# $\mathbb{A}^1$-Homotopy of Chevalley Groups

Matthias Wendt

June 2008

**Abstract**

In this paper, we describe the sheaves of $\mathbb{A}^1$-homotopy groups of a simply-connected Chevalley group $G$. The $\mathbb{A}^1$-homotopy group sheaves can be identified with the sheafification of the unstable Karoubi-Villamayor K-groups.

## Contents



## 1 Introduction

$\mathbb{A}^1$-homotopy theory is an approach towards a homotopy theory of algebraic varieties, which was developed by Morel and Voevodsky, cf. [MV99]. The two characteristic features of this theory are the use of simplicial presheaves, and the fact that the affine line $\mathbb{A}^1$ takes the role of the unit interval in algebraic topology. The goal behind the development is to use the methods of algebraic topology which have already led to a rather deep understanding of differentiable manifolds, and apply these methods to the study of algebraic varieties.

To pursue such a program, computations of homotopy and homology groups of algebraic varieties are needed. Some calculations have already been obtained by Morel: in [Mor07], the homotopy type of $GL_n$ over a field is described, and in [Mor06], the fundamental group of $\mathbb{P}^1$ is computed. In this paper, we supply further descriptions of $\mathbb{A}^1$-homotopy groups of algebraic varieties. It turns out that the homotopy groups that we have been able to compute until now all are in some way related to K-theory.

We now want to give a slightly more detailed overview of the contents of this paper, in which we describe the $\mathbb{A}^1$-homotopy sheaves of a simply-connected Chevalley group $G(\Phi)$, for any reduced and irreducible root system $\Phi$ which is not $A_1$ or $G_2$. These sheaves are determined by the following procedure, which was first applied by Morel in [Mor07]: We consider the simplicial presheaves $\mathrm{Sing}_\bullet^{\mathbb{A}^1}(G(\Phi))$, where $\mathrm{Sing}_\bullet^{\mathbb{A}^1}$ is the singular complex of simplices $\mathbb{A}^n \to G(\Phi)$. Using a collection of more or less well-known matrix factorization and patching results, it is possible to show that this presheaf satisfies the affine Brown-Gersten property and has affine $\mathbb{A}^1$-invariance. These two properties allow to show that the $\mathbb{A}^1$-homotopy groups over an affine smooth scheme $X$ are already determined by the simplicial set $\mathrm{Sing}_\bullet^{\mathbb{A}^1}(G(\Phi))(X)$. The latter complex is known to compute the unstable Karoubi-Villamayor K-groups. We state the main result of the paper, which are proven in Corollary 4.10:

THEOREM 1 *Let $\Phi$ be a root system not equal to $A_1$ or $G_2$, let $R$ be an excellent Dedekind ring and let $S = \mathrm{Spec}\, R$.*

*Then the $S$-group schemes $G(\Phi)$ represent the unstable Karoubi-Villamayor K-groups $KV_n(\Phi, -)$ in the $\mathbb{A}^1$-local model structure on $\Delta^{op}P\mathcal{S}hv(\mathrm{Sm}_S)$. Therefore, for any smooth affine $S$-scheme $U$, we obtain isomorphisms*

$$\pi_n^{\mathbb{A}^1}(G(\Phi), I)(U) \cong KV_{n+1}(\Phi, U).$$

These results are known by the work of Morel [Mor07] in the case where the group $G$ is the general linear group $GL_n$ over a base field $k$.

To be a little more concrete, this implies that for all non-symplectic root systems $\Phi$ other than $A_1$ and $G_2$ and an infinite field $k$, we have an identification

$$\pi_1^{\mathbb{A}^1}(G(\Phi))(\mathrm{Spec}\, k) = H_1^{\mathbb{A}^1}(G(\Phi), \mathbb{Z})(\mathrm{Spec}\, k) = K_2^M(k).$$

On the other hand, for $n \geq 2$, we have

$$\pi_1^{\mathbb{A}^1}(Sp_{2n})(\mathrm{Spec}\, k) = H_1^{\mathbb{A}^1}(Sp_{2n}, \mathbb{Z})(\mathrm{Spec}\, k) = K_2(C_n, k),$$

which by the results of Matsumoto is different from $K_2^M$ if $k$ is not algebraically closed. This follows from the thesis of Jardine, cf. [Jar81, Theorems 4.2.4 and 4.2.5].

As a particular application of Theorem 1, we exhibit a couple of $\mathbb{A}^1$-local fibre sequences resulting from the structure theory of algebraic groups over a field. These allow to extend the description of homotopy group sheaves to split linear algebraic groups over a field. It turns out that for $n \geq 2$, the group sheaves $\pi_n(G)$ are products of Karoubi-Villamayor K-groups of the semi-simple part of $G$ and therefore do not depend on anything in the solvable radical.

One final note concerning the topology: The results as stated in Theorem 1 hold for the Zariski topology over a Dedekind ring. Using Morel's results from [Mor07] they also hold for the Nisnevich topology over a field.

**Structure of the Paper:** In Section 2, we will repeat some of the basic definitions like unstable Karoubi-Villamayor K-groups and an important characterization of Nisnevich neighbourhoods over a Dedekind ring. The main technical tool behind the results is the Brown-Gersten property which will occupy Section 3. Then we will prove the main results in Section 4. Finally, Section 5 discusses the promised applications of the main result.

**Acknowledgements:** The results presented here are taken from my PhD thesis [Wen07] which was supervised by Annette Huber-Klawitter. I would like to use the opportunity to thank her for her encouragement and interest in my work. A large portion of this paper is inspired by the techniques developed in Fabien Morel's preprint [Mor07] and applied there to prove the $GL_n$-case of the results presented here. I would also like to thank Fabien Morel for some interesting conversations concerning homotopy types of algebraic groups and their classifying spaces, as well as making me aware of some mistakes in an earlier version of this paper.

## 2 Preliminaries and Notation

We will mostly work in the category $\mathrm{Sm}_S$ of smooth, finite type schemes over a finite type base $S$, which is assumed to be a field or a Dedekind ring.

**$\mathbb{A}^1$-Homotopy Theory:** We do not want to dive into the depths of homotopical algebra, therefore we take the definitions of model categories and their properties for granted, but see e.g. [GJ99, Hir03].

The general definition of $\mathbb{A}^1$-homotopy theory is due to Morel and Voevodsky [MV99], and we give a brief sketch of the construction. Consider the category of simplicial presheaves $\Delta^{op}\mathcal{PS}hv(\mathrm{Sm}_S)$ on the category of smooth schemes $\mathrm{Sm}_S$. This category has a model structure where the cofibrations are monomorphisms, the weak equivalences are those morphism which induce weak equivalences of simplicial sets on the stalks, and the fibrations are given by the right lifting property. The topologies which we will put on $\mathrm{Sm}_S$ are either the Zariski topology, whose coverings are surjective collections of open subsets, or the Nisnevich topology, whose elementary coverings are pullback squares

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow{\scriptstyle p} \\ U & \underset{i}{\hookrightarrow} & X \end{array}$$

with $p$ an étale morphism and $i$ an open embedding such that $p$ restricts to an isomorphism over $X \setminus U$.

Note that the same constructions still work if the model category in which the presheaves take their values is not in the category $\Delta^{op}\mathcal{S}et$ of simplicial sets but any other proper and simplicial model category. We will need this in Section 3, where we replace simplicial sets by chain complexes of abelian groups.

The most commonly occuring symbol in the sequel will be the $\mathbb{A}^1$-*singular complex* $\mathrm{Sing}_\bullet^{\mathbb{A}^1}(F)$ associated to a presheaf $F$, which is the simplicial presheaf given by

$$U \mapsto F(\Delta^\bullet \times U),$$

for any smooth scheme $U$. The object $\Delta^\bullet$ above is the standard cosimplicial object formed by the affine spaces

$$\Delta^n_{\mathbb{Z}} = \operatorname{Spec} \mathbb{Z}[t_0, \ldots, t_n]/\sum_i t_i = 1.$$

Since we are interested in homotopy groups, all the simplicial presheaves are actually pointed, although it will never be mentioned. All the presheaves are presheaves coming from linear groups, and we choose the identity matrix $I$ as the base point.

**Nisnevich Neighbourhoods:** In this paragraph we repeat a structure theorem for Nisnevich neighbourhoods in the category of smooth finite type schemes over a Dedekind ring, which is needed both for the Brown-Gersten property for Chevalley groups. Originally, this result for the case of varieties over a field appeared in a paper of Lindel [Lin81]. It since has been extended to the case of excellent discrete valuation rings by Dutta in [Dut99, Theorem 1.3]. A similar form of the extension to discrete valuation rings can be found already in [Pop89, Proposition 2.1].

Theorem 2.1 *Let $(A, \mathfrak{m}, K)$ be a regular local ring with maximal ideal $\mathfrak{m}$ and residue field $K$ of dimension $d+1$, essentially of finite type over an excellent discrete valuation ring $(V, \pi)$ such that $A/\mathfrak{m}$ is separably generated over $V/\pi V$. Let $a \in \mathfrak{m}^2$ be such that $a \notin \pi A$. Then there exists a regular local ring $(B, \mathfrak{n}, K) \hookrightarrow (A, \mathfrak{m}, K)$ with maximal ideal $\mathfrak{n}$, which has the same residue field as $A$, and satisfies the following properties:*

*(i) $B$ is the localization of a polynomial ring $W[x_1, \ldots, x_d]$ at a maximal ideal of the form $(\pi, f(x_1), x_2, \ldots, x_d)$ where $f$ is a monic irreducible polynomial in $W[x_1]$ and $(W, \pi) \subseteq A$ is an excellent discrete valuation ring. Moreover, $A$ is an étale neighbourhood of $B$.*

*(ii) There exists an element $h \in B \cap aA$ such that $B/hB \to A/aA$ is an isomorphism, and furthermore $hA = aA$.*

**Chevalley Groups:** The original construction of split simple simply-connected group schemes from Lie algebras over $\mathbb{C}$ is due to Chevalley, cf. [Che55], an overview of the construction of Chevalley groups over fields is given in Jardine [Jar81]. We shortly recall the definition over a general commutative ring, following [PV96].

Let $G$ be a connected simple Lie group over $\mathbb{C}$, with Lie algebra $\mathfrak{g} = L(G)$ and corresponding root system $\Phi$, which is reduced and irreducible. Furthermore, let $V$ be a finite-dimensional complex vector space, and $\pi : G \times V \to V$ be a representation of $V$ with differential $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$. Denoting by $\Lambda(\pi)$ the set of weights of the representation $\pi$, there always exists an admissible base $\{v^\lambda\}_{\lambda \in \Lambda(\pi)}$ such that a $\mathbb{Z}$-form of $L(G)$ acts on the lattice spanned by the admissible base. A choice of such an admissible base allows to identify $V \cong \mathbb{C}^n$ and $GL(V) \cong GL_n(\mathbb{C})$. Moreover, restricting to the $\mathbb{Z}$-forms, we obtain a sub-Hopf-algebra $\mathbb{Z}[G]$ in the algebra $\mathbb{C}[G]$ of complex-valued regular functions on $G$. This algebra defines an affine group scheme $G(\Phi)$ by setting

$$A \mapsto G(\Phi, A) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A)$$

for any commutative ring $A$. Note that there is a choice of admissible lattice on which the group $G(\Phi, A)$ depends.

> However, we always choose the lattice of all weights of the representation, which yields a simply-connected group. This group is also called *universal Chevalley group of type* $\Phi$.

For any $t \in A$ and $\alpha \in \Phi$, we can define the *elementary matrices* $x_\alpha(t) = \exp(t\pi(e_\alpha)) \in GL(V_A)$, where $e_\alpha$ is the element of $L(G)_A$ corresponding to the root $\alpha$. This yields a group homomorphism $X_\alpha : A \to G(\Phi, A) : t \in A \mapsto x_\alpha(t)$. The subgroup of $G(\Phi, A)$ generated by elements of this form is called the *elementary group* $E(\Phi, A)$ for the root system $\Phi$.

Note that another way to define the groups above is to first define elementary matrices as exponentials of nilpotent elements in the Chevalley algebra over $\mathbb{Z}$, and then to define the Chevalley group $G(\Phi)$ via the algebra of functions on the elementary group $E(\Phi, \mathbb{Z})$.

**Karoubi-Villamayor K-Theory:** Karoubi and Villamayor [KV69] used the definition of $K_1$ as the quotient of $GL_n$ modulo the elementary matrices $E_n$ together with a definition of loop rings to define higher algebraic K-groups. We recall the definition from [Ger73].

Definition 2.2 (Karoubi-Villamayor K-theory) *Let $A$ be any commutative ring, and let $\Phi$ be an irreducible and reduced root system. Then we can define $K_1(\Phi, A) = G(\Phi, A)/E(\Phi, A)$.*

*The* path ring *of $A$ is defined to be $EA = \ker(A[t] \to A : t \mapsto 0)$. It comes with an augmentation $\epsilon : EA \to A : \sum a_i t^i \mapsto \sum a_i$. Then the* loop ring *of $A$ is defined to be $\Omega A = \ker \epsilon$. The functor $E$ admits the structure of a cotriple on the category of rings, yielding a simplicial ring $E^\bullet A$. We then apply the Chevalley group functor $G(\Phi, -)$ to obtain a simplicial group $G(\Phi, E^\bullet A)$.*

*The* Karoubi-Villamayor K-groups *of $R$ and $\Phi$ are then given by*

$$KV_{n+2}(\Phi, A) = \widehat{\pi}_n(G(\Phi, E^\bullet A)), n \geq -1$$

*where the $\widehat{\pi}$ is the usual homotopy groups for $n \geq 1$, and some slight modification of the usual homotopy groups for $n = -1, 0$. For more details, cf. [Ger73, Definition 3.1] or [Jar83, p. 194].*

Then a result of Rector, cf. [Ger73, Theorem 3.13] resp. [Jar83, Theorem 3.8], shows that there are isomorphisms of groups

$$KV_{n+1}(\Phi, A) \cong \pi_n \operatorname{Sing}_\bullet^{\mathbb{A}^1}(G(\Phi))(\operatorname{Spec} A).$$

The latter presentation is the one we will use in the sequel.

## 3 Recollections on the Affine Brown-Gersten Property

In this section, we recall the basics of the Brown-Gersten formalism. This formalism first appeared in [BG73] and was designed to show Zariski hypercover descent for K-theory. In [MV99], this formalism was extended to the Nisnevich topology. The final step, and the most important for this work was achieved by Morel in [Mor07], where the affine Brown-Gersten property was introduced, together with the affine replacement of a simplicial presheaf. This allows to determine the $\mathbb{A}^1$-homotopy groups of a simplicial sheaf, at least over smooth affine schemes, under rather weak conditions. The purpose of this section is to explain the basics of this theory, which will be used later to determine the $\mathbb{A}^1$-homotopy of Chevalley group schemes.

The situation we will consider in this section is the following: Let $\mathcal{C}$ be any model category, and let $T$ be a site with enough points. We consider the Jardine model structure on the category of presheaves $T^{op} \to \mathcal{C}$, where a cofibration is a monomorphism, a weak equivalence of $\mathcal{C}$-presheaves is a morphism which induces $\mathcal{C}$-weak equivalences on the stalks, and fibrations are determined by the right lifting property. Because we are talking about homotopy pullbacks and homotopy limits, we assume that $\mathcal{C}$ is a proper and simplicial model category.

The examples we will use in the sequel are simplicial presheaves and presheaves of complexes of abelian groups, both considered on the site $\mathrm{Sm}_S$ of smooth schemes over a base scheme $S$ with either the Zariski or Nisnevich topology.

Definition 3.1 *Let $T$ be a category with a cd-structure $P$. A $\mathcal{C}$-presheaf $F$ satisfies the* Brown-Gersten property *for $P$ if for any distinguished square*

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow{\scriptstyle i} \\ U & \xrightarrow[j]{} & X, \end{array}$$

*the induced square*

$$\begin{array}{ccc} F(X) & \longrightarrow & F(V) \\ \downarrow & & \downarrow \\ F(U) & \longrightarrow & F(U \times_X V) \end{array}$$

*is a homotopy cartesian square in the respective model category $\mathcal{C}$.*

Remark 3.2 *Recall that for sheaves of complexes, the above condition reduces to the requirement that the sequence $F(X) \to F(U) \oplus F(V) \to F(U \times_X V)$ is a distinguished triangle in the derived category of complexes.*

The following proposition was proven for the cases of simplicial presheaves and presheaves of spectra in [CHSW08, Theorem 3.4] as a generalization of [MV99, Proposition 3.1.16].

Proposition 3.3 *Let $T$ be a category with a bounded, complete and regular cd-structure $P$. A $\mathcal{C}$-presheaf $X$ has the Brown-Gersten property for $P$ if and only if it is quasi-fibrant, i.e. for any fibrant replacement $X \to \tilde{X}$ and any $U \in T$, there are induced weak equivalences $X(U) \to \tilde{X}(U)$.*

The idea for proving the result for a general proper and simplicial model category $\mathcal{C}$ is to use the homotopy cartesian squares provided by Definition 3.1 and show that one can obtain *B.G.-functors*, cf. [MV99, Lemma 3.1.17 and 3.1.18]. The fact that there is a model category of $\mathcal{C}$-presheaves which behaves almost exactly as the model structure on simplicial presheaves implies that the argument of [BG73] and [MV99] goes through.

**The affine Brown-Gersten Property:** The result in Proposition 3.3 can be modified using the notion of B.G.-classes [MV99, Definition 3.1.12]: If the Brown-Gersten property holds for the simplicial presheaf $F$ and for a B.G.-class $\mathcal{A}$ of objects of the site $\mathrm{Sm}_S$, then a fibrant replacement $\tilde{F}$ of $F$ induces weak equivalences $F(U) \to \tilde{F}(U)$ for any object $U$ in the B.G.-class. This applies e.g. to the B.G.-classes of smooth quasi-projective schemes or smooth quasi-affine schemes. Unfortunately, it does not apply to the class of affine schemes, because affine schemes do not form a B.G.-class.

The right way around this difficulty was presented in [Mor07], where Morel introduced the affine Brown-Gersten property and the affine replacement for a simplicial sheaf. These constructions allow to replace a simplicial presheaf $F$ which satisfies the affine Brown-Gersten property by another simplicial presheaf $F^{af}$ which satisfies the Brown-Gersten property for all smooth schemes. Moreover, the induced morphism $F(U) \to F^{af}(U)$ is a weak equivalence for every affine smooth scheme $U$.

The following definitions and results are due to Morel [Mor07]. We adapt the proofs to work in a slightly more general situation. The following definition can be found in [Mor07, Definition A.1.7].

Definition 3.4 *A $\mathcal{C}$-presheaf on the site $\mathrm{Sm}_S$ of smooth schemes with either Zariski or Nisnevich topology satisfies the* affine Brown-Gersten property *if the condition of Definition 3.1 is satisfied for all distinguished squares in which all schemes are additionally affine.*

Remark 3.5 *Note that a $\mathcal{C}$-presheaf on $\mathrm{Sm}_S$ has the affine Brown-Gersten property for the Nisnevich topology iff the condition of Definition 3.1 is satisfied for each square of the form*

$$\begin{array}{ccc} A & \longrightarrow & A_f \\ \downarrow & & \downarrow \\ B & \longrightarrow & B_f \end{array}$$

*where $B$ is an étale $A$-algebra such that $A/f \to B/f$ is an isomorphism.*

Before we can give the definition of affine replacement, we introduce some notation necessary for the remainder of this section. We will consider homotopy limits in the model category $\mathcal{C}$, the homotopy limit of the diagram $I : \mathcal{D} \to \mathcal{C}$ will be denoted by $\operatorname{holim}_{\mathcal{D}} I$. For a definition of homotopy limits in simplicial model categories, cf. [Hir03, Definition 18.1.8]. For a functor $F : \mathcal{E} \to \mathcal{D}$ and a diagram $I : \mathcal{D} \to \mathcal{C}$, there is an induced diagram $F^\dagger I : \mathcal{E} \to \mathcal{C}$, [Hir03, Definition 11.1.5]. The point in the unusual notation $F^\dagger I$ is that the functors along which we want to change the diagrams will be $p_*$ and $p^*$ in agreement with algebraic geometry notation.

Our index categories will always be categories of smooth affine schemes: In the following, we denote by $\mathrm{Sm}_S^{af}/X$ the category whose objects are morphisms of $S$-schemes $U \to X$ such that $U$ is an affine $S$-scheme.

Now we are ready to give the definition of affine replacement as given by Morel, cf. [Mor07, Section A.2].

Definition 3.6 *Let $F$ be a $\mathcal{C}$-presheaf on $\mathrm{Sm}_S$. The* affine replacement *of $F$ is denoted by $F^{af}$ and is defined by the following formula:*

$$F^{af}(X) = \operatorname*{holim}_{Y \in \mathrm{Sm}_S^{af}/X} \mathrm{Ex}_{\mathcal{C}}^\infty F(Y).$$

*Note that $\mathrm{Ex}_{\mathcal{C}}^\infty$ in the above formula is a functorial fibrant replacement in the model category $\mathcal{C}$.*

*The presheaf structure of $F^{af}$ can be described as follows: For a morphism $g : U \to X$, there is an induced functor $g_* : \mathrm{Sm}_S^{af}/U \to \mathrm{Sm}_S^{af}/X$ by composing the structure morphism $Y \to U$ with $g$. The diagram*

$$J : (\mathrm{Sm}_S^{af}/U)^{op} \to \mathcal{C} : (Y \to U) \mapsto \mathrm{Ex}_{\mathcal{C}}^\infty F(Y)$$

*is equal to the induced diagram $(g_*)^\dagger I$ which first considers the scheme $Y \to U$ as an $X$-scheme and then evaluates the diagram*

$$I : (\mathrm{Sm}_S^{af}/X)^{op} \to \mathcal{C} : (Y \to X) \mapsto \mathrm{Ex}_{\mathcal{C}}^\infty F(Y)$$

*Therefore, cf. [Hir03, Proposition 19.1.8], there is an induced natural morphism*

$$F^{af}(g) : F^{af}(X) = \operatorname*{holim}_{\mathrm{Sm}_S^{af}/X} I \to \operatorname*{holim}_{\mathrm{Sm}_S^{af}/U} (g_*)^\dagger I = \operatorname*{holim}_{\mathrm{Sm}_S^{af}/U} J = F^{af}(U).$$

*This provides $F^{af}$ with the structure of a simplicial presheaf: It is clear that for* $\mathrm{id} : X \to X$, *the morphism*

$$\operatorname*{holim}_{\mathrm{Sm}_S^{af}/X} \mathrm{id} : \operatorname*{holim}_{\mathrm{Sm}_S^{af}/X} I \to \operatorname*{holim}_{\mathrm{Sm}_S^{af}/X} I$$

*is the identity. For the composition of $g_1 : X \to Y$ and $g_2 : Y \to Z$, we note that the induced diagrams $((g_1)_*)^\dagger(((g_2)_*)^\dagger I)$ and $((g_2 \circ g_1)_*)^\dagger I$ are equal, and therefore the composition $F^{af}(g_1) \circ F^{af}(g_2)$ equals $F^{af}(g_2 \circ g_1)$.*

**The Jouanolou Trick:** One of the essential tools which make the affine replacement useful is the Jouanolou trick, which shows that any quasi-projective scheme “has the $\mathbb{A}^1$-homotopy type” of an affine scheme. We therefore provide the definition of affine vector bundle torsors, and the statement that vector bundle torsors exist on quasi-projective schemes. The definition of affine vector bundle torsors is taken from [Wei89, Definition 4.2].

Definition 3.7 *Let $X$ be a scheme. Then an* affine vector bundle torsor *over $X$ is an affine scheme $U$ and an affine morphism $U \to X$, such that there exists a vector bundle $E \to X$ and $U$ is an $E$-torsor over $X$. We will speak of a* vector bundle torsor *if the total space $U$ is not required to be affine.*

The existence of affine vector bundle torsors over quasi-projective schemes was proven in [Jou73, Lemme 1.5].

Proposition 3.8 *Let $S$ be an affine scheme, and let $X$ be a quasi-projective $S$-scheme. Then there exists a vector bundle $E \to X$ and an $E$-torsor $U$ over $X$, such that $U$ is affine.*

The basic idea of the proof is to show that the Stiefel variety

$$GL_{n+1}/(\mathbb{G}_m \times GL_n)$$

provides an affine vector bundle torsor over $\mathbb{P}^n$, and pulling this torsor back along an embedding $X \hookrightarrow \mathbb{P}^n$.

**The Affine Replacement:** The main technical step we need for the remainder of our paper is the following result of Morel [Mor07, Theorem A.2.2] which allows to compute sections of homotopy group sheaves over smooth affine schemes.

Theorem 3.9 ([Mor07], Theorem A.2.2) *Let $F$ be a $\mathcal{C}$-presheaf on the category $\mathrm{Sm}_S$, which satisfies affine $\mathbb{A}^1$-invariance and the affine Brown-Gersten property. Then the affine replacement satisfies the Brown-Gersten property for the Zariski topology on $\mathrm{Sm}_S$. Moreover, its fibrant replacement $\tilde{F}$ is $\mathbb{A}^1$-local and we have $\mathcal{C}$-weak equivalences*

$$F(U) \to \tilde{F}(U)$$

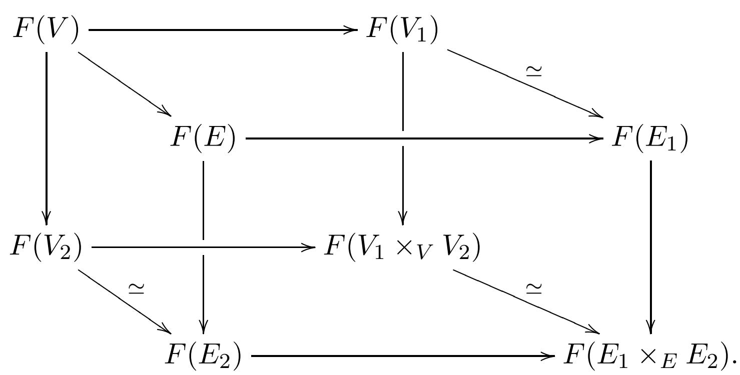
*for any smooth affine $U$.*

For the convenience of the reader, we give the details of the proof which was sketched in [Mor07]. The proof basically proceeds along the lines of the work of Weibel in [Wei89]. Therefore, we need an intermediate result showing that the affine replacement has some kind of homotopy invariance, i.e. any vector bundle torsor induces a weak equivalence on the affine replacement. This will be done in the following proposition. We assume in the following that the base scheme $S$ is affine.

Proposition 3.10 *Let $F$ be a $\mathcal{C}$-presheaf on the category of smooth schemes, which satisfies the affine Brown-Gersten property and affine $\mathbb{A}^1$-invariance. Then an (affine) vector bundle torsor $U \to X$ over a smooth scheme $X$ induces a weak equivalence $F^{af}(X) \to F^{af}(U)$, where $F^{af}$ is the affine replacement of $F$.*

Proof: The proof will actually show that we do not need the total space $U$ to be affine. We proceed in several steps: In Step (i), we show that the affine Brown-Gersten property and $\mathbb{A}^1$-invariance imply homotopy invariance for general vector bundles over affine schemes. In Step (ii), we construct a morphism $p^*$, and Steps (iii) and (iv) show that this morphism induces a weak equivalence $F^{af}(U) \to F^{af}(X)$. In Step (v), we give an argument that the composition $p^* \circ p_*$ also induces a weak equivalence $F^{af}(U) \to F^{af}(U)$. Steps (vi) and (vii) then use these facts to show that $p_*$ induces the required weak equivalence $F^{af}(X) \to F^{af}(U)$.

(i) We prove that for any affine vector bundle torsor $E \to V$ over a smooth affine scheme $V$, the morphism $F(V) \to F(E)$ is a weak equivalence. First note, that an affine vector bundle torsor over an affine scheme $V$ is a vector bundle with affine total space $E$, cf. [Wei89, Definition 4.2]. If the vector bundle $E \to V$ is in fact trivial, the assertion follows from affine $\mathbb{A}^1$-invariance. Furthermore, every vector bundle over $V$ trivializes over a Zariski covering $V_i \to V$, which induces a covering $E_i = V_i \times_V E \to E$. An inductive process allows to assume that the covering consists of only two elements $V_1, V_2 \to V$. Consider the following diagram:




The front and back face of the cube are homotopy cartesian by the affine Brown-Gersten property. Moreover, the vector bundle trivializes over $F(V_i)$ and $F(V_1 \times_V V_2)$, therefore the induced morphisms are weak equivalences. Since $\mathcal{C}$ is proper, the glueing lemma holds [GJ99, Corollary II.8.13], and therefore we conclude that $F(V) \to F(E)$ is a weak equivalence.

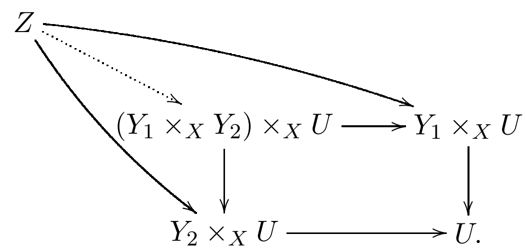
(ii) Consider a vector bundle torsor $p : U \to X$ over a general smooth scheme $X$. Then there exists a functor

$$p^* : \mathrm{Sm}_S^{af}/X \to \mathrm{Sm}_S^{af}/U : (Y \to X) \mapsto (Y \times_X U \to U).$$

The scheme $Y \times_X U$ is again affine because $p : U \to X$ is a vector bundle torsor, and therefore $Y \times_X U \to Y$ is a vector bundle over an affine base. This morphism induces a morphism $\phi : F^{af}(U) \to \mathrm{holim}_{\mathrm{Sm}_S^{af}/X}(p^*)^\dagger J$, where $J$ is the diagram presenting $F^{af}(U)$ as in Definition 3.6. Therefore we have

$$(p^*)^\dagger J : \mathrm{Sm}_S^{af}/X \to \mathcal{C} : (Y \to X) \mapsto \mathrm{Ex}_{\mathcal{C}}^\infty F(Y \times_X U).$$

(iii) We prove that the morphism $\phi$ constructed in Step (ii) is a weak equivalence by showing that the functor $p^*$ is homotopy right cofinal. The claim then follows from [Hir03, Theorem 19.6.7(2)]. Note that we have to interchange left and right because the diagrams we consider are contravariant functors. Let $Z \to U$ be an object in $\mathrm{Sm}_S^{af}/U$, and consider the category $(Z \downarrow p^*)$ whose objects are morphisms $Z \to p^*(Y) = Y \times_X U$ with $Y \in \mathrm{Sm}_S^{af}/X$. This category is directed, as can be seen from the following diagram:




The same argument works for parallel morphisms $Y_1 \times_X U \rightrightarrows Y_2 \times_X U$: these are induced from morphisms $Y_1 \rightrightarrows Y_2$ and $Y \times_X U$ is an object in $(Z \downarrow p^*)$, where $Y$ is the equalizer of the parallel morphisms. The category $(Z \downarrow p^*)$ is therefore contractible. This implies that the functor $p^*$ is homotopy right cofinal, cf. [Hir03, Definition 19.6.1].

(iv) As in Definition 3.6, we denote by $I$ the diagram presenting $F^{af}(X)$:

$$I : \mathrm{Sm}_S^{af}/X \to \mathcal{C} : (Y \to X) \mapsto \mathrm{Ex}^\infty_{\mathcal{C}} F(Y),$$

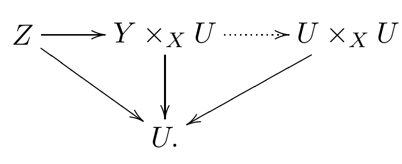
Then there is a morphism of diagrams $(p^*)^\dagger J \to I$, which associates to an object $Z \in \mathrm{Sm}_S^{af}/X$ the morphism $\mathrm{Ex}^\infty_{\mathcal{C}} F(Z \times_X U) \to \mathrm{Ex}^\infty_{\mathcal{C}} F(Z)$ which is induced by the zero-section $Z \to Z \times_X U$. From Step (i), we know that the morphism $F(Z) \to F(Z \times_X U)$ induced by the vector bundle projection is a weak equivalences for any $Z \in \mathrm{Sm}_S^{af}/X$. Note that we have $p \circ z = \mathrm{id}$, where $z$ is the zero-section of the vector bundle $Z \times_X U \to Z$. We then have $F(\mathrm{id}) = F(z) \circ F(p)$ and by the 2-out-of-3-property, we conclude that $F(z)$ is a weak equivalence. Therefore, the morphism

$$\psi : \operatorname*{holim}_{\mathrm{Sm}_S^{af}/X} (p^*)^\dagger J \to \operatorname*{holim}_{\mathrm{Sm}_S^{af}/X} I$$

is a weak equivalence, cf. [Hir03, Theorem 19.4.2]. We therefore obtain a weak equivalence

$$\psi \circ \phi : F^{af}(U) \to F^{af}(X).$$

(v) Now we show that the functor $p^* \circ p_* : \mathrm{Sm}_S^{af}/U \to \mathrm{Sm}_S^{af}/U$ is homotopy right cofinal as well. Therefore, let $Z \to U$ and $Y \to U$ be $U$-schemes, and consider the category $(Z \downarrow p^* \circ p_*)$. Note that the scheme $U \times_X U$ is always contained in $(Z \downarrow p^* \circ p_*)$, by the universal property of the pullback. Moreover, there is a unique morphism from any given $U$-morphism $Z \to (p^* \circ p_*)Y = Y \times_X U$ to $Z \to U \times_X U$, again by the universal property:




Therefore $U \times_X U$ is a terminal object of $(Z \downarrow p^* \circ p_*)$, hence the category $(Z \downarrow p^* \circ p_*)$ is contractible. This implies that the morphism

$$F^{af}(U) = \operatorname*{holim}_{\mathrm{Sm}_S^{af}/U} J \to \operatorname*{holim}_{\mathrm{Sm}_S^{af}/U} (p^* \circ p_*)^\dagger J$$

is a weak equivalence.

(vi) The morphism considered in Step (v) factors as follows:

$$F^{af}(U) = \operatorname*{holim}_{\mathrm{Sm}_S^{af}/U} J \xrightarrow{p^*} \operatorname*{holim}_{\mathrm{Sm}_S^{af}/X} (p^*)^\dagger J \xrightarrow{p_*} \operatorname*{holim}_{\mathrm{Sm}_S^{af}/U} (p^* \circ p_*)^\dagger J.$$

From Step (v), it follows that the composition is a weak equivalence. From Step (iii) it follows that the morphism induced by $p^*$ is a weak equivalence. By the 2-out-of-3-property of the model category $\mathcal{C}$, we find that the morphism induced by $p_*$ is also a weak equivalence.

(vii) Now we need to change the diagrams: We have a commutative square

$$\begin{array}{ccc} \operatorname{holim}_{\mathrm{Sm}_S^{af}/X}(p^*)^\dagger J & \xrightarrow[\simeq]{p_*} & \operatorname{holim}_{\mathrm{Sm}_S^{af}/U}(p^* \circ p_*)^\dagger J \\ \simeq \downarrow & & \downarrow \simeq \\ F^{af}(X) & \xrightarrow[p_*]{} & F^{af}(U). \end{array}$$

The commutativity of the square follows by inspecting the definition of homotopy limits and the definition of the morphisms we consider, cf. [Hir03, Definition 19.1.5 and Proposition 19.1.8(2)]. The horizontal morphism on the top is the one from Step (vi). The vertical morphisms are changes of diagrams as in Step (iv), therefore they are weak equivalences. Therefore, the lower horizontal morphism is also a weak equivalence. ■

Remark 3.11 *(i) The obvious idea for proving that a morphism between homotopy limits is a weak equivalence would be to directly appeal to [Hir03, Theorem 19.6.7(2)]. This however does not work, as the functor $p_*$ fails to be homotopy right cofinal, cf. [Hir03, Definition 19.1.6]. This can be seen from the following argument: The functor $p_*$ does not induce a weak equivalence on homotopy limits if the diagrams do not arise from functors which have affine $\mathbb{A}^1$-invariance and the affine Brown-Gersten property. Since the categories $\mathrm{Sm}_S^{af}/U$ and $\mathrm{Sm}_S^{af}/X$ both have terminal objects, the projection morphisms $F^{af}(U) \to F(U)$ and $F^{af}(X) \to F(X)$ are weak equivalences. For any presheaf $F$ for which $F(U)$ and $F(X)$ are not weakly equivalent, the homotopy limits can not be weakly equivalent either. There do indeed exist functors which do not satisfy affine $\mathbb{A}^1$-invariance.*

*(ii) We want to note that for an affine vector bundle torsor $U \to X$, the functors $p_*$ and $p^*$ are not adjoint, but still there exist morphisms*

$$\mathrm{Hom}_{\mathrm{Sm}_S^{af}/U}(p^*Y, Z) \rightleftarrows \mathrm{Hom}_{\mathrm{Sm}_S^{af}/X}(Y, p_*Z).$$

*for any affine $Y$ in $\mathrm{Sm}_S^{af}(X)$ and any affine $Z$ in $\mathrm{Sm}_S^{af}(U)$. We also obtain natural transformations: $Y \to p_*p^*Y$ is the zero section, the morphism $p^*p_*Y = Y \times_X U \to Y$ is the projection. Therefore the categories $\mathrm{Sm}_S^{af}/U$ and $\mathrm{Sm}_S^{af}/X$ are both contractible.*

We are now ready to complete the proof of Morel's result.

Proof of Theorem 3.9: The main idea of the proof can be found in [Wei89, Proposition 5.3 and Theorem 5.1]. Using Proposition 3.10 above, we are able to bootstrap Weibel's argument. In the following, we will consider a distinguished square for the Zariski topology:

$$\begin{tikzcd} U \times_X V \arrow[r] \arrow[d] & V \arrow[d, hook] \\ U \arrow[r, hook] & X. \end{tikzcd}$$

(i) This is a slight modification of the proof of [Wei89, Proposition 5.3]. We assume that we are in the special situation that $X = \operatorname{Spec} A$ is affine and $U = \operatorname{Spec} A_f$ for some $f \in A$. By Jouanolou's trick, we can replace $V$ by an affine vector bundle torsor $\operatorname{Spec} B \to V$. We want to prove that the upper square in the following diagram is homotopy cartesian:

$$\begin{tikzcd} F^{af}(\operatorname{Spec} A) \arrow[r] \arrow[d] & F^{af}(\operatorname{Spec} A_f) \arrow[d] \\ F^{af}(V) \arrow[r] \arrow[d, "\simeq"'] & F^{af}(V \times_{\operatorname{Spec} A} \operatorname{Spec} A_f) \arrow[d, "\simeq"] \\ F^{af}(\operatorname{Spec} B) \arrow[r] & F^{af}(\operatorname{Spec} B_f). \end{tikzcd}$$

Note that $\operatorname{Spec} B_f \to U \times_X V$ is also an affine vector bundle torsor, and therefore Proposition 3.10 implies that the two lower vertical maps in the above diagram are indeed weak equivalences. Thus the lower square is homotopy cartesian, and by the homotopy pullback lemma it suffices to show that the outer square is homotopy cartesian.

(ii) As in Weibel's proof [Wei89, Proposition 5.3], we consider the ideal $J$ defining the complement of $U \hookrightarrow X$. There exists a $g \in J$ such that $fA + gA = A$. Then we have a diagram as follows:

$$\begin{tikzcd} F(\operatorname{Spec} A) \arrow[rr] \arrow[dd] \arrow[dr] & & F(\operatorname{Spec} A_f) \arrow[dd] \arrow[dr] & \\ & F(\operatorname{Spec} B) \arrow[rr] \arrow[dd] & & F(\operatorname{Spec} B_f) \arrow[dd] \\ F(\operatorname{Spec} A_g) \arrow[rr] \arrow[dr, "\simeq"] & & F(\operatorname{Spec} A_{fg}) \arrow[dr, "\simeq"] & \\ & F(\operatorname{Spec} B_g) \arrow[rr] & & F(\operatorname{Spec} B_{fg}). \end{tikzcd}$$

The front and back face are homotopy cartesian by the affine Brown-Gersten property. Moreover, $\operatorname{Spec} B_g \to \operatorname{Spec} A_g$ is a vector bundle, which implies that the two lower diagonal morphisms are indeed weak equivalences and thus the bottom face is homotopy cartesian. From the homotopy pullback lemma in the proper model category $\mathcal{C}$ it follows that the top square is homotopy cartesian as well.

(iii) Assume $X$ is quasi-projective, and let $W \to X$ be an affine vector bundle torsor over $X$ given by the Jouanolou trick. This induces another distinguished square

$$\begin{array}{ccc} W \times_X U \times_X V & \longrightarrow & W \times_X V \\ \downarrow & & \downarrow \\ W \times_X U & \overset{}{\underset{i}{\hookrightarrow}} & W, \end{array}$$

where $W$ is affine and all projections away from $W$ are vector bundle torsors, not necessarily with affine total space. Replacing the original square by the weakly equivalent square

$$\begin{array}{ccc} F^{af}(W = \operatorname{Spec} A) & \longrightarrow & F^{af}(W \times_X V) \\ \downarrow & & \downarrow \\ F^{af}(W \times_X U) & \longrightarrow & F^{af}(W \times_X U \times_X V), \end{array}$$

we can therefore assume that $X = \operatorname{Spec} A$ is affine.

(iv) By making $U$ smaller, we can obtain a distinguished square, in which $U = \operatorname{Spec} A_f$ for some $f \in A$. We obtain the following diagram:

$$\begin{array}{ccccc} F^{af}(X = \operatorname{Spec} A) & \longrightarrow & F^{af}(U) & \longrightarrow & F^{af}(\operatorname{Spec} A_f) \\ \downarrow & & \downarrow & & \downarrow \\ F^{af}(V) & \longrightarrow & F^{af}(U \times_X V) & \longrightarrow & F^{af}(V \times_X \operatorname{Spec} A_f). \end{array}$$

From Step (ii), we know that the outer square is homotopy cartesian. By the homotopy pullback lemma for the proper model category $\mathcal{C}$, it suffices to show that Step (ii) applies to the right square as well. By using a vector bundle torsor $Z \to U$ and repeating the argument of Step (iii) above, we obtain a square which is weakly equivalent to the right square with $U$ replaced by the affine scheme $Z$. Step (ii) now applies, and therefore this square is homotopy cartesian.

(v) For the passage from quasi-projective to all smooth schemes, we note that the arguments of [Wei89, Proposition 6.7] go through to show that $F^{af}$ indeed satisfies the Brown-Gersten property for all smooth schemes.

(vi) We still have to show that the morphism $F^{af}(X) \to F^{af}(X \times \mathbb{A}^1)$ is a weak equivalence for any smooth scheme $X$. For quasi-projective schemes $X$, this follows again via Proposition 3.10 by considering an affine vector bundle torsor $U \to X$ together with its pullback $U \times \mathbb{A}^1 \to X \times \mathbb{A}^1$, which is again an affine vector bundle torsor. Now contemplate the following diagram:

$$\begin{array}{ccc} F^{af}(U) & \overset{\simeq}{\longrightarrow} & F^{af}(U \times \mathbb{A}^1) \\ \simeq\downarrow & & \downarrow\simeq \\ F^{af}(X) & \longrightarrow & F^{af}(X \times \mathbb{A}^1) \end{array}$$

The proof of [Wei89, Theorem 6.11] then yields the general case.

(vii) An argument as in [Mor07, Remark A.1.6 and Lemma A.2.3] finishes the proof. ■

Remark 3.12 *The above result is stated for a field in [Mor07], but the proof obviously works for any affine base scheme. An analogous result holds for the Nisnevich topology, but for the time being needs the restriction that the base is a field and that the simplicial sheaf is a simplicial sheaf* of groups. *The description of the homotopy groups of algebraic groups which we give in this paper will therefore also hold in the Nisnevich topology over a base field.*

*The point where the above argument fails to work for the Nisnevich topology is the beginning of Step (v) above. Making $U$ smaller works only for the Zariski topology, because if $p : V \to X$ is an étale map which is an isomorphism over $X \setminus U$, it is simply not possible to make $U$ smaller...*

Corollary 3.13 *Let $F$ be a $\mathcal{C}$-presheaf which satisfies the affine Brown-Gersten property and affine $\mathbb{A}^1$-invariance, and denote by $\tilde{F}$ a fibrant replacement of $F$. Furthermore, let $X$ be any smooth scheme, and let $U \to X$ be an affine vector bundle torsor over $X$. Then we have a zig-zag of weak equivalences*

$$F^{af}(U) \leftarrow F^{af}(X) \to \tilde{F}(X).$$

In particular, to compute the values of the homotopy groups of a simplicial sheaf over projective space $\mathbb{P}^n$, it suffices to compute the homotopy groups over the Stiefel variety

$$GL_{n+1}/(\mathbb{G}_m \times GL_n).$$

# 4 The Brown-Gersten Property for Chevalley Groups

In this section, we are going to prove our main results, the representability of unstable Karoubi-Villamayor K-theories by Chevalley groups. This result depends essentially on the existence of a matrix factorization

$$G(\Phi, A[t_1, \ldots, t_n]) = G(\Phi, A) \cdot E(\Phi, A[t_1, \ldots, t_n])$$

for any reduced and irreducible root system $\Phi$ not equal to $A_1$ or $G_2$. Such matrix factorizations exist whenever the ring $R$ is smooth and essentially of finite type over an excellent Dedekind ring. Questions of this type are known as the $K_1$-analogue of Serre's conjecture, because the existence of the above matrix factorization is equivalent to affine $\mathbb{A}^1$-invariance of the unstable $K_1$-functor

$$K_1(\Phi, A) = G(\Phi, A)/E(\Phi, A).$$

This question was first considered by Suslin [Sus77] for the case of $\Phi = A_l$, $l \geq 2$ and $A$ a field. The more general question of regular algebras over fields has first been investigated by Vorst [Vor81] for the special case $GL_n$, and in greater generality by Abe [Abe83]. Results for orthogonal groups have also been obtained by Kopeiko and Suslin [KS82]. Grunewald, Mennicke and Vaserstein investigated the case of symplectic groups over a locally principal ideal ring [GMV91]. These are just a couple of references dealing with $K_1$-analogues of Serre's conjecture; for a survey of most important aspects of Serre's conjecture as well as a comprehensive bibliography, I highly recommend the book of Lam [Lam06].

The basic proof structure for all these proofs was probably first conceived by Lindel [Lin81]. We give a short sketch of how the proofs proceed. The final goal is to obtain the matrix factorization over a general regular algebra over a field resp. a Dedekind ring. Using a suitable local-global principle, this question is reduced to the case of regular local rings. The first place where a local-global principle appeared was Quillen's paper [Qui76], and a very general approach to local-global principles has been developed by Bass, Connell and Wright [BCW76]. The next step is a description of the behaviour of the $\mathbb{A}^1$-invariance property in Nisnevich neighbourhoods. By Lindel's result, cf. Theorem 2.1, every regular local algebra can be realized as Nisnevich neighbourhood of a localization of a polynomial ring over the base field resp. Dedekind ring. The question is thus reduced to $\mathbb{A}^1$-invariance over the base scheme.

We recall the local-global principle in Section 4.1, and the base case of the matrix factorization in Section 4.2. These results are all well known, but scattered in the literature, therefore we give a short overview of the relevant papers. The general matrix factorization will then be used in Section 4.3 to prove that the presheaf $\mathrm{Sing}_\bullet^{\mathbb{A}^1}(G(\Phi))$ has the affine Brown-Gersten property, which implies representability for unstable Karoubi-Villamayor K-theory.

> In this chapter, the base scheme $S$ will be the spectrum of a field or of an excellent Dedekind ring $R$. When we speak of $\mathbb{A}^1$-weak equivalence, we are referring to the $\mathbb{A}^1$-local model structure on the category of simplicial presheaves with the Zariski topology. Using [Mor07], the results remain valid for the Nisnevich topology if the base scheme is the spectrum of a field.

## 4.1 The Local-Global Principle

The following form of the local-global principle is given in [Abe83, Theorem 1.15], and generalizes the corresponding result of Suslin for the case $GL_n$. See also the exposition in [Lam06, Chapter VI.1].

THEOREM 4.1 (LOCAL-GLOBAL PRINCIPLE) *Let $\Phi$ be an irreducible and reduced root system not equal to $A_1$ or $G_2$, and let $A$ be a commutative ring. For any element $\sigma \in G(\Phi, A[t], (t))$, the following assertions hold:*

*(i) The Quillen set $Q(\sigma) = \{s \in A \mid \sigma_s \in E(\Phi, A_s[t])\}$ is an ideal in $A$.*

*(ii) If $\sigma_{\mathfrak{m}} \in E(\Phi, A_{\mathfrak{m}}[t])$ for every maximal ideal $\mathfrak{m}$ in $A$, then $\sigma \in E(\Phi, A[t])$.*

COROLLARY 4.2 *Let $\sigma \in G(\Phi, A[t])$ with $\mathrm{rk}\,\Phi \geq 2$, and let*

$$\sigma_{\mathfrak{m}} \in G(\Phi, A_{\mathfrak{m}}) \cdot E(\Phi, A_{\mathfrak{m}}[t])$$

*for every maximal ideal $\mathfrak{m}$ of $A$, then $\sigma \in G(\Phi, A) \cdot E(\Phi, A[t])$.*

The next thing we need is control on the behaviour of the structures we are interested in with respect to Nisnevich neighbourhoods. The prototypical results from the literature are [Lin81, Proposition 1] for the case of projective modules, and [Vor81, Lemma 2.4] for the case of general linear groups. The following patching theorem for the elementary subgroups in the *Nisnevich topology* is the generalization of [Vor81, Lemma 2.4] to arbitrary Chevalley groups, and can be found in [Abe83, Lemma 3.7].

PROPOSITION 4.3 *Let $A \hookrightarrow B$ be a subring, $f \in B$ not a nilpotent element.*

*(i) If $Bf + A = B$ then there exists for every $\alpha \in E(\Phi, B_f)$ some $\beta \in E(\Phi, A_f)$ and $\gamma \in E(\Phi, B)$ such that $\alpha = \gamma_f \beta$.*

*(ii) If moreover $Bf \cap A = Af$ and $f$ is not a zero-divisor in $B$ then there exists for every $\alpha \in G(\Phi, B)$ with $\alpha_f \in E(\Phi, B_f)$ some $\beta \in G(\Phi, A)$ and $\gamma \in E(\Phi, B)$ such that $\alpha = \gamma\beta$.*

REMARK 4.4 *Let $R$ be an excellent Dedekind ring, let $A$ be a smooth $R$-algebra, $B$ be an étale $A$-algebra and $f \in A$ be an element such that $A/f \to B/f$ is an isomorphism. Then $\operatorname{Spec} A_f, \operatorname{Spec} B \to \operatorname{Spec} A$ is an elementary Nisnevich covering, cf. Remark 3.5. In this situation, we have $Bf + A = B$ and $Bf \cap A = Af$.*

*Note that the same also holds for the corresponding polynomial rings, i.e. $B[t_1, \ldots, t_n]f + A[t_1, \ldots, t_n] = B[t_1, \ldots, t_n]$ and $B[t_1, \ldots, t_n]f \cap A[t_1, \ldots, t_n] = A[t_1, \ldots, t_n]f$. Therefore, Proposition 4.3 can be applied to Nisnevich neighbourhoods.*

The following result extends the factorization of matrices from rings to arbitrary localizations, and is therefore some kind of converse of the local-global principle. It has been proven in [Vor81, Lemma 2.1] for the case of $GL_n$, and generalized in [Abe83, Lemma 3.6].

PROPOSITION 4.5 *Let $A$ be a commutative ring and $S$ be a multiplicative subset. If $G(\Phi, A[t_1, \ldots, t_n]) = G(\Phi, A)E(\Phi, A[t_1, \ldots, t_n])$, then also*

$$G(\Phi, A_S[t_1, \ldots, t_n]) = G(\Phi, A_S)E(\Phi, A_S[t_1, \ldots, t_n]).$$

## 4.2 Matrix Factorizations

The base case of the Lindel-Vorst proof scheme is the proof of the matrix factorization for polynomial rings over the base scheme $S$ which in our case is assumed to be the spectrum of an excellent Dedekind ring $R$. This result is basically known, but I could not find a suitable reference for the general case.

PROPOSITION 4.6 *Let $\Phi$ be a reduced and irreducible root system not equal to $A_1$ or $G_2$, and let $R$ be a Dedekind ring. Then we have the following matrix factorization*

$$G(\Phi, R[t_1, \ldots, t_n]) = G(\Phi, R) \cdot E(\Phi, R[t_1, \ldots, t_n]).$$

PROOF: By the local-global principle, cf. Theorem 4.1, we can assume that $R$ is a discrete valuation ring. The result is known for the root systems $A_l$ and $C_l$ over a Dedekind ring, cf. [GMV91, Theorem 1.2]. It was remarked in that paper that the same type of result could be extended to other groups. For the convenience of the reader, we shortly explain how to do this.

We first consider the case of one variable. By the assumption that $R$ is a discrete valuation ring, the maximal ideal spectrum of $R[t]$ has dimension 1, and therefore the absolute stable range condition $ASR_3$ holds, cf. [Ste78, Theorem 1.4]. Furthermore, under the condition $ASR_3$, the inclusions of root systems $B_l \to B_{l+1}$ and $D_l \to D_{l+1}$ induce surjections $K_1(B_l, R) \to K_1(B_{l+1}, R)$ resp. $K_1(D_l, R) \to K_1(D_{l+1}, R)$. Surjective stability for the inclusion of root systems $\Delta \to \Phi$ can be reformulated as the following matrix factorization:

$$G(\Phi, R) = E(\Phi, R) \cdot G(\Delta, R).$$

By an inductive procedure and the identifications $B_2 = C_2$ and $D_3 = A_3$, we can thus extend the matrix factorization

$$G(\Phi, R[t]) = E(\Phi, R[t]) \cdot G(\Phi)$$

to all classical root systems.

The same argument works for the exceptional groups: By [Plo93, Theorem 1], the inclusions $D_5 \to E_6$, $E_6 \to E_7$ and $E_7 \to E_8$ induce surjections under the condition $ASR_4$. Similarly, the inclusions $B_3, C_3 \to F_4$ induce surjections on $K_1$ under the condition $ASR_3$. The matrix factorizations therefore hold for all root systems and one-dimensional polynomial rings over discrete valuation rings.

Finally, we note that all that is needed for the proof of [GMV91, Theorem 1.2] to go through is the local-global principle Theorem 4.1, a variation of Horrock's theorem, [Abe83, Proposition 3.3] resp. [KS82, Theorem 6.9], and the matrix factorization for a polynomial ring in one variable, which we explained above. ■

### 4.3 Representability of Unstable K-Groups

The local-global principle is the chief tool to prove the matrix factorization of Proposition 4.6 for general smooth $R$-algebras essentially of finite type. The proof is basically the one of [Vor81, Theorem 3.3], cf. also [Abe83, Theorem 3.8]. The only modifications concern the passage from $GL_n$ to an arbitrary Chevalley group, as well as a generalization from fields to excellent Dedekind rings.

PROPOSITION 4.7 *Let $\Phi$ be a reduced and irreducible root system not equal to $A_1$ or $G_2$, let $R$ be an excellent Dedekind ring, and let $B$ be a regular $R$-algebra, which is smooth and essentially of finite type. Then we have a factorization*

$$G(\Phi, B[t_1, \dots, t_n]) = G(\Phi, B) \cdot E(\Phi, B[t_1, \dots, t_n]).$$

*As a consequence, the functor $K_1(\Phi, -)$ has affine $\mathbb{A}^1$-invariance, i.e.*

$$K_1(\Phi, B[t_1, \dots, t_n]) = K_1(\Phi, B).$$

PROOF: We claim that it suffices to prove that for any matrix $\sigma(t_1, \dots, t_n) \in G(\Phi, B[t_1, \dots, t_n])$ such that $\sigma(0) = I$, we have $\sigma(t_1, \dots, t_n) \in E(\Phi, B[t_1, \dots, t_n])$.

The factorization then follows, since for $\alpha \in G(\Phi, B[t_1, \dots, t_n])$ we obtain $(\alpha(0, \dots, 0))^{-1}\alpha \in E(\Phi, B[t_1, \dots, t_n])$. Finally, since

$$E(\Phi, B[t_1, \dots, t_n]) \cap G(\Phi, B) = E(\Phi, B),$$

we have

$$\begin{aligned} K_1(\Phi, B[t_1,\dots,t_n]) &= G(\Phi, B[t_1,\dots,t_n])/E(\Phi, B[t_1,\dots,t_n]) \\ &= G(\Phi, B)/E(\Phi, B) \\ &= K_1(\Phi, B). \end{aligned}$$

We now prove that for any matrix $\sigma(t_1,\dots,t_n) \in G(\Phi, B[t_1,\dots,t_n])$ such that $\sigma(0) = I$, we have $\sigma(t_1,\dots,t_n) \in E(\Phi, B[t_1,\dots,t_n])$. The proof is by induction on the Krull dimension of $B$, where the base case of the induction is given by Proposition 4.6.

By the local-global principle Theorem 4.1, we can assume that $R$ and $B$ are local rings, in particular $R$ is a field or a discrete valuation ring. We concentrate on the dvr case. By Theorem 2.1, $B$ is an étale neighbourhood of a localization of a polynomial algebra over a discrete valuation ring, i.e. there is an étale discrete valuation ring extension $R \hookrightarrow W$, a localization $A$ of $W[t_1,\dots,t_{\dim B-1}]$ at a maximal ideal, and an element $f \in A$ such that $Bf + A = B$ and $Bf \cap A = Af$. We are therefore exactly in the situation of Proposition 4.3, as $f$ is not a zero-divisor in $B[t_1,\dots,t_n]$ since $B$ is regular and local. Since $B$ is local, we also have $\dim B_f < \dim B$, and by induction hypothesis, $\sigma_f(t_1,\dots,t_n) \in E(\Phi, B_f[t_1,\dots,t_n])$. By part (ii) of Proposition 4.3, there is a factorization

$$\sigma(t_1,\dots,t_n) = \gamma(t_1,\dots,t_n)\beta(t_1,\dots,t_n)$$

with $\gamma(t_1,\dots,t_n) \in E(\Phi, B[t_1,\dots,t_n])$ and $\beta(t_1,\dots,t_n) \in G(\Phi, A[t_1,\dots,t_n])$. From this we also obtain

$$\sigma(t_1,\dots,t_n) = \gamma(t_1,\dots,t_n)\gamma(0,\dots,0)^{-1}\beta(0,\dots,0)^{-1}\beta(t_1,\dots,t_n),$$

since $I = \alpha(0,\dots,0) = \gamma(0,\dots,0)\beta(0,\dots,0)$. From the induction assumption, the first two factors are in $E(\Phi, B[t_1,\dots,t_n])$. By the base case, we know that the result holds for polynomial rings $W[t_1,\dots,t_n]$, and via Proposition 4.5 it also holds for localizations of polynomial rings, in particular for $A$. Therefore,

$$\beta(0,\dots,0)^{-1}\beta(t_1,\dots,t_n) \in E(\Phi, A[t_1,\dots,t_n]) \hookrightarrow E(\Phi, B[t_1,\dots,t_n]).$$

■

From the matrix factorizations for $\mathrm{Sing}_\bullet^{\mathbb{A}^1} E(\Phi)$ and the affine $\mathbb{A}^1$-invariance for the $K_1$-presheaves, we obtain the affine Brown-Gersten property for the Chevalley groups. The following results were first obtained by Morel [Mor07] for the special case $GL_n$. The technique of proof is the same as in Morel's paper.

THEOREM 4.8 *Let $\Phi$ be a root system not equal to $A_1$ or $G_2$. Then the simplicial presheaf $\mathrm{Sing}_\bullet^{\mathbb{A}^1}(G(\Phi))$ on the site $\mathrm{Sm}_S$ has the affine Brown-Gersten property for the Nisnevich topology.*

PROOF: The result follows Proposition 4.3 by an argument similar to Morel's argument in [Mor07, Theorem 1.3.4]: let the following elementary distinguished square be given:

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A_f & \longrightarrow & B_f. \end{array}$$

Assume furthermore that $A$ and $B$ are integral domains and that $A \to B$ is injective. For any of the rings $A, B$ and $A_f$, we consider the groups $\mathrm{Sing}_\bullet^{\mathbb{A}^1} \tilde{E}(\Phi, R) = \mathrm{Sing}_\bullet^{\mathbb{A}^1} E(\Phi, B_f) \cap \mathrm{Sing}_\bullet^{\mathbb{A}^1} G(\Phi, R)$. Now consider the morphism

$$\mathrm{Sing}_\bullet^{\mathbb{A}^1} \tilde{E}(\Phi, A_f)/\,\mathrm{Sing}_\bullet^{\mathbb{A}^1} \tilde{E}(\Phi, A) \to \mathrm{Sing}_\bullet^{\mathbb{A}^1} E(\Phi, B_f)/\,\mathrm{Sing}_\bullet^{\mathbb{A}^1} \tilde{E}(\Phi, B).$$

We claim that it is an isomorphism. It is a monomorphism, because if $\gamma \in \mathrm{Sing}_\bullet^{\mathbb{A}^1} E(\Phi, A_f)$ becomes the identity in $\mathrm{Sing}_\bullet^{\mathbb{A}^1} E(\Phi, B_f)/\,\mathrm{Sing}_\bullet^{\mathbb{A}^1} E(\Phi, B)$, it has to be in the image of $\mathrm{Sing}_\bullet^{\mathbb{A}^1} E(\Phi, B)$. Since $\mathrm{Sing}_\bullet^{\mathbb{A}^1} E(\Phi, -)$ is a sheaf, $\gamma$ is in the image of $\mathrm{Sing}_\bullet^{\mathbb{A}^1} G(\Phi, A) \cap \mathrm{Sing}_\bullet^{\mathbb{A}^1} E(\Phi, B_f)$.

It is surjective by Proposition 4.3, because any $\alpha \in E(\Phi, B_f[t_1, \ldots, t_n])$ can be factored as $\alpha = \gamma_f \beta$ with $\beta \in E(\Phi, A_f[t_1, \ldots, t_n])$ and $\gamma \in E(\Phi, B[t_1, \ldots, t_n])$. Therefore $[\alpha] \in \mathrm{Sing}_\bullet^{\mathbb{A}^1} E(\Phi, B_f)/\,\mathrm{Sing}_\bullet^{\mathbb{A}^1} E(\Phi, B)$ lifts to $\beta$.

Applying [Mor07, Lemma 1.3.5], we find that the elementary distinguished square induces a homotopy cartesian square:

$$\begin{array}{ccc} \mathrm{Sing}_\bullet^{\mathbb{A}^1} \tilde{E}(\Phi, A) & \longrightarrow & \mathrm{Sing}_\bullet^{\mathbb{A}^1} \tilde{E}(\Phi, B) \\ \downarrow & & \downarrow \\ \mathrm{Sing}_\bullet^{\mathbb{A}^1} \tilde{E}(\Phi, A_f) & \longrightarrow & \mathrm{Sing}_\bullet^{\mathbb{A}^1} E(\Phi, B_f). \end{array}$$

Now the quotient $\mathrm{Sing}_\bullet^{\mathbb{A}^1} G(\Phi, R)/\,\mathrm{Sing}_\bullet^{\mathbb{A}^1} E(\Phi, R)$ is simplicially constant, by the matrix factorization Proposition 4.7. By [Mor07, Lemma 1.3.5] again, we find that $\mathrm{Sing}_\bullet^{\mathbb{A}^1}(G(\Phi))$ has the affine Brown-Gersten property for the Nisnevich topology. ■

Remark 4.9 *It is possible to describe a patching of homotopic morphisms. The proof structure is similar to [Lam06, Theorem VI.1.18]. In fact, the affine Brown-Gersten property is a generalization of the weak pullback property for the unstable $K_1$-functors.*

*Assume given the following morphisms*

$$\alpha \in G(\Phi, A_f[t_1, \ldots, t_n]) \textit{ and } \beta \in G(\Phi, B[t_1, \ldots, t_n])$$

*which are homotopic over $B_f[t_1, \ldots, t_n]$, i.e. there is $H \in G(\Phi, B_f[t_1, \ldots, t_n][t])$ such that $H(t = 0) = \alpha$ and $H(t = 1) = \beta$.*

*We will show that it is possible to modify $\alpha$ and $\beta$ up to homotopy such that they actually are* equal *over $B_f$. Consider the element $\sigma = \beta\alpha^{-1} \in G(\Phi, B_f[t_1, \ldots, t_n])$. Since $H$ is a homotopy between $\alpha$ and $\beta$, $(H \cdot \alpha^{-1})(t = 0) = I$ and $(H \cdot \alpha^{-1})(t = 1) = \sigma$. This implies that $\sigma \in E(\Phi, B_f[t_1, \ldots, t_n])$, by Proposition 4.7.*

*By (i) of Proposition 4.3, we can decompose $\sigma = \beta_0^{-1}\alpha_0$ with*

$$\alpha_0 \in E(\Phi, A_f[t_1, \ldots, t_n]) \textit{ and } \beta_0 \in E(\Phi, B[t_1, \ldots, t_n]).$$

*Then $I = \sigma^{-1}\sigma = \alpha_0^{-1}\beta_0\beta\alpha^{-1}$, and we therefore have the equation $\alpha_0\alpha = \beta_0\beta$ in $G(\Phi, B_f[t_1,\dots,t_n])$. Since $G(\Phi)$ is a sheaf in the Nisnevich topology, there is a global $\gamma \in G(\Phi, A[t_1,\dots,t_n])$ such that $\gamma = \alpha_0\alpha$ in $G(\Phi, A_f[t_1,\dots,t_n])$ and $\gamma = \beta_0\beta$ in $G(\Phi, B[t_1,\dots,t_n])$. Because $\alpha_0 \in E(\Phi, A_f[t_1,\dots,t_n])$, there is an element $H_\alpha \in G(\Phi, A_f[t_1,\dots,t_n][t])$ with $H_\alpha(t=0) = I$ and $H_\alpha(t=1) = \alpha_0$. Then $H \cdot \alpha \in G(\Phi, A_f[t_1,\dots,t_n][t])$ provides a homotopy $\alpha \sim \alpha_0\alpha$. The same argument works for $\beta \sim \beta_0\beta$. It has been shown in [Wen07] how to prove the affine Brown-Gersten property from such a patching result.*

Corollary 4.10 *Let $\Phi$ be a root system not equal to $A_1$ or $G_2$. Then the group schemes $G(\Phi)$ represent the unstable Karoubi-Villamayor K-groups $KV_n(\Phi, -)$ in the $\mathbb{A}^1$-local model structure on $\Delta^{op}P\mathcal{S}hv(\mathrm{Sm}_S)$. The corresponding simplicial resolutions $\mathrm{Sing}_\bullet^{\mathbb{A}^1}(G(\Phi))$ represent the unstable Karoubi-Villamayor K-groups in the simplicial model structure on $\Delta^{op}P\mathcal{S}hv(\mathrm{Sm}_S)$.*

*Therefore, for any smooth affine $S$-scheme $U$, we obtain isomorphisms*

$$\pi_n^{\mathbb{A}^1}(G(\Phi), I)(U) \cong KV_{n+1}(\Phi, U).$$

Proof: By Theorem 3.9 and the affine Brown-Gersten property, cf. Theorem 4.8 above, we conclude that the affine replacement of $\mathrm{Sing}_\bullet^{\mathbb{A}^1}(G(\Phi))$ has the Brown-Gersten property. Moreover, the fibrant replacement of $\mathrm{Sing}_\bullet^{\mathbb{A}^1}(G(\Phi))$ is $\mathbb{A}^1$-local. Then for any smooth affine scheme $U$, we have a weak equivalence

$$\mathrm{Sing}_\bullet^{\mathbb{A}^1}(G(\Phi))(U) \to \mathrm{Ex}_{\mathbb{A}^1}^\infty \mathrm{Sing}_\bullet^{\mathbb{A}^1}(G(\Phi))(U)$$

where $\mathrm{Ex}_{\mathbb{A}^1}^\infty$ denotes the fibrant replacement in the $\mathbb{A}^1$-local model structure on $\mathrm{Sm}_S$ with the Zariski-topology. The Karoubi-Villamayor unstable K-groups are the homotopy groups of the simplicial complex $\mathrm{Sing}_\bullet^{\mathbb{A}^1}(G(\Phi))(U)$, and the homotopy groups $\pi_n^{\mathbb{A}^1}(G(\Phi), I)(U)$ are the homotopy groups of $\mathrm{Ex}_{\mathbb{A}^1}^\infty \mathrm{Sing}_\bullet^{\mathbb{A}^1}(G(\Phi))(U)$. This proves the claim for the simplicial model structure. For the $\mathbb{A}^1$-local model structure, we note that $G(\Phi) \to \mathrm{Sing}_\bullet^{\mathbb{A}^1}(G(\Phi))$ is an $\mathbb{A}^1$-weak equivalence. ■

Remark 4.11 *(i) Over a base field $k$, we can use Morel's result [Mor07, Theorem A.4.2] to conclude that Karoubi-Villamayor K-theory is representable in the $\mathbb{A}^1$-local model structure also for the Nisnevich topology. It seems reasonable to conjecture that the same result can also be obtained for (excellent) Dedekind rings.*

*(ii) From [MV99, Proposition 3.2.9], it follows that for any scheme $X$ which is smooth and essentially of finite type over a Dedekind ring $R$, the pullback of a fibrant replacement of $\mathrm{Sing}_\bullet^{\mathbb{A}^1}(G(\Phi))$ along the structure morphism $X \to \mathrm{Spec}\, R$ is again $\mathbb{A}^1$-local. Therefore, Chevalley groups represent the corresponding unstable Karoubi-Villamayor K-theory in any of the model categories $\Delta^{op}P\mathcal{S}hv(\mathrm{Sm}_S)$ for $S$ smooth over a Dedekind ring.*

Corollary 4.12 *Let $\Phi$ be a root system not equal to $A_1$ or $G_2$. The schemes $G(\Phi)$ are $\mathbb{A}^1$-connected. In particular, the inclusion $E(\Phi) \hookrightarrow G(\Phi)$ is an $\mathbb{A}^1$-weak equivalence.*

PROOF: By Theorem 4.8, we know that the simplicial sheaves $\mathrm{Sing}_\bullet^{\mathbb{A}^1}(G(\Phi))$ satisfy the affine Brown-Gersten property. For a local ring $A$, we have $\mathrm{Sing}_\bullet^{\mathbb{A}^1}(G(\Phi)) = \mathrm{Sing}_\bullet^{\mathbb{A}^1}(E(\Phi))$, [Abe69]. Therefore, the simplicial presheaf $\mathrm{Sing}_\bullet^{\mathbb{A}^1}(G(\Phi))$ is $\mathbb{A}^1$-connected, because the presheaf $\pi_0 \mathrm{Sing}_\bullet^{\mathbb{A}^1}(G(\Phi))$ sheafifies to the point. Therefore, the inclusion $E(\Phi) \hookrightarrow G(\Phi)$ induces isomorphisms

$$\pi_n \mathrm{Sing}_\bullet^{\mathbb{A}^1}(E(\Phi, A)) \xrightarrow{\cong} \pi_n \mathrm{Sing}_\bullet^{\mathbb{A}^1}(G(\Phi, A))$$

for any $n \geq 1$ and is hence an $\mathbb{A}^1$-weak equivalence. ■

# 5 Applications

## 5.1 Fundamental Groups of Split Simply-Connected Groups

In this section, we explain how the result above can be used to compute the fundamental group of a split simply-connected group.

PROPOSITION 5.1 *Let $\Phi$ be a root system not equal to $A_1$ or $G_2$, and let $k$ be a field. Then we have an isomorphism*

$$\pi_1^{\mathbb{A}^1}(G(\Phi))(\mathrm{Spec}\, k) = \begin{cases} K_2^M(k) & \Phi \text{ non-symplectic} \\ H_2(Sp_\infty(k), \mathbb{Z}) & \Phi \text{ symplectic} \end{cases}$$

*By the Hurewicz theorem over fields [Mor06], this also describes the $\mathbb{A}^1$-homology groups $H_1^{\mathbb{A}^1}(G(\Phi), \mathbb{Z})(\mathrm{Spec}\, k)$.*

PROOF: By Corollary 4.10, it suffices to compute $KV_2(\Phi, k)$ for any field $k$. The stabilization results we used in Proposition 4.6 shows that the inclusions of root systems induce isomorphisms on $K_1$ for rings of Krull-dimension 1. By definition of Karoubi-Villamayor K-theory,

$$KV_2(\Phi, k) = KV_1(\Phi, \Omega k) = G(\Phi, \Omega k)/E(\Phi, \Omega k),$$

where $\Omega k$ is the loop ring of $k$, as explained in Definition 2.2. For a field $k$, $\Omega k$ is a noetherian ring of dimension 1, and we obtain isomorphisms of $KV_2$ for the inclusions of root systems in Proposition 4.6. By a similar argument, we find that for all non-symplectic root systems, $KV_2(\Phi, k) \cong KV_2(A_\infty, k)$, and for all symplectic root systems, we have $KV_2(\Phi, k) \cong KV_2(C_\infty, k)$. By the results of Matsumoto [Mat69] and the identification of stable Karoubi-Villamayor K-theory with the plus-construction, we obtain the conclusion. ■

The reader should compare the above statements with Gille's computation of the first Suslin homology groups of semisimple groups, cf. [Gil08].

## 5.2 Homotopy Types of Algebraic Groups

In this section, we provide an applications of the above results: We describe fibre sequences which allow to determine the homotopy types of split algebraic groups over a field.

Let us therefore take a closer look at the homotopy types of algebraic groups. Assuming we are working over a perfect field $k$, there are quite a lot of results which allow to decompose a linear algebraic group into simpler parts. We will show how these decomposition results yield fibre sequences in the $\mathbb{A}^1$-homotopy theory, and that these fibre sequences allow to reduce the computation of the homotopy group sheaves of linear algebraic groups to the case of Chevalley groups which was settled in Corollary 4.10. In particular, it turns out that for any linear algebraic group $G$, all the homotopy group sheaves $\pi_n^{\mathbb{A}^1}(G)$ for $n \geq 2$ are always sheaves of unstable Karoubi-Villamayor K-groups for the root system of the semisimple part of $G$. There is a couple of restriction we have to make, however:

> All algebraic groups in the following section will be smooth and $k$-split, i.e. their maximal torus $T$ is isomorphic to some $\mathbb{G}_m^n$ over $k$. Moreover, we assume that the base field $k$ is perfect.

This assumption is necessary to avoid some subtleties in the non-split case which we are not ready to deal with yet.

**Reduction to Linear Connected Groups:** We start with an algebraic group $G$ over the perfect base field $k$, i.e. $G$ is a group object in the category of $k$-schemes. The theorem of Chevalley then allows to decompose $G$ into a linear algebraic part, and an abelian variety. For a statement of the theorem, and a readable proof we refer to [Con02].

PROPOSITION 5.2 (CHEVALLEY'S THEOREM) *Let $k$ be a perfect field, and let $G$ be a smooth and connected algebraic group over $k$. Then there exists a unique normal linear algebraic closed subgroup $H$ in $G$ for which $G/H$ is an abelian variety. Therefore, there is an exact sequence of algebraic groups*

$$H \longrightarrow G \longrightarrow A$$

*with $H$ a linear algebraic group and $A$ an abelian variety.*

COROLLARY 5.3 *Let $k$ be a perfect field, and let $G$ be an algebraic group over $k$. Then there is an $\mathbb{A}^1$-local fibre sequence*

$$H \xrightarrow{i} G \xrightarrow{p} A,$$

*with $H$ a linear algebraic group and $A$ an abelian variety.*

*The induced morphism on $\pi_0$ is a short left exact sequence of sheaves of groups*

$$0 \longrightarrow \pi_0^{\mathbb{A}^1}(H) \xrightarrow{i_*} \pi_0^{\mathbb{A}^1}(G) \xrightarrow{p_*} A = \pi_0^{\mathbb{A}^1}(A),$$

*and the inclusion of the fibre induces isomorphisms $i_* : \pi_n^{\mathbb{A}^1}(H, x) \to \pi_n^{\mathbb{A}^1}(G, i(x))$ for any $n \geq 1$ and any choice of base point $x \in H$.*

PROOF: First, we note that the sequence $H \to G \to A$ induces a fibre sequence in the simplicial model structure: The morphism $G \to A$ is a principal $H$-bundle with $H$ linear algebraic, and therefore $G \to A$ is locally trivial in the étale topology. Therefore, [Wen07, Example 3.3.10] implies that this sequence is a fibre sequence in the simplicial model structure with the Nisnevich topology.

The base is an abelian variety, therefore it is $\mathbb{A}^1$-rigid and hence local. By [Wen07, Theorem 4.3.10], this fibre sequence is preserved by $\mathbb{A}^1$-localization.

Since the abelian variety $A$ is $\mathbb{A}^1$-local, we have $\pi_n^{\mathbb{A}^1}(A,*) = 0$ for any $n \geq 1$ and any choice of base point. From the long exact sequence for the fibre sequence, we get the isomorphisms claimed. ■

Note that, contrary to what was stated in [Wen07, Corollary 4.4.12], the morphism $\pi_0^{\mathbb{A}^1}(G) \xrightarrow{p_*} A = \pi_0^{\mathbb{A}^1}(A)$ of sheaves in the Nisnevich topology need not be surjective. Only the sheafification in the étale topology is surjective, by local isotriviality. For special groups however, for which $H^1_{Nis} = H^1_{\text{ét}}$, the morphism $\pi_0^{\mathbb{A}^1}(G) \xrightarrow{p_*} A = \pi_0^{\mathbb{A}^1}(A)$ is indeed surjective.

A similar proof yields the following statement which reduces the homotopy type of a linear algebraic group to that of a *connected* group.

Proposition 5.4 *Let $G$ be a linear algebraic group over $k$. Then there exists an $\mathbb{A}^1$-local fibre sequence*

$$G^0 \xrightarrow{i} G \xrightarrow{p} H.$$

*where $H$ is a finite étale group over $k$. Therefore there exists a short left exact sequence of sheaves of groups*

$$0 \longrightarrow \pi_0^{\mathbb{A}^1}(G^0) \xrightarrow{i_*} \pi_0^{\mathbb{A}^1}(G) \xrightarrow{p_*} H = \pi_0^{\mathbb{A}^1}(H),$$

*and the inclusion of the fibre induces isomorphisms $i_* : \pi_n^{\mathbb{A}^1}(G^0, x) \to \pi_n^{\mathbb{A}^1}(G, i(x))$ for any $n \geq 1$ and any choice of base point $x \in G^0$.*

Proof: It is a standard fact that the group of connected components of a linear algebraic group is a finite étale group scheme, cf. [Bor91, Proposition 1.2(b)]. The $\mathbb{A}^1$-rigidity for a finite étale group scheme follows from [MV99, Proposition 4.3.5]. ■

**Homotopy Types of Solvable Groups:** In this paragraph, we will study the homotopy types of split solvable groups. We need to understand a couple of special cases, beginning with unipotent groups. Recall that a smooth connected linear algebraic group is unipotent if it has a filtration $\{1\} = U_n \subseteq U_{n-1} \subseteq \cdots \subseteq U_1 \subseteq U_0 = U$ such that $U_{i+1}$ is normal in $U_i$ and the quotient $U_i/U_{i+1}$ is isomorphic to (a form of) the additive group $\mathbb{G}_a$. Note that such forms only exist over non-perfect fields, since for a perfect field $k$ we have $H^1(k, \mathbb{G}_a) = 0$.

Via an inductive argument, one obtains the following result:

Proposition 5.5 *Let $U$ be a unipotent smooth connected linear algebraic group over a perfect field $k$. Then $U$ is $\mathbb{A}^1$-contractible. Moreover, any principal $U$-bundle $U \to X \to Y$ induces an $\mathbb{A}^1$-weak equivalence $X \xrightarrow{\simeq} Y$.*

In particular, any connected linear algebraic group has the homotopy type of a reductive group, since the morphism $\phi : G \to G/R_uG$ is an $\mathbb{A}^1$-weak equivalence, where $R_uG$ denotes the unipotent radical of $G$.

Next, we will deal with groups of multiplicative type. On the one hand, we know that groups of multiplicative type are $\mathbb{A}^1$-local, they are discrete objects in the $\mathbb{A}^1$-homotopy category. On the other hand, not every torsor under a group of multiplicative type over affine space is extended, i.e. induces an $\mathbb{A}^1$-local fibre sequence. The basic example of this phenomenon is a finite étale group scheme whose order is divisible by the characteristic of the base field. The next proposition will show that this comprises all the difficulties.

We introduce some notation to be used in the next proposition: For a group $T$ of multiplicative type, there is the following exact sequence of groups of multiplicative type

$$1 \to T^0 \to T \to \mu \to 1,$$

where $\mu$ is finite étale and $T^0$ is connected.

Proposition 5.6 *Let $k$ be a field, and let $\phi : G \to H$ be a torsor under a group $T$ of multiplicative type such that $\mu$ has order prime to the characteristic of $k$. Then there is an $\mathbb{A}^1$-local fibre sequence*

$$T \xrightarrow{i} G \xrightarrow{p} H$$

*and the inclusion of the fibre induces isomorphisms $i_* : \pi_n^{\mathbb{A}^1}(G, x) \to \pi_n^{\mathbb{A}^1}(H, p(x))$ for any $n \geq 2$ and any choice of base point $x \in G$.*

Proof: Using étale descent, it is easy to show that a group $T$ of multiplicative type over a field $k$ is $\mathbb{A}^1$-rigid, cf. [Wen07, Example 4.4.3].

It follows from [MV99, Proposition 4.3.1] and our assumption on the order of the finite étale group scheme $\mu$ that the classifying space $B\mu$ is $\mathbb{A}^1$-local.

By the standard theory of classifying spaces, there is a fibre sequence of classifying spaces associated to the exact sequence of groups above:

$$BT^0 \to BT \to B\mu.$$

To show that $BT$ is $\mathbb{A}^1$-local, it suffices now to show $BT^0$ is $\mathbb{A}^1$-local.

First note, that by [MV99, Proposition 4.3.8], the classifying space $B\mathbb{G}_m^n$ of a split torus is $\mathbb{A}^1$-local. To prove $BT^0$ is $\mathbb{A}^1$-local, we will show that $H^1_{\text{ét}}(X, T^0) \to H^1_{\text{ét}}(X \times \mathbb{A}^1, T^0)$ is an isomorphism for any smooth $X$. Let therefore $E \to X \times \mathbb{A}^1$ be a $T^0$-torsor and let $L/k$ be a splitting field for $T^0$. As was noted above, the classifying space of a split torus is local, and so $E_L \to (X \times_k L) \times \mathbb{A}^1$ is a $\mathbb{G}_m^n$-torsor which is extended from $X \times_k L$. Therefore, to present the $T^0$-torsor $E$, we can find a covering $U_i \to X$ and a cocycle of the special form $\rho_{ij} : (U_i \times_X U_j) \times \mathbb{A}^1 \to T^0$. By homotopy invariance of $T^0$ this yields a presentation for a torsor $E_0 \to X$, such that the extension $E_0 \times \mathbb{A}^1 \to X \times \mathbb{A}^1$ is isomorphic to $E$. This shows that the classifying space $BT^0$ is $\mathbb{A}^1$-local. A similar assertion about homotopy invariance of torsors under tori can be found in [CTS87, Lemma 2.4].

The group $G$ is a $T$-torsor over $H$, therefore it induces a simplicial fibre sequence and is thus classified by a morphism into $BT$, the classifying space of $T$-torsors. This is local by the argument above, so [Wen07, Theorem 4.3.10] implies that $T \to G \to H$ is an $\mathbb{A}^1$-local fibre sequence. ■

Using the fact that any solvable group arises as an extension of a group of multiplicative type by a unipotent group, we can finally describe the homotopy type and fibre bundles under connected solvable groups.

Proposition 5.7 *Let $G$ be a connected solvable group over a perfect field $k$. Then there exists a unipotent subgroup $G_u \hookrightarrow G$ such that $G/G_u$ is a connected group of multiplicative type.*

*The projection $\phi : G \to G/G_u$ is an $\mathbb{A}^1$-local weak equivalence. The classifying spaces $BG$ and $B(G/G_u)$ are therefore $\mathbb{A}^1$-weakly equivalent, and therefore any principal $G$-bundle induces an $\mathbb{A}^1$-local fibre sequence.*

Proof: The algebraic assertions about solvable groups can be found in [Bor91, Theorem V.15.4]. The fact that the projection $\phi : G \to G/G_u$ is an $\mathbb{A}^1$-weak equivalence follows from Proposition 5.5. From standard theory on classifying spaces, there is a fibre sequence $BG_u \to BG \to B(G/G_u)$, which together with Proposition 5.6 implies the statement about principal $G$-bundles. ∎

Corollary 5.8 *For any smooth connected linear algebraic group $G$, there is an $\mathbb{A}^1$-local fibre sequence $RG \to G \to G/RG$. In particular, there are induced isomorphisms of sheaves of homotopy groups $i_* : \pi_n^{\mathbb{A}^1}(G, x) \to \pi_n^{\mathbb{A}^1}(G/RG, p(x))$ for any $n \geq 2$ and any choice of base point $x \in G$.*

**Reduction to Chevalley Groups:** The previous corollary reduced the description of homotopy types of algebraic groups over a perfect field to the study of split semi-simple groups. It is still possible to make a semisimple group "even simpler" by decomposing it into an almost direct product of simple groups. We therefore get an $\mathbb{A}^1$-homotopy version of [Jar81, Theorem 3.2.5 and Corollary 3.2.6].

Proposition 5.9 *Let $G$ be a connected smooth semisimple split linear algebraic group over $k$, and let $S_1, \ldots, S_n$ be the finite set of minimal connected normal algebraic subgroups of $G$ of nonzero dimension. Then the multiplication homomorphism*

$$m : S_1 \times \cdots \times S_n \to G : (s_1, \ldots, s_n) \mapsto s_1 \cdots s_n$$

*is a surjective homomorphism with finite, étale, central kernel $P$. Therefore, if the order of $P$ is prime to the characteristic of $k$, the following is an $\mathbb{A}^1$-local fibre sequence:*

$$P \hookrightarrow S_1 \times \cdots \times S_n \xrightarrow{m} G.$$

*In the latter situation, we have a short exact sequence of $\mathbb{A}^1$-homotopy group sheaves:*

$$0 \to \bigoplus_{i=1}^{n} \pi_1^{\mathbb{A}^1}(S_i) \to \pi_1^{\mathbb{A}^1}(G) \to P \to 0,$$

*and for all $l \geq 2$, the multiplication map induces isomorphisms*

$$\bigoplus_{i=1}^{n} \pi_l^{\mathbb{A}^1}(S_i) \xrightarrow{\cong} \pi_l^{\mathbb{A}^1}(G).$$

Proof: The algebraic argument works as in the proof of [Jar81, Theorem 3.2.5]. That the corresponding morphism induces a fibre sequence if the order of $P$ is prime to the characteristic of $k$ follows from Proposition 5.6.

The right exactness of the short exact sequence of $\pi_1$-sheaves follows since the functor $K_1(G, R)$ is trivial for any simple, simply connected group $G$ and semi-local ring $R$, cf. [Abe69]. The group of connected components $\pi_0(S_i)$ can be identified with $K_1$ of the corresponding root system by Corollary 4.10. ∎

Corollary 5.10 *Let $G$ be a smooth, linear and split algebraic group. Then for any $n \geq 2$, we have isomorphisms of sheaves*

$$\pi_n^{\mathbb{A}^1}(G)(U) \cong \bigoplus KV_{n+1}(\Phi, U),$$

*where the direct sum ranges over all irreducible and reduced components of the root system of $G/RG$.*

### 5.3 $\mathbb{A}^1$-Locality for Split Groups

We finally show that the fibrant replacement of $\mathrm{Sing}_\bullet^{\mathbb{A}^1}(G)$ for a split group $G$ is $\mathbb{A}^1$-local. This follows from the fibre sequences established above. It should also follow from these fibre sequences that $\mathrm{Sing}_\bullet^{\mathbb{A}^1}(G)$ has the affine Brown-Gersten property for any split group.

Proposition 5.11 *Let $G$ be a split smooth linear algebraic group over a perfect field $k$. Assume that the fundamental group of its semisimple part has order prime to the characteristic of $k$. Then a fibrant replacement of $\mathrm{Sing}_\bullet^{\mathbb{A}^1}(G)$ in the simplicial model structure is $\mathbb{A}^1$-local.*

Proof: (i) For a simply-connected Chevalley group, this is Theorem 4.8.

(ii) Any semi-simple group can be written as

$$1 \to \Pi_1(G) \to S_1 \times \cdots \times S_n \to G \to 1,$$

where $S_i$ are the minimal connected normal simple subgroups of $G$ and $\Pi_1(G)$ is the fundamental group of the semisimple group $G$, cf. also Proposition 5.9. This induces an $S_1 \times \cdots \times S_n$-torsor

$$S_1 \times \cdots \times S_n \to G \to B\Pi_1(G).$$

This induces a fibre sequence

$$\mathrm{Sing}_\bullet^{\mathbb{A}^1}(S_1 \times \cdots \times S_n) \to \mathrm{Sing}_\bullet^{\mathbb{A}^1}(G) \to \mathrm{Sing}_\bullet^{\mathbb{A}^1}(B\Pi_1(G)).$$

But by our assumptions, $\Pi_1(G)$ is a finite étale group scheme of order prime to the characteristic of the base, therefore $B\Pi_1(G)$ is $\mathbb{A}^1$-local, and we have a weak equivalence

$$\mathrm{Sing}_\bullet^{\mathbb{A}^1}(B\Pi_1(G)) \simeq B\Pi_1(G).$$

Now we apply the fibrant replacement to this fibre sequence. Then the base is a fibrant and $\mathbb{A}^1$-local simplicial presheaf, and the same holds for the fibre, by Step (i). Therefore $\mathrm{Sing}_\bullet^{\mathbb{A}^1}(G)$ becomes $\mathbb{A}^1$-local after fibrant replacement.

(iii) Now any reductive group $G$ sits in an extension

$$1 \to RG \to G \to G/RG \to 1,$$

where the group $G/RG$ is semisimple. Applying Corollary 5.8, we obtain a fibre sequence

$$\mathrm{Sing}_\bullet^{\mathbb{A}^1}(RG) \to \mathrm{Sing}_\bullet^{\mathbb{A}^1}(G) \to \mathrm{Sing}_\bullet^{\mathbb{A}^1}(G/RG).$$

After a fibrant replacement, the fibre is $\mathbb{A}^1$-local, because it is weakly equivalent to a torus, and the base is $\mathbb{A}^1$-local by Step (ii). Therefore, the total space is $\mathbb{A}^1$-local.

(iv) Now for any connected group $G$, there is a simplicial weak equivalence $\mathrm{Sing}_\bullet^{\mathbb{A}^1}(G) \simeq \mathrm{Sing}_\bullet^{\mathbb{A}^1}(G/R_uG)$, where $R_uG$ denotes the unipotent radical. This finishes the proof. ∎

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Matthias Wendt, Mathematisches Institut, Albert-Ludwigs-Universität Freiburg, Eckerstrasse 1, 79104, Freiburg im Breisgau, Germany
*E-mail address:* `matthias.wendt@math.uni-freiburg.de`