COMPARING NATURAL VOLUME FORMS ON Gl_n

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ABSTRACT. There are two natural choices for a volume form on the algebraic group $\mathrm{Gl}_n/\mathbb{Q}$: the first is the integral form (unique up to sign), the other is the product of the primitive classes in algebraic de Rham cohomology. We work out the explicit comparision factor between the two.

1. INTRODUCTION

Consider the group Gl_n and its Lie algebra $\mathfrak{gl}_n$. Both are defined over $\mathbb{Z}$. The isomorphism

$$H^i(\mathfrak{gl}_n, \mathbb{Q}) \to H^i_{\mathrm{dR}}(\mathrm{Gl}_n, \mathbb{Q})$$

([Ho] Lemma 4.1) can be used to define an integral structure in algebraic de Rham cohomology as the image of integral Lie algebra cohomology.

Definition 1.1. Let

$$\rho^{\mathrm{dR}}_{\mathbb{Z}} \in H^{n^2}_{\mathrm{dR}}(\mathrm{Gl}_n, \mathbb{Q})$$

be the image of a generator of

$$H^{n^2}(\mathfrak{gl}_n, \mathbb{Z}) = \bigwedge^{n^2} \mathfrak{gl}_n^*$$

where $\mathfrak{gl}_n^*$ is the $\mathbb{Z}$-dual of the integral Lie algebra $\mathfrak{gl}_n$.

Note that $\rho^{\mathrm{dR}}_{\mathbb{Z}}$ is only well-defined up to sign.

Let $p^{\mathrm{dR}}_i \in H^{2i-1}_{\mathrm{dR}}(\mathrm{Gl}_n, \mathbb{Q})$ be the primitive element normalized as suspension of the universal Chern class $c^{\mathrm{dR}}_i \in H^{2i}(B\mathrm{Gl}_n, \mathbb{Q})$.

Definition 1.2. We call

$$\omega^{\mathrm{dR}} = p^{\mathrm{dR}}_1 \wedge \cdots \wedge p^{\mathrm{dR}}_n \in H^{n^2}_{\mathrm{dR}}(\mathrm{Gl}_n, \mathbb{Q})$$

the *Borel element*.

The Borel element occurs in his definition of a regulator on higher algebraic K-theory of number fields. In [Bo2] he relates it to special values of Dedekind ζ-functions of number fields, at least up to a rational factor.

The purpose of this note is to verify the following comparison result:

Proposition 1.3.

$$\omega^{\mathrm{dR}} = \pm \left(\prod_{j=1}^{n} (j-1)! \right) \rho_{\mathbb{Z}}^{\mathrm{dR}}$$

Our strategy is to use the comparison isomorphism between de Rham cohomology and singular cohomology, which is compatible with Leray spectral sequences, products and Chern classes. The structure of singular cohomology of $\mathrm{Gl}_n(\mathbb{C})$ with integral coefficients is well-known and in particular the product of the primitive classes is an integral generator of $H_{\mathrm{sing}}^{n^2}(\mathrm{Gl}_n(\mathbb{C}), \mathbb{Z})$. It remains to compare it with $\rho_{\mathbb{Z}}^{\mathrm{dR}}$. This is done by integrating the differential form $\rho_{\mathbb{Z}}^{\mathrm{dR}}$ over a fundamental cycle, i.e., over $\mathrm{U}(n)$.

The interest for this result comes from an ongoing joint project of the first author and G. Kings relating the unkown rational factor in Borel's work to the Bloch-Kato conjecture for Dedekind-ζ-functions.

2. Singular cohomology

Definition 2.1. Let $E\mathrm{Gl}_n$ be the simplicial scheme with $E_n\mathrm{Gl}_n = \mathrm{Gl}_n^k$ with boundary maps given by projections and degeneracies by diagonals. It carries a natural diagonal operation of Gl_n. The classifying space $B\mathrm{Gl}_n$ is the quotient of $E\mathrm{Gl}_n$ by this action.

We view $\mathrm{Gl}_n(\mathbb{C})$ etc. as topological spaces with the analytic topology. Let $\mathrm{U}(n)$ be the unitary group as real Lie group.

Proposition 2.2 (Borel). *Let $c_j^{\mathrm{sing}} \in H_{\mathrm{sing}}^{2j}(B\mathrm{Gl}_n(\mathbb{C}), \mathbb{Q})$ be the universal j-th Chern class in singular cohomology. Let*

$$s_j : H_{\mathrm{sing}}^{2j}(B\mathrm{Gl}_n(\mathbb{C}), \mathbb{Z}) \to H_{\mathrm{sing}}^{2j-1}(\mathrm{Gl}_n(\mathbb{C}), \mathbb{Z})$$

be the suspension map. Let $p_j^{\mathrm{sing}} = s_j(c_j^{\mathrm{sing}})$. Then:

(1)
$$H_{\mathrm{sing}}^*(B\mathrm{Gl}_n(\mathbb{C}), \mathbb{Z}) = \mathbb{Z}[c_1^{\mathrm{sing}}, c_2^{\mathrm{sing}}, \ldots, c_n^{\mathrm{sing}}]$$
as graded algebras.

(2) *With $P_n = \bigoplus_{j=1}^{n} \mathbb{Z} p_j^{\mathrm{sing}}$ we have*

$$H_{\mathrm{sing}}^*(\mathrm{Gl}_n(\mathbb{C}), \mathbb{Z}) = \bigwedge_{\mathbb{Z}}^{*} P_n$$

as graded Hopf-algebras.

Proof. Let S^i be the i-sphere. Integral cohomology of the group is computed as

$$H^*(S^{2n+1} \times S^{2n-1} \times \cdots \times S^1, \mathbb{Z})$$

in [Bo1] Proposition 9.1. This means it is an exterior algebra on generators $y_1, \ldots, y_n$. By loc. cit. Proposition 19.1 (b) $H^*_{\mathrm{sing}}(B\mathrm{Gl}_n(\mathbb{C}), \mathbb{Z})$ is a polynomial algebra on the same generators.

Let T be the diagonal torus of $\mathrm{Gl}_n(\mathbb{C})$ and W the Weyl group (i.e. the symmetric group). Then we have ([Hu] Ch. 18, Theorem 3.2)

$$H^*_{\mathrm{sing}}(B\mathrm{Gl}_n(\mathbb{C}), \mathbb{Z}) = H^*_{\mathrm{sing}}(BT(\mathbb{C}), \mathbb{Z})^W = \mathbb{Z}[c_1^{\mathrm{sing}}, \ldots, c_n^{\mathrm{sing}}]$$

This implies that y_i can be identified with the universal Chern class c_i^{sing}. □

Remark 2.3. This is the statement in the form usually used in algebraic topology. From the point of view of complex or algebraic geometry it would be more natural to view c_j as an element of $H^{2j}_{\mathrm{sing}}(B\mathrm{Gl}_n(\mathbb{C}), (2\pi i)^j \mathbb{Z})$. There is a hidden choice of i or orientation on $\mathbb{C}$ behind the translation from one point of view to the other.

Corollary 2.4. *The product*

$$\omega_{\mathrm{sing}} = p_1^{\mathrm{sing}} \wedge \ldots p_n^{\mathrm{sing}}$$

is a generator of $H^{n^2}_{\mathrm{sing}}(\mathrm{Gl}_n(\mathbb{C}), \mathbb{Z})$. It is the dual of the fundamental class of

$$[\mathrm{U}(n)] \in H_{n^2}^{\mathrm{sing}}(\mathrm{U}(n), \mathbb{Z}) \cong H_{n^2}^{\mathrm{sing}}(\mathrm{Gl}_n(\mathbb{C}), \mathbb{Z})$$

Proof. The first statement is contained in the proposition. $\mathrm{U}(n) \subset \mathrm{Gl}_n(\mathbb{C})$ is a homotopy equivalence. $\mathrm{U}(n)$ is compact, orientable and connected, hence $H_{n^2}^{\mathrm{sing}}(\mathrm{U}(n), \mathbb{Z})$ is generated by the manifold $\mathrm{U}(n)$ itself ([GH] Theorem 22.24). □

3. De Rham cohomology

Definition 3.1. Let X be a smooth algebraic variety over $\mathbb{Q}$. Its *algebraic de Rham cohomology* is defined as

$$H^j_{\mathrm{dR}}(X) = H^j(X, \Omega^*_X)$$

the hypercohomology of the algebraic de Rham complex.

Recall that there is a natural isomorphism of functors

$$\sigma : H^j_{\mathrm{sing}}(X(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \to H^j_{\mathrm{dR}}(X) \otimes_{\mathbb{Q}} \mathbb{C}$$

It is induced by the inclusion $\mathbb{Z}_X \to \mathbb{C}_X$ of sheaves for the analytic topology on $X(\mathbb{C})$ and the quasi-isomorphism

$$\mathbb{C}_X \to \Omega^{\mathrm{an},*}_X$$

with the holomorphic de Rham complex (holomorphic Poincaré Lemma) on the one hand and the comparison between algebraic and holomophic de Rham cohomology on the other hand. In particular, σ is compatible with products.

Proposition 3.2. *Let $c_j^{\mathrm{dR}} \in H_{\mathrm{dR}}^{2j}(B\mathrm{Gl}_n)$ be the universal j-th Chern class in algebraic de Rham cohomology. Let*

$$s_j : H_{\mathrm{dR}}^{2j}(B\mathrm{Gl}_n) \to H_{\mathrm{dR}}^{2j-1}(\mathrm{Gl}_n)$$

be the suspension map. Let $p_j^{\mathrm{dR}} = s_j(c_j^{\mathrm{dR}})$. Then:

(1)
$$H_{\mathrm{dR}}^*(B\mathrm{Gl}_n) = \mathbb{Q}[c_1^{\mathrm{dR}}, c_2^{\mathrm{dR}}, \ldots, c_n^{\mathrm{dR}}]$$

as graded algebras.
(2) *With $P_n = \bigoplus_{j=1}^n \mathbb{Q}p_j^{\mathrm{dR}}$ we have*

$$H_{\mathrm{dR}}^*(\mathrm{Gl}_n) = \bigwedge \mathbb{Q}^* P_n$$

as graded Hopf-algebras.

Proof. There are different arguments for this fact. Once algebraicity of the Chern classes is known, the result follows directly from Proposition 2.2 and the existence of the comparison isomorphism. $\qquad\square$

Proposition 3.3. *The comparison isomorphism σ is compatible with Chern classes. More precisely,*

$$\sigma((2\pi i)^j c_j^{\mathrm{sing}}) = c_j^{\mathrm{dR}}$$

Proof. Recall that the j-th Chern class is the j-th elementary symmetric polynomial in the 1-st Chern class of diagonal torus (splitting principle). Hence it suffices to consider the case $j = 1$.

For singular cohomology (or rather cohomology of sheaves on $\mathrm{Gl}_1(\mathbb{C}) = \mathbb{C}^*$) consider the exact sequence of sheaves

$$0 \to \mathbb{Z} \xrightarrow{2\pi i} \mathcal{O}^{\mathrm{an}} \xrightarrow{\exp} \mathcal{O}^{\mathrm{an}*} \to 1$$

c_1^{sing} is the image of the invertible function z (the coordinate function of $\mathbb{C}^*$) under the connecting homomorphism. For algebraic or holomorphic de Rham cohomology consider the morphism of complexes

$$\mathcal{O}^*[-1] \to \Omega^* \quad f \mapsto \frac{df}{f}$$

$c_1^{\mathrm{dR}} = \frac{dz}{z}$ is the image of the invertible function z under this morphism of complexes. The two constructions are nearly (but not quite) compatible with the definition of the comparison functor σ which asks for $\mathbb{Z}$ to be naturally embedded into the constant functions $\mathbb{C} \subset \mathcal{O}^{\mathrm{an}}$. This gives the factor $2\pi i$ as claimed. $\qquad\square$

Corollary 3.4. *Let as before $\omega^{\mathrm{dR}} = p_1^{\mathrm{dR}} \wedge \cdots \wedge p_n^{\mathrm{dR}} \in H_{\mathrm{dR}}^{n^2}(\mathrm{Gl}_n)$. Then*

$$\sigma((2\pi i)^{\frac{n(n+1)}{2}} \omega^{\mathrm{sing}}) = \omega^{\mathrm{dR}}$$

Proof. By Proposition 3.3 and compatibility of σ with the suspension map we have

$$\sigma((2\pi i)^j p_j^{\text{sing}}) = p_j^{\text{dR}}$$

Moreover, σ is compatible with products. $\qquad\qquad\square$

Lemma 3.5. *For $i, j = 1, \ldots, n$ let z_{ij} be the natural coordinate on $n \times n$-matrices. Recall that $\rho_{\mathbb{Z}}^{\text{dR}}$ is the integral generator of $H^{n^2}(\mathfrak{gl}_n, \mathbb{Z}) \subset H_{\text{dR}}^{n^2}(\text{Gl}_n)$. Then*

$$\rho_{\mathbb{Z}} = \frac{1}{\det^n} \bigwedge_{i,j=1}^{n} dz_{ij}$$

Proof. The complex $\Omega^*(\text{Gl}_{n,\mathbb{Q}})$ is quasi-isomorphic to $\bigwedge^* \mathfrak{gl}_{n,\mathbb{Q}}^*$ where elements in $\mathfrak{gl}_{n,\mathbb{Q}}^*$ are viewed as left-invariant differential forms (see [Ho] Lemma 4.1). The differential form in the statement is clearly Gl_n-invariant. In order to check that it is an integral basis, it suffices to restrict to the tangent space of $1 \in \text{Gl}_n$. There it is the standard generator. $\qquad\square$

Remark 3.6. All computations are up to sign, hence we do not have to specify a prefered ordering of the coordinates.

4. A VOLUME COMPUTATION

Proposition 4.1. *Let z_{ij} be the standard holomorphic coordinates on $\text{Gl}_n(\mathbb{C})$. Then*

$$\int_{\text{U}(n)} \frac{1}{\det^n} \bigwedge_{ij=1}^{n} dz_{ij} = \pm \prod_{\nu=0}^{n-1} \frac{(2\pi \, i)^{\nu+1}}{\nu!}$$

Before going into the proof, we review integration of differential forms over fibres of a bundle, thereby fixing notation. Consider $p : X \to Y$ a fibre bundle with smooth compact fibres of dimension c and a C_∞-volume form ω on Y. Recall the definition of the volume form $\int_p \omega$ on Y: for every $y \in Y$ and tangent vectors $v_1, \ldots, v_q \in T_y Y$, the volume form $\omega[v_1, \ldots, v_q]$ on $p^{-1}(y)$ assigns to all $x \in p^{-1}y$ and $w_1, \ldots, w_c \in T_x p^{-1}(y)$ the value

$$\omega[v_1, \ldots, v_q](w_1, \ldots, w_c) = \omega(w_1, \ldots, w_c, \tilde{v}_1, \ldots, \tilde{v}_q)$$

where $\tilde{v}_i$ is a preimage of v_i. The form $\omega[v_1, \ldots, v_q]$ is independent of the choice of these $\tilde{v}_i$. Then

$$\left(\int_p \omega\right)(v_1, \ldots, v_q) = \int_{p^{-1}(y)} \omega[v_1, \ldots, v_q]$$

Proof of Proposition 4.1: Recall

$$\rho_{\mathbb{Z}} = \frac{1}{\det^n} \bigwedge_{ij=1}^{n} dz_{ij}$$

We argue by induction on n. For $n = 1$ we have

$$\int_{S^1} \frac{dz}{z} = 2\pi i$$

by Cauchy's formula.

Suppose now the formula holds true for n. We abbreviate the value by $\pm C(n)$. The claim reads

$$C(n+1) = \pm \frac{(2\pi i)^{n+1}}{n!} C(n)$$

We consider the diagram

$$
\begin{array}{ccc}
A \longmapsto \mathrm{diag}(1, A) & & (z_0, \ldots, z_n) \\
& & \Big\| \\
\mathrm{U}(n) \hookrightarrow \mathrm{U}(n+1) \longrightarrow \mathbb{C}^{n+1} \ni (x_0 + iy_0, \ldots, x_n + iy_n) \\
\Big\downarrow {\scriptstyle p} \qquad\qquad\qquad \wr \Big\uparrow \\
S^{2n+1} \hookrightarrow \mathbb{R}^{2n+2} \ni (x_0, y_0, = \ldots, x_n, y_n)
\end{array}
$$

with the left vertical p given by application to the first vector of the standard basis $\vec{a}_0 = (1, 0, \ldots, 0)^\top \in \mathbb{R}^{2n+2}$.

We integrate $\rho_{\mathbb{Z}}$ over the fibres. The resulting form $\int_p \rho_{\mathbb{Z}}$ is $\mathrm{U}(n+1)$-invariant and uniquely determined by its value in $\vec{a}_0$, which we are going to compute. Let $\vec{v}_1, \ldots, \vec{v}_{2n+1} \in T_{\vec{a}_0} S^{2n+1}$ be tangent vectors. Then $\rho_{\mathbb{Z}}[v_1, \ldots, v_{2n+1}]$ is an $\mathrm{U}(n)$-invariant form and uniquely determined by its value in the unit matrix E.

We choose as basis of the tangent space of S^{2n+1} in $\vec{a}_0$ the other vectors in the standard basis of $\mathbb{R}^{2n+2}$ and denote them

$$\vec{b}_0, \vec{a}_1, \vec{b}_1, \ldots, \vec{a}_n, \vec{b}_n.$$

Let ρ_0 be the unique $\mathrm{U}(n+1)$-equivariant form on S^{2n+1} with

$$\rho_0(\vec{b}_0, \vec{a}_1, \ldots, \vec{b}_n) = 1$$

For later use, we record that the surface of the unit ball in dimension $2n + 2$ is computed by

$$(*) \qquad\qquad \int_{S^{2n+1}} \rho_0 = 2\frac{\pi^{n+1}}{n!}$$

We have to choose preimages in the tangent space $T_E \mathrm{U}(n+1)$ for these vectors. We can use arbitrary hermitian matrices A_ν for $1 \le \nu \le n$ und B_ν for $0 \le \nu \le n$ such that

$$A_\nu \vec{a}_0 = \vec{a}_n \qquad B_\nu \vec{a}_0 = \vec{b}_\nu.$$

A simple choice are the complex matrices

$$
\begin{aligned}
A_\nu &= E_{\nu 0} - E_{0\nu} & \nu &\geq 1 \\
B_\nu &= iE_{\nu 0} + iE_{0\nu} & \nu &\geq 1 \\
B_0 &= iE_{00}
\end{aligned}
$$

Here we are using the usual notation for the standard basis of the matrix ring over $\mathbb{C}$ but with indices starting from 0.

It is more convenient to pass to complexified tangent spaces. $T_{\vec{a}_0}^{\mathbb{C}} S^{2n+1}$ has the simpler basis

$$
\begin{aligned}
\vec{v}_0 &= i\vec{b}_0 \\
\vec{v}_\nu &= (\vec{a}_\nu - i\vec{b}_\nu)/2 & \nu &= 1, \ldots, n \\
\vec{v}_{n+\nu} &= (\vec{a}_\nu + i\vec{b}_\nu)/2 & \nu &= 1, \ldots, n
\end{aligned}
$$

Its lift to $T_E^{\mathbb{C}} U(n+1)$ is given by

$$
\begin{aligned}
\tilde{v}_0 &= -E_{00} \\
\tilde{v}_\nu &= +E_{\nu 0} & \nu &= 1, \ldots, n \\
\tilde{v}_{n+\nu} &= -E_{0\nu} & \nu &= 1, \ldots, n
\end{aligned}
$$

By evaluating in a standard basis of $T_E^{\mathbb{C}} U(n)$ we get

$$
\rho_{\mathbb{Z}}[\vec{v}_0, \ldots, \vec{v}_{2n}] = \pm dz_{11} \wedge \ldots \wedge dz_{nn} = \rho_{\mathbb{Z}}^{U(n)}
$$

By inductive hypothesis this implies

$$
\left(\int_p \rho_{\mathbb{Z}} \right) (\vec{v}_0, \ldots, \vec{v}_{2n}) = \pm C(n)
$$

We now translate back to the original basis. We easily find for $\nu \geq 1$

$$
\vec{v}_\nu \wedge \vec{v}_{n+\nu} = \frac{1}{4}(\vec{a}_\nu - i\vec{b}_\nu) \wedge (\vec{a}_\nu + i\vec{b}_\nu) = \frac{i}{2}\vec{a}_\nu \wedge \vec{b}_\nu
$$

and hence

$$
\bigwedge_{i=0}^{2n} \vec{v}_i = \pm(\vec{b}_0 \wedge \vec{a}_1 \wedge \vec{b}_1 \wedge \ldots \wedge \vec{a}_n \wedge \vec{b}_n) \cdot \frac{i^{n+1}}{2^n} \Rightarrow
$$

$$
\left(\int_p \rho_{\mathbb{Z}} \right) (\vec{b}_0, \vec{a}_1, \vec{b}_1, \ldots, \vec{b}_n) = \pm C(n) i^{n+1} 2^n \Rightarrow
$$

$$
\int_p \rho_{\mathbb{Z}} = \pm C(n) i^{n+1} 2^n \rho_0
$$

Together with equation (*) for the unit sphere this yields

$$C(n+1) = \int_{U(n+1)} \rho_{\mathbb{Z}} = \int_{S^{2n+1}} \left(\int_p \rho_{\mathbb{Z}} \right)$$

$$= \pm C(n) i^{n+1} 2^n \int_{S^{2n+1}} \rho_0$$

$$= \pm C(n) \frac{(2\pi i)^{n+1}}{n!}$$

This proves the claim.   □

5. Proof of the main result

Proof of Proposition 1.3. We want to compare the elements $\rho_{\mathbb{Z}}^{\mathrm{dR}}$ (see Lemma 3.5) and ω^{dR} (see Corollary 3.4) in $H_{\mathrm{dR}}^{n^2}(\mathrm{Gl}_n)$. Let $\alpha \in \mathbb{Q}^*$ such that

$$\omega_{\mathrm{dR}} = \alpha \rho_{\mathbb{Z}}^{\mathrm{dR}}$$

By Corollary 3.4

$$\sigma(\omega^{\mathrm{sing}}) = (2\pi i)^{-\frac{n(n+1)}{2}} \alpha \rho_{\mathbb{Z}}^{\mathrm{dR}}$$

The comparison isomorphism between singular cohomology and holomorphic de Rham cohomology can be reformulated as integration. By Corollary 2.4, Lemma 3.5 and Proposition 4.1, this means

$$(2\pi i)^{\frac{n(n+1)}{2}} \alpha^{-1} = \int_{U(n)} \frac{1}{\det^n} \wedge_{i,j=1}^n dz_{ij} = \pm \prod_{\nu=0}^{n-1} \frac{(2\pi i)^{\nu+1}}{\nu!}$$

where z_{ij} are the holomorphic coordinates on the space on $n \times n$ matrices. Hence

$$\alpha = \pm \prod_{\nu=0}^{n-1} \frac{1}{\nu!}$$

as claimed.   □

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