THE PERIOD ISOMORPHISM IN TAME GEOMETRY

ANNETTE HUBER

ABSTRACT. We describe singular homology of a manifold X via simplices $\sigma : \Delta_d \to X$ that satisfy Stokes' formula with respect to all differential forms. The notion is geared to the case of tame geometry (definable manifolds with respect to an o-minimal structure), where it gives a description of the period pairing with de Rham cohomology via definable σ's.

In this note we close a gap in the literature on the period pairing. If X is a differentiable manifold, there is a canonical isomorphism between de Rham cohomology and singular cohomology. It is induced from the *period pairing* between de Rham cohomology and singular homology. The pairing has a good description by integration

$$(\sigma, \omega) \mapsto \int_{\Delta_d} \sigma^* \omega.$$

In order for this formula to make sense, the map σ has to have good regularity properties. A good choice is to restrict to smooth maps. If X is a definable manifold for some o-minimal structure, e.g., if $X \subset \mathbb{R}^N$ is semi-algebraic, then the integral is absolutely convergent without any regularity assumptions. In [HMS17], this was used to give an alternative description of the set of period numbers in terms of semi-algebraic sets. Indeed, such a description is used as a definition for the notion of a period number in [KZ01].

There are two problems that were not addressed in [HMS17]:

(1) in order to get a well-defined pairing on homology, we need to establish Stokes' formula for semi-algebraic σ;
(2) in order to show that the two pairings agree, we need to compare smooth and semi-algebraic σ's.

The same problems also appear in the setting of exponential periods treated in [CHH20], where it was side-stepped, see also Remark 5.5 below. We now present a conceptually clean solution. As in [CHH20], we use input from the structure theory of definable sets: the existence of triangulations that are globally of class C^1 shown by Czapla-Pawlucki in [CP18]. (An alternative is to apply the panel meating method of Ohmoto-Shiota [OS17] instead. It allows us to reparametrise a given semi-algebraic simplex as a C^1-map. There are some problems with this approach, see [CHH20, Section 7].)

In the present note, we solve the two problems by introducing the notion of a simplex *satisfying Stokes*. They are C^1 along open faces, all periods integrals converge and satisfy Stokes' formula. We show that the complex built from these simplices computes singular homology.

Date: August 9, 2022.

We also check the transformation rule and Stoke's formula for definable simplices without any regularity assumptions. This allows us to make the comparison we wanted.

Acknowledgement: This note can be read as an addendum to the paper [CHH20] with Johan Commelin and Philipp Habegger. I thank them for many discussions on issues related with period integrals and tame geometry. I am also indebted to my collegue Sebastian Goette for help with integrability computations. The construction in the appendix is joint work with Johan Commelin. I thank him for allowing me to add it to the paper.

1. Set-up

Following [War83], we define the *standard d-simplex* as

$$\Delta_d = \left\{(a_1,\dots,a_d)\in\mathbb{R}^d | a_i\geq 0, \sum a_i\leq 1\right\}.$$

By an *open face* of Δ_d, we mean the interior of a face (of any dimension) of Δ_d. Throughout we are going to consider continuous maps

$$\sigma:\Delta_d\to\mathbb{R}^N$$

whose restriction to each open face is of class C^1.

Most arguments center on the following extension, see [War83, p. 194]:

$$\begin{aligned}&\hat{\sigma}:\Delta_{d+1}\to\mathbb{R}^N;\\ &(a_0,a_1,\dots,a_d)\mapsto A\sigma(a_1/A,\dots,a_d/A),\quad A=\sum_{i=0}^d a_i.\end{aligned}$$

The simplex $\hat{\sigma}$ has a vertex at 0 and the opposite face equal to σ. It interpolates linearly in between. The map is again continuous (even for $A\to 0$) and C^1 on all open faces. This simplicial version of a homotopy can also be described via

$$\begin{aligned}\bar{\sigma}&:[0,1]\times\Delta_d\to\mathbb{R}^N\\ \bar{\sigma}&:(t,b_1,\dots,b_d)\mapsto(1-t)\sigma(b_1,\dots,b_d).\end{aligned}$$

Consider

$$\begin{aligned}&q\colon[0,1]\times\Delta_d\to\Delta_{d+1}\\ &(t,b_1,\dots,b_d)\mapsto((1-t)(1-\sum_{i=1}^d b_i),(1-t)b_1,\dots,(1-t)b_d).\end{aligned}$$

We have $\bar{\sigma}=\hat{\sigma}\circ q$ because $A=\sum_{i=0}^d a_i=1-t$.

Note that q admits the partial inverse $i:\Delta_{d+1}\smallsetminus 0\to[0,1)\times\Delta_d$ by

$$i:(a_0,a_1,\dots,a_d)\mapsto(A,a_1/A,\dots a_d/A)$$

It is a diffeomorphism.

2. Finite volume

Let $\sigma : \Delta_d \to \mathbb{R}^N$ be as fixed in the last section.

Definition 2.1. We say that σ has *finite volume* if

$$\int_{\Delta_d} \sigma^*\omega$$

converges absolutely for every continuous d-form ω on $\sigma(\Delta_d)$.

Remark 2.2. The pull-back of ω to the interior of Δ_d is a continuous d-form. The integral exists locally. Global existence, i.e, convergence when approaching the boundary, is the issue. The condition is equivalent to integrability (in the measure theoretic sense) of $\sigma^*\omega$, in other words, it is L^1.

Lemma 2.3. *It suffices to check the assumption for the standard d-forms* $\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_d}$ *for all* $\{i_1, \ldots, i_d\} \subset \{1, \ldots, N\}$.

Proof. We write

$$\omega = \sum_I a_I \, \mathrm{d}x_I$$

where the sum is over multi-indices of length d. By assumption the a_I are continuous on $\sigma(\Delta)$, in particular bounded. The pull-back is

$$\sigma^*(\omega) = a_i \circ \sigma \cdot \sigma^*(\mathrm{d}x_I).$$

By assumption, all $\sigma^*(\mathrm{d}x_I)$ are integrable. The $a_i \circ \sigma$ are bounded and continuous. This makes the sum integrable. □

Remark 2.4. In particular, it does not matter if the convergence condition is imposed for C^∞ differential forms or continuous forms. If $d = N$, then our condition is indeed equivalent to finiteness of $\mathrm{vol}(\sigma(\Delta_n))$.

Example 2.5.
(1) If σ is C^1 globally on Δ_d, then $\sigma^*(\mathrm{d}x_I)$ is C^0, in particular integrable.
(2) If σ is semi-algebraic, or more generally definable in some o-minimal structure, then

$$\int_{\Delta_d} \sigma^*(\mathrm{d}x_I) = \int_{\sigma(\Delta_d)} \mathrm{d}x_I$$

converges, see [CHH20, Theorem 3.22] (actually a lot easier).

The notion extends immediately to manifolds.

Definition 2.6. Let X be an N-dimensional C^1-manifold with corners, $\sigma : \Delta_d \to X$ continuous and C^1 on all open faces. We say that σ has *finite volume*, if there is a subdivison of σ such that the pieces are contained in a single chart each and have finite volume there.

Corollary 2.7. *The condition is independent of the choice of subdivison and charts.*

Proof. Independence of the subdivision is obvious. Assume without loss of generality that $\sigma(\Delta_d)$ is contained in two charts $\phi_i : U_i \to V_i \subset \mathbb{R}^N$. Let $\phi_{12} = \phi_2 \circ \phi_1^{-1}$ be the transition map where it is defined. By assumption it is C^1.

Suppose that $\phi_1 \circ \sigma$ has finite volume. We have to check that $\phi_2 \circ \sigma$ has finite volume. By Lemma 2.3 it suffices to consider

$$\sigma^* \phi_2^*(\omega) = \sigma^* \phi_1^* \phi_{12}^* \omega$$

for all smooth forms ω on V_2. The pull-back $\phi_{12}^*\omega$ is a C^0-form on V_1. By assumption, its pull-back to Δ_d is integrable. □

Lemma 2.8. *A measurable differential form ω on Δ_{d+1} is integrable if and only if $q^*\omega$ is integrable on $[0,1] \times \Delta_d$.*

Proof. The map q is a diffeomorphism up to a set of measure 0. The sign of the Jacobian determinant cannot change because it does not vanish for diffeomorphisms. One of the integrals is finite if and only if the other is. □

We need to check that the notion is stable under homotopies.

Proposition 2.9. *If σ has finite volume, then so does the cone $\hat{\sigma}$.*

Proof. By the last lemma it suffices to establish finiteness for $\bar{\sigma}$. Let $\omega = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{d+1}$ and ω_j the wedge product with the factor $\mathrm{d}x_j$ dropped. We compute:

$$\frac{\partial}{\partial t}\bar{\sigma}_j = -\sigma_j$$
$$\frac{\partial}{\partial a_i}\bar{\sigma}_j = (1-t)\frac{\partial}{\partial a_i}\sigma_j$$

These are the entries of the Jacobian matrix. Hence

$$\bar{\sigma}^*\omega = \sum_{j=1}^{d+1} -(-1)^j (1-t)^d \sigma_j \sigma^*(\omega_j).$$

It suffices to treat each summand separately. By assumption $\sigma^*(\omega_j)$ is integrable. The function σ_j is continuous, making $\sigma_j\sigma^*(\omega_j)$ integrable on Δ_d. By Fubini this makes $(1-t)^d\sigma_j\sigma^*(\omega_j)$ integrable on $[0,1] \times \Delta_d$. □

Corollary 2.10. *Let X be a C^1-manifold with corners. Then the complex of singular simplices σ such that all faces have finite volume computes singular homology.*

Proof. The assumption is stable under the boundary map, so we get a well-defined complex. To see that the subcomplex computes singular homology, we go through the argument in [War83, Section 5.31].

Locally in each ball, the complex is contractible. The simplicial homotopy in [War83, p. 194]is given by $\sigma \mapsto \hat{\sigma}$. The faces of $\hat{\sigma}$ are σ and faces of the form $\hat{\tau}$ for a face τ of σ. By the proposition, they all have finite volume, making the homotopy well-defined. □

3. Stokes

Let X be a C^1-manifold with corners.

Definition 3.1. We say that $\sigma : \Delta_d \to X$ *satisfies Stokes* if σ and $\partial\sigma$ have finite volume and for every smooth $(d-1)$-form ω on a neighbourhood of $\sigma(\Delta_d)$ we have the formula

$$\int_{\Delta_d} \sigma^*(\,\mathrm{d}\omega) = \int_{\partial\Delta_d} \sigma^*(\omega).$$

Again we want to check that the condition is well-behaved under our homotopies.

Lemma 3.2. *Assume that σ and $\partial\sigma$ have finite volume. Let ω be a $(d+1)$-form on a neighbourhood of $\hat{\sigma}(\Delta_{d+1})$. Then $\hat{\sigma}$ satisfies Stokes on Δ_{d+1} if and only if $\bar{\sigma}$ satisfies Stokes on $[0,1]\times\Delta_{d+1}$.*

Proof. We compare the contributions on $[0,1]\times\Delta_d$ and Δ_{d+1}. By the transformation formula and the arguments that we used in order to show convergence, they match up. The only exception is the face $\{1\}\times\Delta_d$ which does not show up in $\partial\Delta_{d+1}$. However, its image in $\mathbb{R}^N$ is constant, hence the pull-back of $\mathrm{d}\omega$ to this face vanishes. It does not contribute to the sum in Stokes' formula. $\square$

We work on $I\times\Delta_d$ from now on (where $I=[0,1]$) and think of the first coordinate as time. The computation becomes clearer when we allow slightly more generality than $\bar{\sigma}$.

Lemma 3.3. *Let $\sigma:\Delta_d\to\mathbb{R}^n$ be continuous and C^1 on all open faces, $f:I\to\mathbb{R}$ a C^1-function, and $\tau: I\times\Delta_d\to\mathbb{R}^n$ given by*

$$\tau(t,b_1,\dots,b_d)=f(t)\sigma(b_1,\dots,b_n).$$

Let

$$\eta=h\,\mathrm{d}x_1\wedge\dots\wedge\,\mathrm{d}x_d$$

with a continuous function h. Then

$$\tau^*\eta=A+B$$

where

$$\begin{aligned}A&=\tau^*(h)f^d\omega\\ B&=\,\mathrm{d}t\wedge\tau^*C\\ C&=h\sum_i(-1)^{i-1}x_i\frac{\partial f}{\partial t}\,\mathrm{d}x_1\wedge\dots\wedge\,\mathrm{d}\dot{x}_i\dots\wedge\,\mathrm{d}x_d.\end{aligned}$$

The restriction to the faces of $I\times\Delta_d$ are

$$\tau^*\eta|_{\{0,1\}\times\Delta_d}=A|_{\{0,1\}\times\Delta_d}$$

and

$$\tau^*\eta_{I\times F}=B|_{I\times F}$$

for all $(d-1)$-faces F of Δ_d.

Proof. We have $\tau_i=f(t)\sigma_i$ and hence

$$\mathrm{d}\tau_i=\frac{\partial f}{\partial t}\sigma_i\,\mathrm{d}t+f(t)\,\mathrm{d}\sigma_i$$

This implies

$$\begin{aligned}&\mathrm{d}\tau_1\wedge\dots\wedge\,\mathrm{d}\tau_d\\ &=f(t)^d\,\mathrm{d}\sigma_1\wedge\dots\wedge\mathrm{d}\sigma_d+\sum_i(-1)^{i-1}\frac{\partial f}{\partial t}f(t)^{d-1}\sigma_i\,\mathrm{d}t\wedge\mathrm{d}\sigma_1\wedge\dots\wedge\mathrm{d}\dot{\sigma}_i\dots\wedge\mathrm{d}\sigma_d\end{aligned}$$

where $\dot{\sigma}_i$ means that we omit this factor. We introduce

$$\omega=\,\mathrm{d}\sigma_1\wedge\dots\wedge\,\mathrm{d}\sigma_d,\quad \omega_i=\,\mathrm{d}\sigma_1\wedge\dots\wedge\,\mathrm{d}\dot{\sigma}_i\dots\wedge\,\mathrm{d}\sigma_d$$

This allows us to write

$$\mathrm{d}\tau_1 \wedge \cdots \wedge \mathrm{d}\tau_d = f(t)^d \omega + \sum_i (-1)^{i-1} \frac{\partial f}{\partial t} f(t)^{d-1} \sigma_i \,\mathrm{d}t \wedge \omega_i$$

and hence

$$\tau^*\eta = (h \circ \tau) \left(f(t)^d \omega + \sum_i (-1)^{i-1} \frac{\partial f}{\partial t} f(t)^{d-1} \sigma_i \,\mathrm{d}t \wedge \omega_i \right).$$

We define the firsts summand as A and the second of B. We then have

$$B = \mathrm{d}T \wedge \tau^* C$$

as claimed.

We now restrict to faces. The restriction of $\tau^*\eta$ to $t = 0$ is

$$\tau^*(h)|_{\{0\}\times\Delta_d} \mathrm{d}(f(0)\sigma_1) \wedge \cdots \wedge \mathrm{d}(f(0)\sigma_d) = \tau^*(h)|_{\{0\}\times\Delta_d} f(0)^d \omega = A|_{\{0\}\times\Delta_d}.$$

Let F be a $(d-1)$-face of Δ_d. Then

$$A|_{I\times F} = (\tau^*(h) f^d))|_{I\times F} \omega|_F = 0$$

because ω is a d-form on a $(d-1)$-dimensional face. □

Proposition 3.4. *If σ satisfies Stokes, so does $\hat{\sigma}$.*

Proof. As in the previous section, it suffices to consider $\bar{\sigma}$. We write $\mathrm{d} = \mathrm{d}_t + \mathrm{d}_s$ for the decomposition into the time and space derivative on $I \times \Delta_d$. Passing to barycentric subdivisions if necessary, we may assume that $\sigma(\Delta_d)$ is contained in a chart. Without loss of generality $X = \mathbb{R}^N$. Every smooth d-form on $\mathbb{R}^N$ decomposes as

$$\sum_I h_I dx_I.$$

It suffices to consider each summand separately. Without loss of generality it suffices to verify the formula for

$$\eta = h \,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_d$$

with smooth h. We apply the last lemma to $f(t) = (1-t)$ and $\tau = \bar{\sigma}$.

We claim that

$$\int_{I\times\Delta_d} \mathrm{d}\tau^*\eta = \int_{\partial I\times\Delta_d} \tau^*\eta.$$

Under the decomposition of $\tau^*\eta$ in the last lemma, this is equivalent to

$$\int_{I\times\Delta_d} \mathrm{d}A = \int_{(\partial I)\times\Delta_d} A$$

and

$$\int_{I\times\Delta_d} \mathrm{d}B = \int_{I\times(\partial\Delta_d)} B.$$

We have

$$\mathrm{d}A = \mathrm{d}_t(\tau^* h \cdot f^d) \wedge \omega = \frac{\partial(\tau^* h \cdot f^d)}{\partial t} \,\mathrm{d}t \wedge \omega.$$

The partial derivative is continuous on I because f and h are C^1 (actually smooth) and $\frac{\partial \tau_i}{\partial t} = \frac{\partial f}{\partial t}\sigma_i$ is continuous. (Note that σ_i does not depend on t!) By assumption ω is integrable and independent of t. This makes $\mathrm{d}t \wedge \omega$ and then dA integrable and we may apply Fubini. We first integrate in time direction, then in the spatial direction. By the fundamental theorem of calculus we may evaluate

$$\begin{aligned}\int_{I\times\Delta_d} \mathrm{d}A &= \int_{\Delta_d}\int_I d_t(\tau^* h f^d)\wedge\omega\\ &= \int_{\Delta_d}(\tau^* h f^d)|_{\{1\}\times\Delta_d}\omega - \int_{\Delta_d}(\tau^* h f^d)|_{\{0\}\times\Delta_d}\omega\\ &= \int_{(\partial I)\times\Delta} A.\end{aligned}$$

We write

$$B = \mathrm{d}t\wedge\tau^* C,\quad \mathrm{d}B = \mathrm{d}t\wedge\tau^*\,\mathrm{d}_s C.$$

Again we may apply Fubini because $dB = d\tau^*\eta - dA$ is integrable. This time we take the integral in the spatial direction first. For fixed t, the differential form C^t on $\mathbb{R}^N$ is smooth. By assumption, σ satisfies Stokes, hence

$$\int_{\{t\}\times\Delta_d}(\tau^*\,\mathrm{d}_s C)^t = \int_{\Delta_d}\sigma^* d_s C^t = \int_{\partial\Delta_d}\sigma^* C^t.$$

This yields

$$\int_{I\times\Delta_d}\mathrm{d}B = \int_{I\times(\partial\Delta_d)} B.$$

□

We have treated differential forms on $\mathbb{R}^N$ above, but the notion immediately generalises to smooth manifolds.

Corollary 3.5. *Let X be a C^1-manifold with corners. The complex with simplices σ s.t. all faces satisfy Stokes computes singular homology.*

Proof. The same argument as for the proof of Corollary 2.10 shows that the subcomplex computes singular homology. □

4. The tame case

We fix an o-minimal structure on $\mathbb{R}^N$, e.g., the theory of semi-algebraic sets defined over a subfield of $\mathbb{R}$. This means that we have chosen a system of *definable* subsets of $\mathbb{R}^n$ for all n, satisfying certain axioms, see [vdD98]. Our discussion was chosen to apply to this case.

In [CHH20, Chapter 3], we introduced the notion of a definable C^p-manifold (with corners) for $\infty \geq p \geq 0$ and the basics of integration theory for differentiable forms. We will restrict to manifolds without boundary or corners in this chapter for ease of exposition. Everything would work in the general case.

Let X be a definable manifold and $G \subset X$ a definable subset of dimension d. We denote by $\mathrm{Reg}_d(G)$ the locus where G is a C^p-submanifold of X. By [CHH20, Lemma 3.8], the subset is definable and the complement has dimension strictly less than d. A *pseudo-orientation* on G is the choice of a definable open subset $U \subset \mathrm{Reg}_d(G)$ such that $\dim(G - U) < \dim G$ together with an orientation on U, see [CHH20, Definition 3.14]. Two such pseudo-orientations are equivalent if they agree on the intersection of the open sets. If ω is a continuous d-form on G, then we can define

$$\int_G \omega := \int_U \omega.$$

The value only depends on the equivalence class of the pseudo-orientation. If G is compact, then the integral is absolutely convergent, see [CHH20, Theorem 3.22].

We now show that standard properties of integration extend. Our first aim is the transformation rule.

Definition 4.1. Let $p \geq 1$. Let X_1, X_2 be definable C^p-manifolds, $G_1 \subset X_1$ a definable subset of dimension d. Let $f : G_1 \to X_2$ be a continuous definable map. We put

$$\mathrm{Reg}(f) = \{x \in \mathrm{Reg}_d(G_1) | f \text{ is } C^p \text{ near } x\}.$$

Lemma 4.2. *The subset* $\mathrm{Reg}(f) \subset G_1$ *is open, definable and* $\dim(G_1 - \mathrm{Reg}(f)) < d$.

Proof. The condition is open. By [vdDM96, B.7] the set is definable. It remains to check the dimension property. By [CHH20, Lemma 3.8], the set $G_1 - \mathrm{Reg}_d(G_1)$ has dimension smaller than d. We replace G_1 by $\mathrm{Reg}_d(G_1)$. By [vdDM96, C.2] there is an open subset V (indeed, a union of cells) of G_1 such that $\dim(G_1 - V) < d$ and $f|_V$ is C^p. □

Definition 4.3. Let $p \geq 1$. Let $G_1 \subset X_1$ and $G_2 \subset X_2$ be pseudo-oriented definable subsets of dimension d in definable C^p-manifolds, with orientations defined on U_1 and U_2. A continuous definable map $f : G_1 \to G_2$ is *compatible with orientations* if there is a definable open $U \subset U_1 \cap \mathrm{Reg}(f) \cap f^{-1}(U_2)$ such that the map $U \to U_2$ is orientable and $\dim(f(G_1 - U)) < d$.

Remark 4.4. Here $\mathrm{Reg}(f)$ refers to the regularity locus of the composition $G_1 \to G_2 \to X_2$. On the set $U_1 \cap f^{-1}(U_2) \cap \mathrm{Reg}(f)$, the induced map is C^p. By admitting the smaller set U, the notion becomes independent of the choice of representative for the pseudo-orientations.

Proposition 4.5. *Let* $f : G_1 \to G_2$ *be a continuous definable map between pseudo-oriented definable subsets of definable* C^p*-manifolds. Then the transformation rule holds, i.e., for any continuous differential form* ω *on* G_2*, we have*

$$\int_{G_1} f^*\omega = \int_{f(G_1)} \omega$$

where $f(G_1)$ *is pseudo-oriented as a subset of* G_2.

Remark 4.6. If $f(G_1)$ has empty interior in G_2, then the statement has to be understood as saying that the left hand side vanishes, see [CHH20, Remark 3.15].

Proof. Neither value changes if we replace G_1 by an open subset such that the complement has measure 0. Without loss of generality, $G_1 = \mathrm{Reg}(f)$ and G_1 is oriented. Let $U \subset G_1$ be as in the definition of compatiblity of f with orientation. By the usual transformation formula for differentiable maps, we have

$$\int_U f^*\omega = \int_{f(U)} \omega.$$

Let $G' = G_1 - U$. As $\dim(f(G')) < d$, we also have

$$\int_{f(U)} \omega = \int_{f(G_1)} \omega.$$

Moreover, all fibres of $f : G' \to X_2$ have positive dimension, hence the Jacobian does not have full rank. This implies $f^*\omega|_{G'} = 0$ and hence

$$\int_U f^*\omega = \int_{G_1} f^*\omega.$$

□

Lemma 4.7. *Let $f : G_1 \to G_2$ be a definable homeomorphism between definable subsets of definable manifolds. Given a pseudo-orientation on G_1 there is a unique pseudo-orientation on G_2 making f compatible with orientations, and conversely.*

Proof. We may remove the complements of $\mathrm{Reg}_d(G_1)$ and $\mathrm{Reg}_d(G_2)$ as well as $\mathrm{Reg}(f)$ and $\mathrm{Reg}(f^{-1})$ from the situation. So without loss of generality G_1 and G_2 are manifolds and f is a diffeomorphism. We can then use f to transport the orientation. □

Remark 4.8. The result is completely standard for oriented manifolds, even with boundary. Even though we usually think of orientations in terms of the tangent bundle, it is actually a completely topological notion that can be determined in terms of homology, see [Hat02, Section 3.3].

Proposition 4.9. *Let X be definable manifold with corners, and $\sigma : \Delta_d \to X$ be a definable continuous map, ω a C^1-form on a neighbourhood of $\sigma(\Delta_d)$. Then Stokes' formula holds:*

$$\int_{\Delta_d} \sigma^*(\,\mathrm{d}\omega) = \int_{\partial\Delta_d} \sigma^*\omega.$$

Proof. We begin with the case $X = \mathbb{R}^N$. Let $\Gamma \subset \Delta_d \times \mathbb{R}^N$ be the graph of σ. We choose a definable triangulation of Γ that is globally C^1. It exists by Czapla-Pawlucki [CP18]. The projection to the first factor is a triangulation (K, Φ) of Δ^d such that both Φ and $\sigma \circ \Phi$ are globally C^1. The orientation on Δ_d defines an orientation on $|K|$ and all simplices in K. By the transformation rule for Φ

$$\int_{|K|} \Phi^*\sigma^*\,\mathrm{d}\omega = \int_{\Delta_d} \sigma^*\,\mathrm{d}\omega, \quad \int_{\partial|K|} \Phi^*\sigma^*\omega = \int_{\partial\Delta_d} \sigma^*\omega.$$

Let

$$\tau = \sigma \circ \Phi : \Delta_d \to \mathbb{R}^N$$

be a simplex in this triangulation. The map is C^1 (globally, not only on open faces). By Stokes' theorem (in the C^1-version of Whitney of [Whi57], see also [CHH20, Theorem 1.4]) we have

$$\int_{\Delta_d} \tau^*(\,\mathrm{d}\omega) = \int_{\partial\Delta_d} \tau^*(\omega).$$

We sum over all d-simplices in K. The open subset $\bigcup_\tau \tau(\Delta_d)$ is open and dense in $|K|$, hence the sum is

$$\int_{|K|} \Phi^*\sigma^*\,\mathrm{d}\omega.$$

Now consider the sum over the boundaries. Two things can happen: If F is an open $(d-1)$-simplex of K, then it is either fully contained in the interior of $|K|$ or fully contained in $\partial|K|$. In the first case, there is a second d-simplex with face F, but opposite orientation. These contributions cancel. In the second case, F is part of a triangulation of $\partial|K|$. The sum gives

$$\sum_{K_{d-1}} \int_{\Delta_{d-1}} \tau^*\omega = \int_{\partial|K|} \tau^*\omega.$$

Putting the equalities together, we have Stokes' formula for σ in the affine case.

For general X, we may apply repeated barycentric subdivison such that the image of smaller simplex is contained in a single chart. □

5. Period isomorphisms

We are now ready to show that the period pairing for definable manifolds can be computed via definable simplices. Throughout this section, let X be smooth definable manifold with corners.

Definition 5.1. We set:

- $S_d^{\mathrm{sing}}(X)$ the free abelian group with basis continuous maps $\sigma : \Delta_d \to X$;
- $S_d^{\infty}(X)$ the free abelian group with basis smooth maps $\sigma : \Delta_d \to X$;
- $S_d^{\mathrm{Stokes}}(X)$ the free abelian group with basis continuous maps $\sigma : \Delta_d(X)$ which are C^1 on all open faces and such that all faces have finite volume (see Definition 2.1) and satisfy Stokes (see Definition 3.1);
- $S_d^{\mathrm{def},C^1}(X)$ the free abelian group with basis definable continuous maps $\sigma : \Delta_d \to X$ which are C^1 on all open faces;
- $S_d^{\mathrm{def}}(X)$ the free abelian group with basis definable continuous maps $\sigma : \Delta_d \to X$;
- $A^d(X)$ the space of all smooth d-forms on X.

In each case, the groups organise into a complex with the standard differential for singular homology, and de Rham cohomology, respectively. The complexes $S_*^{\mathrm{def}}(X)$ are functorial for all continuous definable maps between definable manifolds with corners. In applications, it is often helpful to pass to a subcomplex with a finite basis, even if functoriality is lost. This is were simplicial homology comes in. Definable triangulations exist, see the discussion in the appendix.

Definition 5.2. Let (K,Φ) be a definable triangulation of X. We set

- $S_d^{\Delta}(X)$ the free abelian group with basis the elements of K_d.

We obtain a complex $S_*^{\Delta}(X)$ with the differential of simplicial homology. It computes singular homology of X.

Corollary 5.3. *The inclusions*

$$\begin{tikzcd}[column sep=tiny]
 & & S^{\mathrm{sing}}_*(X) & & \\
 & S^{\mathrm{def}}_*(X) \arrow[ur, hook] & & S^{\mathrm{Stokes}}_*(X) \arrow[ul, hook'] & \\
S^{\Delta}_*(X) \arrow[ur, hook] & & S^{\mathrm{def},C^1}_* \arrow[ul, hook'] \arrow[ur, hook] & & S^{\infty}_*(X) \arrow[ul, hook']
\end{tikzcd}$$

are natural quasi-isomorphisms.

Proof. All complexes compute singular homology. For the Stokes version we pointed this out before. The case of $S^{\infty}_*(X)$ is in [War83, Section 5.31]. The same argument also gives the definable case. This uses that fact that if σ is definable, then so is $\hat{\sigma}$. $\square$

The *period pairing*

$$S^{\infty}_d(X) \times A^d(X) \to \mathbb{R}, \quad (\sigma, \omega) \to \int_{\Delta_d} \sigma^*\omega$$

induces a pairing of complexes by Stoke's theorem. Stokes's theorem also holds for $S^{\mathrm{Stokes}}_*(X)$, its subcomplex $S^{\mathrm{def},C^1}(X)$ and for $S^{\mathrm{def}}_*(X)$. We also get well-defined pairings of complexes in these cases.

Theorem 5.4. *Let X be a definable manifold with corners and (K, h) a definable triangulation of X. Then the period pairing can be computed by integration on continuous definable simplices. In other workds, the pairing extends to a pairing of complexes of $A^*(X)$ with $S^{\mathrm{Stokes}}_*(X)$, $S^{\mathrm{def},C^1}_*(X)$, $S^{\mathrm{def}}_*(X)$ and $S^{\Delta}_*(X)$ in a compatible way with the quasi-isomorphisms*

$$\begin{tikzcd}[column sep=tiny]
 & S^{\mathrm{def}}_*(X) & & S^{\mathrm{Stokes}}_*(X) & \\
S^{\Delta}_*(X) \arrow[ur, hook] & & S^{\mathrm{def},C^1}_* \arrow[ul, hook'] \arrow[ur, hook] & & S^{\infty}_*(X) \arrow[ul, hook']
\end{tikzcd}$$

Proof. In each case, the pairing is given by integration. This means that the pairings are compatible.

To make the pairings well-defined as pairings of complexes, we have to check Stokes' formula. For $S^{\mathrm{Stokes}}_*(X)$ it holds by assumption. For its subcomplex $S^{\mathrm{def},C^1}(X)$ and the full $S^{\mathrm{def}}_*(X)$ it holds by Proposition 4.9. $\square$

Remark 5.5. In [CHH20], we describe singular homology with simplices $\sigma : \Delta_d \to X$ which are globally C^1-simplices, not only on open faces. No definability assumptions are made. We still need to show that every homology class is represented by definable simplices. For this fact we exploit the full strength of [CP18] to construct a definable C^1-triangulation of X, see [CHH20, Proposition 7.6]. Theorem 5.4 is conceptually clearer and more flexible.

Appendix A. Existence of triangulations

This appendix is joint work with Johan Commelin. As in Section 4 we work in the setting of definable manifolds in a fixed o-minimal structure. The existence of triangulations is known for definable sets. We extend this to the manifold setting.

Remark A.1. In the semi-algebraic case, every semi-algebraic manifold is affine, that is, there is a semi-algebraic homeomorphism onto a subset of $\mathbb{R}^n$. This is due to [Rob83]. In particular, a semi-algebraic triangulation exists. By [CHH20, Proposition 7.6] the result can be strengthened to make the triangulation globally C^1 or (with the same proof) C^p along all open faces. Note that these additional regularity facts are ot needed in the main text.

It is likely that Robson's result generalises to all o-minimal structures, but we are not aware of a reference. We use an alternative argument instead.

Lemma A.2. *Let $p \geq 0$. Let X_1 and X_2 be compact definable subsets of some ambient definable C^p-manifold with corners. Denote by X the union $X_1 \cup X_2$ and by $B = X_1 \cap X_2$. Let $A \subset X$ be definable subset. Assume that X_1 has a definable triangulation relative to B and $A \cap X_1$ which is C^p on all open faces of simplices. Assume that X_2 is affine. Then there is a definable triangulation of X relative to Arelative to A which is C^p on all open faces of simplices.*

Proof. We start with a definable triangulation $\mathcal{T}_1 = (h_1, K_1)$ of X_1 relative to B which is C^p on open faces.

Note that for every set of vertices $v_0, \dots, v_n \in K_1$ there is at most one open n-simplex with these vertices because this is the case for a simplicial complex. We will write $(v_0, \dots, v_n)$ for this simplex and $(h_1(v_0), \dots, h_1(v_n))$ for its image in X_1. Without loss of generality we may assume that for every simplex $(v_0, \dots, v_n)$ in K_1 such that $h_1(v_0), \dots, h_1(v_n)$ lie in B the entire simplex $(v_0, \dots, v_n)$ lies in B (pass to the barycentric subdivision if necessary). Now choose a triangulation $\mathcal{T}_2 = (h_2, K_2)$ of X_2 relative to the images of elements of $\mathcal{T}_1|_B$ that is C^p on closed simplices. It exists by [vdD98, Chapter 8] because X_2 is affine. Again we may assume that if the images of the vertices of a simplex are in B, then so is the image of the simplex. On B, the triangulation $\mathcal{T}_2$ "subdivides" $\mathcal{T}_1$. It remains to modify $\mathcal{T}_1$ on $X_1 \smallsetminus B$ such that the triangulations become compatible.

We will now construct a set $K \subset K_1 \times K_2$ of simplices, as follows. For every simplex

$$\sigma = (v_0, \dots, v_m, b_0, \dots, b_n) \in K_1$$

with $h_1(b_0), \dots, h_1(b_n)$ in B and $h_1(v_0), \dots, h_1(v_m) \notin B$, and for every simplex $\tau = (w_0, \dots, w_s) \in \mathcal{T}_2|_{h_1(b_0), \dots, h_1(b_n))}$ we add $(v_0, \dots, v_m, w_0, \dots, w_s)$ to K.

We make some remarks about this construction:

- The condition $\tau \in \mathcal{T}_2|_{(h_1(b_0), \dots, h_1(b_n))}$ is meaningful, because the image of the entire simplex $(h_1(b_0), \dots, h_1(b_n))$ is contained in B, by assumption. In particular, we get a triangulation of $h_1(\bar{\sigma})$ that is also compatible with the same construction on the faces of σ.
- By taking $m = 0$, we see that K contains all simplices of K_2.

- Similarly, by taking $n = 0$, we see that K contains all simplices of K_1 that do not have faces in B.

The next step is to define the triangulation map $h : |K| \to X$. We do this on closed simplices such that the definition is compatible with restriction to the faces. We fix σ and τ.

Since X_1 is compact, the closed simplex $\bar{\sigma} \in K_1$ can be identified with the standard simplex. We are given a $\tilde{k}$-semi-algebraic map $h_1|_{\bar{\sigma}} \colon \bar{\Delta}_{m+n+1} \to X_1$ that is a homeomorphism onto its image. The simplex τ gives a $\tilde{k}$-semi-algebraic map $h_2|_{\bar{\tau}} \colon \bar{\Delta}_s \to B$, again a homeomorphism onto its image. Consider $g = h_1|_{\bar{\sigma}}^{-1} \circ h_2|_{\bar{\tau}} \colon \bar{\Delta}_s \to \bar{\Delta}_{m+n+1}$. We define a new map $h : \bar{\Delta}_{s+m+1} \to X_1$ by mapping

$$\sum_{i=0}^{s+m+1} a_i e_i \mapsto h_1|_{\bar{\sigma}} \left(\sum_{i=0}^{m} a_i e_i + ag \left(\frac{1}{a} \sum_{i=0}^{s} a_{i+m+1} e_i \right) \right).$$

where the scaling factor a is defined to be $\sum_{i=0}^{s-1} a_{i+m+1}$. For this, we check the limit when a tends to 0. The value of g is bounded, hence $ag(\cdot)$ tends to 0, when $a \to 0$. We apply the continuous function h_1, so the limit is

$$h_1|_{\bar{\sigma}} \left(\sum_{i=0}^{m} a_i e_i \right).$$

The map h takes the vertex v_i (identified with e_i in the formula) to $h_1(v_i)$ and the vertex w_j (identified with e_{j+m+1} in the formula) to $h_2(w_j)$. The map is clearly definable and C^p on all open faces. □

Proposition A.3. *Let X be a compact semi-algebraic C^p-manifold with corners, $A_1, \dots, A_M$ definable subsets of X. Then there is a definable triangulation of X relative to $A_1, \dots, A_M$ which is C^p on open faces and such that every simplex is contained in an affine chart.*

Proof. Let $U_1, \dots, U_n \subset X$ be an open cover of an atlas, $\phi_i \colon U_i \to V_i$ the charts with $V_i \subset \mathbb{R}^{n_i}_{\geq 0} \times \mathbb{R}^{m_i}$ open definable. In particular, the transition maps are C^p and definable.

For every $P \in X$ there is a compact semi-algebraic neighbourhood X_P contained in one of the U_i. Finitely many of these suffice to cover X. Let $X_1, \dots, X_m$ be such a cover. We start with a definable triangulation on X_1 relative to $A_i \cap X_1$ and relative to all $X_1 \cap X_I$ for $X_I = \bigcap_{i \in I} X_i$ for all $I \subset \{2, \dots, n\}$, and assume that is C^p on all open faces. By the preceding proposition, we obtain a definable triangulation on $X_1 \cup X_2$ relative to $A_i \cap (X_1 \cup X_2)$ and $(X_1 \cup X_2) \cap X_I$ for all I, that is also C^p on all open faces. We proceed inductively until we have found the desired triangulation of X. □

Remark A.4. In the case of an o-minimal structure where every definable subset of $\mathbb{R}^N$ admits a partition into smooth cells, e.g., in the semi-algebraic case, the construction gives a triangulation of compact manifolds by simplices which are smooth on all faces.

References

[CHH20] Johan Commelin, Philipp Habegger, and Annette Huber. Exponential periods and o-minimality, 2020. revised 2022, arXiv:2007.08280.

[CP18] Małgorzata Czapla and Wiesław Pawłucki. Strict C^1-triangulations in o-minimal structures. *Topol. Methods Nonlinear Anal.*, 52(2):739–747, 2018.

[Hat02] A. Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[HMS17] Annette Huber and Stefan Müller-Stach. *Periods and Nori motives*, volume 65 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer, Cham, 2017. With contributions by Benjamin Friedrich and Jonas von Wangenheim.

[KZ01] M. Kontsevich and D. Zagier. Periods. In *Mathematics unlimited—2001 and beyond*, pages 771–808. Springer, Berlin, 2001.

[OS17] Toru Ohmoto and Masahiro Shiota. C^1-triangulations of semialgebraic sets. *J. Topol.*, 10(3):765–775, 2017.

[Rob83] Robert Robson. Embedding semi-algebraic spaces. *Math. Z.*, 183(3):365–370, 1983.

[vdD98] L. van den Dries. *Tame Topology and O-minimal Structures*, volume 248 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1998.

[vdDM96] Lou van den Dries and Chris Miller. Geometric categories and o-minimal structures. *Duke Math. J.*, 84(2):497–540, 1996.

[War83] F. W. Warner. *Foundations of differentiable manifolds and Lie groups*, volume 94 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1983. Corrected reprint of the 1971 edition.

[Whi57] Hassler Whitney. *Geometric integration theory*. Princeton University Press, Princeton, N. J., 1957.