrm{an}} \subset \mathbb{R}^N$$

and pick a representative $\Gamma \in \partial\gamma$ which is a rational combination $\Gamma = \sum_k a_k \Gamma_k$ of $\widetilde{\mathbb{Q}}$-semi-algebraic simplices Γ_k (by Proposition 6.2.2). Then the images of the Γ_k are $\widetilde{\mathbb{Q}}$-semi-algebraic sets

$$G_k := \operatorname{im}\Gamma_k \subset \mathbb{R}^N$$

by Fact 6.1.3. Applying Proposition 6.2.3 to the system $\{G_k, D_i^{\mathrm{an}}\}$ yields triangulations of the G_k's, i.e. we can write

$$G_k = \bigcup_l \triangle_{k,l}$$

where the $\triangle_{k,l}$ are (homeomorphic images of) oriented closed simplices of dimension $d-1$ whose interiors are disjoint. They enjoy the additional property that each $\triangle_{k,l}$ lies entirely in one of the D_i^{an} (because the same $\triangle_{k,l}$ triangulate also the D_i^{an}'s). The induced triangulation on

$$G_k^i := G_k \cap D_i^{\mathrm{an}}$$

gives us a singular chain Γ_k^i on D_i^{an} and hence a singular homology class

$$\gamma_k^i \in \mathrm{H}_{d-1}^{\mathrm{sing}}(D_i^{\mathrm{an}}, \cup_{i \neq j} D_i^{\mathrm{an}} \cap D_j^{\mathrm{an}}; \mathbb{Q})$$

which we call $\partial\gamma \cap D_i^{\mathrm{an}}$.

We will denote the equivalence class represented by the quadruple $(X_0, D_0, \omega_0, \gamma)$ by $[X_0, D_0, \omega_0, \gamma]$.

Proposition 7.3.2. *We have an evaluation morphism*

$$\mathrm{ev} : \mathcal{P}_+ \to \quad \mathbb{C}$$
$$[X_0, D_0, \omega_0, \gamma] \mapsto \quad \int_\gamma \omega.$$

Proof. On page 44, we have already seen that this map is well-defined on the level of representatives

$$\widetilde{\mathrm{ev}} : (X_0, D_0, \omega_0, \gamma) \mapsto \int_\gamma \omega,$$

so it remains to show that the map ev factors through the relations imposed in Definition 7.3.1.

- (linearity)

$$\widetilde{\mathrm{ev}}(X_0, D_0, q_1\omega_{0,1} + q_2\omega_{0,2}, \gamma) = \int_\gamma q_1\omega_1 + q_2\omega_2 =$$

$$= q_1 \int_\gamma \omega_1 + q_2 \int_\gamma \omega_2 = q_1\widetilde{\mathrm{ev}}(X_0, D_0, \omega_{0,1}, \gamma) + q_2\widetilde{\mathrm{ev}}(X_0, D_0, \omega_{0,2}, \gamma),$$

$$\widetilde{\mathrm{ev}}(X_0, D_0, \omega_0, q_1\gamma_1 + q_2\gamma_2) = \int_{q_1\gamma_1 + q_2\gamma_2} \omega =$$

$$= q_1 \int_{\gamma_1} \omega + q_2 \int_{\gamma_2} \omega = q_1\widetilde{\mathrm{ev}}(X_0, D_0, \omega_0, \gamma_1) + q_2\widetilde{\mathrm{ev}}(X_0, D_0, \omega_0, \gamma_2),$$

- (change of variables)

$$\widetilde{\mathrm{ev}}(X_1, D_1, f^*\omega_2, \gamma_1) = \int_{\gamma_1} f^*\omega_2 = \int_{f_*\gamma_1} \omega_2 = \widetilde{\mathrm{ev}}(X_2, D_2, \omega_2, f_*\gamma_1),$$

- (Stoke's formula)

$$\widetilde{\mathrm{ev}}(X_0, D_0, d\omega_0, \gamma) = \int_\gamma d\omega = \int_{\partial\gamma} \omega =$$
$$= \sum_i \int_{\partial\gamma \cap D_i^{\mathrm{an}}} \omega = \sum_i \widetilde{\mathrm{ev}}(D_i, \cup_{i \neq j} D_i \cap D_j, \omega_0, \partial\gamma \cap D_i)$$

$\square$

We state some easy properties of $\mathcal{P}_+$.

Proposition 7.3.3 (Basic properties of $\mathcal{P}_+$, [K, p. 63]).

1. *The set $\mathcal{P}_+$ is a $\mathbb{Q}$-algebra.*

2. $\overline{\mathbb{Q}} \subset \mathrm{im}(\mathrm{ev})$.

Proof (cf. [K, p. 63]). First fact: Let $[X_i, D_i, \omega_i, \gamma_i] \in \mathcal{P}_+$, $i = 1, 2$ and define their product to be

$$[X_1 \times_\mathbb{Q} X_2, D_1 \times_\mathbb{Q} X_2 \cup X_1 \times_\mathbb{Q} D_2, p_1^*\omega_1 \wedge p_2^*\omega_2, \gamma_1 \otimes \gamma_2] \in \mathcal{P}_+,$$

where $p_i : X_1 \times_\mathbb{Q} X_2 \to X_i$, $i = 1, 2$, are the natural projections. By the Fubini formula
$$\int_{\gamma_1 \otimes \gamma_2} p_1^*\omega_1 \wedge p_2^*\omega_2 = \left(\int_{\gamma_1} \omega_1 \right) \cdot \left(\int_{\gamma_2} \omega_2 \right),$$
i.e. the map ev becomes a $\mathbb{Q}$-algebra-morphism.
Second fact: Let $\alpha \in \overline{\mathbb{Q}}$, $p \in \mathbb{Q}[t]$ its minimal polynomial and $X_0 := \mathrm{Spec}\,\mathbb{Q}[t]/(p) = \mathrm{Spec}\,\mathbb{Q}(\alpha)$. We may regard α as a closed point of X via

$$\mathrm{Spec}\,\mathbb{C} \to X$$
$$\mathbb{C} \leftarrow \mathbb{C}[t]/(p)$$
$$\alpha \leftarrow\!\shortmid t.$$

Any closed point x of X gives an element of zeroth homology $\{x\} \in \mathrm{H}_0^{\mathrm{sing}}(X^{\mathrm{an}}; \mathbb{Q})$. Now $t \in \Omega_{X_0}^0$ and we get
$$\mathrm{ev}[X, \varnothing, t, \{\alpha\}] = \alpha.$$

$\square$

Note that the $\mathbb{Q}$-algebra morphism $\mathrm{ev} : \mathcal{P}_+ \to \mathbb{P}$ is surjective as a consequence of the definition of abstract periods (cf. Definition 5.2.1).

Conjecture 7.3.4 ([K, p. 62]). *It is conjectured that the map* $\mathrm{ev} : \mathcal{P}_+ \to \mathbb{P}$ *is injective, i.e. that all identities between integrals can be proved using standard rules only.*

This would give us a $\mathbb{Q}$-algebra-isomorphism $\mathcal{P}_+ \cong \mathbb{P}$.

For the remainder of this section, we assume Conjecture 7.3.4.

We adjoin formally $(2\pi i)^{-1}$ to $\mathcal{P}_+$

$$\mathcal{P} := \mathcal{P}_+[\tfrac{1}{(2\pi i)}] \overset{?}{=} \mathbb{P}[\tfrac{1}{(2\pi i)}]$$

and define a coproduct on $\mathcal{P}$.

Definition 7.3.5 (The triple coproduct Δ on $\mathcal{P}$, [K, p. 63]).
Let $\alpha := [X_0, D_0, \omega_0, \gamma]$ be an effective period represented by an quadruple $(X_0, D_0, \omega_0, \gamma)$. We write P for the period matrix of (X_0, D_0) as defined in Definition 5.1.1 and assume $\mathrm{ev}(\alpha) = P_{i,j}$, the entry in P at position (i,j). We define

$$\Delta(P_{i,j}) := \sum_{k,l} P_{i,k} \otimes (P^{-1})_{k,l} \otimes P_{l,j} \in \mathcal{P} \otimes_{\mathbb{Q}} \mathcal{P} \otimes_{\mathbb{Q}} \mathcal{P}.$$

Here we used Remark 5.1.2 about the determinant of P: $\det P$ being a period in $\mathbb{P}$ times a power of $(2\pi i)$. Thus P^{-1} has coefficients in $\mathcal{P} = \mathcal{P}_+[\tfrac{1}{2\pi i}] \overset{?}{=} \mathbb{P}[\tfrac{1}{2\pi i}]$. We adopt the convention

$$\Delta\left(\frac{P_{i,j}}{(2\pi i)^n}\right) := \sum_{k,l} \frac{P_{i,k}}{(2\pi i)^n} \otimes (2\pi i)^n (P^{-1})_{k,l} \otimes \frac{P_{l,j}}{(2\pi i)^n} \quad for \quad n \in \mathbb{N}_0.$$

Extending the map Δ in a $\mathbb{Q}$-linear way to all of $\mathcal{P}$ yields the triple coproduct

$$\Delta : \mathcal{P} \to \mathcal{P} \otimes_{\mathbb{Q}} \mathcal{P} \otimes_{\mathbb{Q}} \mathcal{P}.$$

Remark 7.3.6. It is far from being obvious that Δ is well-defined. In his paper [K], Kontsevich states that the proof of the correctness of Definition 7.3.5 relies on an unpublished result by M. Nori [K, Thm. 6, p. 63].

From Nori's theorem, also the following proposition would follow.

Proposition 7.3.7. *Modulo Conjecture 7.3.4 and Nori's theorem, the triple coproduct on $\mathcal{P}$ induces the structure of a torsor on $\mathrm{Spec}\,\mathcal{P}$.*

We calculate a triple coproduct in Example 8.1.

8 Examples

8.1 First Example: $\mathbb{A}^1_{\mathbb{C}} \setminus \{0\}$

First part (cf. [K, p. 63]): Let

$$X_0 := \mathbb{A}^1_{\mathbb{Q}} \setminus \{0\} = \operatorname{Spec} \mathbb{Q}[t, t^{-1}]$$

be the affine line with the point 0 deleted and

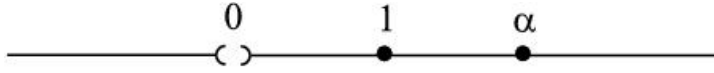$$D_0 := \{1, \alpha\} \quad \text{with} \quad \alpha \neq 0, 1$$

a divisor on X_0.

$$\text{Figure 4: The algebraic pair } (X_0, D_0)$$

Then the *singular homology* of the pair $(X^{\mathrm{an}}, D^{\mathrm{an}}) = (\mathbb{C}^\times, \{1, \alpha\})$ (cf. Definition 4.8.1) is generated by a small loop σ turning counter-clockwise around 0 once and the interval $[1, \alpha]$.

In order to compute the *algebraic deRham cohomology* of (X_0, D_0) (cf. Definition 3.2.6), we first note that

$$\mathrm{H}^\bullet_{\mathrm{dR}}(X_0, D_0/\mathbb{Q}) = \mathrm{h}^\bullet \Gamma(X_0; \widetilde{\Omega}^\bullet_{X_0, D_0/\mathbb{Q}}),$$

since X_0 is affine and the sheaves $\widetilde{\Omega}^p_{X_0, D_0/\mathbb{Q}}$ are quasi-coherent, hence acyclic for the global section functor by [Ha, Thm. III.3.5, p. 215].
We spell out the complex $\Gamma(X_0; \widetilde{\Omega}^\bullet_{X_0, D_0/\mathbb{Q}})$ in detail (cf. page 15)

$$0$$
$$\uparrow$$
$$\Gamma(X_0; \widetilde{\Omega}^1_{X_0, D_0/\mathbb{Q}}) = \Gamma\big(X_0; \Omega^1_{X_0/\mathbb{Q}} \oplus \bigoplus_j i_* \mathcal{O}_{D_j}\big) = \mathbb{Q}[t, t^{-1}]dt \oplus \underset{1}{\mathbb{Q}} \oplus \underset{\alpha}{\mathbb{Q}}$$
$$\uparrow d$$
$$\Gamma(X_0; \mathcal{O}_{X_0}) = \mathbb{Q}[t, t^{-1}]$$

and observe that the map "evaluation-at-a-point"

$$\mathbb{Q}[t, t^{-1}] \twoheadrightarrow \underset{1}{\mathbb{Q}} \oplus \underset{\alpha}{\mathbb{Q}}$$
$$f(t) \mapsto \big(f(1), f(\alpha)\big)$$

is surjective with kernel

$$(t-1)(t-\alpha)\mathbb{Q}[t, t^{-1}] = \operatorname{span}_{\mathbb{Q}}\{t^{n+2} - (\alpha+1)t^{n+1} + \alpha t^n \,|\, n \in \mathbb{Z}\}.$$

Differentiation maps this kernel to

$$\operatorname{span}_{\mathbb{Q}}\{(n+2)t^{n+1} - (n+1)(\alpha+1)t^n - n\alpha t^{n-1} \,|\, n \in \mathbb{Z}\}dt.$$

Therefore we get

$$
\begin{aligned}
\mathrm{H}^1_{\mathrm{dR}}(X_0, D_0; \mathbb{Q}) &= \Gamma(X_0; \widetilde{\Omega}_{X_0, D_0/\mathbb{Q}}) / \Gamma(X_0; \mathcal{O}_{X_0}) \\
&= \mathbb{Q}[t, t^{-1}]dt \oplus \underset{1}{\mathbb{Q}} \oplus \underset{\alpha}{\mathbb{Q}} / d(\mathbb{Q}[t, t^{-1}]) \\
&= \mathbb{Q}[t, t^{-1}]dt / \operatorname{span}_{\mathbb{Q}}\{(n+2)t^{n+1} - (n+1)(\alpha+1)t^n - n\alpha t^{n-1}\}dt.
\end{aligned}
$$

By the last line, we see that the class of $t^n dt$ in $\mathrm{H}^1_{\mathrm{dR}}(X_0, D_0; \mathbb{Q})$ for $n \neq -1$ is linearly dependent of

- $t^{n-1}dt$ and $t^{n-2}dt$, and

- $t^{n+1}dt$ and $t^{n+2}dt$,

hence linearly dependent of $\frac{dt}{t}$ and dt by an induction argument. Hence $\mathrm{H}^1_{\mathrm{dR}}(X_0, D_0; \mathbb{Q})$ is spanned by

$$
\frac{dt}{t} \quad \text{and} \quad \frac{1}{\alpha - 1}dt.
$$

We obtain the following *period matrix* P for (X_0, D_0) (cf. Definition 5.1.1)

$$
\begin{array}{c|cc}
 & \frac{1}{\alpha-1}dt & \frac{dt}{t} \\
\hline
[1, \alpha] & 1 & \ln \alpha \\
\sigma & 0 & 2\pi i
\end{array}
\tag{39}
$$

Let us the compute the *triple coproduct* of $\ln \alpha$ (cf. Definition 7.3.5) from this example. We have

$$
P^{-1} = \begin{pmatrix} 1 & \frac{-\ln \alpha}{2\pi i} \\ 0 & \frac{1}{2\pi i} \end{pmatrix}
$$

and thus get for the triple coproduct (cf. [K, p. 63])

$$
\Delta(\ln \alpha) = \ln \alpha \otimes \tfrac{1}{2\pi i} \otimes 2\pi i - 1 \otimes \tfrac{\ln \alpha}{2\pi i} \otimes 2\pi i + 1 \otimes 1 \otimes \ln \alpha.
\tag{40}
$$

Second part: Now consider the degenerate configuration $\alpha = 1$, i.e. $D = 2 \cdot \{1\}$ with $D_1 = D_2 = \{1\}$. Although D_0 is not a divisor with normal crossings anymore, the machinery developed in Section 3 still works.
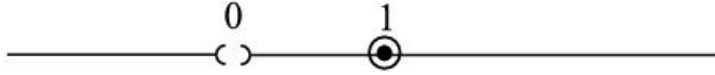
Figure 5: The algebraic pair (X_0, D_0)

Repeating the calculations done for the general case gives

$$
\mathrm{H}^1_{\mathrm{dR}}(X_0, D_0/\mathbb{Q}) = \mathbb{Q}\,\frac{dt}{t} \oplus \mathbb{Q}\,(1_{D_1} + 0_{D_2}),
$$

where c_{D_i} is the constant function equal to c on the irreducible component D_i of D, $i = 1, 2$. Computing the smooth singular cohomology group $\mathrm{H}^1_{\infty}(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{Q})$ (cf. Definition 4.11.3) in a similar fashion gives

$$
\mathrm{H}^1_{\infty}(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{Q}) = \mathbb{Q}\,\sigma \oplus \mathbb{Q}\,(D_1 - D_2),
$$

and we find that the period matrix of (X_0, D_0) is precisely the limit of (39) for $\alpha \to 1$

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$$\begin{array}{c|cc} & 1_{D_1} + 0_{D_2} & \frac{dt}{t} \\ \hline D_1 - D_2 & 1 & 0 \\ \sigma & 0 & 2\pi i \end{array}$$

Third part: Finally, we let

$$D_0 := \{1, \alpha, \beta\} \quad \text{with} \quad \alpha \neq 0, 1 \quad \text{and} \quad \beta \neq 0, 1, \alpha,$$

but keep $X_0 := \mathbb{A}^1_{\mathbb{Q}} \setminus \{0\} = \operatorname{Spec} \mathbb{Q}[t, t^{-1}]$.
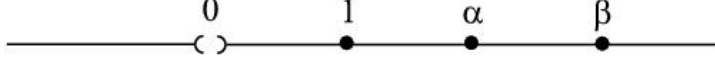

Figure 6: The algebraic pair (X_0, D_0)

Then $\mathrm{H}^{\mathrm{sing}}_1(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{Q})$ is generated by the loop σ from the first part and the intervals $[1, \alpha]$ and $[\alpha, \beta]$. Hence the differential forms $\frac{dt}{t}$, dt and $2t\,dt$ give a basis of $\mathrm{H}^1_{\mathrm{dR}}(X_0, D_0; \mathbb{Q})$: If they were linearly dependent, the period matrix P would not be of full rank

$$\begin{array}{c|ccc} & \frac{dt}{t} & dt & 2t\,dt \\ \hline \sigma & 2\pi i & 0 & 0 \\ [1, \alpha] & \ln \alpha & \alpha - 1 & \alpha^2 - 1 \\ [\alpha, \beta] & \ln\left(\frac{\beta}{\alpha}\right) & \beta - \alpha & \beta^2 - \alpha^2. \end{array}$$

Observe that $\det P = 2\pi i(\alpha - 1)(\beta - \alpha)(\beta - 1) \neq 0$.

We can compute the triple coproduct of $\ln \alpha$ again. We have

$$P^{-1} = \begin{pmatrix} \frac{1}{2\pi i} & 0 & 0 \\ \frac{\ln \beta(\alpha^2-1) - \ln \alpha(\beta^2-1)}{2\pi i(\beta-\alpha)(\alpha-1)(\beta-1)} & \frac{\alpha+\beta}{(\alpha-1)(\beta-1)} & \frac{\alpha+1}{(\alpha-\beta)(\beta-1)} \\ \frac{-\ln \beta(\alpha-1) + \ln \alpha(\beta-1)}{2\pi i(\beta-\alpha)(\alpha-1)(\beta-1)} & \frac{-1}{(\alpha-1)(\beta-1)} & \frac{-1}{(\alpha-\beta)(\beta-1)} \end{pmatrix},$$

and therefore get for the triple coproduct

$$
\begin{aligned}
\Delta(\ln \alpha) = {}& \ln \alpha \otimes \frac{1}{2\pi i} \otimes 2\pi i \\
& + (\alpha - 1) \otimes \frac{-\ln \beta(\alpha^2 - 1) + \ln \alpha(\beta^2 - 1)}{2\pi i(\beta - \alpha)(\alpha - 1)(\beta - 1)} \otimes 2\pi i \\
& + (\alpha - 1) \otimes \frac{\alpha + \beta}{(\alpha - 1)(\beta - 1)} \otimes \ln \alpha \\
& + (\alpha - 1) \otimes \frac{\alpha + 1}{(\alpha - \beta)(\beta - 1)} \otimes \ln\left(\frac{\beta}{\alpha}\right) \\
& + (\alpha^2 - 1) \otimes \frac{\ln \beta(\alpha - 1) - \ln \alpha(\beta - 1)}{2\pi i(\beta - \alpha)(\alpha - 1)(\beta - 1)} \otimes 2\pi i \\
& + (\alpha^2 - 1) \otimes \frac{-1}{(\alpha - 1)(\beta - 1)} \otimes \ln \alpha \\
& + (\alpha^2 - 1) \otimes \frac{-1}{(\alpha - \beta)(\beta - 1)} \otimes \ln\left(\frac{\beta}{\alpha}\right) \\
= {}& \ln \alpha \otimes \frac{1}{2\pi i} \otimes 2\pi i - 1 \otimes \frac{\ln \alpha}{2\pi i} \otimes 2\pi i + 1 \otimes 1 \otimes \ln \alpha,
\end{aligned}
$$

i.e. we obtain the same result as in (40).

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8.2   Second Example: Quadratic Forms

Let

$$Q(\underline{x}): \quad \begin{array}{ccc} \mathbb{Q}^3 & \longrightarrow & \mathbb{Q} \\ \underline{x} = (x_0, x_1, x_2) & \mapsto & \underline{x}\, A\, \underline{x}^T \end{array}$$

be a quadratic form with $A \in \mathbb{Q}^{3 \times 3}$ being a regular, symmetric matrix.
The zero-locus of $Q(\underline{x})$

$$\overline{X}_0 := \{\underline{x} \in \mathbb{Q}P^2 \,|\, Q(\underline{x}) = 0\}$$

is a *quadric* or non-degenerate *conic*. We are interested in its affine piece

$$X_0 := \overline{X}_0 \cap \{x_0 \neq 0\} \subset \mathbb{Q}^2 \subset \mathbb{Q}P^2.$$

We show that we can assume $Q(\underline{x})$ to be of a particular nice form. A vector $v \in \mathbb{Q}^3$ is called Q-*anisotropic*, if $Q(v) \neq 0$. Since char $\mathbb{Q} \neq 2$, there exist such vectors; just suppose the contrary:

$$\begin{array}{llll} Q(1,0,0) = 0 & \text{gives} & A_{11} = 0, \\ Q(0,1,0) = 0 & \text{gives} & A_{22} = 0, \\ Q(1,1,0) = 0 & \text{gives} & 2 \cdot A_{12} = 0 \end{array}$$

and A would be degenerate. In particular

$$Q(1, \lambda, 0) = Q(1,0,0) + 2\lambda Q(1,1,0) + \lambda^2 Q(0,1,0)$$

will be different form zero for almost all $\lambda \in \mathbb{Q}$. Hence, we can assume that $(1,0,0)$ is anisotropic after applying a coordinate transformation of the form

$$x'_0 := x_0, \quad x'_1 := -\lambda x_0 + x_1, \quad x'_2 := x_2.$$

After another affine change of coordinates, we can also assume that A is a diagonal matrix by [Se, Ch. IV, § 1, 2, Thm. 1, p. 30]. An inspection of the proof of this result reveals that we can choose this coordinate transformation such that the x_0-coordinate is left unaltered. (Just take for e_1 the anisotropic vector $(1,0,0)$ in the proof.) Such a transformation does not change the isomorphy-type of X_0, and we can take X_0 to be cut out by an equation of the form

$$ax^2 + by^2 = 1 \quad \text{for} \quad a, b \in \mathbb{Q}^\times$$

with affine coordinates $x := \frac{x_1}{x_0}$ and $y := \frac{x_2}{x_0}$.

Since X_0 is affine, and hence the sheaves $\Omega^p_{X_0/\mathbb{Q}}$ are $\Gamma(X_0; ?)$-acyclic by [Ha, Thm. III.3.5, p. 215], we can compute its *algebraic deRham cohomology* (cf. Definition 3.2.3) by

$$\mathrm{H}^\bullet_{\mathrm{dR}}(X_0/\mathbb{Q}) = \mathrm{h}^\bullet \Gamma(X_0; \Omega^\bullet_{X_0/\mathbb{Q}});$$

so we write down the complex $\Gamma(X_0; \Omega^\bullet_{X_0/\mathbb{Q}})$ in detail

$$0$$
$$\uparrow$$
$$\Gamma(X_0; \Omega^1_{X_0/\mathbb{Q}}) = \mathbb{Q}[x,y]/(ax^2 + by^2 - 1)\{dx, dy\} \,/\, (ax\,dx + by\,dy)$$
$$d \uparrow$$
$$\Gamma(X_0; \mathcal{O}_{X_0}) = \mathbb{Q}[x,y]/(ax^2 + by^2 - 1).$$

Obviously, $\mathrm{H}^1_{\mathrm{dR}}(X_0/\mathbb{Q})$ can be presented with generators $x^n y^m dx$ and $x^n y^m dy$ for $m, n \in \mathbb{N}_0$ modulo numerous relations. We easily get

- $y^m\,dy = d\,\frac{y^{m+1}}{m+1} \sim 0$

- $x^n\,dx = d\,\frac{x^{n+1}}{n+1} \sim 0$

$n \geq 1$
- $x^n y^m\,dy = \frac{-n}{m+1} x^{n-1} y^{m+1}\,dx + d\,\frac{x^n y^{m+1}}{m+1}$
$$\sim \frac{-n}{m+1} x^{n-1} y^{m+1}\,dx \quad \text{for} \quad n \geq 1$$

- $x^n y^{2m}\,dx = x^n \left(\frac{1-ax^2}{b}\right)^m dx$

- $x^n y^{2m+1}\,dx = x^n \left(\frac{1-ax^2}{b}\right)^m y\,dx$

- $xy\,dx = \frac{-x^2}{2}\,dy + d\,\frac{x^2 y}{2}$
$$\sim \frac{by^2-1}{2a}\,dy$$
$$= \frac{b}{2a} y^2\,dy - \frac{1}{2a}\,dy \sim 0$$

$n \geq 2$
- $x^n y\,dx = \frac{-b}{a} x^{n-1} y^2\,dy + x^n y\,dx + \frac{b}{a} x^{n-1} y^2\,dy$
$$= \frac{-b}{a} x^{n-1} y^2\,dy + \frac{x^{n-1}y}{2a}\,d(ax^2 + by^2 - 1)$$
$$= \frac{-b}{a} x^{n-1} y^2\,dy + d\left(\frac{(x^{n-1}y)(ax^2+by^2-1)}{2a}\right)$$
$$\sim \frac{-b}{a} x^{n-1} y^2\,dy$$
$$= \left(x^{n+1} - \frac{x^{n-1}}{a}\right) dy$$
$$= \left(-(n+1)x^n y + \frac{n-1}{a} x^{n-2} y\right) dx + d\left(x^{n+1}y - \frac{x^{n-1}}{a}y\right)$$

$\Rightarrow \ x^n y\,dx \sim \frac{n-1}{(n+2)a} x^{n-2} y\,dx \quad \text{for} \quad n \geq 2.$

Thus we see that all generators are linearly dependent of $y\,dx$

$$\mathrm{H}^1_{\mathrm{dR}}(X_0/\mathbb{Q}) = \mathrm{h}^1\Gamma(X_0; \Omega^\bullet_{X_0/\mathbb{Q}})$$
$$= \mathbb{Q}\ y\,dx.$$

What about X, the base change to $\mathbb{C}$ of X_0? We use the symbol $\sqrt{}$ for the principal branch of the square root. Over $\mathbb{C}$, the change of coordinates

$$u := \sqrt{a}x - i\sqrt{b}y, \quad v := \sqrt{a}x + i\sqrt{b}y$$

gives

$$X = \mathrm{Spec}\,\mathbb{C}[x,y]/(ax^2 + by^2 - 1)$$
$$= \mathrm{Spec}\,\mathbb{C}[u,v]/(uv - 1)$$
$$= \mathrm{Spec}\,\mathbb{C}[u, u^{-1}]$$
$$= \mathbb{A}^1_{\mathbb{C}} \setminus \{0\}.$$

Hence the *first singular homology group* $\mathrm{H}^{\mathrm{sing}}_\bullet(X^{\mathrm{an}}; \mathbb{Q})$ of X^{an} is generated by

$$\sigma : [0,1] \to X^{\mathrm{an}}, s \mapsto u = e^{2\pi i s},$$

i.e. a circle with radius 1 turning counter-clockwise around $u = 0$ once.

The *period matrix* (cf. Definition 5.1.1) consists of a single entry

$$\int_\sigma y\,dx = \int_\sigma \frac{v-u}{2i\sqrt{b}}\,d\,\frac{u+v}{2\sqrt{a}}$$
$$\overset{\text{Stokes}}{=} \int_\sigma \frac{v\,du - u\,dv}{4i\sqrt{ab}}$$
$$= \frac{1}{2i\sqrt{ab}}\int_\sigma \frac{du}{u}$$
$$= \frac{\pi}{\sqrt{ab}}$$
$$= \frac{\pi}{\sqrt{\text{discriminant}}}.$$

The denominator squared is nothing but the *discriminant* of the quadratic form Q (cf. [Se, Ch. IV, §1, 1, p. 27])

$$\text{disc}\,Q := \det A \in \mathbb{Q}^\times/\mathbb{Q}^{\times 2}.$$

This is an important invariant, that distinguishes some, but not all isomorphy classes of quadratic forms. Since $\text{disc}\,Q$ is well-defined modulo $(\mathbb{Q}^\times)^2$, it makes sense to write

$$\mathrm{H}^1_{\mathrm{dR}}(X_0/\mathbb{Q}) = \mathbb{Q}\,\frac{\pi}{\sqrt{\text{disc}\,Q}} \subset \mathrm{H}^1_{\mathrm{sing}}(X^{\mathrm{an}};\mathbb{Q})\otimes_\mathbb{Q} \mathbb{C}.$$

8.3 Third Example: Elliptic Curves

In this example, we give a brief summary of Chapter VI in Silverman's classical textbook [Sil] on elliptic curves. His Chapter VI deals with elliptic curves over $\mathbb{C}$, but some results generalize easily to the case of elliptic curves over $\mathbb{Q}$.
An *elliptic curve* E_0 is an one-dimensional, nonsingular, complete scheme defined over a field k, which has genus 1, where the *geometric genus* p_g is defined as

$$p_g(E_0) := \dim_k \Gamma(E_0; \Omega^1_{E_0/k}).$$

For simplicity, we assume $k = \mathbb{Q}$. It can be shown, using the Riemann-Roch theorem, that such an elliptic curve E_0 can be given as the zero locus in $\mathbb{Q}P^2$ of a *Weierstraß-equation* (cf. [Sil, Ch. III, §3, p. 63])

$$Y^2 Z = 4X^3 - 60G_4 X Z^2 - 140 G_6 Z^3 \tag{41}$$

with coefficients $G_4, G_6 \in \mathbb{Q}$ and projective coordinates X, Y and Z.
The base change $E := E_0 \times_\mathbb{Q} \mathbb{C}$ of E_0 gives us a complex torus E^{an} (see [Sil, Ch. VI, §5, Thm. 5.1, p. 161]), i.e. an isomorphism

$$E^{\mathrm{an}} \cong \mathbb{C}/\Lambda_{\omega_1,\omega_2} \tag{42}$$

in the complex analytic category (cf. Subsection 2.1), with

$$\Lambda_{\omega_1,\omega_2} := \omega_1 \mathbb{Z} \oplus \omega_2 \mathbb{Z}$$
$$\text{for } \omega_1, \omega_2 \in \mathbb{C} \text{ linearly independent over } \mathbb{R},$$
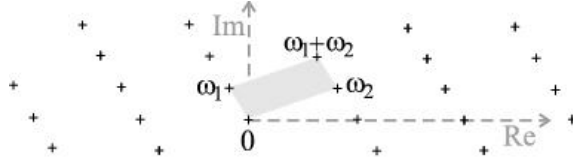
being a lattice of full rank (cf. Figure 7).

Figure 7: The lattice $\Lambda_{\omega_1, \omega_2} \subset \mathbb{C}$

Thus all elliptic curves over $\mathbb{C}$ are diffeomorphic to the standard torus $S^1 \times S^1$ (cf. Figure 8), but carry different complex structures as the parameter $\tau := \omega_2/\omega_1$ varies. We can describe the isomorphism (42) quite explicitly using periods. Let α and β be a basis of

$$\mathrm{H}_1^{\mathrm{sing}}(E^{\mathrm{an}}; \mathbb{Z}) = \mathrm{H}_1^{\mathrm{sing}}(S^1 \times S^1; \mathbb{Z}) = \mathbb{Z}\,\alpha \oplus \mathbb{Z}\,\beta.$$

The $\mathbb{Q}$-vector space $\Gamma(E_0; \Omega^1_{E_0/\mathbb{Q}})$ is spanned by the so-called *invariant differential* [Sil, p. 48]

$$\omega = \frac{d(X/Z)}{Y/Z}.$$

One can show (cf. [Sil, Ch. VI, §5, Prop. 5.2, p. 161]), that the (abstract) periods (cf. Definition 5.2.1)

$$\omega_1 := \int_\alpha \omega \quad \text{and} \quad \omega_2 := \int_\beta \omega$$

are $\mathbb{R}$-linearly independent and that the map

$$
\begin{aligned}
E^{\mathrm{an}} &\to \quad \mathbb{C}/\Lambda_{\omega_1, \omega_2} \\
P &\mapsto \int_O^P \omega \ \text{ modulo } \Lambda_{\omega_1, \omega_2}
\end{aligned}
\tag{43}
$$

is an isomorphism (here $O = [0 : 1 : 0]$ denotes a base-point on E).

Note that under the natural projection $\pi : \mathbb{C} \to \mathbb{C}/\Gamma_{\omega_1, \omega_2}$ any meromorphic function f on the torus $\mathbb{C}/\Gamma_{\omega_1, \omega_2}$ lifts to a doubly-periodic function $\pi^* f$ on the complex plane $\mathbb{C}$ with periods ω_1 and ω_2

$$f(x + n\omega_1 + m\omega_2) = f(x) \quad \text{for all} \quad n, m \in \mathbb{Z} \quad \text{and} \quad x \in \mathbb{C}.$$

This example is possibly the origin of the term "period" for the elements of the sets $\mathbb{P}_p$, $\mathbb{P}_a$, and $\mathbb{P}_n$.

The inverse map $\mathbb{C}/\Lambda_{\omega_1, \omega_2} \to E^{\mathrm{an}}$ for the isomorphism (43) can be described in terms of the *Weierstraß-$\wp$-function* of the lattice $\Lambda := \Lambda_{\omega_1, \omega_2}$

$$\wp(z) = \wp(z, \Lambda) := \frac{1}{z^2} + \sum_{\substack{\omega \in \Lambda \\ \omega \neq 0}} \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2}$$
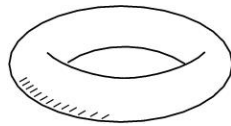

Figure 8: The standard torus $S^1 \times S^1$

and takes the form (cf. [Sil, Prop. 3.6, p. 158])

$$\mathbb{C}/\Lambda_{\omega_1,\omega_2} \to E^{\mathrm{an}} \subset \mathbb{C}P^2_{\mathrm{an}}$$
$$z \mapsto [\wp(z) : \wp'(z) : 1].$$

The defining coefficients G_4, G_6 of E_0 can be recovered from $\Lambda_{\omega_1,\omega_2}$ by computing *Eisenstein series* (cf. [Sil, Ch. VI, §3, Thm. 3.5, p. 157])

$$G_{2k} := \sum_{\substack{\omega \in \Lambda \\ \omega \neq 0}} \omega^{-2k} \quad \text{for} \quad k = 2, 3.$$

Thus the periods ω_1 and ω_2 determine the elliptic curve E_0 uniquely. However, they are not invariants of E_0, since they depend on the chosen Weierstraß-equation (41) of E_0. It is proved in [Sil, p. 50], that a change of coordinates which preserves the shape of (41), must be of the form

$$X' = u^2 X, \quad Y' = u^3 Y, \quad Z' = Z \quad \text{for} \quad u \in \mathbb{Q}^\times.$$

In the new parameterization X', Y', Z', we have

$$G'_4 = u^4 G_4, \quad G'_6 = u^6 G_6,$$
$$\omega' = u^{-1}\omega$$
$$\omega'_1 = u^{-1}\omega_1 \quad \text{and} \quad \omega'_2 = u^{-1}\omega_2.$$

Hence $\tau = \omega_2/\omega_1$ is a true invariant of E_0, which allows us to distinguish non-isomorphic elliptic curves E over $\mathbb{C}$. If we consider both τ and the image of ω_1 in $\mathbb{C}^\times/\mathbb{Q}^\times$, we can even distinguish elliptic curves over $\mathbb{Q}$.

We may ask whether the isomorphy-type of E_0 is already encoded in its cohomology. The answer is yes if we consider also the pure Hodge structure living on cohomology. By the Hodge theorem [GH, p. 116], the (hypercohomology) spectral sequence

$$\mathrm{H}^p(E^{\mathrm{an}}; \Omega^q_{E^{\mathrm{an}}}) \Rightarrow \mathrm{H}^n_{\mathrm{dR}}(E^{\mathrm{an}}; \mathbb{C})$$

splits, yielding a decomposition of the first classical deRham cohomology group (cf. Subsection 3.1)

$$\mathrm{H}^1_{\mathrm{dR}}(E^{\mathrm{an}}; \mathbb{C}) = \Gamma(E^{\mathrm{an}}; \Omega^1_{E^{\mathrm{an}}}) \oplus \mathrm{H}^1(E^{\mathrm{an}}; \mathcal{O}_{E^{\mathrm{an}}}).$$

This Hodge decomposition already lives on the algebraic deRham cohomology group $\mathrm{H}^1_{\mathrm{dR}}(E_0/\mathbb{Q})$ (cf. Definition 3.2.3)

$$\mathrm{H}^1_{\mathrm{dR}}(E_0/\mathbb{Q}) = \Gamma(E_0; \Omega^1_{E_0/\mathbb{Q}}) \oplus \mathrm{H}^1(E_0; \mathcal{O}_{E_0}). \tag{44}$$

In order to see this, observe that the morphism of spectral sequences

$$\mathrm{H}^p(E_0; \Omega^q_{E_0/\mathbb{Q}}) \otimes_{\mathbb{Q}} \mathbb{C} \Rightarrow \mathrm{H}^n_{\mathrm{dR}}(E_0; \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$$
$$\downarrow \qquad\qquad\qquad \downarrow$$
$$\mathrm{H}^p(E^{\mathrm{an}}; \Omega^q_{E^{\mathrm{an}}}) \Rightarrow \mathrm{H}^n_{\mathrm{dR}}(E^{\mathrm{an}}; \mathbb{C})$$

is indeed an isomorphism by the Base Change Proposition 3.5.1, the GAGA-theorem 2.2.3 and the Comparison Theorem 4.10.1. Hence the spectral sequence

$$\mathrm{H}^p(E_0; \Omega^q_{E_0/\mathbb{Q}}) \Rightarrow \mathrm{H}^n_{\mathrm{dR}}(E_0; \mathbb{Q})$$

splits as well and we obtain the decomposition (44).

Pairing $\mathrm{H}_1^{\mathrm{sing}}(E^{\mathrm{an}}; \mathbb{Z}) = \mathbb{Z}^2$ with some $\omega' \in \Gamma(E^{\mathrm{an}}; \Omega^1_{E^{\mathrm{an}}}) = \mathbb{Q}\,\omega$ gives us $\Lambda_{u\cdot\omega_1, u\cdot\omega_2} \subset \mathbb{C}$, where we write $\omega' = u \cdot \omega$ with $u \in \mathbb{Q}^\times$. That is, we can recover $\Lambda_{\omega_1, \omega_2}$ up to a rational multiple from the integral singular homology $\mathrm{H}_1^{\mathrm{sing}}(E^{\mathrm{an}}; \mathbb{Z})$ of E^{an} plus the algebraic deRham cohomology of E_0 equipped with its Hodge decomposition (44), i.e. we can recover the isomorphy-type of E_0 from this data.

8.4 Fourth Example: A ζ-value

Earlier we raised the question of how to write $\zeta(2)$ as a period and found the identity (cf. Proposition 5.3.5)

$$\zeta(2) = \int_{0 \le x \le y \le 1} \frac{dx \wedge dy}{(1-x)\,y}.$$

The problem was that this identity did not give us a valid representation of $\zeta(2)$ as a naïve period (cf. Definition 5.3.2), since the pole locus of the integrand and the domain of integration are not disjoint. We show how to circumvent this difficulty. First we define

$$Y_0 := \mathbb{A}_{\mathbb{Q}}^2 \quad \text{with coordinates } x \text{ and } y,$$
$$Z_0 := \{x = 1\} \cup \{y = 0\},$$
$$X_0 := Y_0 \setminus Z_0,$$
$$D_0 := (\{x = 0\} \cup \{y = 1\} \cup \{x = y\}) \setminus Z_0,$$
$$\triangle := \{(x,y) \in Y^{\mathrm{an}} \mid x, y \in \mathbb{R}, \, 0 \le x \le y \le 1\} \quad \text{a triangle in } Y^{\mathrm{an}}, \quad \text{and}$$
$$\omega_0 := \frac{dx \wedge dy}{(1-x)\,y},$$

thus getting

$$\zeta(2) = \int_{\triangle} \omega,$$

with $\omega_0 \in \Gamma(X_0, \Omega^2_{X_0/\mathbb{Q}})$ and $\partial\triangle \subset D^{\mathrm{an}} \cup \{(0,0), (1,1)\}$, see Figure 9.
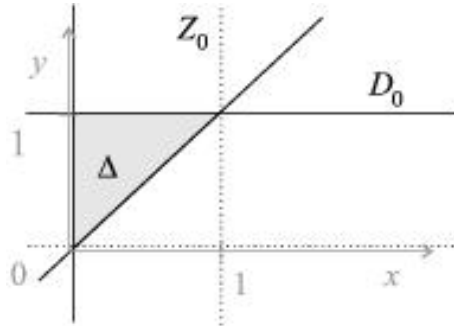

Figure 9: The configuration $Z_0, D_0, \triangle$

Now we blow up Y_0 in the points $(0,0)$ and $(1,1)$ obtaining $\pi_0 : \widetilde{Y}_0 \to Y_0$. We denote the strict transform of Z_0 by $\widetilde{Z}_0$, $\pi_0^* \omega_0$ by $\widetilde{\omega}_0$ and $\widetilde{Y}_0 \setminus \widetilde{Z}_0$ by $\widetilde{X}_0$. The "strict transform" $\overline{\pi_{\mathrm{an}}^{-1}(\triangle \setminus \{(0,0), (1,1)\})}$ will be called $\widetilde{\triangle}$ and (being $\widetilde{\mathbb{Q}}$-semi-algebraic hence triangulable — cf. Theorem 6.2.3) gives rise to a singular chain

$$\widetilde{\gamma} \in \mathrm{H}_2^{\mathrm{sing}}(\widetilde{X}^{\mathrm{an}}, \widetilde{D}^{\mathrm{an}}; \mathbb{Q}).$$

Since π is an isomorphism away from the exceptional locus, this exhibits $\zeta(2)$ as an (abstract) period (cf. Definition 5.2.1)

$$\zeta(2) = \int_{\triangle} \omega = \int_{\widetilde{\gamma}} \widetilde{\omega} \in \mathbb{P}_a = \mathbb{P},$$
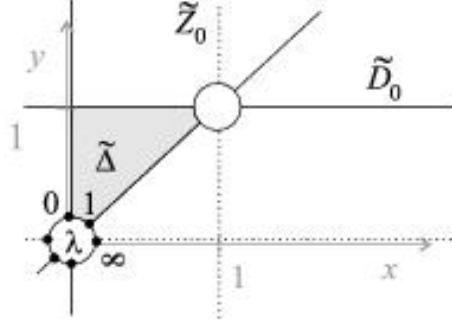
see Figure 10.

Figure 10: The configuration $\widetilde{Z}_0, \widetilde{D}_0, \widetilde{\triangle}$

We will conclude this example by writing out $\widetilde{\omega}_0$ and $\widetilde{\triangle}$ more explicitely. Note that $\widetilde{Y}_0$ can be described as the subvariety of

$$\mathbb{A}^2_{\mathbb{Q}} \times \mathbb{Q}P^1 \times \mathbb{Q}P^1 \quad \text{with coordinates} \quad (\widetilde{x}, \widetilde{y}; \lambda_0 : \lambda_1; \mu_0 : \mu_1)$$

cut out by

$$\widetilde{x}\lambda_0 = \widetilde{y}\lambda_1 \quad \text{and} \quad (\widetilde{x} - 1)\mu_0 = (\widetilde{y} - 1)\mu_1.$$

With this choice of coordinates π_0 takes the form

$$\pi_0 : \quad \begin{array}{ccc} \widetilde{Y}_0 & \to & Y_0 \\ (\widetilde{x}, \widetilde{y}; \lambda_0 : \lambda_1; \mu_0 : \mu_1) & \mapsto & (\widetilde{x}, \widetilde{y}) \end{array}$$

and we have $\widetilde{X}_0 := \widetilde{Y}_0 \setminus (\{\lambda_0 = 0\} \cup \{\mu_1 = 0\})$. We can embed $\widetilde{X}_0$ into affine space

$$\widetilde{X}_0 \to \mathbb{A}^4_{\mathbb{Q}}$$

$$(\widetilde{x}, \widetilde{y}; \lambda_0 : \lambda_1; \mu_0 : \mu_1) \mapsto (\widetilde{x}, \widetilde{y}, \frac{\lambda_1}{\lambda_0}, \frac{\mu_0}{\mu_1})$$

and so have affine coordinates $\widetilde{x}$, $\widetilde{y}$, $\lambda := \frac{\lambda_1}{\lambda_0}$ and $\mu := \frac{\mu_0}{\mu_1}$ on $\widetilde{X}_0$.

Now, on $\widetilde{X}_0 \setminus \pi_0^{-1}(1, 1)$, the form $\widetilde{\omega}_0$ is given by

$$\widetilde{\omega}_0 = \frac{d\widetilde{x} \wedge d\widetilde{y}}{(1 - \widetilde{x})\,\widetilde{y}} = \frac{d(\lambda\widetilde{y}) \wedge d\widetilde{y}}{(1 - \widetilde{x})\,\widetilde{y}} = \frac{d\lambda \wedge d\widetilde{y}}{1 - \widetilde{x}},$$

while on $\widetilde{X}_0 \setminus \pi_0^{-1}(0, 0)$ we have

$$\widetilde{\omega}_0 = \frac{d\widetilde{x} \wedge d\widetilde{y}}{(1 - \widetilde{x})\,\widetilde{y}} = \frac{d\widetilde{x} \wedge d(\widetilde{y} - 1)}{(1 - \widetilde{x})\,\widetilde{y}} = \frac{d\widetilde{x} \wedge d(\mu(\widetilde{x} - 1))}{(1 - \widetilde{x})\,\widetilde{y}} = \frac{-d\widetilde{x} \wedge d\mu}{\widetilde{y}}.$$

The region $\widetilde{\triangle}$ is given by

$$\widetilde{\triangle} = \{(\widetilde{x}, \widetilde{y}, \lambda, \mu) \in \widetilde{X}^{\mathrm{an}} \mid \widetilde{x}, \widetilde{y}, \lambda, \mu \in \mathbb{R}, \ 0 \le \widetilde{x} \le \widetilde{y} \le 1, \ 0 \le \lambda \le 1, \ 0 \le \mu \le 1\}.$$

8.5 Fifth Example: The Double Logarithm Variation of Mixed Hodge Structures

The previous example fits into a more general framework.

8.5.1 Multiple Polylogarithms

Define the *hyperlogarithm* as the iterated integral

$$\mathrm{I}_n(a_1,\ldots,a_n) := \int_{0 \le t_1 \le \cdots \le t_n \le 1} \frac{dt_1}{t_1 - a_1} \wedge \cdots \wedge \frac{dt_n}{t_n - a_n}$$

with $a_1,\ldots,a_n \in \mathbb{C}$ (cf. [Z1, p. 168]). These integrals specialize to the *multiple polylogarithm* (cf. [*loc. cit.*][1])

$$\mathrm{Li}_{m_1,\ldots,m_n}\left(\frac{a_2}{a_1}, \cdots, \frac{a_n}{a_{n-1}}, \frac{1}{a_n}\right) := (-1)^n \, \mathrm{I}_{\sum m_n}(a_1, \underbrace{0,\ldots,0}_{m_1-1}, \ldots, a_n, \underbrace{0,\ldots,0}_{m_n-1}),$$

which is convergent if $1 < |a_1| < \cdots < |a_n|$ (cf. [G3, 2.3, p. 9]). Alternatively, we can describe the multiple polylogarithm as a power series (cf. [G3, Thm. 2.2, p. 9])

$$\mathrm{Li}_{m_1,\ldots,m_n}(x_1,\ldots,x_n) = \sum_{0 < k_1 < \cdots < k_n} \frac{x_1^{k_1} \cdots x_n^{k_n}}{k_1^{m_1} \cdots k_n^{m_n}} \quad \text{for} \quad |x_i| < 1. \tag{45}$$

Of special interest to us will be the *dilogarithm* $\mathrm{Li}_2(x) = \sum_{k>0} \frac{x^k}{k^2}$ and the *double logarithm* $\mathrm{Li}_{1,1}(x,y) = \sum_{0<k<l} \frac{x^k y^l}{kl}$.

At first, the functions $\mathrm{Li}_{m_1,\ldots,m_n}(x_1,\ldots,x_n)$ only make sense for $|x_i| < 1$, but they can be analytically continued to multivalued meromorphic functions on $\mathbb{C}^n$ (cf. [Z1, p. 2]), for example $\mathrm{Li}_1(x) = -\ln(1-x)$.

General Remarks. Polylogarithms have applications in Zagier's conjecture (cf. [Z1, p. 2]) and are used in the study of certain motivic fundamental groups (cf. [HW, p. 9] and [G2]).

Besides, equation (45) would suggest a connection between values of multiple polylogarithms and multiple ζ-values (cf. [G1, p. 617])

$$\zeta(m_1,\ldots,m_n) := \sum_{0 < k_1 < \ldots < k_m} \frac{1}{k_1^{m_1} \cdots k_n^{m_n}}.$$

Indeed, we have formally

$$-\zeta(2) = \frac{-\pi^2}{6} = -\mathrm{Li}_2(1) = \lim_{b \to 0} \lim_{a \to 1} \mathrm{I}_2(a,b) = \lim_{b \to 0} \lim_{a \to 1} \mathrm{Li}_{1,1}\left(\frac{b}{a}, \frac{1}{b}\right).$$

However, such formal computations are problematic, since a limit may diverge or at least depend on the direction in which the limit is taken, because multiple polylogarithms are multivalued. To deal with this problem, Zagier, Goncharov and others proposed regularization procedures, for which we have to refer to [G3, 2.9–2.10, p. 18–24].

---

[1]Note that the factor $(-1)^n$ is missing in equation (1.2) of [Z1, p. 168].

After these general remarks, we want to study a variation of mixed $\mathbb{Q}$-Hodge structures: the *double logarithm variation*, for which multiple polylogarithms appear as coefficients. For that we will combine results in [G1, p. 620f], [Kj, 4, p. 32f], [Z1], [Z2a] and [Z2b].[2] By computing limit mixed $\mathbb{Q}$-Hodge structures, we will obtain $-\zeta(2)$ as a "period" of a mixed $\mathbb{Q}$-Hodge structure.

8.5.2 The Configuration

Let us consider the configuration

$$\begin{aligned}
Y &:= \mathbb{A}^2_{\mathbb{C}} \quad \text{with coordinates } x \text{ and } y, \\
Z &:= \{x = a\} \cup \{y = b\} \quad \text{with} \quad a \neq 0, 1 \quad \text{and} \quad b \neq 0, 1 \\
X &:= Y \setminus Z \\
D &:= (\{x = 0\} \cup \{y = 1\} \cup \{x = y\}) \setminus Z,
\end{aligned}$$

see Figure 11.
We denote the irreducible components of the divisor D as follows:

$$\begin{aligned}
D_1 &:= \{x = 0\} \setminus \{(0, b)\}, \\
D_2 &:= \{y = 1\} \setminus \{(a, 1)\}, \quad \text{and} \\
D_3 &:= \{x = y\} \setminus \{(a, a), (b, b)\}.
\end{aligned}$$

By projecting from Y^{an} onto the y- or x-axis, we get isomorphisms for the associated complex analytic spaces (cf. Subsection 2.1)

$$D_1^{\mathrm{an}} \cong \mathbb{C} \setminus \{b\}, \quad D_2^{\mathrm{an}} \cong \mathbb{C} \setminus \{a\}, \quad \text{and} \quad D_3^{\mathrm{an}} \cong \mathbb{C} \setminus \{a, b\}.$$
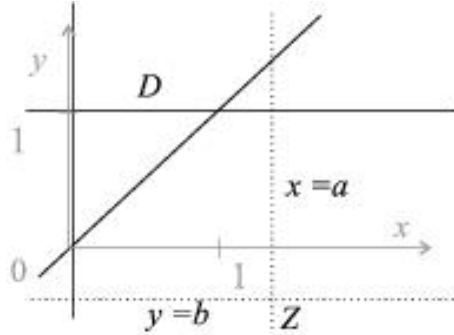
Figure 11: The algebraic pair (X, D)

8.5.3 Singular Homology

We can easily give generators for the second singular homology of the pair $(X^{\mathrm{an}}, D^{\mathrm{an}})$, see Figure 12.

- Let $\alpha : [0, 1] \to \mathbb{C}$ be a smooth path, which does not meet a or b. We define a "triangle"

$$\triangle := \{(\alpha(s), \alpha(t)) \mid 0 \leq s \leq t \leq 1\}.$$

---

[2]Note that [Z2a] plus [Z2b] is just a detailed version of [Z1].

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- Consider the closed curve in $\mathbb{C}$

$$C_b := \left\{ \frac{a}{b + \varepsilon e^{2\pi i s}} \mid s \in [0,1] \right\},$$

which divides $\mathbb{C}$ into two regions: an inner one containing $\frac{a}{b}$ and an outer one. We can choose $\varepsilon > 0$ small enough such that C_b separates $\frac{a}{b}$ from 0 to 1, i.e. such that 0 and 1 are contained in the outer region. This allows us to find a smooth path $\beta : [0,1] \to \mathbb{C}$ from 0 to 1 not meeting C_b. We define a "slanted tube"

$$S_b := \left\{ \left(\beta(t) \cdot (b + \varepsilon e^{2\pi i s}), b + \varepsilon e^{2\pi i s} \right) \mid s, t \in [0,1] \right\}$$

which winds around $\{y = b\}$ and whose boundary components are supported on D_1^{an} (corresponding to $t = 0$) and D_3^{an} (corresponding to $t = 1$). The special choice of β guarantees $S_b \cap Z^{\mathrm{an}} = \varnothing$.

- Similarly, we choose $\varepsilon > 0$ such that the closed curve

$$C_a := \left\{ \frac{b - 1}{a - 1 - \varepsilon e^{2\pi i s}} \mid s \in [0,1] \right\}$$

separates $\frac{b-1}{a-1}$ form 0 and 1. Let $\gamma : [0,1] \to \mathbb{C}$ be a smooth path from 0 to 1 which does not meet C_a. We have a "slanted tube"

$$S_a := \left\{ \left(a + \varepsilon e^{2\pi i s}, 1 + \gamma(t) \cdot (a + \varepsilon e^{2\pi i s} - 1) \right) \mid s, t \in [0,1] \right\}$$

winding around $\{x = a\}$ with boundary supported on D_2^{an} and D_3^{an}.

- Finally, we have a torus

$$T := \{ (a + \varepsilon e^{2\pi i s}, b + \varepsilon e^{2\pi i t}) \mid s, t \in [0,1] \}.$$

The 2-form $ds \wedge dt$ defines an orientation on the unit square $[0,1]^2 = \{(s,t) \mid s,t \in [0,1]\}$. Hence the manifolds with boundary $\triangle$, S_b, S_a, T inherit an orientation; and since they can be triangulated, they give rise to smooth singular chains. By abuse of notation we will also write $\triangle$, S_b, S_a, T for these smooth singular chains. The homology classes of $\triangle$, S_b, S_a and T will be denoted by γ_0, γ_1, γ_2 and γ_3, respectively.
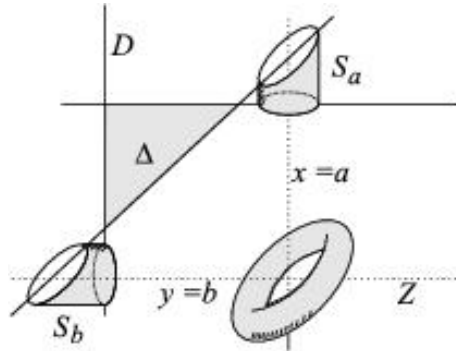

Figure 12: Generators of $\mathrm{H}_2^{\mathrm{sing}}(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{Q})$

An inspection of the long exact sequence in singular homology will reveal that $\gamma_0, \ldots, \gamma_3$ form a system generators (see the following proof)

$$\mathrm{H}_2^{\mathrm{sing}}(D^{\mathrm{an}}; \mathbb{Q}) \longrightarrow \mathrm{H}_2^{\mathrm{sing}}(X^{\mathrm{an}}; \mathbb{Q}) \longrightarrow \mathrm{H}_2^{\mathrm{sing}}(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{Q}) \longrightarrow$$

$$\mathrm{H}_1^{\mathrm{sing}}(D^{\mathrm{an}}; \mathbb{Q}) \xrightarrow{\ i_1\ } \mathrm{H}_1^{\mathrm{sing}}(X^{\mathrm{an}}; \mathbb{Q}).$$

Claim 8.5.1. *With notation as above, we have for the second singular homology of the pair* $(X^{\mathrm{an}}, D^{\mathrm{an}})$

$$\mathrm{H}_2^{\mathrm{sing}}(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{Q}) = \mathbb{Q}\,\gamma_0 \oplus \mathbb{Q}\,\gamma_1 \oplus \mathbb{Q}\,\gamma_2 \oplus \mathbb{Q}\,\gamma_3.$$

Proof. For $c := a$ and $c := b$, the inclusion of the circle $\{c + \varepsilon e^{2\pi i s} \mid s \in [0,1]\}$ into $\mathbb{C}\backslash\{c\}$ is a homotopy equivalence, hence the product map $T \hookrightarrow X^{\mathrm{an}}$ is also a homotopy equivalence. This shows

$$\mathrm{H}_2^{\mathrm{sing}}(X^{\mathrm{an}}; \mathbb{Q}) = \mathbb{Q}\,T,$$

while $\mathrm{H}_1^{\mathrm{sing}}(X^{\mathrm{an}}; \mathbb{Q})$ has rank two with generators

- one loop winding counterclockwise around $\{x = a\}$ once, but not around $\{y = b\}$, thus being homologous to both $\partial S_a \cap D_2^{\mathrm{an}}$ and $-\partial S_a \cap D_3^{\mathrm{an}}$, and

- another loop winding counterclockwise around $\{y = b\}$ once, but not around $\{x = a\}$, thus being homologous to $\partial S_b \cap D_1^{\mathrm{an}}$ and $-\partial S_b \cap D_3^{\mathrm{an}}$.

In order to compute the Betti-numbers b_i of D^{an}, we use the spectral sequence from Proposition 4.4.2 for the closed covering $\{D_i^{\mathrm{an}}\}$

$$E_2^{p,q} := \begin{array}{ccccccc} \cdots & 0 & 0 & & 0 & 0 & \cdots \\ \cdots & 0 & \bigoplus_{i=1}^3 \mathrm{H}_{\mathrm{dR}}^1(D_i^{\mathrm{an}}; \mathbb{C}) & & 0 & 0 & \cdots \\ \cdots & 0 & \ker \delta & & \mathrm{coker}\,\delta & 0 & \cdots \\ \cdots & 0 & 0 & & 0 & 0 & \cdots \end{array} \quad \Rightarrow \quad E_\infty^n := \mathrm{H}_{\mathrm{dR}}^n(D^{\mathrm{an}}; \mathbb{C}),$$

where

$$\delta : \bigoplus_{i=1}^3 \mathrm{H}_{\mathrm{dR}}^0(D_i^{\mathrm{an}}; \mathbb{C}) \longrightarrow \bigoplus_{i<j} \mathrm{H}_{\mathrm{dR}}^0(D_{ij}^{\mathrm{an}}; \mathbb{C}).$$

Note that this spectral sequence degenerates. Since D^{an} is connected, we have $b_0 = 1$, i.e.

$$1 = b_0 = \mathrm{rank}_{\mathbb{C}}\, E_\infty^0 = \mathrm{rank}_{\mathbb{C}}\, E_2^{0,0} = \mathrm{rank}_{\mathbb{C}} \ker \delta.$$

Hence

$$\mathrm{rank}_{\mathbb{C}} \mathrm{coker}\,\delta = \mathrm{rank}_{\mathbb{C}} \mathrm{codomain}\,\delta - \mathrm{rank}_{\mathbb{C}} \mathrm{domain}\,\delta + \mathrm{rank}_{\mathbb{C}} \ker \delta$$
$$= (1 + 1 + 1) - (1 + 1 + 1) + 1 = 1,$$

and so

$$b_1 = \mathrm{rank}_{\mathbb{C}}\, E_\infty^1 = \mathrm{rank}_{\mathbb{C}}\, E_2^{1,0} + \mathrm{rank}_{\mathbb{C}}\, E_2^{0,1}$$

$$= \sum_{i=1}^3 \mathrm{rank}_{\mathbb{C}} \mathrm{H}_{\mathrm{dR}}^1(D_i^{\mathrm{an}}; \mathbb{C}) + \mathrm{rank}_{\mathbb{C}} \mathrm{coker}\,\delta$$

$$= \mathrm{rank}_{\mathbb{C}} \mathrm{H}^1(\mathbb{C} \setminus \{b\}; \mathbb{C}) + \mathrm{rank}_{\mathbb{C}} \mathrm{H}^1(\mathbb{C} \setminus \{a\}; \mathbb{C}) + \mathrm{rank}_{\mathbb{C}} \mathrm{H}^1(\mathbb{C} \setminus \{a, b\}; \mathbb{C}) + 1$$

$$= (1 + 1 + 2) + 1 = 5.$$

We can easily specify generators of $\mathrm{H}_1^{\mathrm{sing}}(D^{\mathrm{an}};\mathbb{Q})$ as follows

$$\mathrm{H}_1^{\mathrm{sing}}(D^{\mathrm{an}};\mathbb{Q}) = \mathbb{Q}\cdot(\partial S_b\cap D_1^{\mathrm{an}})\oplus \mathbb{Q}\cdot(\partial S_a\cap D_2^{\mathrm{an}})\oplus \mathbb{Q}\cdot(\partial S_b\cap D_3^{\mathrm{an}})\oplus \mathbb{Q}\cdot(\partial S_a\cap D_3^{\mathrm{an}})\oplus \mathbb{Q}\cdot\partial\triangle.$$

Clearly $b_2 = \mathrm{rank}_{\mathbb{C}}\,\mathrm{H}_2^{\mathrm{sing}}(D^{\mathrm{an}};\mathbb{Q}) = 0$. Now we can compute $\ker i_1$ and obtain

$$\ker i_1 = \mathbb{Q}\cdot\partial\triangle \oplus \mathbb{Q}\cdot(\partial S_b\cap D_1^{\mathrm{an}} + \partial S_b\cap D_3^{\mathrm{an}})\oplus \mathbb{Q}\cdot(\partial S_a\cap D_2^{\mathrm{an}} + \partial S_a\cap D_3^{\mathrm{an}}).$$

This shows finally

$$\mathrm{rank}_{\mathbb{Q}}\,\mathrm{H}_2^{\mathrm{sing}}(X^{\mathrm{an}}, D^{\mathrm{an}};\mathbb{Q}) = \mathrm{rank}_{\mathbb{Q}}\,\mathrm{H}_2^{\mathrm{sing}}(X^{\mathrm{an}};\mathbb{Q}) + \mathrm{rank}_{\mathbb{Q}}\ker i_1 = 1 + 3 = 4.$$

From these explicit calculations we also derive the linear independence of $\gamma_0 = [\triangle]$, $\gamma_1 = [S_b]$, $\gamma_2 = [S_a]$, $\gamma_3 = [T]$ and Claim 8.5.1 is proved. $\qquad\square$

8.5.4 Smooth Singular Homology

Recall the definition of smooth singular cohomology (cf. Definition 4.11.3). With the various sign conventions made so far (cf. Subsection 4.11.1 and the appendix), the boundary map $\delta : \mathrm{C}_2^\infty(X^{\mathrm{an}}, D^{\mathrm{an}};\mathbb{Q}) \to \mathrm{C}_1^\infty(X^{\mathrm{an}}, D^{\mathrm{an}};\mathbb{Q})$ is given by

$$\delta : \mathrm{C}_2^\infty(X^{\mathrm{an}};\mathbb{Q}) \oplus \bigoplus_{i=1}^{3}\mathrm{C}_1^\infty(D_i^{\mathrm{an}};\mathbb{Q}) \oplus \bigoplus_{i<j}\mathrm{C}_0^\infty(D_{ij}^{\mathrm{an}};\mathbb{Q}) \to \mathrm{C}_1^\infty(X^{\mathrm{an}};\mathbb{Q}) \oplus \bigoplus_{i=1}^{3}(D_i^{\mathrm{an}};\mathbb{Q})$$

$$(\underset{\varnothing}{\sigma}, \underset{1}{\sigma_1}, \underset{2}{\sigma_2}, \underset{3}{\sigma_3}, \underset{12}{\sigma_{12}}, \underset{13}{\sigma_{13}}, \underset{23}{\sigma_{23}}) \mapsto$$
$$(\partial\sigma + \underset{\varnothing}{\sigma_1 + \sigma_2 + \sigma_3}, -\partial\sigma_1 + \underset{1}{\sigma_{12} + \sigma_{13}}, -\partial\sigma_2 - \underset{2}{\sigma_{12} + \sigma_{23}}, -\partial\sigma_3 - \underset{3}{\sigma_{13} - \sigma_{23}}).$$

Thus the following elements of $\mathrm{C}_2^\infty(X^{\mathrm{an}}, D^{\mathrm{an}};\mathbb{Q})$ are cycles

- $\Gamma_0 := (\underset{\varnothing}{\triangle}, \underset{1}{-\partial\triangle \cap D_1^{\mathrm{an}}}, \underset{2}{-\partial\triangle \cap D_2^{\mathrm{an}}}, \underset{3}{-\partial\triangle \cap D_3^{\mathrm{an}}}, \underset{12}{D_{12}^{\mathrm{an}}}, \underset{13}{-D_{13}^{\mathrm{an}}}, \underset{23}{D_{23}^{\mathrm{an}}})$,

- $\Gamma_1 := (\underset{\varnothing}{S_b}, \underset{1}{-\partial S_b \cap D_1^{\mathrm{an}}}, \underset{2}{0}, \underset{3}{-\partial S_b \cap D_3^{\mathrm{an}}}, \underset{12}{0}, \underset{13}{0}, \underset{23}{0})$,

- $\Gamma_2 := (\underset{\varnothing}{S_a}, \underset{1}{0}, \underset{2}{-\partial S_a \cap D_2^{\mathrm{an}}}, \underset{3}{0}, \underset{12}{-\partial S_a \cap D_3^{\mathrm{an}}}, \underset{13}{0}, \underset{23}{0})$ and

- $\Gamma_3 := (\underset{\varnothing}{T}, \underset{1}{0}, \underset{2}{0}, \underset{3}{0}, \underset{12}{0}, \underset{13}{0}, \underset{23}{0})$.

Under the isomorphism $\mathrm{H}_2^\infty(X^{\mathrm{an}}, D^{\mathrm{an}};\mathbb{Q}) \xrightarrow{\sim} \mathrm{H}_2^{\mathrm{sing}}(X^{\mathrm{an}}, D^{\mathrm{an}};\mathbb{Q})$ the classes of these cycles $[\Gamma_0]$, $[\Gamma_1]$, $[\Gamma_2]$, $[\Gamma_3]$ are mapped to γ_0, γ_1, γ_2, γ_3, respectively (cf. Proposition 4.11.5).

8.5.5 Algebraic DeRham Cohomology of $(X^{\mathrm{an}}, D^{\mathrm{an}})$

Recall the definition of the complex $\widetilde{\Omega}_{X,D/\mathbb{C}}^\bullet$ on page 15. We consider

$$\Gamma(X;\widetilde{\Omega}_{X,D/\mathbb{C}}^2) = \Gamma(X;\Omega_{X/\mathbb{C}}^2) \oplus \bigoplus_{i=1}^{3}\Gamma(D_i;\Omega_{D_i/\mathbb{C}}^1) \oplus \bigoplus_{i<j}\Gamma(D_{ij};\mathcal{O}_{D_{ij}})$$

together with the following cycles of $\Gamma(X;\widetilde{\Omega}_{X,D/\mathbb{C}}^2)$

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- $\omega_0 := (\underset{\varnothing}{\tfrac{dx\wedge dy}{(x-a)(y-b)}}, \underset{1}{0}, \underset{2}{0}, \underset{3}{0}, \underset{12}{0}, \underset{13}{0}, \underset{23}{0})$,

- $\omega_1 := (\underset{\varnothing}{0}, \underset{1}{\tfrac{-dy}{y-b}}, \underset{2}{0}, \underset{3}{0}, \underset{12}{0}, \underset{13}{0}, \underset{23}{0})$,

- $\omega_2 := (\underset{\varnothing}{0}, \underset{1}{0}, \underset{2}{\tfrac{-dx}{x-a}}, \underset{3}{0}, \underset{12}{0}, \underset{13}{0}, \underset{23}{0})$, and

- $\omega_3 := (\underset{\varnothing}{0}, \underset{1}{0}, \underset{2}{0}, \underset{3}{0}, \underset{12}{0}, \underset{13}{0}, \underset{23}{1})$.

By computing the (transposed) period matrix $P_{ij} := \langle \Gamma_j, \omega_i \rangle$ (cf. Definition 5.1.1) and checking its non-degeneracy, we will show that $\omega_0, \ldots, \omega_3$ span $\mathrm{H}^2_{\mathrm{dR}}(X, D/\mathbb{C})$.

Claim 8.5.2. *Let X and D be as in Subsection 8.5.2. Then the second algebraic de Rham cohomology group $\mathrm{H}^2_{\mathrm{dR}}(X, D/\mathbb{C})$ of the pair (X, D) is generated by the cycles $\omega_0, \ldots, \omega_3$ considered above.*

8.5.6 Period Matrix of (X, D)

Easy calculations give us for the (transposed) period matrix $P := (\langle \Gamma_j, \omega_i \rangle)^3_{i,j=1}$ (cf. Definition 5.1.1)

	Γ_0	Γ_1	Γ_2	Γ_3
ω_0	1	0	0	0
ω_1	$\mathrm{Li}_1(\tfrac{1}{b})$	$2\pi i$	0	0
ω_2	$\mathrm{Li}_1(\tfrac{1}{a})$	0	$2\pi i$	0
ω_3	$?$	$2\pi i \mathrm{Li}_1(\tfrac{b}{a})$	$2\pi i \ln\left(\tfrac{a-b}{1-b}\right)$	$(2\pi i)^2$;

for example (cf. Subsection 4.11.5)

- $P_{1,1} = \langle \Gamma_1, \omega_1 \rangle = \int_{-\partial S_b \cap D^{\mathrm{an}}_1} \tfrac{-dy}{y-b}$
$$= \int_{|y-b|=\varepsilon} \tfrac{dy}{y-b}$$
$$= 2\pi i,$$

- $P_{3,3} = \langle \Gamma_3, \omega_3 \rangle = \int_T \tfrac{dx}{x-a} \wedge \tfrac{dy}{y-b}$
$$= \left(\int_{|x-a|=\varepsilon} \tfrac{dx}{x-a}\right) \cdot \left(\int_{|y-b|=\varepsilon} \tfrac{dy}{y-b}\right) \quad \text{by Fubini}$$
$$= (2\pi i)^2,$$

- $P_{1,0} = \langle \Gamma_0, \omega_1 \rangle = \int_{-\partial\triangle \cap D^{\mathrm{an}}_1} \tfrac{-dy}{y-b}$
$$= \int_0^1 \tfrac{-\alpha(t)}{\alpha(t)-b}$$
$$= -[\ln(\alpha(t)) - b]_0^1$$
$$= -\ln\left(\tfrac{1-b}{-b}\right)$$
$$= -\ln\left(1 - \tfrac{1}{b}\right)$$
$$= \mathrm{Li}_1\left(\tfrac{1}{b}\right), \quad \text{and}$$

- $P_{3,1} = \langle \Gamma_1, \omega_3 \rangle = \int_{S_b} \tfrac{dx}{x-a} \wedge \tfrac{dy}{y-b}$
$$= \int_{[0,1]^2} \tfrac{d\big(\beta(t)\cdot(b+\varepsilon e^{2\pi i s})\big)}{\beta(t)\cdot(b+\varepsilon e^{2\pi i s})-a} \wedge \tfrac{d\big(b+\varepsilon e^{2\pi i s}\big)}{\varepsilon e^{2\pi i s}}$$

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$$= \int_{[0,1]^2} \frac{b + \varepsilon e^{2\pi i s}}{\beta(t) \cdot (b + \varepsilon e^{2\pi i s}) - a} d\beta(t) \wedge 2\pi i ds$$

$$= -\int_0^1 \left[\frac{a \ln\big(\beta(t) \cdot (b + \varepsilon e^{2\pi i s}) - a\big) - 2\pi i \beta(t) b s}{\beta(t) \cdot (-\beta(t) b + a)} \right]_0^1 d\beta(t)$$

$$= -2\pi i \int_0^1 \frac{d\beta(t)}{\beta(t) - \frac{a}{b}}$$

$$= -2\pi i \left[\ln\big(\beta(t) - \tfrac{a}{b}\big) \right]_0^1$$

$$= -2\pi i \ln\left(\frac{1 - \frac{a}{b}}{-\frac{a}{b}} \right)$$

$$= -2\pi i \ln\big(1 - \tfrac{a}{b}\big)$$

$$= 2\pi i \operatorname{Li}_1\big(\tfrac{b}{a}\big).$$

Obviously the period matrix P is non-degenerate and so Claim 8.5.2 is proved.
What about the entry $P_{3,0}$? We want to show that $\langle \Gamma_0, \omega_3 \rangle = \operatorname{Li}_{1,1}\big(\tfrac{b}{a}, \tfrac{1}{b}\big)$, where $\operatorname{Li}_{1,1}(x, y)$ is an analytic continuation of the double logarithm defined for $|x|, |y| < 1$ in Subsection 8.5.1.

8.5.7 Analytic Continuation of the Double Logarithm $\operatorname{Li}_{1,1}(x, y)$

We describe this analytic continuation in detail. Our approach is similar to the one taken in [G3, 2.3, p. 9], but differs from that in [Z2a, p. 7].
Let $B^{\mathrm{an}} := (\mathbb{C} \setminus \{0, 1\})^2$ be the parameter space and choose a point $(a, b) \in B^{\mathrm{an}}$. For $\varepsilon > 0$ we denote by $D_\varepsilon(a, b)$ the polycylinder

$$D_\varepsilon(a, b) := \{(a, b) \in B^{\mathrm{an}} \mid |a' - a| < \varepsilon, |b' - b| < \varepsilon\}.$$

If $\alpha : [0, 1] \to \mathbb{C}$ is a smooth path from 0 to 1 passing through neither a nor b, then there exists an $\varepsilon > 0$ such that $\operatorname{im} \alpha$ does not meet any of the discs

$$D_{2\varepsilon}(a) := \{a' \in \mathbb{C} \mid |a' - a| < 2\varepsilon\}, \quad \text{and}$$
$$D_{2\varepsilon}(b) := \{b' \in \mathbb{C} \mid |b' - b| < 2\varepsilon\}.$$

Hence the power series (46) below

$$\frac{1}{\alpha(s) - a'} \frac{1}{\alpha(t) - b'} = \frac{1}{\alpha(s) - a} \frac{1}{1 - \frac{a' - a}{\alpha(s) - a}} \frac{1}{\alpha(t) - b} \frac{1}{1 - \frac{b' - b}{\alpha(t) - b}}$$

$$= \sum_{k,l=0}^{\infty} \underbrace{\frac{1}{(\alpha(s) - a)^{k+1}(\alpha(t) - b)^{l+1}}}_{c_{k.l}} (a' - a)^k (b' - b)^l \qquad (46)$$

has coefficients $c_{k,l}$ satisfying

$$|c_{k,l}| < \left(\frac{1}{2\varepsilon} \right)^{k+l+2}.$$

In particular, (46) converges uniformly for $(a', b') \in D_\varepsilon(a, b)$ and we see that the integral

$$\mathrm{I}_2^\alpha(a', b') := \int_{0 \leq s \leq t \leq 1} \frac{d\alpha(s)}{\alpha(s) - a'} \wedge \frac{d\alpha(t)}{\alpha(t) - b'}$$

$$= \sum_{k,l=0} \left(\int_{0 \leq s \leq t \leq 1} \frac{d\alpha(s)}{(\alpha(s) - a)^{k+1}} \wedge \frac{d\alpha(t)}{(\alpha(t) - b)^{l+1}} \right) (a' - a)^k (b' - b)^l$$

defines an analytic function of $D_\varepsilon(a, b)$. In fact, by the same argument we get an analytic function I_2^α on all of $(\mathbb{C} \setminus \mathrm{im}\,\alpha)^2$.

Now let $\alpha_r : [0, 1] \to \mathbb{C} \setminus (D_{2\varepsilon}(a) \cup D_{2\varepsilon}(b))$ with $r \in [0, 1]$ be a smooth homotopy of paths from 0 to 1, i.e. $\alpha_r(0) = 0$ and $\alpha_r(1) = 1$ for all $r \in [0, 1]$. We show

$$I_2^{\alpha_0}(a', b') = I_2^{\alpha_1}(a', b') \quad \text{for all} \quad (a', b') \in D_\varepsilon(a, b).$$

Define a subset $\Gamma \subset \mathbb{C}^2$

$$\Gamma := \{(\alpha_r(s), \alpha_r(t)) \mid 0 \le s \le t \le 1, r \in [0, 1]\}.$$

The boundary of Γ is built out of five components (each being a manifold with boundary)

- $\Gamma_{s=0} := \{(0, \alpha_r(t)) \mid r, t \in [0, 1]\}$,

- $\Gamma_{s=t} := \{(\alpha_r(s), \alpha_r(s)) \mid r, s \in [0, 1]\}$,

- $\Gamma_{t=1} := \{(\alpha_r(s), 1) \mid r, s \in [0, 1]\}$,

- $\Gamma_{r=0} := \{(\alpha_0(s), \alpha_0(t) \mid 0 \le s \le t \le 1\}$,

- $\Gamma_{r=1} := \{(\alpha_1(s), \alpha_1(t) \mid 0 \le s \le t \le 1\}$.

Let $(a', b') \in D_\varepsilon(a, b)$. Since the restriction of $\frac{dx}{x-a'} \wedge \frac{dy}{y-b'}$ to $\Gamma_{s=0}$, $\Gamma_{s=t}$ and $\Gamma_{t=1}$ is zero, we get by Stoke's theorem

$$
\begin{aligned}
0 = \int_\Gamma 0 = \int_\Gamma d\frac{dx}{x-a'} &\wedge \frac{dy}{y-b'} \\
&= \int_{\partial\Gamma} \frac{dx}{x-a'} \wedge \frac{dy}{y-b'} \\
&= \int_{\Gamma_{r=1} - \Gamma_{r=0}} \frac{dx}{x-a'}\frac{dy}{y-b'} \\
&= I_2^{\alpha_1}(a', b') - I_2^{\alpha_0}(a', b').
\end{aligned}
$$

For each pair of smooth paths $\alpha_0, \alpha_1 : [0, 1] \to \mathbb{C}$ from 0 to 1, we can find a homotopy α_r relative to $\{0, 1\}$ between both paths. Since $\mathrm{im}\,\alpha_r$ is compact, we also find a point $(a, b) \in B^{\mathrm{an}} = (\mathbb{C} \setminus \{0, 1\})^2$ and an $\varepsilon > 0$ such that $\mathrm{im}\,\alpha_r$ does not meet $D_{2\varepsilon}(a, b)$ or $D_{2\varepsilon}(a, b)$. Then $I_2^{\alpha_0}$ and $I_2^{\alpha_1}$ must agree on $D_\varepsilon(a, b)$. By the identity principle for analytic functions of several complex variables [Gun, A, 3, p. 5], the functions $I_2^\alpha(a', b')$, each defined on $(\mathbb{C} \setminus \mathrm{im}\,\alpha)^2$, patch together to give a multivalued analytic function on $B^{\mathrm{an}} = (\mathbb{C} \setminus \{0, 1\})^2$.

Now assume $1 < |b| < |a|$, then we can take $\alpha = \mathrm{id} : [0, 1] \to \mathbb{C}$, $s \mapsto s$, and obtain

$$I_2^{\mathrm{id}}(a, b) = I_2(a, b) = \mathrm{Li}_{1,1}\left(\frac{b}{a}, \frac{1}{y}\right),$$

where $\mathrm{Li}_{1,1}(x, y)$ is the double logarithm defined for $|x|, |y| < 1$ in Subsection 8.5.1. Thus we have proved the following lemma.

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Lemma 8.5.3. *The integrals*

$$\mathrm{I}_2^\alpha\left(\frac{1}{xy},\frac{1}{y}\right) = \int_{0\le s\le t\le 1} \frac{d\alpha(s)}{\alpha(s)-\frac{1}{xy}} \wedge \frac{d\alpha(t)}{\alpha(t)-\frac{1}{y}}$$

with $\alpha : [0,1] \to \mathbb{C}$ *a smooth path from* 0 *to* 1, *and* $\frac{1}{xy}, \frac{1}{b} \in \mathbb{C} \setminus \operatorname{im}\alpha$, *defined above on page 83, provide a genuine analytic continuation of* $\mathrm{Li}_{1,1}(x,y)$ *to a multivalued function which is defined on* $\{(x,y) \in \mathbb{C}^2 \,|\, x,y \ne 0, xy \ne 1, y \ne 1\}$.

Definition 8.5.4 (Double logarithm). *We call the analytic continuation from Lemma 8.5.3 the* double logarithm *as well and continue to use the notation* $\mathrm{Li}_{1,1}(x,y)$.

This gives us finally $P_{3,0} = \mathrm{Li}_{1,1}\left(\frac{b}{a},\frac{1}{b}\right)$.

8.5.8 Properties of the Double and the Dilogarithm

Similarly, one can analytically continue the dilogarithm $\mathrm{Li}_2(x)$ to a multivalued meromorphic function on $\mathbb{C}$.

Since analytic continuations are uniquely determined (as multivalued functions) our Definition 8.5.4 must agree with the ones given by Goncharov in [G3, p. 9] or by Zhao in [Z2b, 4, p. 4f]. Hence we are allowed to cite various facts concerning the double and the dilogarithm.

Lemma 8.5.5 (Monodromy of the double logarithm, [Z2a, Thm. 5.3, p. 11]). *Let* $\sigma = \{1 + \varepsilon e^{2\pi i s}, b) \,|\, s \in [0,1]\}$ *be a loop in* $B^{\mathrm{an}} = (\mathbb{C} \setminus \{0,1\})^2$ *winding counterclockwise around* $\{a = 1\}$ *once, but not around* $\{a = 0\}$, $\{b = 0\}$ *or* $\{b = 1\}$. *Then*

$$\int_\sigma d\mathrm{Li}_1\left(\frac{1}{a}\right) = -2\pi i, \quad \text{and}$$

$$\int_\sigma d\mathrm{Li}_{1,1}\left(\frac{b}{a},\frac{1}{b}\right) = -2\pi i \ln\left(\frac{1-b+\varepsilon}{1-b}\right).$$

Lemma 8.5.6 (Relation with the dilogarithm, [G1, p. 620]). *We have the following identity of multivalued functions*

$$\mathrm{Li}_{1,1}\left(b,\frac{1}{b}\right) = -\mathrm{Li}_2\left(\frac{1}{1-b}\right).$$

Lemma 8.5.7 (Monodromy of the dilogarithm, [H, Prop. 2.2, p. 7]). *If* $\sigma = \{1 + \varepsilon e^{2\pi i s} \,|\, s \in [0,1]\}$ *is a circle in* $\mathbb{C} \setminus \{1\}$ *of radius* ε *winding counterclockwise around* 1 *once, then*

$$\int_\sigma d\mathrm{Li}_2(c) = 2\pi i \,\mathrm{Li}_1(-\varepsilon).$$

8.5.9 The Mixed $\mathbb{Q}$-Hodge Structure on $\mathrm{H}^2_{\mathrm{sing}}(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{Q})$

According to Deligne (cf. [D3, 8.3.8, p. 43]) the cohomology group $\mathrm{H}^2_{\mathrm{sing}}(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{Q})$ carries a *mixed $\mathbb{Q}$-Hodge structure*, short $\mathbb{Q}$-MHS, consisting of

- an increasing *weight filtration* $W_\bullet$ on $\mathrm{H}^2_{\mathrm{sing}}(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{Q})$ and

- a decreasing *Hodge filtration* $F^\bullet$ on $\mathrm{H}^2_{\mathrm{sing}}(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{C})$.

Deligne's *mixed Hodge theory* takes up several publications, [D2] and [D3] being of central importance. For a brief introduction, see [GM, Ch. 6] and [Dur].

Using the same ideas as in [Kj, 4.2, p. 38f], we can compute this $\mathbb{Q}$-MHS and obtain the following lemma.

Lemma 8.5.8. *Let X and D be as in Subsection 8.5.2 and let X^{an}, D^{an} be the associated complex analytic spaces (cf. Subsection 2.1).*
The cohomology group $\mathrm{H}^2_{\mathrm{sing}}(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{Q})$ carries a $\mathbb{Q}$-MHS with weight filtration $W_\bullet$ given by

$$
\begin{aligned}
W_4 &= \mathrm{H}^2_{\mathrm{sing}}(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{Q}) = \mathbb{Q}\gamma_0^* \oplus \mathbb{Q}\gamma_1^* \oplus \mathbb{Q}\gamma_2^* \oplus \mathbb{Q}\gamma_3^* \supset \\
W_3 &= W_2 = \mathbb{Q}\gamma_0^* \oplus \mathbb{Q}\gamma_1^* \oplus \mathbb{Q}\gamma_2^* \supset \\
W_1 &= W_0 = \mathbb{Q}\gamma_0^* \supset \\
W_{-1} &= 0,
\end{aligned}
$$

where the γ_j^* are the duals of the γ_j, $j = 1, \ldots, 4$ defined in Subsection 8.5.3, and Hodge filtration $F^\bullet$ given by

$$
\begin{aligned}
F^0 &= \mathrm{H}^2_{\mathrm{sing}}(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{C}) = \mathbb{C}\omega_0 \oplus \mathbb{C}\omega_1 \oplus \mathbb{C}\omega_2 \oplus \mathbb{C}\omega_3 \supset \\
F^1 &= \mathbb{C}\omega_1 \oplus \mathbb{C}\omega_2 \oplus \mathbb{C}\omega_3 \supset \\
F^2 &= \mathbb{C}\omega_3 \supset \\
F^3 &= 0,
\end{aligned}
$$

where we used the cycles $\omega_0, \ldots, \omega_3$ from Subsection 8.5.5 and identified

$$
\mathrm{H}^2_{\mathrm{dR}}(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{C}) \cong \mathrm{H}^2_{\mathrm{sing}}(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{C})
$$

using the Comparison Theorem 4.10.1.

We illustrate the proof of Lemma 8.5.8 with two exemplary calculations.

- We want to show that $\gamma_3^* \notin W_3 \mathrm{H}^2_{\mathrm{sing}}(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{Q})$. From the inclusion $T \hookrightarrow X^{\mathrm{an}}$, we get an element $[T]^* \in \mathrm{H}^2_{\mathrm{sing}}(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{Q})$, which is the preimage of γ_3^* under the connecting morphism

$$
\mathrm{H}^2_{\mathrm{sing}}(X^{\mathrm{an}}; \mathbb{Q}) \longrightarrow \mathrm{H}^2_{\mathrm{sing}}(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{Q}).
$$

This map induces a map of $\mathbb{Q}$-MHS (cf. [GM, Ch. 6, Thm. 2.4, p. 147]), which is strictly compatible with the weight and Hodge filtrations (cf. [GM, Ch. 6, 1.3, p. 143]). In particular, it will be sufficient to show

$$
[T]^* \notin W_3 \mathrm{H}^2_{\mathrm{sing}}(X^{\mathrm{an}}; \mathbb{Q}).
$$

We consider first

$$
U_c := \mathbb{A}^1_{\mathbb{C}} \setminus \{c\} \quad \text{for} \quad c \in \mathbb{C} \quad \text{with standard coordinate } t.
$$

Let σ_c be the loop $\{c + \varepsilon e^{2\pi i s} \,|\, s \in [0,1]\}$ winding around c and σ_c^* its dual. Since the weight-one-part of $\mathrm{H}^1_{\mathrm{sing}}(U_c; \mathbb{Q})$ is zero (cf. [GM, Ch. 6, 2.5.1, p. 147])

$$
W_1 \mathrm{H}^1_{\mathrm{sing}}(U_c^{\mathrm{an}}; \mathbb{Q}) = \mathrm{H}^1_{\mathrm{sing}}(\overline{U}_c^{\mathrm{an}}; \mathbb{Q}) = \mathrm{H}^1_{\mathrm{sing}}(\mathbb{C}P^1_{\mathrm{an}}; \mathbb{Q}) = 0,
$$

we see $\sigma_c^* \in W_2 \mathrm{H}^1_{\mathrm{sing}}(U_c^{\mathrm{an}}; \mathbb{Q})$, but $\sigma_c^* \notin W_1 \mathrm{H}^1_{\mathrm{sing}}(U_c^{\mathrm{an}}; \mathbb{Q})$, i.e. σ_c^* has weight two. We have $X = U_a \times U_b$ and the Künneth isomorphism (cf. [Br, Ch. VI, Thm. 1.6, p. 320])

$$\mathrm{H}^2_{\mathrm{sing}}(X^{\mathrm{an}}; \mathbb{Q}) = \bigoplus_{k+l=2} \mathrm{H}^k_{\mathrm{sing}}(U_a^{\mathrm{an}}; \mathbb{Q}) \otimes \mathrm{H}^l_{\mathrm{sing}}(U_b^{\mathrm{an}}; \mathbb{Q})$$

maps $[T]^*$ to $\sigma_a^* \otimes \sigma_b^*$. The Künneth isomorphism is an isomorphism of $\mathbb{Q}$-MHS (cf. [D3, Prop. 8.2.10, p. 40], [GM, Ch. 6, Thm. 2.1, p. 145]), hence $[T]^*$ has weight four, i.e. $[T]^* \notin W_3 \mathrm{H}^2_{\mathrm{sing}}(X^{\mathrm{an}}; \mathbb{Q})$.

- We want to show $[\omega_1] \in F^1 \mathrm{H}^1_{\mathrm{sing}}(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{Q})$. The connecting morphism of the long exact sequence in analytic deRham cohomology 4.3.1

$$\delta : \mathrm{H}^1_{\mathrm{dR}}(D^{\mathrm{an}}; \mathbb{C}) \longrightarrow \mathrm{H}^2_{\mathrm{dR}}(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{C})$$

is indeed a morphism of $\mathbb{Q}$-MHS (see [D3, Prop. 8.3.9, p. 43], [GM, Ch. 6, Thm. 2.4, p. 147]); and so we have a strict morphism of filtered complexes (cf. [GM, Ch. 6, 1.3, p. 143])

$$\delta : (\mathrm{H}^1_{\mathrm{dR}}(D^{\mathrm{an}}; \mathbb{C}), F^\bullet) \longrightarrow (\mathrm{H}^2_{\mathrm{dR}}(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{C}), F^\bullet).$$

Now the class of

$$\omega_1' := \left(\frac{-dy}{y-b}_{\!\!1}, 0_{\,2}, 0_{\,3} \right) \in \Gamma(D^{\mathrm{an}}; \widetilde{\Omega}^1_{D^{\mathrm{an}}}) = \bigoplus_{i=1}^{3} \Gamma(D_i^{\mathrm{an}}; \Omega^1_{D_i^{\mathrm{an}}})$$

is mapped to $[\omega_1]$ under δ; hence it suffices to prove

$$[\omega_1'] \in F^1 \mathrm{H}^1_{\mathrm{sing}}(D^{\mathrm{an}}; \mathbb{C}).$$

Using a similar argument for the morphism of $\mathbb{Q}$-MHS (cf. [GM, Ch. 6, Thm. 2.1, p. 145])

$$i^* : (\mathrm{H}^1_{\mathrm{sing}}(D^{\mathrm{an}}; \mathbb{Q}), W_\bullet, F^\bullet) \longrightarrow (\mathrm{H}^1_{\mathrm{sing}}(D_1^{\mathrm{an}}; Q), W_\bullet, F^\bullet)$$

induced by the inclusion $i : D_1^{\mathrm{an}} \hookrightarrow D^{\mathrm{an}}$, we reduce to the claim

$$\left[\frac{-dy}{y-b} \right] \in F^1 \mathrm{H}^1_{\mathrm{sing}}(D_1^{\mathrm{an}}; \mathbb{C}).$$

From the map

$$j : D_1 \hookrightarrow \mathbb{C}P^1$$
$$(0, y) \mapsto [1 : y - b]$$

we get an isomorphism

$$j : D_1 \cong \mathbb{C}P^1 \setminus \{[1 : 0], [0 : 1]\}.$$

The divisor $[1 : 0] + [0 : 1]$ on $\mathbb{C}P^1$ is just the inverse image of the very ample divisor $[1 : 0 : 0]$ on $\mathbb{C}P^2$ under the 2-uple embedding by [Ha, Ex. 7.6.1, p. 155], hence very ample itself. This allows us to apply Proposition 9.2.4 from [D3, p. 49], which states that:

For Z a very ample divisor on a smooth projective variety Y over $\mathbb{C}$ and ω a closed differential q-form on $Y \setminus Z$, we have $[\omega] \in F^p \mathrm{H}^q_{\mathrm{sing}}(Y^{\mathrm{an}} \setminus Z^{\mathrm{an}}; \mathbb{C})$ if and only if ω has a pole of order less or equal $q - p + 1$ along Z.

Thus we obtain

$$j_* \left[\frac{-dy}{y-b} \right] \in F^1 \mathrm{H}^1_{\mathrm{sing}}(\mathbb{C}P^1_{\mathrm{an}} \setminus \{[1:0],[0:1]\}; \mathbb{C}),$$

i.e.

$$\left[\frac{-dy}{y-b} \right] \in F^1 \mathrm{H}^1_{\mathrm{sing}}(D_1^{\mathrm{an}}; \mathbb{C}),$$

and our claim is proved.

Alternatively, we could have argued that D_1 is isomorphic to U_b (as defined in the previous calculation)

$$j : D_1 \xrightarrow{\sim} U_b$$
$$(0, y) \mapsto y.$$

Since $\mathrm{H}^1_{\mathrm{sing}}(U_b^{\mathrm{an}}; \mathbb{Q})$ carries a pure $\mathbb{Q}$-Hodge structure (see page 86) and the corresponding Hodge decomposition enjoys the symmetry (cf. [GH, p. 116])

$$\mathrm{H}^q(U_b^{\mathrm{an}}; \Omega^p_{U_b^{\mathrm{an}}}) \cong \mathrm{H}^p(U_b^{\mathrm{an}}; \Omega^q_{U_b^{\mathrm{an}}}),$$

it follows from the fact that $\mathrm{H}^1_{\mathrm{dR}}(U_b^{\mathrm{an}}; \mathbb{C})$ is one-dimensional with generator $j_* \left[\frac{-dy}{y-b} \right]$, that $\frac{-dy}{y-b}$ has type $(1,1)$.

For aesthetic reasons, we will dualize the $\mathbb{Q}$-MHS on $\mathrm{H}^2_{\mathrm{sing}}(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{Q})$; thus getting a $\mathbb{Q}$-MHS on the homology group $\mathrm{H}_2^{\mathrm{sing}}(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{Q})$. Recall that the dual weight and Hodge filtrations of a $\mathbb{Q}$-MHS $(M_{\mathbb{Q}}, W_\bullet, F^\bullet)$ is defined as (cf. [GM, Ch. 6, 1.5, p. 143], [D2, 1.1.6, p. 7])

$$W_p M_{\mathbb{Q}}^\vee := (M_{\mathbb{Q}}/W_{-1-p} M_{\mathbb{Q}})^\vee, \quad \text{and}$$
$$F^p M_{\mathbb{C}}^\vee := (M_{\mathbb{C}}/F^{1-p} M_{\mathbb{C}})^\vee, \quad \text{for} \quad p \in \mathbb{Z}.$$

Let us summarize our results so far.

Proposition 8.5.9. *Let X, D be as in Subsection 8.5.2 and let X^{an}, D^{an} be the associated complex analytic spaces (cf. Subsection 2.1). Then the homology group $\mathrm{H}_2^{\mathrm{sing}}(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{Q})$ of the pair (X, D) carries a $\mathbb{Q}$-MHS $(W_\bullet, F^\bullet)$. If γ_0, γ_1, γ_2, γ_3 is the basis of $\mathrm{H}_2^{\mathrm{sing}}(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{Q})$ defined in Subsection 8.5.3 and ω_0^*, ω_1^*, ω_2^*, ω_3^* the basis of $\mathrm{H}_2^{\mathrm{sing}}(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{C})$ dual to the one considered in Subsection 8.5.5, we find*

(i) a (transposed) period matrix P expressing the γ_j in terms of the ω_i^*

$$(\gamma_0, \gamma_1, \gamma_2, \gamma_3) = \begin{pmatrix} \omega_0^* \\ \omega_1^* \\ \omega_2^* \\ \omega_3^* \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ \mathrm{Li}_1(\frac{1}{b}) & 2\pi i & 0 & 0 \\ \mathrm{Li}_1(\frac{1}{a}) & 0 & 2\pi i & 0 \\ \mathrm{Li}_{1,1}(\frac{b}{a}, \frac{1}{b}) & 2\pi i \mathrm{Li}_1(\frac{b}{a}) & 2\pi i \ln\left(\frac{a-b}{1-b}\right) & (2\pi i)^2 \end{pmatrix},$$

(ii) the weight filtration in terms of the $\{\gamma_j\}$

$$W_p\mathrm{H}_2^{\mathrm{sing}}(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{Q}) = \begin{cases} 0 & \text{for} \quad p \leq -5 \\ \mathbb{Q}\gamma_3 & \text{for} \quad p = -4, -3 \\ \mathbb{Q}\gamma_1 \oplus \mathbb{Q}\gamma_2 \oplus \mathbb{Q}\gamma_3 & \text{for} \quad p = -2, -1 \\ \mathbb{Q}\gamma_0 \oplus \mathbb{Q}\gamma_1 \oplus \mathbb{Q}\gamma_2 \oplus \mathbb{Q}\gamma_3 & \text{for} \quad p \geq 0, \end{cases}$$

(iii) the Hodge filtration in terms of the $\{\omega_i^*\}$

$$F^p\mathrm{H}_2^{\mathrm{sing}}(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{Q}) = \begin{cases} \mathbb{C}\omega_0^* \oplus \mathbb{C}\omega_1^* \oplus \mathbb{C}\omega_2^* \oplus \mathbb{C}\omega_3^* & \text{for} \quad p \leq -2 \\ \mathbb{C}\omega_0^* \oplus \mathbb{C}\omega_1^* \oplus \mathbb{C}\omega_2^* & \text{for} \quad p = -1 \\ \mathbb{C}\omega_0^* & \text{for} \quad p = 0 \\ 0 & \text{for} \quad p \geq 1. \end{cases}$$

Remark 8.5.10. This $\mathbb{Q}$-MHS resembles very much the $\mathbb{Q}$-MHS considered in [G1, 2.2, p. 620] and [Z2a, 3.2, p. 6]. Nevertheless a few differences are note-worthy:

- Goncharov defines the weight filtration slightly different, for example his lowest weight is -6.

- The entry $P_{3,2} = 2\pi i \ln\left(\frac{a-b}{1-b}\right)$ of the period matrix P differs by $(2\pi i)^2$, or put differently, the basis $\{\gamma_0, \gamma_1, \gamma_2 - \gamma_3, \gamma_3\}$ is used.

8.5.10 A Variation of Mixed $\mathbb{Q}$-Hodge Structures

Up to now, the parameters a and b of the configuration (X, D) have been fixed. By varying a and b, we obtain a family of configurations: Equip $\mathbb{A}_{\mathbb{C}}^2$ with coordinates a and b and let

$$B := \mathbb{A}_{\mathbb{C}}^2 \setminus (\{a = 0\} \cup \{a = 1\} \cup \{b = 0\} \cup \{b = 1\})$$

be the parameter space. Take another copy of $\mathbb{A}_{\mathbb{C}}^2$ with coordinates x and y and define total spaces

$$\underline{X} := \underset{(a,b,x,y)}{(B \times \mathbb{A}_{\mathbb{C}}^2)} \setminus (\{x = a\} \cup \{y = b\}), \quad \text{and}$$

$$\underline{D} := \text{``}B \times D\text{''} = \underline{X} \cap (\{x = 0\} \cup \{y = 1\} \cup \{x = y\}).$$

We now have a projection

$$\begin{array}{ccc} \underline{D} & \hookrightarrow & \underline{X} \\ & \searrow & \downarrow \pi \\ & & B \end{array} \qquad \begin{array}{c} (a, b, x, y) \\ \downarrow \\ (a, b) \end{array} \quad ,$$

whose fibre over a closed point $(a, b) \in B$ is precisely the configuration (X, D) for the parameter choice a, b.

π *is flat.* Since open immersions and structure morphisms to the spectrum of a field are flat and flatness is stable under base extension and composition [Ha, Prop. 9.2, p. 254], we see that π and $\pi_{|\underline{D}}$ define flat families. Note that these families could be

defined over $\mathbb{Z}$ or $\mathbb{Q}$ already. This justifies the use of the term period on the preceding pages: For $(a,b) \in B^{\mathrm{an}}$ with $a,b \in \mathbb{Q}$, the fibre over (a,b) of π is the base change to $\mathbb{C}$ of a pair of varieties (X_0, D_0) defined over $\mathbb{Q}$ and the "differential forms" $\omega_0, \omega_1, \omega_2$ ω_3 span $\mathrm{H}^2_{\mathrm{dR}}(X_0, D_0/\mathbb{Q})$. Hence the complex numbers P_{ij} computed above are indeed periods in the sense of definitions 5.1.1, 5.2.1, 5.3.2, if $a, b \in \mathbb{Q}$.

In the following, we identify $\mathrm{H}^{\mathrm{sing}}_2(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{C})$ with $\mathbb{C}^4$ by mapping the ω^*_i to a standard basis of $\mathbb{C}^4$. For each choice of parameters $(a,b) \in B^{\mathrm{an}}$, this defines a $\mathbb{Q}$-lattice of full rank in $\mathbb{C}^4$ together with weight and Hodge filtrations that turn $\mathbb{C}^4$ into a $\mathbb{Q}$-MHS. Note that the $\mathbb{Q}$-lattice varies when we move (a,b) inside B^{an}. Assigning to each $(a,b) \in B^{\mathrm{an}}$ this $\mathbb{Q}$-MHS defines not merely a "family of $\mathbb{Q}$-MHS", but a *good unipotent variation of mixed $\mathbb{Q}$-Hodge structures*. Again we have to refer to the literature for the definition of this concept. See for example [HZ], [St, p. 37f] or [Z2b, 3.1, p. 4f]. For explicitness, we pick a branch for each entry P_{ij} of the (transposed) period matrix P.

Proposition 8.5.11 ([Z1, Thm. 4.1, p. 184]). *The assignment*

$$B^{\mathrm{an}} \ni (a,b) \mapsto (V_{\mathbb{Q}}, W_\bullet, F^\bullet),$$

where

$$V_{\mathbb{Q}} := \mathrm{span}_{\mathbb{Q}}\{s_0, \ldots, s_3\},$$

$$V_{\mathbb{C}} := \mathbb{C}^4 \quad \text{with standard basis } e_0, \ldots, e_3,$$

$$s_0 := \begin{pmatrix} 1 \\ \mathrm{Li}\left(\frac{1}{b}\right) \\ \mathrm{Li}_1\left(\frac{1}{a}\right) \\ \mathrm{Li}_{1,1}\left(\frac{b}{a}, \frac{1}{b}\right) \end{pmatrix}, \ s_1 := \begin{pmatrix} 0 \\ 2\pi i \\ 0 \\ 2\pi i \mathrm{Li}_1\left(\frac{b}{a}\right) \end{pmatrix}, \ s_2 := \begin{pmatrix} 0 \\ 0 \\ 2\pi i \\ 2\pi i \ln\left(\frac{a-b}{1-b}\right) \end{pmatrix}, \ s_3 := \begin{pmatrix} 0 \\ 0 \\ 0 \\ (2\pi i)^2 \end{pmatrix},$$

$$W_p V_{\mathbb{Q}} = \begin{cases} 0 & \text{for} \quad p \leq -5 \\ \mathbb{Q}s_3 & \text{for} \quad p = -4, -3 \\ \mathbb{Q}s_1 \oplus \mathbb{Q}s_2 \oplus \mathbb{Q}s_3 & \text{for} \quad p = -2, -1 \\ V_{\mathbb{Q}} & \text{for} \quad p \geq 0, \quad \text{and} \end{cases}$$

$$F^p V_{\mathbb{C}} = \begin{cases} V_{\mathbb{C}} & \text{for} \quad p \leq -2 \\ \mathbb{C}e_0 \oplus \mathbb{C}e_1 \oplus \mathbb{C}e_2 & \text{for} \quad p = -1 \\ \mathbb{C}e_0 & \text{for} \quad p = 0 \\ 0 & \text{for} \quad p \geq 1 \end{cases}$$

defines a good unipotent variation of $\mathbb{Q}$-MHS on B^{an}.

Note that the Hodge filtration $F^\bullet$ does not depend on $(a,b) \in B^{\mathrm{an}}$.

One of the main characteristics of good unipotent variations of $\mathbb{Q}$-MHS is that they can be extended to a compactification of the base space (if the complement is a divisor with normal crossings).

The algorithm for computing these extensions, so called *limit mixed $\mathbb{Q}$-Hodge structures*, can be found for example in [H, 7, p. 24f] and [Z2b, 4, p. 12].

8.5.11 Limit Mixed $\mathbb{Q}$-Hodge Structures

In a first step, we extend the variation from Proposition 8.5.11 to the divisor $\{a = 1\}$ minus the point $(1, 0)$ and then in a second step we extend it to the point $(1, 0)$. We will use the notation of Proposition 8.5.11. In particular, we assume that a branch has been picked for each entry P_{ij} of P. We will follow [Z2b, 4.1, p. 14f] very closely. *First step:* Let σ be the loop from Lemma 8.5.5 winding counterclockwise around $\{a = 1\}$ once, but not around $\{a = 0\}$, $\{b = 0\}$ or $\{b = 1\}$. If we analytically continue an entry P_{ij} of P along σ we possibly get a second branch of the same multivalued function. In fact, by Lemma 8.5.5, the matrix resulting from analytic continuation of every entry along σ will be of the form

$$P \cdot T_{\{a=1\}},$$

where

$$T_{\{a=1\}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is the *monodromy matrix* corresponding to σ. The *local monodromy logarithm* is defined as

$$N_{\{a=1\}} = \frac{\ln T_{\{a=1\}}}{2\pi i} = \frac{1}{2\pi i} \sum_{n=1}^{\infty} \frac{-1}{n} \left(\begin{pmatrix} 1 & & \\ & 1 & 1 \\ & & 1 \end{pmatrix} - T_{\{a=1\}} \right)^n$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{-1}{2\pi i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We want to extend our $\mathbb{Q}$-MHS along the tangent vector $\frac{\partial}{\partial a}$, i.e. we introduce a local coordinate $t := a - 1$ and compute the *limit period matrix*

$$P_{\{a=1\}} := \lim_{t \to 0} P \cdot e^{-\ln(t) \cdot N_{\{a=1\}}}$$

$$= \lim_{t \to 0} \begin{pmatrix} 1 & 0 & 0 & 0 \\ \mathrm{Li}_1\left(\frac{1}{b}\right) & 2\pi i & 0 & 0 \\ \mathrm{Li}_1\left(\frac{1}{1+t}\right) & 0 & 2\pi i & 0 \\ \mathrm{Li}_{1,1}\left(\frac{b}{1+t}, \frac{1}{b}\right) & 2\pi i \mathrm{Li}_1\left(\frac{b}{1+t}\right) & 2\pi i \ln\left(\frac{1-b+t}{1-b}\right) & (2\pi i)^2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{\ln(t)}{2\pi i} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \lim_{t \to 0} \begin{pmatrix} 1 & 0 & 0 & 0 \\ \mathrm{Li}_1\left(\frac{1}{b}\right) & 2\pi i & 0 & 0 \\ \mathrm{Li}_1\left(\frac{1}{1+t}\right) + \ln(t) & 0 & 2\pi i & 0 \\ \mathrm{Li}_{1,1}\left(\frac{b}{1+t}, \frac{1}{b}\right) + \ln\left(\frac{1-b+t}{1-b}\right) \cdot \ln(t) & 2\pi i \mathrm{Li}_1\left(\frac{b}{1+t}\right) & 2\pi i \ln\left(\frac{1-b+t}{1-b}\right) & (2\pi i)^2 \end{pmatrix}$$

$$\overset{(*)}{=} \begin{pmatrix} 1 & 0 & 0 & 0 \\ \mathrm{Li}_1\left(\frac{1}{b}\right) & 2\pi i & 0 & 0 \\ 0 & 0 & 2\pi i & 0 \\ -\mathrm{Li}_2\left(\frac{1}{1-b}\right) & 2\pi i \mathrm{Li}_1(b) & 0 & (2\pi i)^2 \end{pmatrix}.$$

Here we used at $(*)$

- $P_{\{a=1\}2,0} = \lim_{t \to 0} \mathrm{Li}_1\left(\frac{1}{1+t}\right) + \ln(t)$

$$= \lim_{t \to 0} -\ln\left(1 - \frac{1}{1+t}\right) + \ln(t)$$

$$= \lim_{t \to 0} -\ln(t) + \ln(1+t) + \ln(t)$$

$$= 0, \quad \text{and}$$

- $P_{\{a=1\}3,0} = \lim_{t \to 0} \mathrm{Li}_{1,1}\left(\frac{b}{1+t}, \frac{1}{b}\right) + \ln\left(\frac{1-b+t}{1-b}\right) \cdot \ln(t)$

$$= \mathrm{Li}_{1,1}\left(b, \frac{1}{b}\right) \quad \text{by L'Hospital}$$

$$= -\mathrm{Li}_2\left(\frac{1}{1-b}\right) \quad \text{by Lemma 8.5.6.}$$

The vectors s_0, s_1, s_2, s_3 spanning the $\mathbb{Q}$-lattice of the limit $\mathbb{Q}$-MHS on $\{a = 1\} \setminus \{(1,0)\}$ are now given by the columns of the limit period matrix

$$s_0 = \begin{pmatrix} 1 \\ \mathrm{Li}_1\left(\frac{1}{b}\right) \\ 0 \\ -\mathrm{Li}_2\left(\frac{1}{1-b}\right) \end{pmatrix}, \quad s_1 = \begin{pmatrix} 0 \\ 2\pi i \\ 0 \\ 2\pi i\,\mathrm{Li}_1(b) \end{pmatrix}, \quad s_2 = \begin{pmatrix} 0 \\ 0 \\ 2\pi i \\ 0 \end{pmatrix}, \quad s_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ (2\pi i)^2 \end{pmatrix}.$$

The weight and Hodge filtration of the limit $\mathbb{Q}$-MHS can be expressed in terms of the s_j and the standard basis vectors e_i of $\mathbb{C}^4$ as in Proposition 8.5.11 using the same formulae as given there. This gives us a variation of $\mathbb{Q}$-MHS on the divisor $\{a = 1\} \setminus \{(1,0)\}$. This variation is actually (up to signs) an extension of Deligne's famous *dilogarithm variation* considered for example in [Kj], 4.2, p. 38f]. In loc. cit. the geometric origin of this variation is explained in detail.

Second step: We now extend this variation along the tangent vector $\frac{-\partial}{\partial b}$ to the point $(1,0)$, i.e. we write $b = -t$ with a local coordinate t. Let σ be the loop in $\{a = 1\} \setminus \{(1,0)\}$ from Lemma 8.5.7 winding counterclockwise around $(1,0)$ once, but not around $(1,1)$. Then by Lemma 8.5.7 the monodromy matrix corresponding to σ is given by

$$T_{(1,0)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

hence the local monodromy logarithm is given by

$$N_{(1,0)} = \frac{\ln T_{(1,0)}}{2\pi i} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{2\pi i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus we get for the limit period matrix

$$P_{(1,0)} := \lim_{t \to 0} P_{\{a=1\}} \cdot e^{-\ln(t) \cdot N_{(1,0)}}$$

$$= \lim_{t \to 0} \begin{pmatrix} 1 & 0 & 0 & 0 \\ \mathrm{Li}_1\left(\frac{-1}{t}\right) & 2\pi i & 0 & 0 \\ 0 & 0 & 2\pi i & 0 \\ -\mathrm{Li}_2\left(\frac{1}{1+t}\right) & 2\pi i\,\mathrm{Li}_1(-t) & 0 & (2\pi i)^2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{-\ln(t)}{2\pi i} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \lim_{t \to 0} \begin{pmatrix} 1 & 0 & 0 & 0 \\ \mathrm{Li}_1\left(\frac{-1}{t}\right) - \ln(t) & 2\pi i & 0 & 0 \\ 0 & 0 & 2\pi i & 0 \\ -\mathrm{Li}_2\left(\frac{1}{1+t}\right) - \mathrm{Li}_1(-t) \cdot \ln(t) & 0 & 0 & (2\pi i)^2 \end{pmatrix}$$

$$\overset{(*)}{=} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2\pi i & 0 & 0 \\ 0 & 0 & 2\pi i & 0 \\ -\zeta(2) & 0 & 0 & (2\pi i)^2 \end{pmatrix}.$$

Here we used at $(*)$

- $P_{(1,0)_{1,0}} = \lim_{t\to 0} \mathrm{Li}_1\left(\frac{-1}{t}\right) - \ln(t)$
  $= \lim_{t\to 0} -\ln\left(1 + \frac{1}{t}\right) - \ln(t)$
  $= \lim_{t\to 0} -\ln(1 + t) + \ln(t) - \ln(t)$
  $= 0, \quad$ and

- $P_{(1,0)_{3,0}} = \lim_{t\to 0} -\mathrm{Li}_2\left(\frac{1}{1+t}\right) - \mathrm{Li}_1(-t) \cdot \ln(t)$
  $= \lim_{t\to 0} \mathrm{Li}_2\left(\frac{1}{1+t}\right) + \ln(1 + t) \cdot \ln(t)$
  $= -\mathrm{Li}_2(1) \quad$ by L'Hospital
  $= -\zeta(2).$

As in the previous step, the vectors s_0, s_1, s_2, s_3 spanning the $\mathbb{Q}$-lattice of the limit $\mathbb{Q}$-MHS are given by the columns of the limit period matrix $P_{(1,0)}$ and weight and Hodge filtrations by the formulae in Proposition 8.5.11.
So we obtained $-\zeta(2)$ as a "period" of a limit $\mathbb{Q}$-MHS.

A   Sign Conventions and Related Material

We adopt some sign conventions from [Eb, A 3.12, p. 657f]. Let $\mathfrak{A}$ be an abelian category.

- Any $\mathfrak{A}$-object A gives rise to an $\mathfrak{A}$-complex A[0] concentrated in degree zero with $(A[0])^0 = A$. Similarly, we define A[−1].

- For $\mathfrak{A}$-complexes $A^\bullet$ with differential $d_{A^\bullet}$, we have an i-th *cohomology functor* $h^i A^\bullet := \ker d_{A^\bullet}^i / \operatorname{im} d_{A^\bullet}^{i-1}$.

- If $A^\bullet$ is a complex of $\mathfrak{A}$-objects with differential $d_{A^\bullet}$, we write $A^\bullet[j]$ for the *complex shifted by j* (to the left) with $A^i[j] := A^{i+j}$ and $d_{A^\bullet[j]} := (-1)^j d_{A^\bullet}$.

- If $f : B^\bullet \to A^\bullet$ is a morphism of $\mathfrak{A}$-complexes, we define the *mapping cone* M_f of f by

$$M_f := A^\bullet[-1] \oplus B^\bullet$$

  with differential

$$d_{M_f} := \begin{pmatrix} d_{A^\bullet[-1]} & f \\ 0 & d_{B^\bullet} \end{pmatrix},$$

  i.e.

$$d_{M_f}^i : A^{i-1} \oplus B^i \to A^i \oplus B^{i+1}$$
$$(a, b) \quad \mapsto (-d_{A^\bullet}^{i-1} a + f(b), d_{B^\bullet}^i b).$$

- By a *double complex* $A^{\bullet,\bullet}$ of $\mathfrak{A}$-objects, we mean a doubly indexed family of $\mathfrak{A}$-objects $\{A^{p,q}\}_{p,q\in\mathbb{Z}}$ together with vertical differentials

$$d_{\mathrm{I}}^{p,q} : A^{p,q} \to A^{p+1,q}$$

  and horizontal differentials

$$d_{\mathrm{II}}^{p,q} : A^{p,q} \to A^{p,q+1},$$

  which satisfy for all $p, q \in \mathbb{Z}$

$$d_{\mathrm{I}}^{p+1,q} \circ d_{\mathrm{I}}^{p,q} = 0,$$
$$d_{\mathrm{II}}^{p,q+1} \circ d_{\mathrm{II}}^{p,q} = 0 \quad \text{and}$$
$$d_{\mathrm{I}}^{p,q+1} \circ d_{\mathrm{II}}^{p,q} = d_{\mathrm{II}}^{p+1,q} \circ d_{\mathrm{I}}^{p,q}.$$

- If $A^{\bullet,\bullet}$ is a double complex, we define its *total complex*

$$A^\bullet := \operatorname{tot} A^{\bullet,\bullet}$$

  by $A^n := \bigoplus_{p+q=n} A^{p,q}$ with differential

$$d_{A^\bullet}^n = \bigoplus_{p+q=n} \left(d_{\mathrm{I}}^{p,q} + (-1)^p d_{\mathrm{II}}^{p,q} \right) : A^n \to A^{n+1}.$$

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Glossary of Notations

$\|X\|$	set of closed points of a variety X, 4
$\widetilde{M}$	sheaf associated to a module M, 5
$\langle\,\cdot\,,\,\cdot\,\rangle$	homology-cohomology-pairing, 41
$\mathrm{A}[0]$	complex concentrated in degree zero with $(\mathrm{A}[0])^0 = \mathrm{A}$, 94
$\mathrm{A}^{\bullet}[j]$	complex shifted by j, 94
an	functor yielding the associated complex analytic space, 4
$\mathbb{A}^n_k$	affine n-space over a field k, 42
$\mathbb{C}P^n$	complex projective space in the category of schemes, 5
$\mathbb{C}P^n_{\mathrm{an}}$	complex projective space in the category of complex analytic spaces, 5
$\mathbb{C}_X$	constant sheaf with fibre $\mathbb{C}$ on a space X, 10
$\mathrm{C}^p_{\mathrm{sing}}(M;\mathbb{C})$	$\mathbb{C}$-vector space of singular p-cochains of a space M, 11
$\mathcal{C}^p_{\mathrm{sing}}(M;\mathbb{C})$	sheaf of singular p-cochains of a space M, 11
$\check{\mathcal{C}}^p(\mathfrak{U},?)$	Čech functor, 18
$\check{C}^p(\mathfrak{U};\mathcal{F})$	Čech group of a sheaf $\mathcal{F}$, 18
$\mathcal{C}^p_{\mathrm{sing}}(X^{\mathrm{an}},D^{\mathrm{an}};\mathbb{C})$	sheaf of singular p-cochains of a pair $(X^{\mathrm{an}}, D^{\mathrm{an}})$, 32
$\mathrm{C}^\infty_p(M;\mathbb{C})$	$\mathbb{C}$-vector space of smooth singular p-chains of a complex manifold M, 34
$\mathrm{C}^{\mathrm{sing}}_p(M;\mathbb{C})$	$\mathbb{C}$-vector space of singular p-chains of a space M, 34
$\mathrm{C}^p_\infty(M;\mathbb{C})$	$\mathbb{C}$-vector space of smooth singular p-cochains of a complex manifold M, 34
$\mathcal{C}^p_\infty(M;\mathbb{C})$	sheaf of smooth singular p-cochains of a complex manifold M, 34

$\mathcal{C}_\infty^{\bullet;\bullet}(D^{\mathrm{an}};\mathbb{C})$	double complex of sheaves of smooth singular cochains of $D^{\mathrm{an}\bullet}$, 35
$\mathcal{C}_\infty^{\bullet}(D^{\mathrm{an}};\mathbb{C})$	complex of sheaves of smooth singular cochains of a divisor D^{an}, 35
$\mathcal{C}_\infty^{\bullet}(X^{\mathrm{an}},D^{\mathrm{an}};\mathbb{C})$	complex of sheaves of smooth singular cochains of a pair $(X^{\mathrm{an}},D^{\mathrm{an}})$, 35
$D_{i_0\dots i_p}$	intersection of $D_{i_0},\cdots,D_{i_p}$, 12
D_I	intersection of D_i for $i\in I$, 12
$D^\bullet$	simplicial scheme associated to a divisor D, 13
$\mathfrak{D}$	closed covering $(D_i^{\mathrm{an}})_{i=1}^r$, 30
$\triangle^\circ$	interior of a simplex or a point, 53
$\triangle_p^{\mathrm{std}}$	standard p-simplex, 53
Δ	triple coproduct, 66
$\mathcal{E}_M^\bullet$	classical deRham complex of a complex manifold M, 10
ev	evaluation morphism $\mathcal{P}_+\to\mathbb{C}$, 64
$\mathcal{G}_{\mathcal{F}}^\bullet$	Godement resolution of a sheaf $\mathcal{F}$, 8
$\mathcal{G}_{\mathcal{F}^\bullet}^\bullet$	Godement resolution of a complex of sheaves $\mathcal{F}^\bullet$, 8
Γ_p	top-dimensional part of a p-chain $\Gamma\in\mathrm{C}_p^\infty(X^{\mathrm{an}},D^{\mathrm{an}};\mathbb{Q})$, 44
$\mathrm{H}_{\mathrm{dR}}^\bullet(M;\mathbb{C})$	analytic deRham cohomology of a complex manifold M, 11
$\mathrm{H}^\bullet(M;\mathbb{C})$	complex cohomology of a space M, 11
$\mathrm{H}_{\mathrm{dR}}^\bullet(X/k)$	algebraic deRham cohomology of a smooth variety X, 12
$\mathrm{H}_{\mathrm{dR}}^\bullet(D/k)$	algebraic deRham cohomology of a divisor D, 14
$\mathrm{H}_{\mathrm{dR}}^\bullet(X,D/k)$	algebraic deRham cohomology of a pair (X,D), 15
$\check{\mathrm{H}}^p(\mathfrak{U};\mathcal{F})$	Čech cohomology of a sheaf $\mathcal{F}$, 18
$\mathcal{H}^p(\mathcal{F}^\bullet)$	sheafified hypercohomology of a complex of sheaves $\mathcal{F}^\bullet$, 20
$\mathrm{H}_{\mathrm{dR}}^\bullet(D^{\mathrm{an}};\mathbb{C})$	analytic deRham cohomology of a divisor D^{an}, 24
$\mathrm{H}_{\mathrm{dR}}^\bullet(X^{\mathrm{an}},D^{\mathrm{an}};\mathbb{C})$	analytic deRham cohomology of a pair $(X^{\mathrm{an}},D^{\mathrm{an}})$, 24
$\mathrm{H}^\bullet(X^{\mathrm{an}},D^{\mathrm{an}};\mathbb{C})$	complex cohomology of a pair $(X^{\mathrm{an}},D^{\mathrm{an}})$, 28
$\mathrm{H}_{\mathrm{sing}}^\bullet(M;\mathbb{C})$	singular cohomology of a space M, 32
$\mathrm{H}_{\mathrm{sing}}^\bullet(X^{\mathrm{an}},D^{\mathrm{an}};\mathbb{C})$	singular cohomology of a pair $(X^{\mathrm{an}},D^{\mathrm{an}})$, 32
$\mathrm{H}_\infty^\bullet(M;\mathbb{C})$	smooth singular cohomology of a complex manifold M, 34

Index

Erklärung

Ich versichere, dass ich die vorliegende Arbeit selbstständig und nur unter Verwendung der angegebenen Quellen und Hilfsmittel angefertigt habe.
Diese Arbeit wurde in dieser oder ähnlicher Form noch keiner anderen Prüfungsbehörde vorgelegt.

Ort   Datum   Unterschrift