On the Motivic Fundamental Groups of Smooth Toric Varieties

Matthias Wendt

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Abstract

In this short note, we want to investigate combinatorial descriptions of the motivic fundamental group $\pi_1^{\mathbb{A}^1}(X(\Delta))$ for a smooth toric variety associated to the fan Δ. This reduces the computation of the fundamental group of the toric variety $X(\Delta)$ to a computation of the $\mathbb{A}^1$-localization of an explicitly given sheaf of groups. As a corollary, a smooth toric variety for which the irrelevant ideal in the Cox ring has codimension ≥ 3 has a torus as $\mathbb{A}^1$-fundamental group.

1 Introduction

The goal of this paper is to give a sample application of the theory of fibre sequences developed in my thesis [Wen07]. More precisely, I will exhibit some fibre sequences describing the fundamental groups of smooth toric varieties. The main result can be stated as follows:

THEOREM 1 *Let N be a lattice, and Δ be a regular fan in $N \otimes_{\mathbb{Z}} \mathbb{R}$. Let $X(\Delta)$ be the smooth toric variety corresponding to the fan Δ. The group of $\mathbb{A}^1$-connected components is given by*

$$\pi_0^{\mathbb{A}^1}(X(\Delta)) = \mathbb{G}_m^{\operatorname{rk} N/N'},$$

where N' is the sublattice of N generated by Δ.

The fundamental group $\pi_1^{\mathbb{A}^1}(X(\Delta))$ sits in an exact sequence

$$1 \to \pi_1^{\mathbb{A}^1}(X(\Delta, \mathbb{Z}^{\Delta(1)})) \to \pi_1^{\mathbb{A}^1}(X(\Delta)) \to \operatorname{Hom}(A_{n-1}(X(\Delta)), \mathbb{G}_m) \to 1,$$

and $\pi_1^{\mathbb{A}^1}(X(\Delta, \mathbb{Z}^{\Delta(1)}))$ can be described as follows

$$\pi_1^{\mathbb{A}^1}(X(\Delta, \mathbb{Z}^{\Delta(1)})) = \ker\left(\pi_1^{\mathbb{A}^1}(DJ(\Delta)) \to \mathbb{G}_m^{\Delta(1)}\right).$$

The space $DJ(\Delta)$ is an analogue of the topological Davis-Januszkiewicz spaces associated to the fan Δ, and its $\mathbb{A}^1$-fundamental group is given as follows:

$$\pi_1^{\mathbb{A}^1}(DJ(\Delta)) \cong L_{\mathbb{A}^1}\left(\bigstar_{\rho \in \Delta(1)} \mathbb{G}_m(U)/R\right),$$

where $L_{\mathbb{A}^1}$ is the $\mathbb{A}^1$-localization functor on the category of sheaves of groups, and associates to each sheaf of groups a stronlgy $\mathbb{A}^1$-invariant sheaf of groups, and R is the subgroup generated by commutators

$$(u_1, \rho_1)(u_2, \rho_2)(u_1^{-1}, \rho_1)(u_2^{-1}, \rho_2)$$

whenever ρ_1 and ρ_2 span a cone $\eta \in \Delta(2)$.

One of the main corollaries of the above theorem is that the fundamental group of a smooth toric variety over a general regular base depends only on $\Delta(1)$ and $\Delta(2)$. In particular the fundamental group of $\mathbb{P}^n$, $n \geq 2$, is isomorphic to the sheaf of groups $\mathbb{G}_m$ for any regular base scheme, not just over a field.

The remaining problem in the above theorem is, that the fundamental group of the spaces $DJ(\Delta)$ is a graph product in the category of *strongly* $\mathbb{A}^1$*-invariant sheaves of groups*. However, computing coproducts in this category is not at all easy. For example, the free product $\mathbb{G}_m * \mathbb{G}_m$ in the category of strongly $\mathbb{A}^1$-invariant sheaves of groups has an *abelian* commutator subgroup, which by Morel's work [Mor06] can be identified as K_2^{MW}. The above theorem therefore reduces the complete computation of the fundamental group of a smooth toric variety to the $\mathbb{A}^1$-localization of a sheaf of groups which we can describe explicitly.

It is also interesting to note, that some of the work on homotopy types of complements of subspace arrangements and the fundamental group of real toric varieties is reflected in the above theorem: the fundamental group of a real toric variety has the same graph product representation, only using $\mathbb{Z}/2\mathbb{Z}$ instead of the group of units $\mathbb{G}_m$. As we will see later, some of the proofs from classical algebraic topology directly carry over, once we have some general results on the theory of fibre sequences developed in [Wen07].

Note also that the general idea of this paper carries through to arbitrary $\mathbb{G}_m$-cellular homotopy types: If the homotopy type is cellular, then its fundamental groupoid can be expressed as a homotopy colimit of copies of $\mathbb{G}_m$ in the category of strongly $\mathbb{A}^1$-invariant sheaves of groups. This however is a rather useless statement as the cellularity assumption does not give any information on the diagram presentation. For toric varieties, we can explicitly write down diagrams in terms of the fans.

Finally, I would like to draw the readers attention to the paper of Asok and Doran [AD07], where similar results on the fundamental group of toric varieties are obtained by different methods. The restriction to base fields in their paper allows stronger results than we can achieve here.

Remarks on Non-Smooth and Non-Split Cases: Our results only apply to smooth toric varieties which contain a *split* torus densely, and we can only give some indications what goes wrong in the more general situations.

First, the torus covering space from Lemma 4.5 does only exist in this form if the fan is regular. If the fan is still simplicial, it might be possible to find a covering of $X(\Delta)$ by a principal homogeneous space under a more general group of multiplicative type, taking care of the étale fundamental group of the quotient singularities of $X(\Delta)$. However, if the fan is not even simplicial, then there is a morphism of toric varieties $X(\Delta, \mathbb{Z}^{\Delta(1)}) \to X(\Delta)$, but it will not be a principal homogeneous space any more. Similarly, the description of the connected components of Lemma 4.2 fails in general: what one can show is that for a quotient singularity with group G, the connected components of $X(\Delta)$ embed into the Nisnevich sheafification of $H^1_{\text{ét}}(-, G)$, but it is not clear which G-torsors over a henselian local ring $\mathcal{O}^h_{U,u}$ are realized as pullbacks of the quotient presentation of $X(\Delta)$. Summing up, in the case of singular toric varieties $X(\Delta)$, it is not possible to reduce a description of the fundamental group to the Davis-Januszkiewicz spaces.

In the non-split case, there are yet other complications to the description of the connected components of $X(\Delta)$: the Brauer group $H^2(X(\Delta), \mathbb{G}_m)$ resp. its nontrivial part $H^2(X(\Delta), \mathbb{G}_m)/H^2(k, \mathbb{G}_m)$ start appearing, since the latter group can be identified with the Galois cohomology group $H^1(\mathrm{Gal}(E/k), \mathrm{Pic}(X(\Delta)))$. These elements may give rise to nontrivial connected components, since (in the non-split but smooth case) the torus covering from Lemma 4.5 is classified by a morphism to the classifying space BT, where T is the (non-split!) torus associated to the Galois-module $\mathrm{Pic}(X(\Delta))$. On connected components this classifying morphism $X(\Delta) \to BT$ yields a morphism

$$\pi_0^{\mathbb{A}^1}(X(\Delta)) \to \pi_0^{\mathbb{A}^1}(BT) \cong H^1_{\text{ét}}(-, \mathrm{Pic}(X(\Delta))).$$

Beyond that general nonsense, there is nothing I can say at the moment about the nature of this map. Finally, also the definition of the Davis-Januszkiewicz spaces would have to be adjusted to deal with non-split tori.

Structure of the Paper: After some preliminaries and notations introduced in Section 2, we explain in Section 3 what the $\mathbb{A}^1$-fundamental groupoid actually is, and how one can prove the van-Kampen theorem. We finally compute the fundamental groupoid of a toric variety in Section 4. Some additional homotopy decompositions of certain toric varieties are considered in Section 5.

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2 Preliminaries and Notation

$\mathbb{A}^1$-Homotopy Theory: We mostly use the notation from [MV99]. We let S be a regular finite type base scheme, and denote by Sm_S the category of smooth finite type schemes over S. All the simplicial presheaves mentioned in the following paper are presheaves on the category Sm_S with either the Zariski or Nisnevich topology.

Toric Varieties: The definition of a fan Δ and of the corresponding toric variety can be found in [Ful93].

The following introduces the standard notation for lattices, fans etc. Usually, the lattice in which the fans live will be denoted by $N \cong \mathbb{Z}^n$, $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ denotes the dual lattice, and $\langle -, - \rangle : M \times N \to \mathbb{Z}$ the duality pairing. We let Δ be a fan in $N \times_{\mathbb{Z}} \mathbb{R}$, whose cones will usually be denoted by σ. The set of cones in Δ of dimension k will be denoted by $\Delta(k)$. Finally, we denote the toric variety corresponding to the fan Δ by $X(\Delta, N)$. In the special case of a cone σ, we write $X(\sigma, N)$ for the corresponding affine toric variety. If the lattice is clear from the context, we will simply write $X(\Delta)$. The big torus of $X(\Delta)$ corresponding to the cone (0) is usually denoted by $\mathbb{T}$.

A cone σ generated by vectors $v_1, \dots, v_n$ is called a *simplex* if the vectors $v_1, \dots, v_n$ are linearly independent. A fan Δ is called *simplicial*, if all its cones are simplices.

A vector $v \in \mathbb{Z}^n$ is called *primitive* if its coordinates are coprime. A cone σ generated by vectors $v_1, \dots, v_r$ is called *regular* if the vectors $v_1, \dots, v_r$ are primitive, and there exist further primitive vectors $v_{r+1}, \dots, v_n$ such that $\det(v_1, \dots, v_n) = \pm 1$. A fan is called *regular* if all its cones are regular.

The notion of a Δ-collection was defined in [Cox95a]. It allows to have a concrete interpretation of the functor represented by a toric variety, and can also be used to describe smooth toric varieties over arbitrary base schemes.

Definition 2.1 (Δ-Collection) *Let X be a scheme, and Δ be a fan. A* Δ-collection on X *consists of line bundles L_ρ and sections $u_\rho \in H^0(X, L_\rho)$, indexed by $\Delta(1)$, and isomorphisms $c_m : \otimes_\rho L_\rho^{\otimes \langle m, n_\rho \rangle} \cong \mathcal{O}_X$, indexed by $m \in M$, such that:*

(i) $c_m \otimes c_{m'} = c_{m+m'}$ *for all $m, m' \in M$, and*

(ii) $u_\rho \in H^0(X, L_\rho)$ *gives $u_\rho : \mathcal{O}_X \to L_\rho$, which induces $u_\rho^* : L_\rho^{-1} \to \mathcal{O}_X$. Then the map*

$$\sum_{\sigma \in \Delta_{\max}} \otimes_{\rho \not\subseteq \sigma} u_\rho^* : \bigoplus_{\sigma \in \Delta_{\max}} \otimes_{\rho \not\subseteq \sigma} L_\rho^{-1} \to \mathcal{O}_X$$

is a surjection.

Note that the above definition implies that the whole theory of coverings of toric varieties by orbits under the big torus $\mathbb{T}$ still works even over general base schemes.

3 Homotopy Colimits and the Van-Kampen Theorem

In this section, we will discuss the van-Kampen theorem in the context of simplicial presheaves as well as in the context of $\mathbb{A}^1$-homotopy theory. For simplicial presheaves, the statement of the van-Kampen theorem carries over verbatim from the classical situation.

The Fundamental Groupoid: We first discuss the fundamental groupoid in $\mathbb{A}^1$-homotopy theory. We begin with the fundamental groupoid of a simplicial presheaf:

Definition 3.1 (Fundamental Groupoid) *Let $\mathcal{T}$ be a site, and let X be a fibrant simplicial presheaf on $\mathcal{T}$. The* fundamental groupoid *of X is the sheaf of groupoids $U \mapsto \Pi_1 X(U)$, i.e. the object $U \in \mathcal{T}$ is assigned the fundamental groupoid of the simplicial set $X(U)$. Similarly, one can define the fundamental groupoid of any simplicial presheaf as the fundamental groupoid of a fibrant replacement.*

For a morphism of simplicial sheaves $f : X \to Y$, there is an induced morphism of fibrant replacements $\operatorname{Ex}^\infty f : \operatorname{Ex}^\infty X \to \operatorname{Ex}^\infty Y$. This also induces an obvious morphism of fundamental groupoid sheaves $\Pi_1 f : \Pi_1 X \to \Pi_1 Y$.

Definition 3.2 *Let $\mathcal{C}$ be a small category. A* category fibred in groupoids *over $\mathcal{C}$ is a small category $\mathcal{D}$ and a functor $P : \mathcal{D} \to \mathcal{C}$, such that for any morphism $f : C' \to C$ in $\mathcal{C}$ and any lifting $D \in \mathcal{D}$ of C, there is a lifting of f, i.e. a morphism $\overline{f} : D' \to D$ with $F(\overline{f}) = f$, which is unique up to unique isomorphism.*

Now the next lemma explains how to translate between a sheaf of groupoids and a category fibred in groupoids. For some applications it is better to consider the fundamental groupoid as a category fibred in groupoids.

Lemma 3.3 *Let $\mathcal{T}$ be a small site. Let $\mathcal{G}rpd/\mathcal{T}$ denote the category of all small categories fibred in groupoids over $\mathcal{T}$, and let $\mathcal{T}^{op}\mathcal{G}rpd$ denote the category of presheaves of groupoids. Then there are functors*

$$F : \mathcal{G}rpd/\mathcal{T} \to \mathcal{T}^{op}\mathcal{G}rpd : (P : \mathcal{D} \to \mathcal{T}) \mapsto \Big(U \in \mathcal{T} \mapsto P^{-1}(\mathrm{id}_U)\Big)$$

and

$$G : \mathcal{T}^{op}\mathcal{G}rpd \to \mathcal{G}rpd/\mathcal{T} : (Q : \mathcal{T}^{op} \to \mathcal{G}rpd \to \mathcal{C}at) \mapsto \int_{\mathcal{T}} Q,$$

where $\int_{\mathcal{T}} Q$ denotes the Grothendieck construction, which is defined as follows: The objects of $\int_{\mathcal{T}} Q$ are pairs of objects (U, V) with $U \in \mathcal{T}$ and $V \in Q(U)$, and a morphism $(U, V) \to (U', V')$ is given by a morphism $f : U \to U'$ in $\mathcal{T}$ and a morphism $g : U \to Q(f)(U')$. The functor $\int_{\mathcal{T}} Q \to \mathcal{T} : (U, V) \mapsto U$ makes $\int_{\mathcal{T}} Q$ a category fibred in groupoids over $\mathcal{T}$.

These functors induce an equivalence of the categories $\mathcal{G}rpd/\mathcal{T}$ and $\mathcal{T}^{op}\mathcal{G}rpd$.

The above is well-known and can be found e.g. in [Tho79].

It is therefore possible to either view the fundamental groupoid as a sheaf of groupoids or as a category fibred in groupoids, living over the site $\mathcal{T}$. The reason why one should consider the fundamental groupoid as a fibred category is, that there is also a reasonable Grothendieck topology on the fundamental groupoid. This is important for possible definitions of homology with local coefficients for simplicial presheaves.

Note that colimits in the category of groupoids are the sheafifications of the corresponding pointwise operations.

Van-Kampen Theorem for Simplicial Presheaves: The van-Kampen theorem for simplicial sets, as e.g. given in [GJ99, p. 144], can be used to describe the fundamental group(oid) of a homotopy pushout of spaces $X = \mathrm{hocolim}(U \leftarrow U \times_X V \to V)$ in terms of the fundamental groupoids of the spaces U, V and $U \times_X V$. There is also a generalization to more general homotopy colimits instead of open coverings given in [Dro04].

As most of the homotopy theory statements, the van-Kampen theorem holds for simplicial presheaves. This can be seen by constructing a suitable comparison morphism and checking on points that it is a weak equivalence. We therefore obtain the following version of the van-Kampen theorem, which is basically a sheafification of [Dro04, Theorem 1.1].

Theorem 3.4 (van-Kampen Theorem) *Let $\mathcal{T}$ be a site, and let $\mathcal{I}$ be a small category. For any $\mathcal{I}$-diagram $\mathcal{X}$ of simplicial presheaves on $\mathcal{T}$, there is a natural equivalence of sheaves of groupoids*

$$\Pi_1 \operatorname{hocolim}_{\mathcal{I}} \mathcal{X} \xrightarrow{\simeq} \operatorname{hocolim}_{\mathcal{I}} \Pi_1 \mathcal{X}.$$

If the homotopy colimit $\operatorname{hocolim}_{\mathcal{I}} \mathcal{X}$ *is a π_0-connected simplicial presheaf, this yields a corresponding isomorphism of groups.*

Proof: We assume for simplicity that the site $\mathcal{T}$ has enough points.

Let $\mathcal{X} : \mathcal{I} \to \Delta^{op}\mathcal{S}hv(\mathcal{T})$ be a diagram of simplicial presheaves. Applying the functor Π_1, this induces a corresponding diagram of sheaves of groupoids $\Pi_1\mathcal{X} : \mathcal{I} \to \mathcal{T}^{op}\mathcal{G}rpd$. Therefore we have a comparison morphism

$$\operatorname{hocolim}_{\mathcal{I}} \Pi_1 \mathcal{X} \to \Pi_1 \operatorname{hocolim}_{\mathcal{I}} \mathcal{X}.$$

This morphism is a weak equivalence, for the following reasons: For any point p of $\mathcal{T}$, we have the following weak equivalence of simplicial sets $p^*(\operatorname{hocolim}_{\mathcal{I}} \mathcal{X}) \simeq \operatorname{hocolim}_{\mathcal{I}} p^*\mathcal{X}$. Then by the result of Farjoun [Dro04], the homotopy colimit $\operatorname{hocolim}_{\mathcal{I}} p^*\mathcal{X}$ of simplicial sets becomes a homotopy colimit of groupoids, i.e. we have a weak equivalence $\Pi_1 \operatorname{hocolim}_{\mathcal{I}} p^*\mathcal{X} \simeq \operatorname{hocolim}_{\mathcal{I}} \Pi_1 p^*\mathcal{X}$. Finally, we put together the above assertions to obtain a chain of weak equivalences of groupoids

$$\begin{aligned} p^* \operatorname{hocolim} \Pi_1\mathcal{X} &\simeq \operatorname{hocolim} p^*\Pi_1\mathcal{X} \simeq \operatorname{hocolim} \Pi_1 p^*\mathcal{X} \\ &\simeq \Pi_1 \operatorname{hocolim} p^*\mathcal{X} \simeq \Pi_1 p^* \operatorname{hocolim} \mathcal{X} \\ &\simeq p^*\Pi_1 \operatorname{hocolim} \mathcal{X}. \end{aligned}$$

The same statement also obtains for the categories fibred in groupoids, since the Grothendieck construction commutes with homotopy colimits of groupoids, by Thomason's theorem [Tho79]. ■

The following can also be found in [Dro04, Theorem 4.3 and Corollary 4.4].

Corollary 3.5 *Let X be a simplicial presheaf, and let $X_i \hookrightarrow X$ be an $\mathcal{I}$-indexed set of simplicial sub-presheaves of X, such that $X = \bigcup X_i$. Let $\mathcal{X}(3)$ be the diagram of the simplicial presheaves $X_i \cap X_j \cap X_k$ for all $i, j, k \in \mathcal{I}$ and morphisms given by inclusions. Then we have*

$$\Pi_1 X \simeq \operatorname{hocolim} \Pi_1 \mathcal{X}(3).$$

In particular, if the diagram $\mathcal{X}(3)$ consists of pointed and connected simplicial presheaves, then $\Pi_1 X \cong \operatorname{colim} \Pi_1\mathcal{X}(3)$. In that case, the group sheaf $\pi_1 X$ can be written as the quotient sheaf

$$\pi_1 X \cong \big(\bigstar_{\mathcal{I}} \pi_1 X_i\big)/R,$$

where R are the relations coming from inclusions $X_i \cap X_j \hookrightarrow X_i$.

The proof carries over verbatim from [Dro04] using the Postnikov decomposition from [MV99, Definition 2.1.29].

The Local Van-Kampen Theorem: Now $\mathbb{A}^1$-homotopy theory is not simply homotopy theory of simplicial sheaves, there is also a localization involved. This implies that not all sheaves of groupoids can appear as fundamental groupoids of $\mathbb{A}^1$-local simplicial presheaves. The groupoid sheaf notion corresponding to $\mathbb{A}^1$-locality of simplicial presheaves is the strong $\mathbb{A}^1$-invariance, as used e.g. in [Mor06].

DEFINITION 3.6 *Let Sm_S be the site of smooth schemes over a regular finite type scheme S, equipped with the Nisnevich topology. A sheaf of groups (resp. groupoids) G is called* $\mathbb{A}^1$-invariant *if for any smooth scheme $U \in \mathrm{Sm}_S$ the morphism $G(U) \to G(U \times \mathbb{A}^1)$ induced by the projection $U \times \mathbb{A}^1 \to U$ is an isomorphism (resp. a weak equivalence of groupoids).*

It is called strongly $\mathbb{A}^1$-invariant *if the maps*

$$H^i_{\mathrm{Nis}}(U, G) \to H^i_{\mathrm{Nis}}(U \times \mathbb{A}^1, G)$$

are isomorphisms for $i = 0, 1$. In case of a sheaf of groupoids, we again only require a weak equivalence of groupoids for $i = 0$.

These are the relevant algebraic counterparts of $\mathbb{A}^1$-local simplicial presheaves. The inclusion $\mathcal{G}rpd_{\mathbb{A}^1} \hookrightarrow \mathcal{G}rpd$ of the subcategory of strongly $\mathbb{A}^1$-invariant sheaves of groups resp. groupoids into the category of sheaves of groupoids has a left adjoint $G \mapsto L_{\mathbb{A}^1} G$ which we also refer to as $\mathbb{A}^1$-localization. This follows from [Mor06, Remark 4.11] by noting that the Postnikov tower argument does not depend on the base scheme being a field. In particular, we have the following:

LEMMA 3.7 *Let $\mathcal{I}$ be a small category, and let $\mathcal{G}$ be an $\mathcal{I}$-indexed set of strongly $\mathbb{A}^1$-invariant sheaves of groupoids. Then the $\mathbb{A}^1$-local homotopy colimit $\mathrm{hocolim}_{\mathcal{I}}^{\mathbb{A}^1} \mathcal{G}$ of $\mathcal{G}$ is computed as*

$$\operatorname*{hocolim}_{\mathcal{I}, \mathbb{A}^1} \mathcal{G} \simeq L_{\mathbb{A}^1} \operatorname*{hocolim}_{\mathcal{I}} \mathcal{G}.$$

The same argument as given in [Mor06] then implies the following slight variation of Morel's van-Kampen theorem [Mor06, Theorem 4.12]:

THEOREM 3.8 *Let $\mathcal{I}$ be a small category, and let $\mathcal{X}$ be an $\mathcal{I}$-diagram of simplicial presheaves. Then there is a natural equivalence of strongly $\mathbb{A}^1$-invariant sheaves of groupoids*

$$\Pi_1^{\mathbb{A}^1} \operatorname*{hocolim}_{\mathcal{I}} \mathcal{X} \simeq L_{\mathbb{A}^1} \operatorname*{hocolim}_{\mathcal{I}} \Pi_1 \mathcal{X}.$$

In particular, for a simplicial presheaf X and a set of simplicial sub-presheaves $X_i \hookrightarrow X$ such that $X = \bigcup X_i$ such that all triple intersections $X_i \cap X_j \cap X_k$ are π_0-connected, we have the following presentation:

$$\pi_1^{\mathbb{A}^1} X \cong L_{\mathbb{A}^1}\big((\bigstar_{\mathcal{I}} \pi_1 X_i)/R\big),$$

where R are the relations coming from inclusions $X_i \cap X_j \hookrightarrow X_i$.

The latter formulation is the one we will apply in the sequel.

Toric Varieties as Homotopy Colimits: In [DI05], it was shown that the usual covering of a toric variety $X(\Delta)$ by the affine toric varieties U_σ for cones $\sigma \in \Delta$ yields a homotopy colimit description of $X(\Delta)$:

For a fan Δ, we denote by $\mathcal{I}(\Delta)$ the category whose objects are the cones σ of Δ, and whose morphisms are inclusions of cones $\sigma \subseteq \tau$. Then there is an $\mathcal{I}(\Delta)$-diagram

$$\mathcal{D}(\Delta) : \mathcal{I}(\Delta) \to \mathcal{S}hv(\mathrm{Sm}_S) : \sigma \mapsto X(\sigma),$$

where every $X(\sigma)$ is $\mathbb{A}^1$-weakly equivalent to a torus $\mathbb{G}_m^{c(\sigma)}$ with $c(\sigma)$ denoting the codimension of σ in $N \otimes_{\mathbb{Z}} \mathbb{R}$.

Then there is a weak equivalence

$$X(\Delta) \simeq \operatorname*{hocolim}_{\sigma\in\Delta} \mathcal{D}(\Delta)$$

It also follows that $X(\Delta)$ is $\mathbb{A}^1$-weakly equivalent to the homotopy colimit with the same index diagram and $X(\sigma)$ replaced by the torus $\mathbb{G}_m^{c(\sigma)}$.

Another way to describe the toric variety $X(\Delta)$ as a homotopy colimit is via the $\mathcal{X}(3)$-description from Corollary 3.5, applied to the covering of $X(\Delta)$ by the open subschemes $X(\sigma)$ for the cones $\sigma \in \Delta$.

The problem in directly applying the van-Kampen theorem to this homotopy colimit presentation is, that almost all of the spaces in the diagram are not $\mathbb{A}^1$-connected. We will use a different approach in Section 4: instead of the toric variety, we will use a homotopy colimit of classifying spaces of tori to obtain a presentation of the fundamental group of $X(\Delta)$.

4 The Fundamental Group of a Toric Variety

In this section, we will finally describe the fundamental group of a toric variety. We start with a description of the connected components. For a connected toric variety, there is a nice $\mathbb{A}^1$-covering given by the homogeneous coordinate ring of Cox. The homotopy type of this object can be described using analogues of the Davis-Januszkiewicz spaces. Finally, the fundamental group of the Davis-Januszkiewicz space associated to a fan Δ can be computed using the van-Kampen theorem.

Preliminaries on Connectedness: We first start with a statement about connectedness of complements of codimension ≥ 2 subvarieties in $\mathbb{A}^n$. Examples of such situations are given by the homogeneous coordinate rings of Cox [Cox95b].

Lemma 4.1 *Let $Z \hookrightarrow \mathbb{A}^n$ be the union of the codimension two coordinate subspaces of $\mathbb{A}^n$. Then $\mathbb{A}^n \setminus Z$ is $\mathbb{A}^1$-connected.*

Proof: By [MV99, Corollary 2.3.22], it suffices to show that

$$\pi_0(\mathrm{Sing}_\bullet^{\mathbb{A}^1}(\mathbb{A}^n \setminus Z)(R)) = 0$$

for any henselian local ring R: This implies that a local and fibrant replacement of $\mathrm{Sing}_\bullet^{\mathbb{A}^1}(\mathbb{A}^n \setminus Z)(R)$ is $\mathbb{A}^1$-connected, but this is equivalent to a local and fibrant replacement of $\mathbb{A}^n \setminus Z$.

Now let a morphism $\operatorname{Spec} R \to \mathbb{A}^n \setminus Z$ be given. Interpreting the space $\mathbb{A}^n \setminus Z$ as a toric variety and using the description via Δ-collections, such a morphism is given by n elements $f_1, \ldots, f_n \in R$ no $(n-1)$-element subset of which has a common zero. Let f_i be the element with the lowest index such that f_i is not invertible. Now f_i is a morphism $\operatorname{Spec} R \to \mathbb{A}^1$, and there is a homotopy from f_i to a morphism $f_i' : \operatorname{Spec} R \to \mathbb{A}^1$ which factors through $\mathbb{G}_m$, i.e. f_i' is invertible. Applying this homotopy does not change the $f_1, \ldots, f_{i-1}$, they thus remain invertible. We can therefore assume that all f_i are invertible.

Using the same homotopy trick as above, we can move f_1 by a homotopy to the constant morphism $1 \in R$: This is simply composing the morphism $f_1 : \operatorname{Spec} R \to \mathbb{A}^1$ with a contraction of $\mathbb{A}^1$. Although this might pass through functions f which have zeros, the assumption that we have moved all other functions f_i to invertible functions makes sure that the previous homotopy will be a homotopy of morphisms $\operatorname{Spec} R \to \mathbb{A}^n \setminus Z$. This process is then repeatedly applied to all the functions f_i, implying that the morphism $(f_1, \ldots, f_n) : \operatorname{Spec} R \to \mathbb{A}^n \setminus Z$ we started with is actually $\mathbb{A}^1$-homotopic (through a chain of elementary $\mathbb{A}^1$-homotopies) to $(1, \ldots, 1)$. ■

The Connected Components of $X(\Delta)$: In the following, we will show that connectedness of the fan implies $\mathbb{A}^1$-connectedness of the corresponding toric variety. For non-connected fans, we can split the toric variety into a connected factor and a torus. This completely describes the connected components of $X(\Delta)$.

LEMMA 4.2 *Let N be a lattice, and let Δ be a regular fan such that* $\operatorname{span} \Delta(1) = N \otimes_{\mathbb{Z}} \mathbb{R}$. *Then $X(\Delta)$ is $\mathbb{A}^1$-connected.*

PROOF: Analoguous to the proof of Lemma 4.1, it suffices to show

$$\pi_0(\operatorname{Sing}_\bullet^{\mathbb{A}^1}(X(\Delta))(R)) = 0$$

for any henselian local ring R.

Now let R be any local ring. We are going to show that $\operatorname{Sing}_\bullet^{\mathbb{A}^1}(X(\Delta))$ is connected in the following steps: (i) deals with the big torus $\mathbb{T}$ in $X(\Delta)$, and (ii) produces paths from orbits to the torus.

(i) Since Δ is regular, and the one-dimensional cones $\rho \in \Delta(1)$ generate $N \otimes_{\mathbb{Z}} \mathbb{R}$, there are generators v_ρ of the cones $\rho \in \Delta(1)$ which generate N. Let $v_{i_1}, \ldots, v_{i_n}$ be such a set of generators of N, and consider the fan $\widetilde{\Delta}$ consisting of the one-dimensional cones corresponding to the generators $v_{i_1}, \ldots, v_{i_n}$. This is a subfan of Δ, and we find that the inclusion $\mathbb{T} \to X(\Delta)$ factors into the following inclusions $\mathbb{T} \hookrightarrow X(\widetilde{\Delta}) \hookrightarrow X(\Delta)$. The toric variety $X(\widetilde{\Delta})$ is isomorphic to the complement of the codimension 2 coordinate subspaces in $\mathbb{A}^n$, and by Lemma 4.1, it is $\mathbb{A}^1$-connected. Therefore, any two morphisms $f_0, f_1 : \operatorname{Spec} R \to \mathbb{T} \to X(\Delta)$ can be connected by a homotopy $H : \operatorname{Spec} R[t] \to X(\widetilde{\Delta}) \hookrightarrow X(\Delta)$.

(ii) Let $f : \operatorname{Spec} R \to X(\Delta)$ be a general morphism. Since $\operatorname{Spec} R$ is local, there is an affine open subset $X(\sigma)$ such that f factors through the inclusion $X(\sigma) \hookrightarrow X(\Delta)$. As in the proof of [DI05, Lemma 5.2 and 5.3], we can split $X(\sigma) \cong X(\sigma') \times \mathbb{G}_m^{\operatorname{codim}(\sigma)}$, and there is a homotopy $H_{\sigma'} : X(\sigma') \times \mathbb{A}^1 \to X(\sigma')$ with $H_{\sigma'}(0) = (0, \ldots, 0)$ and $H_{\sigma'}(1) = \mathrm{id}$. Composing this homotopy with the morphism $f : \operatorname{Spec} R \to X(\sigma)$, we can obtain a chain of homotopies from f to a morphism $g : \operatorname{Spec} R \to X(\sigma)$ which lands in the big torus $\mathbb{T}$.

This implies that for any two morphisms $f, g : \operatorname{Spec} R \to X(\Delta)$, there is a chain of homotopies connecting them: first move both f and g into the big torus, using (ii). Then using (i), we can connect f and g. ∎

In the following, we will occasionally call a fan *connected* if $\Delta(1)$ spans $N \otimes_{\mathbb{Z}} \mathbb{R}$.

REMARK 4.3 *A general fan which contains a cone of maximal dimension is also $\mathbb{A}^1$-connected, since the morphism $\mathbb{T} \hookrightarrow X(\Delta)$ factors through an affine toric variety $X(\sigma)$, which is contractible, cf. [DI05, Lemma 5.2].*

For a simplicial variety, we can embed the torus $\mathbb{T}$ into a quotient of $X(\widetilde{\Delta})$, but it is not clear if this is $\mathbb{A}^1$-connected. It may well be that the group of connected components in this case is something like a subsheaf of $H^1_{\acute{e}t}(-, N/N')$ where N is the lattice in which Δ lives, and N' is the sublattice that is generated by minimal generators of one-dimensional cones in Δ.

LEMMA 4.4 *Let Δ be a regular fan, let $N' = N \cap \operatorname{span} \Delta(1)$ be the sub-lattice of N generated by Δ. Then there is a splitting*

$$X(\Delta, N) \cong X(\Delta, N') \times \mathbb{G}_m^{\operatorname{rk}(N/N')}.$$

In particular, $\pi_0(X(\Delta, N)) \cong \mathbb{G}_m^{\operatorname{rk}(N/N')}$, and the inclusion $X(\Delta, N') \hookrightarrow X(\Delta, N)$ induces isomorphisms

$$\pi_n^{\mathbb{A}^1}(X(\Delta, N')) \xrightarrow{\cong} \pi_n^{\mathbb{A}^1}(X(\Delta, N))$$

for all $n \geq 1$.

PROOF: Almost by definition, N/N' is free, and there is a sequence of fans

$$(\Delta, N') \to (\Delta, N) \to (\{0\}, N/N'),$$

which also has a splitting $(\{0\}, N/N') \to (\Delta, N)$ and therefore the corresponding sequence of toric varieties has a splitting. This implies the statements about the homotopy groups.

From Lemma 4.2, the toric variety $X(\Delta')$ will be $\mathbb{A}^1$-connected. ∎

Note that the above statements do not depend on the base being a field.

The Torus Covering Space: In the following, we assume that the fan Δ is connected, hence the toric variety $X(\Delta)$ is $\mathbb{A}^1$-connected.

Recall from [Cox95b] the description of the homogeneous coordinate ring. The free abelian group of $\mathbb{T}$-equivariant Weil divisors can be identified with $\mathbb{Z}^{\Delta(1)}$, and there is a morphism $M \to \mathbb{Z}^{\Delta(1)}$ mapping a rational function $\chi^m : \mathbb{T} \to \mathbb{G}_m$ for $m \in M$ to the divisor $\operatorname{div}(\chi^m) = -\sum_\rho \langle m, n_\rho \rangle D_\rho$ in $\mathbb{Z}^{\Delta(1)}$. Here, the D_ρ are the irreducible $\mathbb{T}$-equivariant Weil divisors corresponding to minimal generators of the one-dimensional cones $\rho \in \Delta(1)$. The divisor class group $A_{n-1}(X(\Delta))$ is then given by the short exact sequence

$$0 \to M \to \mathbb{Z}^{\Delta(1)} \to A_{n-1}(X(\Delta)) \to 0.$$

The homogeneous coordinate ring is the polynomial ring $S = k[x_\rho \mid \rho \in \Delta(1)]$. This ring is $A_{n-1}(X(\Delta))$-graded by associating to each monomial $\prod_\rho x_\rho^{a_\rho}$ the image of the tuple of powers $(a_1, \dots, a_d) \in \mathbb{Z}^{\Delta(1)}$ in $A_{n-1}(X)$ as its degree.

Using [Cox95b, Proposition 1.1], we can also interpret the homogeneous coordinate ring as

$$\bigoplus_{D \in A_{n-1}(X(\Delta))} H^0(X(\Delta), \mathscr{O}(D)).$$

In this ring, there is one ideal of special interest, which generalizes the "irrelevant" ideal in the homogeneous coordinates of projective space. This ideal is generated by the monomials $x^{\hat{\sigma}} = \prod_{\rho \notin \sigma(1)} x_\rho$.

Note that the Cox ring can also be written as the toric variety associated to $(\Delta, \mathbb{Z}^{\Delta(1)})$, where we replace the lattice N by the lattice which is generated by all the one-dimensional cones. This will obviously be smooth and quasi-affine, and can be identified with the Cox ring. For a simplicial fan, this will be a torsor under the torus $\mathbb{G}_m^{\Delta(1)-n}$, where $n = \operatorname{rk} N$.

Lemma 4.5 *Let Δ be a regular fan. Then the following is an $\mathbb{A}^1$-local fibre sequence*

$$\operatorname{Hom}(A_{n-1}(X(\Delta)), \mathbb{G}_m) \to X(\Delta, \mathbb{Z}^{\Delta(1)}) \to X(\Delta).$$

Proof: From homotopy distributivity [Wen07, Corollary 3.1.12], it follows that we can restrict to affine toric varieties $X(\sigma) \hookrightarrow X(\Delta)$: Indeed, if

$$\operatorname{Hom}(A_{n-1}(X(\Delta)), \mathbb{G}_m) \to X(\sigma, \mathbb{Z}^{\Delta(1)}) \to X(\sigma)$$

is a fibre sequence of simplicial presheaves for every cone σ, then the corresponding homotopy colimit

$$\operatorname{Hom}(A_{n-1}(X(\Delta)), \mathbb{G}_m) \to \operatorname*{hocolim}_{\sigma \in \Delta} X(\sigma, \mathbb{Z}^{\Delta(1)}) \to \operatorname*{hocolim}_{\sigma \in \Delta} X(\sigma)$$

is also a fibre sequence of simplicial presheaves with fibre $\operatorname{Hom}(A_{n-1}(X(\Delta)), \mathbb{G}_m)$. From the homotopy colimit presentation of toric varieties, it follows that this fibre sequence is weakly equivalent to the fibre sequence

$$\operatorname{Hom}(A_{n-1}(X(\Delta)), \mathbb{G}_m) \to X(\Delta, \mathbb{Z}^{\Delta(1)}) \to X(\Delta).$$

The fact that the latter fibre sequence is $\mathbb{A}^1$-local follows by [Wen07, Theorem 4.3.10], since the base scheme S is assumed to be regular, and therefore the classifying space of the split torus $\operatorname{Hom}(A_{n-1}(X(\Delta)), \mathbb{G}_m)$ is $\mathbb{A}^1$-local. Note that we have assumed that Δ is regular, and therefore $A_{n-1}(X(\Delta)) \cong \operatorname{Pic}(X(\Delta))$ is free abelian of rank $\Delta(1) - \operatorname{rk} N$.

From [DG70, Exposé VIII, Proposition 4.1], we obtain conditions when the above covering is a torsor under a torus: For a regular fan Δ, the description of $X(\Delta, \mathbb{Z}^{\Delta(1)})$ as

$$\bigoplus_{D \in A_{n-1}(X(\Delta))} H^0(X(\Delta), \mathscr{O}(D))$$

implies that Condition (a) in [DG70, Exposé VIII, Proposition 4.1] is satisfied. Since we are dealing with invertible sheaves, we also have isomorphisms

$$H^0(X(\sigma), \mathscr{O}(D)) \otimes H^0(X(\sigma), \mathscr{O}(E)) \to H^0(X(\sigma), \mathscr{O}(D+E))$$

induced from the isomorphisms $\mathscr{O}(D) \otimes_{\mathscr{O}} \mathscr{O}(E) \xrightarrow{\cong} \mathscr{O}(D+E)$, which is Condition (b). It then follows that $X(\Delta, \mathbb{Z}^{\Delta(1)})$ is a $\mathrm{Hom}(A_{n-1}(X(\Delta)), \mathbb{G}_m)$-torsor over $X(\Delta)$ and therefore a fibre sequence. ■

REMARK 4.6 *The above can be generalized to simplicial fans Δ. We do not explicitly need the statement from [Cox95b, Theorem 2.1] that $X(\Delta)$ is actually the geometric quotient of $X(\Delta, \mathbb{Z}^{\Delta(1)})$. All we need is that $X(\sigma, \mathbb{Z}^{\Delta(1)})$ is a torsor under the group* $\mathrm{Hom}(A_{n-1}(X(\Delta)), \mathbb{G}_m)$ *over $X(\sigma)$. For a general base field, one can give conditions under which $X(\Delta)$ is the geometric quotient of $X(\Delta, \mathbb{Z}^{\Delta(1)})$ under the* $\mathrm{Hom}(A_{n-1}(X(\Delta)), \mathbb{G}_m)$*-action, cf. [ANHS02].*

From Lemma 4.1, we know that $X(\Delta, \mathbb{Z}^{\Delta(1)})$ is $\mathbb{A}^1$-connected, and from the long exact homotopy sequence we obtain an extension of homotopy group sheaves

$$1 \to \pi_1^{\mathbb{A}^1}(X(\Delta, \mathbb{Z}^{\Delta(1)})) \to \pi_1^{\mathbb{A}^1} X(\Delta) \to \mathrm{Hom}(A_{n-1}(X(\Delta)), \mathbb{G}_m) \to 1.$$

It thus suffices to compute the fundamental group $\pi_1^{\mathbb{A}^1}(X(\Delta, \mathbb{Z}^{\Delta(1)}))$ in terms of the fan Δ. This is what we will do in the following paragraphs.

EXAMPLE 4.7 Let N be the lattice generated by $e_1, \dots, e_n$, and let $e_0 = -\sum_{i=1}^n e_i$. The fan Δ given by the cones generated by n-element subsets of $\{e_0, \dots, e_n\}$ describes the toric variety $X(\Delta) \cong \mathbb{P}^n$. The corresponding fan Δ in $\mathbb{Z}^{\Delta(1)}$ consisting of cones generated by n-element subsets of $\{e_0, \dots, e_n\}$, and the toric variety $X(\Delta, \mathbb{Z}^{\Delta(1)})$ is isomorphic to $\mathbb{A}^{n+1} \setminus \{0\}$. In this case, the fibre sequence from Lemma 4.5 is the usual quotient presentation of projective space:

$$\mathbb{G}_m \to \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n.$$

□

REMARK 4.8 *The homogeneous coordinate ring which appeared as total space of the above torus bundle can also be defined for more general schemes. One example one could look at are the Cox rings of del Pezzo surfaces, which in favourable cases can also be described by explicit equations. If we can derive a presentation of the corresponding fundamental group from these equations, similar computations could be done for del Pezzo surfaces.*

The Davis-Januszkiewicz Spaces $DJ(\Delta)$: As already mentioned, the cellular presentation of a toric variety $X(\Delta)$ as a homotopy colimit of tori is not suitable as input for the van-Kampen theorem. The key idea is to produce for a fan Δ a simplicial presheaf $DJ(\Delta)$ whose fundamental group can be computed via the van-Kampen theorem. On the other hand, these spaces have the property that they sit in a fibre sequence

$$X(\Delta, \mathbb{Z}^{\Delta(1)}) \to DJ(\Delta) \to B\mathbb{G}_m^{\Delta(1)},$$

which then allows to compute the fundamental group of $X(\Delta, \mathbb{Z}^{\Delta(1)})$.

This procedure has been applied in classical algebraic topology, where it was used e.g. for studying the homotopy type of complements of subspace arrangements, cf. [GT06]. Similar ideas also appear in the computation of the fundamental group of a real toric variety, cf. [Uma04]. The definition of the Davis-Januszkiewicz spaces as a union of classifying spaces of tori was first given by Buchstaber and Panov, cf. [BP02, Definition 6.27]. The following is an $\mathbb{A}^1$-homotopy version of their definition:

DEFINITION 4.9 *Let Δ be a fan. We denote by BT the classifying space of the split torus $\mathbb{G}_m^{\Delta(1)} = \mathrm{Hom}(\mathbb{Z}^{\Delta(1)}, \mathbb{G}_m)$, which can also be identified with the product of $\Delta(1)$ copies of $\mathbb{P}^\infty$. Then we define the* Davis-Januszkiewicz space $DJ(\Delta)$ *as follows: For any cone $\sigma \in \Delta$, we denote by*

$$BT(\sigma) = \{(x_1, \dots, x_d) \in BT \mid x_\rho = * \text{ if } \rho \notin \sigma(1)\}$$

the subsheaf of BT consisting of the copies of $B\mathbb{G}_m$ in BT indexed by those $\rho \in \Delta(1)$ which are not in $\sigma(1)$. Note that $BT(\sigma) \simeq B\mathbb{G}_m^{\dim(\sigma)}$, but we are also interested in the embedding $BT(\sigma) \hookrightarrow BT$. Then $DJ(\Delta)$ is defined as the following union of classifying spaces of tori:

$$DJ(\Delta) = \bigcup_{\sigma \in \Delta} BT(\sigma) \hookrightarrow BT = B\mathbb{G}_m^{\Delta(1)}.$$

EXAMPLE 4.10 Let Δ be a fan given by the "boundary" of an n-cone. Then $DJ(\Delta)$ is the *fat wedge* $W(B\mathbb{G}_m, \dots, B\mathbb{G}_m)$. Examples of such a situation are the fans presenting the projective spaces $\mathbb{P}^{n-1}$: there are n cones of dimension $(n-1)$, glued together like the boundary of an n-cone. □

EXAMPLE 4.11 Let Δ be the fan in $\mathbb{Z}^n$ given by the n one-dimensional cones generated by the e_i. Then $DJ(K)$ is the n-fold wedge $B\mathbb{G}_m \vee \cdots \vee B\mathbb{G}_m$. □

Writing both $X(\Delta, \mathbb{Z}^{\Delta(1)})$ and $DJ(\Delta)$ as unions of corresponding subspaces, we obtain the following:

PROPOSITION 4.12 *The following is an $\mathbb{A}^1$-local fibre sequence:*

$$X(\Delta, \mathbb{Z}^{\Delta(1)}) \to DJ(\Delta) \to B\mathbb{G}_m^{\Delta(1)}.$$

PROOF: (i) First, let σ be a cone in $\Delta(k)$. Then the following is a fibre sequence:

$$\mathbb{G}_m^{\Delta(1)-k} \to DJ(\sigma) \simeq B\mathbb{G}_m^k \to B\mathbb{G}_m^{\Delta(1)}.$$

This follows, since the morphism $DJ(\sigma) \to B\mathbb{G}_m^{\Delta(1)}$ is a product of the identity $B\mathbb{G}_m^k \to B\mathbb{G}_m^k$ and the base-point inclusion $* \to B\mathbb{G}_m^{\Delta(1)-k}$. The homotopy fibre of the former is trivial, whereas for the latter, it is the torus $\mathbb{G}_m^{\Delta(1)-k}$.

(ii) From Corollary 3.5, we can translate the description of $DJ(\Delta)$ as a union of $BT(\sigma)$ into a description as a homotopy colimit.

The space $X(\Delta, \mathbb{Z}^{\Delta(1)})$ can also be presented as a union: All the cones of $X(\Delta, \mathbb{Z}^{\Delta(1)})$ lie in the positive orthant, therefore we have an open immersion $X(\Delta, \mathbb{Z}^{\Delta(1)}) \hookrightarrow \mathbb{A}^{\Delta(1)}$. Then $X(\Delta, \mathbb{Z}^{\Delta(1)})$ can be written as the following union inside $\mathbb{A}^{\Delta(1)}$:

$$X(\Delta, \mathbb{Z}^{\Delta(1)}) = \bigcup_{\sigma \in \Delta} T(\sigma),$$

where $T(\sigma) = \{(x_1, \dots, x_d) \in \mathbb{A}^{\Delta(1)} \mid x_\rho \neq 0 \text{ if } \rho \in \sigma(1)\}$. Note that the latter is exactly the description of the irrelevant ideal in the homogeneous coordinate ring of [Cox95b]. Using Corollary 3.5 again, we can also translate this presentation of $X(\Delta, \mathbb{Z}^{\Delta(1)})$ into a homotopy colimit presentation. Note that the index categories of the corresponding homotopy colimit presentations for $DJ(\Delta)$ and $X(\Delta, \mathbb{Z}^{\Delta(1)})$ are the same.

(iii) Puppe's theorem for simplicial presheaves, cf. [Wen07, Proposition 3.1.16], now implies that the homotopy fibre of the morphism

$$DJ(\Delta) \simeq \operatorname*{hocolim}_{\sigma_1,\sigma_2,\sigma_3 \in \Delta} BT(\sigma_1 \cap \sigma_2 \cap \sigma_3) \to B\mathbb{G}_m^{\Delta(1)}$$

is the homotopy colimit of the corresponding fibres:

$$X(\Delta, \mathbb{Z}^{\Delta(1)}) \simeq \operatorname*{hocolim}_{\sigma_1,\sigma_2,\sigma_3 \in \Delta} T(\sigma_1 \cap \sigma_2 \cap \sigma_3).$$

Thus we have established the following fibre sequence of simplicial presheaves:

$$X(\Delta, \mathbb{Z}^{\Delta(1)}) \to DJ(\Delta) \to B\mathbb{G}_m^{\Delta(1)}.$$

(iv) Since the base scheme S is regular, the classifying space $B\mathbb{G}_m^{\Delta(1)}$ is $\mathbb{A}^1$-local. Using [Wen07, Theorem 4.3.10], the fibre sequence from (iii) is then an $\mathbb{A}^1$-local fibre sequence of simplicial presheaves. ■

Example 4.13 Let Δ be the fan presenting $\mathbb{P}^n$. Then the fan Δ in $\mathbb{Z}^{\Delta(1)}$ describes the toric variety $\mathbb{A}^{n+1} \setminus \{0\}$. The $\mathbb{A}^1$-local fibre sequence from Proposition 4.12 is then

$$\mathbb{A}^{n+1} \setminus \{0\} \simeq \mathbb{G}_m * \cdots * \mathbb{G}_m \to W(B\mathbb{G}_m, \dots, B\mathbb{G}_m) \to B\mathbb{G}_m \times \cdots \times B\mathbb{G}_m,$$

where the dots signify $n+1$ copies of the spaces involved. □

Example 4.14 Similarly, for the fan consisting of n one-dimensional cones in $\mathbb{Z}^n$, we obtain the following fibre sequence:

$$\mathbb{A}^n \setminus Z \to B\mathbb{G}_m \vee \cdots \vee B\mathbb{G}_m \to B\mathbb{G}_m \times \cdots \times B\mathbb{G}_m.$$

In the above, the space Z is the union of the codimension two coordinate planes given by $\{x_i = 0, x_j = 0\}$ for $i \neq j$ and $i, j \in \{1, \dots, n\}$. □

For the Davis-Januszkiewicz spaces, we can use the van-Kampen theorem to compute their fundamental groups:

Proposition 4.15 *Let Δ be a fan. Then the $\mathbb{A}^1$-fundamental group of $DJ(\Delta)$ is given as follows:*

$$\pi_1^{\mathbb{A}^1}(DJ(\Delta)) \cong L_{\mathbb{A}^1}\left(\bigstar_{\rho \in \Delta(1)} \mathbb{G}_m(U)/R\right),$$

where $L_{\mathbb{A}}^1$ is the $\mathbb{A}^1$-localization functor on the category of sheaves of groups, and associates to each sheaf of groups a stronlgy $\mathbb{A}^1$-invariant sheaf of groups, and R is the subgroup generated by commutators

$$(u_1, \rho_1)(u_2, \rho_2)(u_1^{-1}, \rho_1)(u_2^{-1}, \rho_2)$$

whenever ρ_1 and ρ_2 span a cone $\eta \in \Delta(2)$.

Proof: We can apply the van-Kampen theorem to the description of $DJ(\Delta)$ given in Definition 4.9. It follows that we have a colimit presentation

$$\bigstar^{\mathbb{A}^1}_{i,j}\pi_1^{\mathbb{A}^1}(BT(\sigma_i\cap\sigma_j))\rightrightarrows\bigstar^{\mathbb{A}^1}_{i}\pi_1^{\mathbb{A}^1}(BT(\sigma_i))\to\pi_1^{\mathbb{A}^1}(DJ(\Delta))\to 1.$$

The fundamental group sheaves appearing in the above are

$$\pi_1^{\mathbb{A}^1}(BT(\sigma_i))(U)=\prod_{\rho\in\sigma_i(1)}\mathbb{G}_m(U).$$

We consider the presheaf of groups $\pi_1(DJ(\Delta))$, whose sections over a smooth scheme U over S are given by the free product of the groups $\prod_{\rho\in\sigma_i(1)}\mathbb{G}_m(U)$, with relations given by the inclusions $\prod_{\rho\in\sigma(1)}\mathbb{G}_m(U)\hookrightarrow\prod_{\rho\in\tau(1)}\mathbb{G}_m(U)$ for an inclusion of cones $\sigma<\tau$.

We can give an alternative presentation of the above groups, by defining a group $W(U,\Delta)$ as the following graph product of groups: The graph has vertices $\rho\in\Delta(1)$ and edges $\eta\in\Delta(2)$, and the groups associated to each vertex are copies of $\mathbb{G}_m(U)$. The group $W(U,\Delta)$ has as elements the words

$$(u_1,\rho_1)(u_2,\rho_2)\cdots(u_n,\rho_n),\qquad u_i\in\mathbb{G}_m(U),\qquad \rho_i\in\Delta(1),$$

which are subject to the relations $(u_i,\rho_i)(u_j,\rho_j)=(u_j,\rho_j)(u_i,\rho_i)$ whenever ρ_i and ρ_j span a cone η in $\Delta(2)$.

We describe morphisms between these presentations: Let $(a_1,\sigma_1)\cdots(a_n,\sigma_n)$ be a word in $\pi_1(DJ(\Delta))$. We map every tuple $(a_{i,1},\dots,a_{i,k})\in\prod_{\rho\in\sigma_i(1)}\mathbb{G}_m(U)$ to the word $(a_{i,1},\rho_1)\cdots(a_{i,k},\rho_k)$ in the graph product $W(U,\Delta)$. This is well-defined, since the relations in $\pi_1(DJ(\Delta))$ come from inclusions of cones $\tau<\sigma$, and all we need to check is that (a,τ) and (b,σ) commute if $\tau<\sigma$. This indeed holds, since also in the graph product, all the elements (a_i,ρ_i) for $\rho_i\in\sigma(1)$ commute by definition.

There is a morphism in the other direction as well: Let $(a_1,\rho_1)\cdots(a_n,\rho_n)$ be a word in the graph product $W(U,\Delta)$. There are copies of $\mathbb{G}_m$ in the presentation of $\pi_1(DJ(\Delta))$ for every one-dimensional cone, so $(a_1,\rho_1)\cdots(a_n,\rho_n)$ can be interpreted also as a word in $\pi_1(DJ(\Delta))$. We only need to check that (a_i,ρ_i) and (a_j,ρ_j) commute in $\pi_1(DJ(\Delta))$ if ρ_i and ρ_j span a cone in Δ. But then there is a copy of $\mathbb{G}_m\times\mathbb{G}_m$ in the presentation of $\pi_1(DJ(\Delta))$, and inclusions $\mathbb{G}_m\to\mathbb{G}_m\times\mathbb{G}_m$. Evaluating the word $(a_i,\rho_i)(a_j,\rho_j)(a_i^{-1},\rho_i)(a_j^{-1},\rho_j)$ in $\mathbb{G}_m\times\mathbb{G}_m$, it is equal to $(1,1)$, therefore (a_i,ρ_i) and (a_j,ρ_j) commute in $\pi_1(DJ(\Delta))$ and the morphism of groups is well-defined.

Finally, it is easy to check that the above morphisms are inverses of each other, and therefore $\pi_1(DJ(\Delta))(U)$ and $W(U,\Delta)$ are isomorphic groups.

The above identification is an identification for groups, which is compatible with the restriction morphisms. This follows since the diagram over which we take the colimit of the groups is always the same, only the group changes along the morphism $\mathbb{G}_m(U)\to\mathbb{G}_m(V)$. This implies that the two presheaves we can write down for the two different presentations are the isomorphic. The same then follows for their sheafifications.

This means that the fundamental group sheaf we computed with the van-Kampen theorem is the graph product in the category of sheaves of groups appearing in the statement of the proposition. The $\mathbb{A}^1$-local fundamental group $\pi_1^{\mathbb{A}^1}(DJ(\Delta))$ is then the $\mathbb{A}^1$-localization of the non-local fundamental group. ■

Remark 4.16 *Note that the fundamental group of a real toric variety also has a graph product presentation, cf. [Uma04, Proposition 2.3]. The only change is that instead of the group of units $\mathbb{G}_m(\mathbb{R})$, one uses the group $\mathbb{Z}/2\mathbb{Z} = \pi_0(\mathbb{G}_m(\mathbb{R}))$.*

Corollary 4.17 *Under the assumptions of Proposition 4.15, the $\mathbb{A}^1$-fundamental group of $X(\Delta, \mathbb{Z}^{\Delta(1)})$ is given as follows:*

$$\pi_1^{\mathbb{A}^1}(X(\Delta, \mathbb{Z}^{\Delta(1)})) \cong \ker\left(\pi_1^{\mathbb{A}^1}(DJ(\Delta)) \to \mathbb{G}_m^{\Delta(1)}\right)$$

Moreover, the sheaf of groups $\pi_1^{\mathbb{A}^1}(X(\Delta, \mathbb{Z}^{\Delta(1)}))$ is the commutator subgroup sheaf of $\pi_1^{\mathbb{A}^1}(DJ(\Delta))$.

Proof: From Proposition 4.12, we have the $\mathbb{A}^1$-local fibre sequence

$$X(\Delta, \mathbb{Z}^{\Delta(1)}) \to DJ(\Delta) \to B\mathbb{G}_m^{\Delta(1)},$$

and the associated long exact homotopy sequence is

$$\cdots \to \pi_2^{\mathbb{A}^1}(B\mathbb{G}_m^{\Delta(1)}) \to \pi_1^{\mathbb{A}^1}(X(\Delta, \mathbb{Z}^{\Delta(1)})) \to \pi_1^{\mathbb{A}^1}(DJ(\Delta)) \to$$
$$\to \pi_1^{\mathbb{A}^1}(B\mathbb{G}_m^{\Delta(1)}) \to \pi_0^{\mathbb{A}^1}(X(\Delta, \mathbb{Z}^{\Delta(1)})) \to \cdots$$

From Lemma 4.1, we know that $\pi_0^{\mathbb{A}^1}(X(\Delta, \mathbb{Z}^{\Delta(1)})) = 0$, and the homotopy groups of $B\mathbb{G}_m^{\Delta(1)}$ are given by

$$\pi_n^{\mathbb{A}^1}(B\mathbb{G}_m^{\Delta(1)})(U) = \begin{cases} \mathrm{Pic}(U)^{\Delta(1)} & n = 0 \\ \mathbb{G}_m^{\Delta(1)} & n = 1 \\ 0 & \text{otherwise,} \end{cases}$$

cf. [MV99, Proposition 4.3.8]. The presentation given in the corollary follows from this.

The remark on commutator subgroups can be seen as follows: The morphism $\pi_1^{\mathbb{A}^1}(DJ(\Delta)) \to \mathbb{G}_m^{\Delta(1)}$ is the morphism of graph products induced by the inclusion of the graph $(\Delta(1), \Delta(2))$ into the clique on $\Delta(1)$. Including more edges means that more commutators are factored out, and therefore the group sheaf $\mathbb{G}_m^{\Delta(1)}$ is the maximal abelian quotient of the graph product $\pi_1^{\mathbb{A}^1}(DJ(\Delta))$. ■

Some Corollaries: We first note that even without being able to explicitly compute the $\mathbb{A}^1$-localization of the above graph product, we can still state the following corollary:

Corollary 4.18 *Let Δ be a connected regular fan. Then the motivic fundamental group $\pi_1^{\mathbb{A}^1}(X(\Delta))$ only depends on the graph whose vertices are the elements of $\Delta(1)$ and whose edges are the elements of $\Delta(2)$.*

The description of the fundamental group of the homogeneous coordinate ring $X(\Delta, \mathbb{Z}^{\Delta(1)})$ allows to give conditions when it is simply connected. In this case, $X(\Delta, \mathbb{Z}^{\Delta(1)})$ is exactly the universal covering of $X(\Delta)$.

The basic condition is a codimension conditions: It turns out that the homogeneous coordinate ring is simply connected if the codimension of the irrelevant ideal is strictly greater than 2. This condition has also been obtained by Asok and Doran in [AD07].

Corollary 4.19 *Let Δ be a connected regular fan such that the following holds: For any two $v_i, v_j \in \Delta(1)$, the cone generated by v_i and v_j is in $\Delta(2)$. Then we have an isomorphism*

$$\pi_1^{\mathbb{A}^1}(X(\Delta)) \cong \mathbb{G}_m^{\Delta(1)-n},$$

where $n = \operatorname{rk} N$. The universal covering of $X(\Delta)$ is then given by $X(\Delta, \mathbb{Z}^{\Delta(1)})$.

Proof: By Lemma 4.5, it suffices to show that $X(\Delta, \mathbb{Z}^{\Delta(1)})$ is simply connected. From the assumptions it follows that the graph product is indexed by a clique, and therefore reduces to a direct product of commutative groups. Therefore

$$\pi_1^{\mathbb{A}^1}(DJ(\Delta)) \cong L_{\mathbb{A}^1}\Big(\prod_{\rho\in\Delta(1)} \mathbb{G}_m\Big) \cong \prod_{\rho\in\Delta(1)} \mathbb{G}_m,$$

with the morphism $\pi_1^{\mathbb{A}^1}(DJ(\Delta)) \to B\mathbb{G}_m^{\Delta(1)}$ being the identity. Corollary 4.17 then implies that $X(\Delta, \mathbb{Z}^{\Delta(1)})$ is simply connected. ■

Note that the above corollary does not use the $\mathbb{A}^1$-local van-Kampen theorem. We only use that the simplicial fundamental group of the Davis-Januszkiewicz space can be computed to be a torus, which is already $\mathbb{A}^1$-local.

Example 4.20 The space $\mathbb{A}^n \setminus \{0\}$ is a model of the join $\mathbb{G}_m * \cdots * \mathbb{G}_m$ of n copies of $\mathbb{G}_m$, cf. Example 4.10. If $n \geq 3$, then Proposition 4.15 implies that

$$\pi_1^{\mathbb{A}^1}(W(B\mathbb{G}_m, \ldots, B\mathbb{G}_m)) \cong \pi_1^{\mathbb{A}^1}(B\mathbb{G}_m \times \cdots \times B\mathbb{G}_m).$$

From Corollary 4.17, we find that $\mathbb{A}^n \setminus \{0\}$ is simply connected for $n \geq 3$. Note again that this statement holds in general, i.e. for a general regular base scheme S, the scheme $\mathbb{A}^n_S \setminus \{0\}$ is simply connected for any $n \geq 3$.

Over a field, we can also see that $\mathbb{A}^n \setminus \{0\}$ is simply connected by applying the $\mathbb{A}^1$-connectivity theorem to the obviously $(n-2)$-connected model $\Sigma_s^{n-1}\mathbb{G}_m^{\wedge n}$. □

Remark 4.21 *The condition in Corollary 4.19 for $X(\Delta, \mathbb{Z}^{\Delta(1)})$ to be simply connected is one of the conditions given in [Uma04, Theorem 1.1] under which the fundamental group $\pi_1(X(\Delta)(\mathbb{R}))$ of a real toric variety is abelian. The other condition can not be generalized as easily: its proof uses the fact that $\mathbb{Z}/2\mathbb{Z}*\mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}$.*

Note however that at least in the case of $\mathbb{A}^2 \setminus \{0\}$ the conclusion still obtains: It follows from the work of Morel that the fundamental group of $\mathbb{A}^2 \setminus \{0\}$ is abelian, in fact equal to the abelian group K_2^{MW}.

In Corollary 4.17 above, we identified the group $\pi_1^{\mathbb{A}^1}(\mathbb{A}^2 \setminus \{0\})$ as the commutator subgroup of $\mathbb{G}_m * \mathbb{G}_m$, and we therefore find that the commutator subgroup sheaf of $\mathbb{G}_m *_{\mathbb{A}^1} \mathbb{G}_m$ is in fact abelian.

5 Some Wedge Decompositions

In this section, we want to describe some homotopy splittings of spaces appearing frequently when dealing with toric varieties. First, we repeat the probably well-known decomposition of suspensions of products of simplicial presheaves. Then we show how a simplicial presheaf version of Ganea's theorem can be used to describe the homotopy groups of suspensions of tori. Finally, we note that certain for certain toric varieties, the Cox ring can be decomposed into a wedge of spheres.

Suspensions of Tori:

LEMMA 5.1 *Let $X_1, \dots, X_n$ be simplicial presheaves. Then there is a weak equivalence*

$$\Sigma(X_1 \times \cdots \times X_n) \simeq \bigvee_{k=1}^{n} \Big(\bigvee_{1 \le i_1 < \cdots < i_k \le n} \Sigma X_{i_1} \wedge \cdots \wedge X_{i_k} \Big).$$

PROOF: For two pointed spaces X and Y, there is a weak equivalence

$$\Sigma(X \times Y) \simeq \Sigma X \vee \Sigma Y \vee (\Sigma X \wedge Y).$$

This is a consequence of the same fact for simplicial sets, by checking on points, cf. [Mor06, p. 78]: The cofibre sequence $X \vee Y \to X \times Y \to X \wedge Y$ splits after one suspension, using the co-H-group structure of $\Sigma(X \times Y)$.

The rest of the statement follows by induction. ■

COROLLARY 5.2 *The homotopy type of the suspension of a split torus $\Sigma\mathbb{G}_m^n$ is given by*

$$\Sigma\mathbb{G}_m^n \simeq \bigvee_{k=1}^{n} \binom{n}{k} S^{k+1,k}.$$

Ganea's Theorem: For a computation of the fundamental group of the suspension of a torus, the following version of Ganea's theorem is however much more suitable. A proof of this proposition can be found in [Wen07].

PROPOSITION 5.3 *Let X be a simplicial presheaf on a site T, and let L_f be a localization functor. Then the following is an f-local fibre sequence:*

$$\Sigma\Omega L_f X \to L_f X \vee L_f X \to L_f X.$$

Similarly, there is a fibre sequence

$$\Omega L_f X * \Omega L_f Y \to L_f X \vee L_f Y \to L_f X \times L_f Y.$$

COROLLARY 5.4 *Let S be a regular scheme. Then there is an $\mathbb{A}^1$-local fibre sequence*

$$\Sigma\Omega B\mathbb{G}_m^n \simeq \Sigma\mathbb{G}_m^n \to B\mathbb{G}_m^n \vee B\mathbb{G}_m^n \to B\mathbb{G}_m^n.$$

The conclusion still obtains for a general group T of multiplicative type over a field k of characteristic zero.

Consequently, we have the following presentation of the fundamental group of $\Sigma\mathbb{G}_m^n$:

$$\pi_1^{\mathbb{A}^1}(\Sigma\mathbb{G}_m^n) = \ker((\mathbb{G}_m^n *_{\mathbb{A}^1} \mathbb{G}_m^n) \to \mathbb{G}_m^n).$$

Note that the problem at the moment is that we are not yet able to compute even simple things like free products in the category of strongly $\mathbb{A}^1$-invariant sheaves of groups. Describing the corresponding usual colimits of the sheaf of groups above therefore only gives non-local models for the corresponding motivic fundamental groups.

Suspensions of tori can be realized be toric varieties. One way to see this is via the fibre sequence above: we presented the suspension of the torus $\mathbb{G}_m^n$ as homotopy fibre:

$$\Sigma\mathbb{G}_m^n \to B\mathbb{G}_m^n \vee B\mathbb{G}_m^n \to B\mathbb{G}_m^n.$$

The morphism $B\mathbb{G}_m^n \vee B\mathbb{G}_m^n \to B\mathbb{G}_m^n$ is not exactly like the morphisms we could interpret as $DJ(\Delta) \to B\mathbb{G}_m^n$. However, the fibre sequence

$$\mathbb{G}_m^n * \mathbb{G}_m^n \to B\mathbb{G}_m^n \vee B\mathbb{G}_m^n \to B\mathbb{G}_m^n \times B\mathbb{G}_m^n$$

can be interpreted as being realized by toric varieties. The homotopy fibre $\mathbb{G}_m^n * \mathbb{G}_m^n$ is the toric variety given by the following fan: Let N be a lattice of rank n, and let the fan Δ consist of two n-cones in $N \times N$, namely the ones generated by $e_1, \dots, e_n$ resp. by $e_{n+1}, \dots, e_{2n}$. Then we also have $B\mathbb{G}_m^n \vee B\mathbb{G}_m^n \simeq DJ(\Delta)$. To obtain the homotopy type of $\Sigma\mathbb{G}_m^n$, we consider the fibre sequence

$$\Omega X * \Omega X \to \Sigma\Omega X \to X,$$

for the special case of $X \simeq B\mathbb{G}_m^n$. Then there is an associated fibre sequence

$$\mathbb{G}_m^n \to \mathbb{G}_m^n * \mathbb{G}_m^n \to \Sigma\mathbb{G}_m^n,$$

which is extended from the previous one by a loop space on the left. Therefore, the toric variety having $\Sigma\mathbb{G}_m^n$ as homotopy type is presented by the fan Δ in N consisting of the two cones $\langle e_1, \dots, e_n\rangle$ and $\langle -e_1, \dots, -e_n\rangle$.

Another way to describe the fan realizing $\Sigma\mathbb{G}_m^n$ is the following: The suspension is the homotopy colimit of the diagram $* \leftarrow \mathbb{G}_m^n \to *$. In the $\mathbb{A}^1$-local category, we can choose a suitable cofibrant resolution of this diagram as $\mathbb{A}^n \leftarrow \mathbb{G}_m^n \to \mathbb{A}^n$. Then we take the ordinary colimit, glueing the two copies of $\mathbb{A}^n$ along $\mathbb{G}_m^n$. This is also exactly the description of the toric variety described in the previous paragraph.

Note also, that the above fibre sequence provides yet another presentation of the fundamental group of $\Sigma\mathbb{G}_m^n$: The long exact sequence for the torus bundle over $\Sigma\mathbb{G}_m^n$ looks as follows:

$$1 \to \pi_1^{\mathbb{A}^1}(\mathbb{G}_m^n * \mathbb{G}_m^n) \to \pi_1^{\mathbb{A}^1}(\Sigma\mathbb{G}_m^n) \to \mathbb{G}_m^n \to 1.$$

Here, we have already made the identifications $\pi_1^{\mathbb{A}^1}(\mathbb{G}_m^n) = 1$, $\pi_0^{\mathbb{A}^1}(\mathbb{G}_m^n) \cong \mathbb{G}_m^n$ and $\pi_0^{\mathbb{A}^1}(\mathbb{G}_m^n * \mathbb{G}_m^n) \cong 1$. This is very close to the presentation of the fundamental group of $\mathbb{P}^1$ as an extension of $\mathbb{G}_m$ by the unramified Milnor-Witt K-group sheaf K_2^{MW}.

We shortly explain how the above description of the suspension of a torus can be used to describe sheaves of groups, whose $\mathbb{A}^1$-localizations are the fundamental groups of $S^{n+1,n}$-spheres. On the one hand, we have an explicit sheaf of groups modelling the fundamental group of a suspended torus:

$$\pi_1(\Sigma\mathbb{G}_m^n) \cong \ker\left(\mathbb{G}_m^n * \mathbb{G}_m^n \to \mathbb{G}_m^n\right).$$

For a smooth scheme U, the sections of $\pi_1(\Sigma\mathbb{G}_m^n)$ can be described as words $(a_1, b_1, \dots, a_n, b_n)$ of n-tuples of units $a_i, b_i \in \mathbb{G}_m^n(U)$ such that the elementwise product $a_1 b_1 \cdots a_n b_n = (1, \dots, 1)$. On the other hand, we have produced a wedge decomposition

$$\Sigma\mathbb{G}_m^n \simeq \bigvee_{k=1}^{n} \binom{n}{k} S^{k+1,k}.$$

Note that in particular, there is only one copy of the highest-dimensional sphere $S^{n+1,n}$ in $\Sigma\mathbb{G}_m^n$. Moreover, we can cover the subspace

$$\bigvee_{k=1}^{n-1}\binom{n}{k}S^{k+1,k}\hookrightarrow\Sigma\mathbb{G}_m^n\simeq\bigvee_{k=1}^{n}\binom{n}{k}S^{k+1,k}$$

by n copies of $\Sigma\mathbb{G}_m^{n-1}$. This follows by using the decomposition $\Sigma(\mathbb{G}_m^{n-1}\times\mathbb{G}_m)\simeq\Sigma\mathbb{G}_m^{n-1}\vee\Sigma\mathbb{G}_m\vee(\Sigma\mathbb{G}_m^n\wedge\mathbb{G}_m)$ inductively as in Lemma 5.1. The morphisms induced on π_1 by the inclusions $\Sigma\mathbb{G}_m^{n-1}\hookrightarrow\Sigma\mathbb{G}_m^n$ are given by mapping the word $(a_1,b_1,\dots,a_n,b_n)$ of $(n-1)$-tuples of units $a_i,b_i\in\mathbb{G}_m^{n-1}(U)$ to the associated word $(a_1',b_1',\dots,a_n',b_n')$ of n-tuples of units $a_i',b_i'\in\mathbb{G}_m^n(U)$, where

$$a_i'=(a_{i,1},a_{i,2},\dots,1,\dots,a_{i,n-1})$$

and the 1 is inserted at the k-th place. If the word $(a_1,b_1,\dots,a_n,b_n)$ in $\pi_1(\Sigma\mathbb{G}_m^{n-1})$ is in the kernel of the fold map, then obviously the word $(a_1',b_1',\dots,a_n',b_n')$ will also be in the kernel of the fold map.

We finally apply the van-Kampen theorem to the cofibre sequence

$$\bigvee_{k=1}^{n-1}\binom{n}{k}S^{k+1,k}\hookrightarrow\Sigma\mathbb{G}_m^n\simeq\bigvee_{k=1}^{n}\binom{n}{k}S^{k+1,k}\to S^{n+1,n},$$

and we find that the simplicial fundamental group of $S^{n+1,n}$ can be described as the quotient of the group sheaf

$$\pi_1(\Sigma\mathbb{G}_m^n)\cong\ker\left(\mathbb{G}_m^n*\mathbb{G}_m^n\to\mathbb{G}_m^n\right).$$

by the normal subgroups generated by the n-tuples of units having at least one 1 in them.

This sheaf of groups is not strongly $\mathbb{A}^1$-invariant in general: In the case of the $S^{3,2}$-sphere, the above description yields the quotient of $\ker(\mathbb{G}_m^2*\mathbb{G}_m^2\to\mathbb{G}_m^2)$ by the normal subgroups generated by pairs of units $(a,1)$ resp. $(1,a)$. This can be identified with the commutator subgroup of $\ker(\mathbb{G}_m*\mathbb{G}_m\to\mathbb{G}_m)$ by mapping the pair (a,b) to the commutator $aba^{-1}b^{-1}$. However, by the work of Morel [Mor06], we know that the $\mathbb{A}^1$-localization of the above sheaf of groups is actually abelian, and equal to the unramified Milnor-Witt K-group sheaf K_2^{MW}. It therefore seems reasonable to conjecture that the $\mathbb{A}^1$-local fundamental groups of the spheres $S^{n+1,n}$ can be described in a similar way like Milnor-Witt K-theory, but with the multiplicative group $\mathbb{G}_m$ replaced by an n-fold direct product of $\mathbb{G}_m$ and suitably adjusted relations.

After that short digression on the homotopy groups of spheres, we will describe one particular homotopy decomposition for a toric variety. This is again an example of a topological result that carries over directly to the $\mathbb{A}^1$-homotopy world, only a slight adjustment in the motivic indexing of the spheres involved is needed. The following result was proven in [GT06] for the case of topological spaces. The proof proceeds along the same lines as in the above reference.

Proposition 5.5 *Let $X_1,\dots,X_n$ be simplicial presheaves on a site T, and let L_f be a localization functor. Then there is a fibre sequence*

$$F_n\to L_fX_1\vee\cdots\vee L_fX_n\to L_fX_1\times\cdots\times L_fX_n,$$

where the homotopy fibre F_n admits the following wedge decomposition

$$F_n \simeq \bigvee_{k=2}^{n} \Big(\bigvee_{1 \le i_1 < \cdots < i_k \le n} (k-1)(\Sigma\Omega L_f X_{i_1} \wedge \cdots \wedge \Omega L_f X_{i_k}) \Big).$$

PROOF: We proceed by induction on n, the base case being $n = 2$. Then $F_2 \cong \Sigma\Omega L_f X_1 \wedge \Omega L_f X_2$, and the above is a fibre sequence by Proposition 5.3.

The following arguments are similar to those of [GT06], since we only compute in the non-local category of simplicial presheaves, where all the homotopy distributivity tools are available. Now assume the proposition holds for F_{n-1} with $n \ge 3$. For any $k \le n$, we denote $M_k = L_f X_1 \vee \cdots \vee L_f X_k$ and $N_k = L_f X_1 \times \cdots \times L_f X_k$. Then $M_n = M_{n-1} \vee L_f X_n$. We plan to apply Puppe's theorem [Wen07, Proposition 3.1.16] to the following diagram of fibre sequences:

$$\begin{array}{ccccc}
F_{n-1} \times \Omega L_f X_n & \longleftarrow & \Omega N_n & \longrightarrow & \Omega N_{n-1} \\
\downarrow & & \downarrow & & \downarrow \\
M_{n-1} & \longleftarrow & * & \longrightarrow & L_f X_n \\
\downarrow & & \downarrow & & \downarrow \\
N_n & \underset{=}{\longleftarrow} & N_n & \underset{=}{\longrightarrow} & N_n.
\end{array}$$

The vertical arrow sequences are fibre sequences, and we want to take the pushouts of the horizontal diagrams. This yields a fibre sequence

$$(F_{n-1} \times \Omega L_f X_n) \cup^h_{\Omega N_n} \Omega N_{n-1} \to M_{n-1} \vee L_f X_n = M_n \to N_n,$$

where the homotopy fibre is F_n. Therefore we have a homotopy pushout diagram:

$$\begin{array}{ccc}
\Omega N_{n-1} \times \Omega L_f X_n & \xrightarrow{h} & F_{n-1} \times \Omega L_f X_n \\
{\scriptstyle g}\downarrow & & \downarrow \\
\Omega N_{n-1} & \longrightarrow & F_n.
\end{array}$$

As described in [GT06], the morphism g is the product projection and $h \simeq * \times \mathrm{id}_{\Omega L_f X_n}$. Then [GT06, Lemma 2] also goes through, since this lemma is only concerned with homotopy colimits. Alternatively, the lemma can be proved by looking at the points of the site. Therefore $F_n \simeq (\Omega N_{n-1} * \Omega L_f X_n) \vee (F_{n-1} \rtimes \Omega L_f X_n)$. Furthermore, by inductive assumption, F_{n-1} is a wedge of suspensions, therefore the cofibre sequence $F_{n-1} \to F_{n-1} \rtimes \Omega L_f X_n \to F_{n-1} \wedge \Omega L_f X_n$ splits, and we have $F_{n-1} \rtimes \Omega L_f X_n \simeq F_{n-1} \vee (F_{n-1} \wedge \Omega L_f X_n)$. Putting together Lemma 5.1, the decomposition of F_{n-1}, we get the required decomposition of F_n.

Finally, the fibre sequence is an $\mathbb{A}^1$-local fibre sequence, since it has a local base. Therefore, the decomposition holds also $\mathbb{A}^1$-locally, because localization commutes with homotopy colimits. ■

COROLLARY 5.6 *The scheme $\mathbb{A}^n \setminus Z$, where Z is the union of all codimension 2 coordinate subspaces, is weakly $\mathbb{A}^1$-equivalent to the following wedge of spheres:*

$$\bigvee_{k=2}^{n} (k-1)\binom{n}{k} S^{k+1,k}.$$

This corollary is an $\mathbb{A}^1$-homotopy version of a result obtained by Grbić and Theriault, cf. [GT06].

COROLLARY 5.7 *Let Δ be a regular fan. Let $\Delta'_1, \ldots, \Delta'_n$ be a decomposition of Δ into subfans of Δ generated by the connected components of the graph $(\Delta(1), \Delta(2))$. Then there is a homotopy decomposition*

$$X(\Delta, \mathbb{Z}^{\Delta(1)}) \simeq \bigvee_{k=2} \Big(\bigvee_{1\leq i_1<\cdots<i_k\leq n} (k-1)(\Sigma\Omega DJ(\Delta'_{i_1}) \wedge \cdots \wedge \Omega DJ(\Delta'_{i_k})) \Big).$$

Note that the assertion about the homotopy type of $\mathbb{A}^n \setminus Z$ above is a special case of the latter corollary. For a fan given by a set of cones which only intersect in (0), the spaces $\Omega DJ(\Delta'_i)$ are again tori, whence several interesting homotopy types can be realized by toric varieties. Note that there are other general conditions under which complements of complex subspace arrangements split as a wedge of spheres, cf. [GT06]. These conditions could also be carried over to the motivic setting.

Finally, we want to note that in case all the above spaces $\Omega DJ(\Delta)$ are actually tori, the corresponding real realization $X(\Delta, \mathbb{Z}^{\Delta(1)})(\mathbb{R})$ is aspherical. General conditions for this to happen have been given in [Uma04]. However, there seems to be no general structure result as yet to describe the motivic homotopy types of $X(\Delta, \mathbb{Z}^{\Delta(1)})$ for Δ a flag-like fan. The fan describing $\mathbb{P}^1 \times \mathbb{P}^1$ provides an example of a flag-like fan such that $X(\Delta, \mathbb{Z}^{\Delta(1)})$ does not split as a wedge of spheres.

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Matthias Wendt, Mathematisches Institut, Albert-Ludwigs-Universität Freiburg, Eckerstrasse 1, 79104, Freiburg im Breisgau, Germany
E-mail address: matthias.wendt@math.uni-freiburg.de