WEBS AND q-HOWE DUALITIES IN TYPES BCD

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ABSTRACT. We define web categories describing intertwiners for the orthogonal and symplectic Lie algebras, and, in the quantized setup, for certain orthogonal and symplectic coideal subalgebras. They generalize the Brauer category, and allow us to prove quantum versions of some classical type **BCD** Howe dualities.

CONTENTS

1. INTRODUCTION

Throughout the whole paper we fix $k, n \in \mathbb{Z}_{\geq 0}$, and we assume that n is even whenever we write $\mathfrak{sp}_n$. We work over a ring containing the function field $\mathbb{C}(q)$ in one variable q over the complex numbers $\mathbb{C}$.

The framework. Consider the following question: Given some Lie algebra $\mathfrak{g}$, can one give a generator-relation presentation for the category of its finite-dimensional representations, or for some well-behaved subcategory?

Maybe the best-known instance of this is the case of the monoidal category generated by the vector representation V of $\mathfrak{sl}_2$, or by the corresponding representation V_q of its quantized enveloping algebra $\mathbf{U}_q(\mathfrak{sl}_2)$. Its generator-relation presentation is known as the *Temperley-Lieb category* and goes back to work of Rumer-Teller-Weyl [RTW32] and Temperley-Lieb [TL71] (the latter in the quantum setting).

In pioneering work, Kuperberg [Kup96] extended this to all rank 2 Lie algebras and their quantum enveloping algebras. However, it was not clear for quite some time how to extend Kuperberg's constructions further (although some partial results were obtained). Then, in seminal work [CKM14], Cautis-Kamnitzer-Morrison gave a generator-relation presentation of the monoidal category generated by (quantum) exterior powers of the vector representation V_q of $\mathbf{U}_q(\mathfrak{gl}_n)$.

Their crucial observation was that a classical tool from representation and invariant theory, known as *skew Howe duality* [How89, How95], can be quantized and used as a

<div align="center">1</div>

device to describe intertwiners of $\mathbf{U}_q(\mathfrak{gl}_n)$. This *skew q-Howe duality* is based on the $\mathbf{U}_q(\mathfrak{gl}_n)$-module decomposition

$$\tag{1.1} \textstyle\bigwedge_q^\bullet(\mathrm{V}_q \otimes \mathbb{C}_q^k) \cong \bigoplus_{a_i \in \mathbb{Z}_{\geq 0}} \textstyle\bigwedge_q^{a_1} \mathrm{V}_q \otimes \cdots \otimes \bigwedge_q^{a_k} \mathrm{V}_q.$$

(Here $\mathbb{C}_q = \mathbb{C}(q)$ and $\bigwedge_q^\bullet$ denotes the quantum exterior algebra in the sense of [BZ08].) Having (1.1), one obtains commuting actions

$$\tag{1.2} \mathbf{U}_q(\mathfrak{gl}_n) \ \circlearrowright \ \bigoplus_{a_i \in \mathbb{Z}_{\geq 0}} \textstyle\bigwedge_q^{a_1} \mathrm{V}_q \otimes \cdots \otimes \bigwedge_q^{a_k} \mathrm{V}_q \ \circlearrowleft \ \mathbf{U}_q(\mathfrak{gl}_k).$$

These two actions generate each other's centralizer, and the bimodule decomposition can be explicitly given. Moreover, by studying the kernel of the $\mathbf{U}_q(\mathfrak{gl}_k)$-action, one can then completely describe the intertwiners of $\mathbf{U}_q(\mathfrak{gl}_n)$. In fact, as explained in [CKM14], they allow a nice diagrammatic interpretation via so-called $\mathbf{A}$-*webs*, which are basically defined by using the $\mathbf{U}_q(\mathfrak{gl}_k)$-action.

The results from [CKM14] were then extended to various other instances. But, to the best of our knowledge, all generalizations so far stay in type $\mathbf{A}$.

The idea which started this paper was to extend Cautis-Kamnitzer-Morrison's approach to types $\mathbf{BCD}$. However, the main obstacle immediately arises: while the quantization of skew Howe duality is fairly straightforward in type $\mathbf{A}$, it is not even clear in other types how one can define commuting actions as in (1.2). The underlying problem hereby is that $\bigwedge_q^\bullet(\mathrm{V}_q \otimes \mathbb{C}_q^k)$ is not flat if V_q is the vector representation in types $\mathbf{BCD}$ (while this holds in type $\mathbf{A}$, cf. [BZ08] and [Zwi09, Corollary 4.26]). This means that $\bigwedge_q^\bullet(\mathrm{V}_q \otimes \mathbb{C}_q^k)$ does not have the same dimension as its classical counterpart $\bigwedge^\bullet(\mathrm{V} \otimes \mathbb{C}^k)$. Hence, there is no hope for an isomorphism as in (1.1) outside type $\mathbf{A}$, and we cannot follow the approach of [CKM14].

To overcome this problem, we consider alternative quantizations of $\mathfrak{so}_n$ and $\mathfrak{sp}_n$, namely as so-called *coideal subalgebras* $\mathbf{U}_q'(\mathfrak{so}_n) \subset \mathbf{U}_q(\mathfrak{gl}_n)$ and $\mathbf{U}_q'(\mathfrak{sp}_n) \subset \mathbf{U}_q(\mathfrak{gl}_n)$, see [Let99] or [KP11]. For their vector representations, the decomposition (1.1) does hold, since they are subalgebras of $\mathbf{U}_q(\mathfrak{gl}_n)$. Hence, we get commuting actions of $\mathbf{U}_q(\mathfrak{gl}_k)$ and of the $\mathbf{A}$-webs. However, since these coideals are proper subalgebras of $\mathbf{U}_q(\mathfrak{gl}_n)$, such commuting actions do not generate each other's centralizer, cf. (1.8). Consequently, the $\mathbf{A}$-web category does not give rise to full functors to the representation categories of the coideal subalgebras $\mathbf{U}_q'(\mathfrak{so}_n)$ and $\mathbf{U}_q'(\mathfrak{sp}_n)$.

In order to get full functors, we define extended web categories, which we call $\cup$- *and $\downarrow$-web categories*, and prove that they act on the representation categories of the coideal subalgebras. We will then show that these extended web categories are closely connected to $\mathbf{U}_q(\mathfrak{so}_{2k})$ and $\mathbf{U}_q(\mathfrak{sp}_{2k})$ (these are the usual quantized enveloping algebras!), recovering some versions of q-Howe duality in types $\mathbf{BCD}$.

Note that our approach goes somehow **the opposite way** with respect to [CKM14]: instead of using q-Howe duality to obtain a web calculus, we use our web categories to prove quantized Howe dualities. The idea of reversing Cautis-Kamnitzer-Morrison's path comes from the paper [QS15], where it was first deployed to quantize a different kind of Howe duality in type $\mathbf{A}$ (in which the vector representation appears together with its dual). This idea was of considerable importance for this work, and indeed many diagrammatic proofs in our paper are inspired by [QS15].

Main results and proof strategy. As before, we denote by V_q the vector representation of $\mathbf{U}_q(\mathfrak{gl}_n)$, as well as of its coideal subalgebras $\mathbf{U}_q'(\mathfrak{so}_n)$ and $\mathbf{U}_q'(\mathfrak{sp}_n)$. We denote by $\bigwedge_q^\bullet \mathrm{V}_q$ the exterior algebra and by $\mathrm{Sym}_q^\bullet \mathrm{V}_q$ the symmetric algebra of V_q.

Quantizing Howe dualities in types **BCD**. As recalled above, the quantum version of skew Howe duality [LZZ11, Theorem 6.16] states that there are commuting actions generating each other's centralizer:

$$(1.3) \qquad \mathbf{U}_q(\mathfrak{gl}_n) \,\circlearrowright\, \underbrace{\textstyle\bigwedge_q^\bullet \mathrm{V}_q \otimes \cdots \otimes \bigwedge_q^\bullet \mathrm{V}_q}_{k \text{ times}} \,\circlearrowleft\, \mathbf{U}_q(\mathfrak{gl}_k).$$

The corresponding bimodule decomposition is multiplicity free and can be explicitly given. An analog statement holds if we replace $\bigwedge_q^\bullet \mathrm{V}_q$ with $\mathrm{Sym}_q^\bullet \mathrm{V}_q$ (although one has to be slightly more careful since the representation becomes infinite-dimensional).

As observed by Howe [How89, How95], in the classical setting there are four versions of (1.3) in types **BCD**. Our main result is a quantization of Howe's **BCD**-dualities. In this quantization, notably, on the right-hand side the enveloping algebras $\mathbf{U}(\mathfrak{sp}_{2k})$ and $\mathbf{U}(\mathfrak{so}_{2k})$ become their quantum enveloping algebras, but on the left-hand side they get replaced by the coideal subalgebras $\mathbf{U}_q'(\mathfrak{so}_n)$ and $\mathbf{U}_q'(\mathfrak{sp}_n)$.

Theorem A. *There are commuting actions:*

$$(1.4) \qquad \mathbf{U}_q'(\mathfrak{so}_n) \,\circlearrowright\, \underbrace{\textstyle\bigwedge_q^\bullet \mathrm{V}_q \otimes \cdots \otimes \bigwedge_q^\bullet \mathrm{V}_q}_{k \text{ times}} \,\circlearrowleft\, \mathbf{U}_q(\mathfrak{so}_{2k}),$$

$$(1.5) \qquad \mathbf{U}_q'(\mathfrak{so}_n) \,\circlearrowright\, \underbrace{\mathrm{Sym}_q^\bullet \mathrm{V}_q \otimes \cdots \otimes \mathrm{Sym}_q^\bullet \mathrm{V}_q}_{k \text{ times}} \,\circlearrowleft\, \dot{\mathbf{U}}_q(\mathfrak{sp}_{2k}),$$

$$(1.6) \qquad \mathbf{U}_q'(\mathfrak{sp}_n) \,\circlearrowright\, \underbrace{\textstyle\bigwedge_q^\bullet \mathrm{V}_q \otimes \cdots \otimes \bigwedge_q^\bullet \mathrm{V}_q}_{k \text{ times}} \,\circlearrowleft\, \mathbf{U}_q(\mathfrak{sp}_{2k}),$$

$$(1.7) \qquad \mathbf{U}_q'(\mathfrak{sp}_n) \,\circlearrowright\, \underbrace{\mathrm{Sym}_q^\bullet \mathrm{V}_q \otimes \cdots \otimes \mathrm{Sym}_q^\bullet \mathrm{V}_q}_{k \text{ times}} \,\circlearrowleft\, \dot{\mathbf{U}}_q(\mathfrak{so}_{2k}).$$

In (1.4) and (1.5) for n odd, and in (1.6) and (1.7), the two actions generate each other's centralizer. Hence, the corresponding bimodule decompositions are multiplicity free. Moreover, all the above de-quantize to the associated classical dualities of Howe.

In (1.4) and (1.5) for n even one has to add an additional intertwiner on the right-hand side in order to get a full action (see Remark 1.2).

The diagram to keep in mind how our q-Howe dualities are related to (1.3) is:

$$(1.8) \qquad \begin{array}{ccccc} \mathbf{U}_q(\mathfrak{gl}_n) & \circlearrowright & \bigoplus_{a_i \in \mathbb{Z}_{\geq 0}} \bigwedge_q^{a_1} \mathrm{V}_q \otimes \cdots \otimes \bigwedge_q^{a_k} \mathrm{V}_q & \circlearrowleft & \mathbf{U}_q(\mathfrak{gl}_k) \\ \cup & & \| & & \cap \\ \mathbf{U}_q'(\mathfrak{so}_n) & \circlearrowright & \underbrace{\bigwedge_q^\bullet \mathrm{V}_q \otimes \cdots \otimes \bigwedge_q^\bullet \mathrm{V}_q}_{k \text{ times}} & \circlearrowleft & \mathbf{U}_q(\mathfrak{so}_{2k}), \end{array}$$

and similarly in the other three cases (1.5), (1.6) and (1.7).

Explaining the strategy. Our main tool are certain diagrams made out of trivalent graphs with edge labels from $\mathbb{Z}_{>0}$, which we call **A**-, $\cup$- and $\bullet$-*webs*.

The **A**-webs where introduced in [CKM14] and assemble into a monoidal category $\mathcal{W}\mathrm{eb}_q^{\wedge}$. The $\cup$- and $\bullet$-webs are introduced in this paper in order to define categories $\mathcal{W}\mathrm{eb}_{q,\mathbf{z}}^{\cup}$ and $\mathcal{W}\mathrm{eb}_{q,\mathbf{z}}^{\bullet}$. These categories are not monoidal, but they come with a left action of the monoidal category $\mathcal{W}\mathrm{eb}_q^{\wedge}$, cf. Remark 1.1.

We will define these web categories in Sections 2, 3 and 4. All the reader needs to know about them at the moment is summarized in Figure 1.
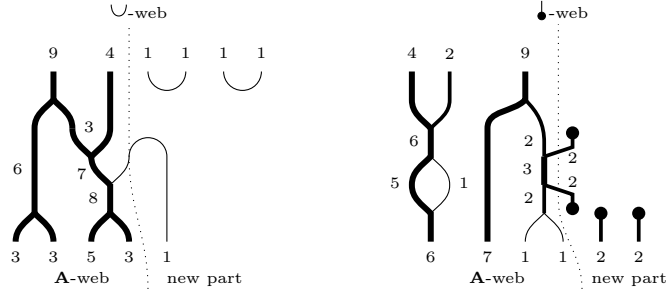
FIGURE 1. Examples of our webs. Both, $\cup$- and $\smalldownarrow$-webs, always consist of an **A**-web to the left and a part with new generators (cup and cap respectively start and end dots) on the right.

Let $\boldsymbol{\mathcal{R}ep}_q(\mathfrak{gl}_n)$, $\boldsymbol{\mathcal{R}ep}'_q(\mathfrak{so}_n)$ and $\boldsymbol{\mathcal{R}ep}'_q(\mathfrak{sp}_n)$ denote the categories of finite-dimensional representations of $\mathbf{U}_q(\mathfrak{gl}_n)$, $\mathbf{U}'_q(\mathfrak{so}_n)$ and $\mathbf{U}'_q(\mathfrak{sp}_n)$, respectively.

Following [CKM14], skew q-Howe duality allows to define a $\mathbf{U}_q(\mathfrak{gl}_n)$-equivariant action of $\mathbf{U}_q(\mathfrak{gl}_k)$ on the k-fold tensor product of $\bigwedge_q^\bullet V_q$'s as in (1.3). This induces a functor $\Phi_{\mathbf{A}}^{\text{ext}} \colon \dot{\mathbf{U}}_q(\mathfrak{gl}_k) \to \boldsymbol{\mathcal{R}ep}_q(\mathfrak{gl}_n)$. By the definition of $\boldsymbol{\mathcal{W}eb}_q^{\boldsymbol{\wedge}}$, this can also be used to define a functor $\Gamma_{\mathbf{A}}^{\text{ext}} \colon \boldsymbol{\mathcal{W}eb}_q^{\boldsymbol{\wedge}} \to \boldsymbol{\mathcal{R}ep}_q(\mathfrak{gl}_n)$. In fact, there is a third functor $\Upsilon_{\mathfrak{gl}} \colon \dot{\mathbf{U}}_q(\mathfrak{gl}_k) \to \boldsymbol{\mathcal{W}eb}_q^{\boldsymbol{\wedge}}$ such that $\Phi_{\mathbf{A}}^{\text{ext}} = \Gamma_{\mathbf{A}}^{\text{ext}} \circ \Upsilon_{\mathfrak{gl}}$. It follows by skew q-Howe duality that all functors $\Phi_{\mathbf{A}}^{\text{ext}}$, $\Gamma_{\mathbf{A}}^{\text{ext}}$ and $\Upsilon_{\mathfrak{gl}}$ are full. The same works in the symmetric case (cf. [RT16] and [TVW15]) where $\bigwedge_q^\bullet V_q$ is replaced by $\text{Sym}_q^\bullet V_q$: again one constructs full functors $\Phi_{\mathbf{A}}^{\text{sym}}$ and $\Gamma_{\mathbf{A}}^{\text{sym}}$ such that $\Phi_{\mathbf{A}}^{\text{sym}} = \Gamma_{\mathbf{A}}^{\text{sym}} \circ \Upsilon_{\mathfrak{gl}}$.

Our goal is to have an analogous picture in types **BCD**: we want to have functors $\Gamma_{\mathbf{BD}}^{\text{ext}}$, $\Gamma_{\mathbf{C}}^{\text{ext}}$, $\Gamma_{\mathbf{BD}}^{\text{sym}}$, $\Gamma_{\mathbf{C}}^{\text{sym}}$, $\Upsilon_{\mathfrak{so}}$ and $\Upsilon_{\mathfrak{sp}}$ and commuting diagrams as in Figure 2.
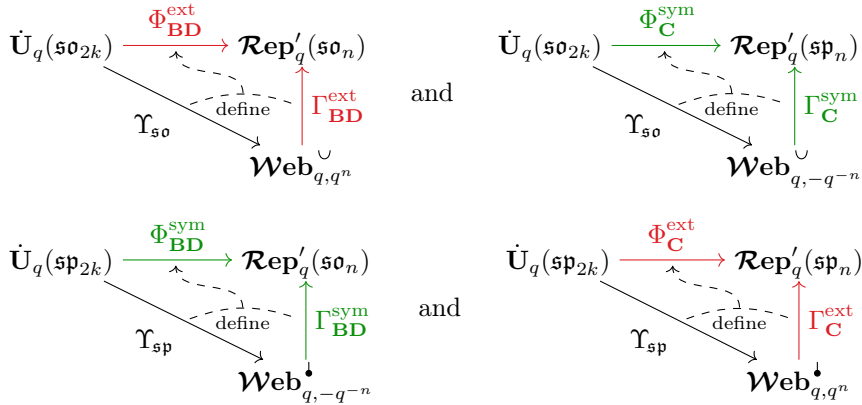

FIGURE 2. Our main commuting diagrams. We call the various Φ's the *Howe functors*, Γ's the *(diagrammatic) presentation functors* and Υ's the *ladderfication functors*.

To summarize (after appropriate parameter substitution in the symmetric case):

Theorem B. *There are ladderfication and presentation functors as in Figure 2. These define the various Howe functors therein and hence, the actions in Theorem A. All of these functors are full in types* **BC**.

As before, fullness in type **D** can be achieved by a slight modification, cf. Remark 1.2. The connection of the various webs and Howe dualities is summarized in Figure 3.
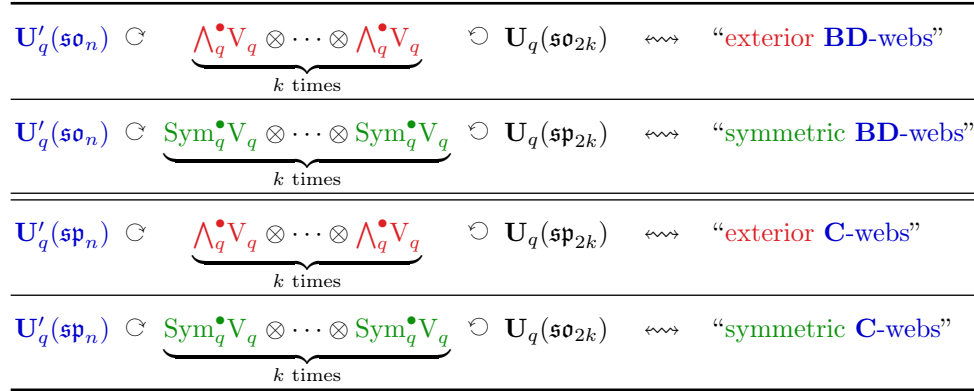
$$\mathbf{U}'_q(\mathfrak{so}_n) \ \circlearrowright \ \underbrace{\textstyle\bigwedge_q^\bullet\mathrm{V}_q \otimes \cdots \otimes \bigwedge_q^\bullet\mathrm{V}_q}_{k \text{ times}} \ \circlearrowleft \ \mathbf{U}_q(\mathfrak{so}_{2k}) \ \leftrightsquigarrow \ \text{``exterior } \mathbf{BD}\text{-webs''}$$

$$\mathbf{U}'_q(\mathfrak{so}_n) \ \circlearrowright \ \underbrace{\mathrm{Sym}_q^\bullet\mathrm{V}_q \otimes \cdots \otimes \mathrm{Sym}_q^\bullet\mathrm{V}_q}_{k \text{ times}} \ \circlearrowleft \ \mathbf{U}_q(\mathfrak{sp}_{2k}) \ \leftrightsquigarrow \ \text{``symmetric } \mathbf{BD}\text{-webs''}$$

$$\mathbf{U}'_q(\mathfrak{sp}_n) \ \circlearrowright \ \underbrace{\textstyle\bigwedge_q^\bullet\mathrm{V}_q \otimes \cdots \otimes \bigwedge_q^\bullet\mathrm{V}_q}_{k \text{ times}} \ \circlearrowleft \ \mathbf{U}_q(\mathfrak{sp}_{2k}) \ \leftrightsquigarrow \ \text{``exterior } \mathbf{C}\text{-webs''}$$

$$\mathbf{U}'_q(\mathfrak{sp}_n) \ \circlearrowright \ \underbrace{\mathrm{Sym}_q^\bullet\mathrm{V}_q \otimes \cdots \otimes \mathrm{Sym}_q^\bullet\mathrm{V}_q}_{k \text{ times}} \ \circlearrowleft \ \mathbf{U}_q(\mathfrak{so}_{2k}) \ \leftrightsquigarrow \ \text{``symmetric } \mathbf{C}\text{-webs''}$$

FIGURE 3. Webs and q-Howe dualities.

Moreover, we will explain in Section 7 how Theorems A and B (in particular, the commuting diagrams from Figure 3) generalize the (quantum) Brauer category.

Some further remarks.

REMARK 1.1. The coideals $\mathbf{U}'_q(\mathfrak{so}_n)$ and $\mathbf{U}'_q(\mathfrak{sp}_n)$ are not Hopf subalgebras of $\mathbf{U}_q(\mathfrak{gl}_n)$, because they do not have an induced coalgebra structure. Hence, $\boldsymbol{\mathcal{R}}\mathbf{ep}'_q(\mathfrak{so}_n)$ and $\boldsymbol{\mathcal{R}}\mathbf{ep}'_q(\mathfrak{sp}_n)$ do not have induced monoidal structures. But since $\mathbf{U}'_q(\mathfrak{so}_n)$ and $\mathbf{U}'_q(\mathfrak{sp}_n)$ are left coideal subalgebras of $\mathbf{U}'_q(\mathfrak{gl}_n)$, there is a left action of $\boldsymbol{\mathcal{R}}\mathbf{ep}_q(\mathfrak{gl}_n)$ on them. In the web language this translates to the left-right partitioning as in Figure 1.

We stress that all these phenomena disappear if one de-quantizes.

REMARK 1.2. Let O_n be the orthogonal group, and V its vector representation. Brauer [Bra37] defined the *Brauer algebra*, which surjects onto $\mathrm{End}_{\mathrm{O}_n}(\mathrm{V}^{\otimes k})$, for all k. But, as Brauer observed (see also [LZ06, § 5.1.3]), if one wants to replace O_n by the special orthogonal group SO_n, then this is not true anymore since:

> ▶ If n is odd, then $\mathrm{End}_{\mathrm{O}_n}(\mathrm{V}^{\otimes k}) = \mathrm{End}_{\mathrm{SO}_n}(\mathrm{V}^{\otimes k})$ for all k.
> ▶ If n is even, then $\mathrm{End}_{\mathrm{O}_n}(\mathrm{V}^{\otimes k}) = \mathrm{End}_{\mathrm{SO}_n}(\mathrm{V}^{\otimes k})$ if and only if $n \geq 2k + 1$.

(Morally, one "Brauer diagram generator" is missing for SO_n if n is even, see also [Gro99] and [LZ16].) As a consequence, surjectivity fails in general for SO_n in type **D**.

We will see in Section 7 that the Brauer algebra is closely related to our web calculus. Hence, to have surjectivity or fullness in general, we would have to add this extra Brauer diagram generator to our web categories. However, since this is not the main point of our construction, we prefer to avoid technicalities. Hence, we obtain slightly weaker statements in type **D** than in types **BC**.

REMARK 1.3. The algebras on the right-hand side of our q-Howe dualities basically define the web categories, while the representation categories of the algebras on the left-hand side are described by the corresponding web categories.

Indeed, our webs have a representation theoretical incarnation via the functors Γ from Figure 2. For example, the start and end dots as in Figure 1 correspond (in the de-quantized setting) to the fact that $\bigwedge^2 V$ (in type **C**) respectively $\mathrm{Sym}^2 V$ (in types **BD**) are not indecomposable, but contain a copy of the trivial module.

Conventions. We work over the ring $\mathbb{C}(q)[\mathsf{z}^{\pm 1}]$, where q and z are transcendental over $\mathbb{C}$. We call q and z *generic parameters*. We also consider specializations of $\mathbb{C}(q)[\mathsf{z}^{\pm 1}]$ obtained by setting z to some non-zero value in the field $\mathbb{C}(q)$. (The cases of overriding importance for us are the specializations of the form $\mathsf{z} = \pm q^{\pm n}$ and there is no harm to think of $\mathsf{z} = \pm q^{\pm n}$ throughout.)

In this setup, let $d_i \in \mathbb{Z}_{\geq 0}$ and set $q_i = q^{d_i}$. The *(z-)quantum number*, the *quantum factorial*, and the *quantum binomial* are given by (here $s \in \mathbb{Z}$ and $t \in \mathbb{Z}_{\geq 0}$)

$$(1.9) \qquad [s]_i = \frac{q_i^s - q_i^{-s}}{q_i - q_i^{-1}} \in \mathbb{C}(q), \quad [\mathsf{z}; s]_i = \frac{\mathsf{z}\, q_i^s - \mathsf{z}^{-1} q_i^{-s}}{q_i - q_i^{-1}} \in \mathbb{C}(q)[\mathsf{z}^{\pm 1}],$$

$$[t]_i! = [t]_i [t-1]_i \dots [1]_i \in \mathbb{C}(q), \quad \begin{bmatrix} s \\ t \end{bmatrix}_i = \frac{[s]_i [s-1]_i \dots [s-t+1]_i}{[t]_i [t-1]_i \dots [1]_i} \in \mathbb{C}(q).$$

By convention, $[0]_i! = 1 = \begin{bmatrix} s \\ 0 \end{bmatrix}_i$. Note that $[0]_i = 0 = \begin{bmatrix} 0 \leq s < t \\ t \end{bmatrix}_i$ and $[-s]_i = -[s]_i$. In case $d_i = 1$ we write $[s] = [s]_1$ etc. for simplicity of notation.

All our categories are assumed to be additive and $\mathbb{K}$-linear, and all our functors are assumed to $\mathbb{K}$-linear (and hence, additive). Which specific choice of $\mathbb{K}$ we mean will be clear from the context.

2. A REMINDER ON THE **A**-WEB CATEGORY

In this section we recall the construction of **A**-*webs* in the spirit of [CKM14].

The A-web category. We start by fixing conventions:

Convention 2.1. For us the composition $\circ$ in diagram categories will be given by vertical stacking, while the monoidal product $\otimes$ will be given by horizontal juxtaposition, and identities are given by parallel vertical strands. We read our diagrams from

bottom to top and left to right, e.g.:

$$
\begin{array}{c} (\mathrm{id} \otimes g) \\ \circ \\ (f \otimes \mathrm{id}) \end{array}
= \circ
\begin{array}{c}
\end{array}
\circ = \begin{array}{c}
\end{array} = \circ
\begin{array}{c}
\end{array}
\circ = \begin{array}{c} (f \otimes \mathrm{id}) \\ \circ \\ (\mathrm{id} \otimes g) \end{array}
$$
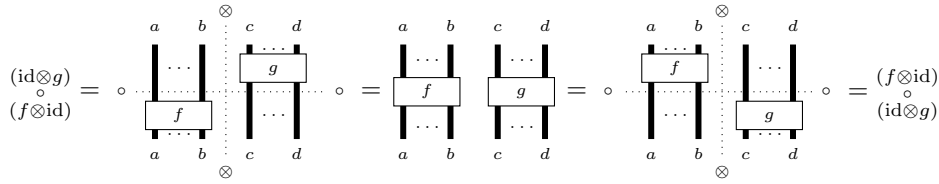
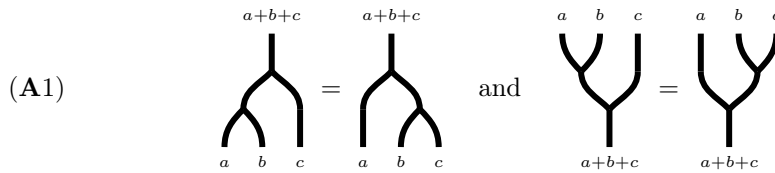Here f, g are some morphisms in the categories in question. Moreover, as in the illustration above, we tend to omit the symbol $\otimes$ between objects.

The monoidal category of **A**-*webs.*

Definition 2.2. The **A**-*web category* $\mathcal{W}\mathrm{eb}_q^{\wedge}$ is the additive closure of the (strict) monoidal, $\mathbb{C}(q)$-linear category generated by objects a for $a \in \mathbb{Z}_{>0}$ (note that this includes the empty sequence $\varnothing$ as an object, which is the monoidal unit), and morphisms

$$
(\mathbf{A}\mathrm{gen}) \qquad \overset{a+b}{\underset{a \quad b}{\bigwedge}}{}^{a+b} : a \otimes b \to a + b \quad \text{and} \quad \overset{a \quad b}{\underset{a+b}{\bigvee}}{}_{a+b}^{a \ b} : a + b \to a \otimes b,
$$
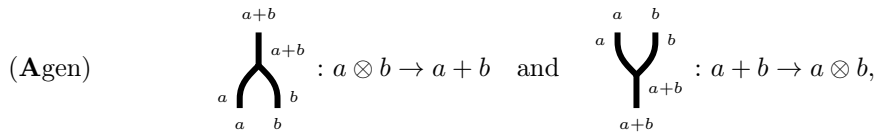
(which we call *merge* and *split*), modulo the relations:

 ▷ *Associativity* and *coassociativity*

$$
(\mathbf{A}1) \qquad \overset{a+b+c}{\bigwedge} = \overset{a+b+c}{\bigwedge} \quad \text{and} \quad \bigvee_{a+b+c} = \bigvee_{a+b+c}
$$

 ▷ The *(thin) square switch*

$$
(\mathbf{A}2) \qquad
\begin{array}{c} a \quad b \\ \\ a \quad b \end{array}
=
\begin{array}{c} a \quad b \\ \\ a \quad b \end{array}
+ [a - b]
\begin{array}{c} a \quad b \\ \\ a \quad b \end{array}
$$
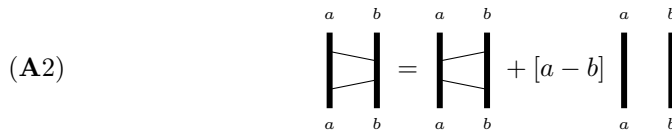
Every diagram representing a morphism in $\mathcal{W}\mathrm{eb}_q^{\wedge}$ will be called an **A**-*web*.

Convention 2.3. We call the label of an edge of an **A**-web the *thickness* of the edge in question. Although we do not allow edges labeled 0 or negative labeled edges, it is convenient in illustrations to allow edges which are potentially zero – these are to be erased to obtain the corresponding **A**-web – or negative – which set the **A**-web to be zero. Edges labeled 1, called *thin*, will play an important role and we illustrate them thinner than arbitrary labeled edges, cf. (**A**2). Moreover, edges of thickness 2 also play a special role and are displayed slightly thicker than thin edges. We sometimes omit to indicate the edges labels: if they are omitted, then they can be recovered from the illustrated ones, or are 1 or 2 whenever they correspond to thinner edges.

Later it will be convenient to consider $\mathcal{W}\mathrm{eb}_q^{\wedge}$ as a $\mathbb{C}(q)[\mathsf{z}^{\pm 1}]$-linear category, denoted by $\mathcal{W}\mathrm{eb}_{q,\mathsf{z}}^{\wedge}$, which can be easily achieved via scalar extension.

REMARK 2.4. Note that the *thick square switches*, i.e.

$$(2.1) \qquad \begin{array}{c} {}^{a-c\ \ b+c}_{+d\ \ -d} \\ \text{[diagram]} \\ {}_{a\ \ b} \end{array} = \sum_e \begin{bmatrix} a-b+c-d \\ e \end{bmatrix} \begin{array}{c} {}^{a-c\ \ b+c}_{+d\ \ -d} \\ {}_{c-e} \\ \text{[diagram]} \\ {}_{d-e} \\ {}_{a\ \ b} \end{array} \quad \text{and} \quad \begin{array}{c} {}^{a+c\ \ b-c}_{-d\ \ +d} \\ \text{[diagram]} \\ {}_{a\ \ b} \end{array} = \sum_e \begin{bmatrix} -a+b-c+d \\ e \end{bmatrix} \begin{array}{c} {}^{a+c\ \ b-c}_{-d\ \ +d} \\ {}_{c-e} \\ \text{[diagram]} \\ {}_{d-e} \\ {}_{a\ \ b} \end{array}$$

where $e \in \mathbb{Z}_{\geq 0}$, as well as the *divided power collapsing*, i.e.

$$(2.2) \qquad \begin{array}{c} {}^{a+c\ \ b-c}_{+d\ \ -d} \\ {}_d \\ \text{[diagram]} \\ {}_c \\ {}_{a\ \ b} \end{array} = \begin{bmatrix} c+d \\ d \end{bmatrix} \begin{array}{c} {}^{a+c\ \ b-c}_{+d\ \ -d} \\ {}_{c+d} \\ \text{[diagram]} \\ {}_{a\ \ b} \end{array} \quad \text{and} \quad \begin{array}{c} {}^{a-c\ \ b+c}_{-d\ \ +d} \\ {}_d \\ \text{[diagram]} \\ {}_c \\ {}_{a\ \ b} \end{array} = \begin{bmatrix} c+d \\ d \end{bmatrix} \begin{array}{c} {}^{a-c\ \ b+c}_{-d\ \ +d} \\ {}_{c+d} \\ \text{[diagram]} \\ {}_{a\ \ b} \end{array}$$

can be deduced from the above relations since we work over $\mathbb{C}(q)$. The example to keep in mind is:

The first step here is called *explosion*. This is a general feature for (many) web categories: the web calculus is basically determined by what happens in the case of thin labels, as the thick ones can be reduced to the thin ones via explosion. We will see this phenomenon turning up later on as well.

Note also that the so-called *digon removals*, i.e.

$$(2.3) \qquad a \underset{a+b}{\overset{a+b}{\bigodot}} b = \begin{bmatrix} a+b \\ b \end{bmatrix} \Big|_{a+b}^{a+b}$$

are special cases of the square switches.

REMARK 2.5. By one of the main results of [CKM14], we have a list of additional relations which we call the **A**-*web Serre relations*. We just give a blueprint example (cf. [CKM14, Lemma 2.2.1]):

Since we work over $\mathbb{C}(q)$, thick versions of these hold as well. We leave it to the reader to write them down, keeping in mind that they are "web versions" of the higher order Serre relations (5.10) of type **A**. (We refer to these specifying s, t as therein.)

The braiding. Recall that $\mathcal{W}\mathbf{eb}_q^{\lambda}$ is a braided category. There is some freedom in the choice of scaling of the braiding. For us the most convenient choice for *thin overcrossings* (left crossing in (2.4)) and *thin undercrossing* (right crossing in (2.4)) is:

$$(2.4) \qquad \overset{1\;\;1}{\underset{1\;\;1}{\bigtimes}} = -q^{-1}\,\Big|\;\Big| \;+\; \overset{1\;1}{\underset{1\;1}{)(}} \quad \text{and} \quad \overset{1\;\;1}{\underset{1\;\;1}{\bigtimes}} = -q\,\Big|\;\Big| \;+\; \overset{1\;1}{\underset{1\;1}{)(}}$$

Recall that a braiding on $\mathcal{W}\mathbf{eb}_q^{\lambda}$ is, via explosion, uniquely determined by specifying (2.4) (see e.g. [QS15, Lemma 5.12]). That is, we also get thick over- and undercrossings and one can inductively compute how these are expressed in terms of the **A**-web generators from (**A**gen).

REMARK 2.6. The category $\mathcal{W}\mathbf{eb}_q^{\lambda}$ has a q-anti-linear (that is, flipping $q \leftrightarrow q^{-1}$) involution Ψ given by switching the crossings and an anti-involution ω given by taking the vertical mirror of a diagram. In particular, it suffices to give relations involving one type of crossing, and we will do so in the following.

We remark that the naturality of the braiding is equivalent to the following *pitchfork* relations, which hold for all values of a, b and c:

$$(2.5) \qquad \overset{b+c\;\;a}{\underset{a\;\;b\;\;\;c}{\bigwedge}} = \overset{b+c\;\;a}{\underset{a\;\;b\;\;\;c}{\bigwedge}} \quad \text{and} \quad \overset{b\;\;\;c\;a}{\underset{a\;\;b+c}{\bigvee}} = \overset{b\;\;\;c\;a}{\underset{a\;\;b+c}{\bigvee}}$$

We additionally need the following relations:

Lemma 2.7. *For all a, b, c the* trivalent twists *hold in* $\mathcal{W}\mathbf{eb}_q^{\lambda}$:

$$(2.6) \qquad \overset{a+b}{\underset{a\;\;\;b}{\bigotimes}} = q^{ab}\,\overset{a+b}{\underset{a\;\;\;b}{\bigwedge}} \quad \text{and} \quad \overset{a\;\;\;b}{\underset{a+b}{\bigotimes}} = q^{ab}\,\overset{a\;\;\;b}{\underset{a+b}{\bigvee}}$$

Proof. These relations are easily verified inductively by using explosion. ∎

3. The $\cup$-web category

Next, we define a web category of which we will see that it describes exterior **BD**-webs as well as symmetric **C**-webs. We call its morphisms $\cup$-*webs*.

Categories with a monoidal action. We will define webs of types **BCD** as morphisms of categories with a left monoidal action of the monoidal category $\mathcal{W}\mathbf{eb}_q^{\lambda}$, as formalized in the following definition.

Definition 3.1. Let $\mathcal{M} = (\mathcal{M}, \otimes, \mathbb{1})$ be a (strict) monoidal category, and $\mathcal{C}$ be a category. A *(left) action of $\mathcal{M}$ on $\mathcal{C}$* is a functor $\otimes\colon \mathcal{M} \times \mathcal{C} \to \mathcal{C}$ with natural isomorphisms $(X \otimes Y) \otimes C \cong X \otimes (Y \otimes C)$ and $\mathbb{1} \otimes C \cong \mathbb{1} \cong C \otimes \mathbb{1}$ for $X, Y \in \mathcal{M}$, $C \in \mathcal{C}$ satisfying the usual coherence conditions (see e.g. [Wei13, Definition IV.4.7]). We will then say that $\mathcal{C}$ is an $\mathcal{M}$-*category*.

The diagrammatic $\cup$-web category.

$\cup$-*webs.* In this section we work over $\mathbb{C}(q)[\mathsf{z}^{\pm 1}]$, if not stated otherwise. For the definition of the quantum numbers see (1.9).

Definition 3.2. The $\cup$-*web category* $\boldsymbol{\mathcal{W}}\mathbf{eb}_{q,\mathsf{z}}^{\cup}$ is the additive closure of the $\mathbb{C}(q)[\mathsf{z}^{\pm 1}]$-linear $\boldsymbol{\mathcal{W}}\mathbf{eb}_{q,\mathsf{z}}^{\wedge}$-category generated by the object $\varnothing$ and by the *cup* and *cap morphisms*

$$(\cup\text{gen}) \qquad \overset{1\quad 1}{\smile} : \varnothing \to 1 \otimes 1 \quad \text{and} \quad \underset{1\quad 1}{\frown} : 1 \otimes 1 \to \varnothing,$$

modulo the following relations:

▷ The *circle removal*

$$(\cup 1) \qquad\qquad\qquad \bigcirc = [\mathsf{z}; 0].$$

▷ The *bubble removal*

$$(\cup 2) \qquad\qquad\qquad \overset{1}{\underset{1}{\big| \bigcirc}} = [\mathsf{z}; -1] \overset{1}{\underset{1}{\big|}}$$

▷ The *lasso move*

$$(\cup 3) \qquad\qquad \overset{1\quad 1}{\underset{1\quad 1}{\Big| \!\! \triangleright}} = \overset{1\quad 1}{\underset{1\quad 1}{\overset{\smile}{\frown}}} + [\mathsf{z}; -2] \overset{1\quad 1}{\underset{1\quad 1}{\big| \ \big|}}$$

▷ The *lollipop relations*

$$(\cup 4) \qquad\qquad\qquad \overset{2}{\underset{}{\big| \bigcirc}} = 0 \quad \text{and} \quad \overset{\bigcirc}{\underset{2}{\big|}} = 0.$$

▷ The *merge-split sliding relations*

$$(\cup 5) \qquad \text{[diagram]} = \text{[diagram]} \quad \text{and} \quad \text{[diagram]} = \text{[diagram]}$$

REMARK 3.3. Thanks to relation ($\cup 4$), it is irrelevant whether we use overcrossings or undercrossings in ($\cup 3$). Moreover, one directly sees that the symmetries Ψ and ω from Remark 2.6 extend to $\boldsymbol{\mathcal{W}}\mathbf{eb}_{q,\mathsf{z}}^{\cup}$ (where we assume that Ψ also flips $\mathsf{z} \leftrightarrow \mathsf{z}^{-1}$). Abusing notation, we denote these symmetries by the same symbols.

REMARK 3.4. Beware that a cup or a cap in a diagram representing a morphism in $\boldsymbol{\mathcal{W}}\mathbf{eb}_{q,\mathsf{z}}^{\cup}$ is only allowed if there are no strands on its right, cf. Figure 1.

Topological versions of the $\cup$-web relations. Next, we give some alternative, topologically more meaningful, relations to our defining relations from above.

Lemma 3.5. *The bubble removal ($\cup$2) is equivalent to*

($\cup$a)
$$\text{[diagram]} = -\mathsf{z}^{-1} \text{[diagram]} \qquad \blacksquare$$

Lemma 3.6. *The lasso move ($\cup$3) is equivalent to*

($\cup$b)
$$\text{[diagram]} = \text{[diagram]} \qquad \blacksquare$$

Lemma 3.7. *The lollipop relations ($\cup$4) are equivalent to*

($\cup$c)
$$\text{[diagram]} = -q^{-1} \text{[diagram]} \quad and \quad \text{[diagram]} = -q^{-1} \text{[diagram]} \qquad \blacksquare$$

Lemma 3.8. *The merge-split sliding relations ($\cup$5) are equivalent to*

($\cup$d)
$$\text{[diagram]} = \text{[diagram]} \quad and \quad \text{[diagram]} = \text{[diagram]} \qquad \blacksquare$$

We omit the proofs of Lemmas 3.5, 3.6, 3.7 and 3.8, which can be easily recovered by expanding the crossings using (2.4). (Verifying the equivalence between ($\cup$3) and ($\cup$b), which inspired the name lasso move, is lengthy, but straightforward.)

Note that, by using the involution Ψ and the anti-involution ω, we obtain many more equivalent relations.

Why $\cup$-webs do not form a monoidal category. One may be tempted to define arbitrary cups and caps as in the following picture:

(3.1)
$$\text{[diagram]} = \text{[diagram]}$$

However, this is dangerous since the diagram

$$\text{[diagram]}$$

would be ambiguous, as it could be any of the following two pictures:

$$\text{[diagram]} \quad \text{or} \quad \text{[diagram]}$$

Unfortunately, these are not equal. (We note that, in the setting of categories with a monoidal action, the first diagram is the correct meaning, and we already used this before, namely in ($\cup$d).)

This problem disappears if one de-quantizes, and the resulting $\cup$-web category at $q = 1$ is a genuine monoidal category. Hence, $\boldsymbol{\mathcal{W}}\mathbf{eb}_{q,\mathbf{z}}^{\cup}$ gives an example of a deformation of a monoidal category which is not monoidal anymore. This is related, as we shall see in Section 5, to the well-understood fact that the quantization of the inclusion $\mathfrak{so}_n \subseteq \mathfrak{gl}_n$ cannot be realized as an inclusion of Hopf algebras, but only as the inclusion of a coideal subalgebra.

Actually, in the de-quantized case the resulting web category is not just monoidal, but also gets a topological flavor by defining *thick cup and cap morphisms* via explosion, cf. Remark 2.4. This is very much in the spirit of the original "web categories" introduced by Kuperberg [Kup96].

Some useful lemmas. Until the end of this section we will work in $\boldsymbol{\mathcal{W}}\mathbf{eb}_{q,\mathbf{z}}^{\cup}$. Our next aim it to derive some diagrammatic relations which, as we will see later, correspond to relations in the quantum group $\mathbf{U}_q(\mathfrak{so}_{2k})$. In the proofs of the following lemmas, we will repeatedly use the defining relations of $\boldsymbol{\mathcal{W}}\mathbf{eb}_{q,\mathbf{z}}^{\cup}$, as well as the braided structure of $\boldsymbol{\mathcal{W}}\mathbf{eb}_{q,\mathbf{z}}^{\curlywedge}$ (in particular, (2.5) and (2.6)). At each step, we will indicate the most important relations that we use.

Lemma 3.9. *For all a, b we have*

$$(3.2)$$

Proof. Using the naturality of the braiding and the defining relations as well as the relations for $\mathbf{A}$-webs, we compute:

$$\overset{(\cup 2)}{\underset{(\cup 3)}{=}} \;\; + [\mathsf{z};-2] \;\; - [a-1]\,[\mathsf{z};-1] \;\; - [a]\,[b-1]\,[\mathsf{z};-1] \\ + [a-1]\,[b-1]\,[\mathsf{z};0]$$

$$\overset{(2.3)}{=} \;\; + [a]\,[b]\,[\mathsf{z};-2] - [a-1]\,[b]\,[\mathsf{z};-1] \\ - [a]\,[b-1]\,[\mathsf{z};-1] + [a-1]\,[b-1]\,[\mathsf{z};0]$$

$$= \;\; + [\mathsf{z};-a-b]$$

The last step is just a tedious calculation with quantum numbers. ∎

Lemma 3.10. *For all a,b we have*

$$(3.3)$$

Proof. We have

$$\overset{(\mathbf{A}2)}{=} \;\; - [a-1] \quad \overset{(\cup 4)}{=}$$

$$\overset{(\cup 4)}{=} \;\; - q^{-1} \quad \overset{(\mathbf{A}1)}{=}$$

∎

Lemma 3.11. *We have*

(3.4)

$$

$$

Proof. We get by the definition of the braiding:

$\blacksquare$

Lemma 3.12. *For all a, b we have*

(3.5)

$$

$$

Proof. By associativity (**A**1), we have

and hence we may assume $a = 0$. Now, we have

$$\overset{(2.5)}{=} q^{b-1} \quad \overset{(3.4)}{=} q^{b-1} \quad \overset{(2.5)}{=} q^{b-1} \quad \overset{(2.6)}{=} \qquad \qquad \blacksquare$$

Lemma 3.13. *For all a, b, c we have*

$$(3.6) \qquad [2] \qquad = \qquad + \qquad$$

Proof. Noticing that (3.6) is equivalent to

$$[2] \qquad = \qquad + \qquad$$

the proof follows from the **A**-web Serre relations (by applying the corresponding relation for $s = 1$, $t = 2$ to the marked part), cf. Remark 2.5. $\blacksquare$

Lemma 3.14. *For all a, b, c we have*

$$(3.7) \qquad [2] \qquad = \qquad + \qquad$$

Proof. First note that

$$\overset{(2.5)}{=} \quad \overset{(\mathbf{A}1)}{=} \quad \overset{(2.3)}{=} \frac{1}{[2]}$$
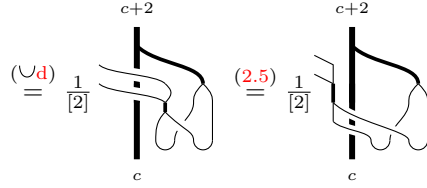
Thus, the statement follows from the thick square switch relations (2.1).   ∎

4. The ↓-web category

In this section, which is structured exactly as the previous one, we define another web category which will play a complimentary role to the ∪-web category, as it describes exterior **C**-webs and symmetric **BD**-webs. We call its morphisms ↓-*webs*.

The diagrammatic ↓-web category.

↓-*webs.* Again, we work over $\mathbb{C}(q)[z^{\pm 1}]$, and we define:

Definition 4.1. The ↓-*web category* $\mathcal{W}\mathrm{eb}_{q,z}^{\downarrow}$ is the additive closure of the $\mathbb{C}(q)[z^{\pm 1}]$-linear $\mathcal{W}\mathrm{eb}_{q,z}^{\wedge}$-category generated by the object $\varnothing$ and by the *start/end dot morphisms*

$$(\text{↓gen}) \qquad \qquad \underset{2}{\overset{}{\big\downarrow}} : \varnothing \to 2 \quad \text{and} \quad \underset{2}{\overset{}{\big\uparrow}} : 2 \to \varnothing,$$

modulo the following relations:

▷ The *barbell removal*

$$(\text{↓}1) \qquad \qquad \big\updownarrow = [z;0]_2\,.$$

▷ The *thin K removal*

$$(\text{↓}2) \qquad \qquad \overset{1}{\underset{1}{\text{K}}} = [z;-1]_2 \; \overset{1}{\underset{1}{\big|}}$$

▷ The *thick K opening*

$$(\text{↓}3) \qquad \qquad \overset{2}{\underset{2}{\text{K}}} = \overset{2}{\underset{2}{\big\updownarrow}} + [z;-2]_2 \; \overset{2}{\big|}$$

▷ The *merge-split sliding relations*

$$(\text{↓}4) \qquad \qquad \text{(diagram)} = \text{(diagram)} \quad \text{and} \quad \text{(diagram)} = \text{(diagram)}$$

where the *cup* and *cap morphisms* are defined as

$$(4.1) \qquad \overset{1 \quad 1}{\smile} = \overset{1 \quad 1}{\bigvee} : \varnothing \to 1 \otimes 1 \quad \text{and} \quad \underset{1 \quad 1}{\frown} = \underset{1 \quad 1}{\bigwedge} : 1 \otimes 1 \to \varnothing.$$

REMARK 4.2. As before for $\cup$-webs, the dot morphisms are only allowed if there are no strands to their right, cf. Remark 3.4 (see also below). Moreover, the category $\boldsymbol{\mathcal{W}}\mathbf{eb}_{q,\mathsf{z}}^{\phi}$ has the same (anti)-involutions as $\boldsymbol{\mathcal{W}}\mathbf{eb}_{q,\mathsf{z}}^{\cup}$ (cf. Remark 2.6), which we, abusing notation, denote also by Ψ and ω.

Topological versions of the ϕ-web relations. For completeness, we give some topologically meaningful versions of the relations above.

Lemma 4.3. *The barbell removal (ϕ1) is equivalent to*

(ϕa) $$\bigcirc = [\mathsf{z};0]\,.$$

Lemma 4.4. *The thin K removal (ϕ2) is equivalent to*

(ϕb)

Lemma 4.5. *The thick K opening (ϕ3) is equivalent to*

(ϕc)

Lemma 4.6. *The following relations hold:*

(ϕd)

Lemma 4.7. *The merge-split sliding relations ($\cup$5) are equivalent to*

(ϕe)

Proof of Lemmas 4.3, 4.4, 4.5, 4.6 and 4.7. Again, the equations can be checked by expanding the crossings using (2.4) (although it requires some time and patience to verify that (ϕc) is equivalent to (ϕ3)). Let us check one as an example, showing that (ϕa) and (ϕ2) imply (ϕb):

(4.2)   ∎

Again, by using Ψ and ω, we obtain many more equivalent relations.

Why ϕ-webs do not form a monoidal category. Again, as for $\cup$-webs, the ϕ-web category is not monoidal. As will become clear later, this is related to the fact that the inclusion $\mathfrak{sp}_n \hookrightarrow \mathfrak{gl}_n$ can only be quantized as an inclusion of a coideal subalgebra. However, de-quantization gives again a genuine monoidal category of $\cup$-webs.

Some more useful lemmas. Until the end of the section we work in $\boldsymbol{\mathcal{W}}\mathbf{eb}_{q,\mathsf{z}}^{\downarrow}$, and we derive some diagrammatic relations which, as we will see later, correspond to relations in the quantum group $\mathbf{U}_q(\mathfrak{sp}_{2k})$.

Lemma 4.8. *For all a we have*

$$(4.3) \qquad \text{[diagram]} = \text{[diagram]} + [\mathsf{z}; -a]_2 \;\Big|$$

Proof. We compute:

$$\text{[diagram]} \overset{(2.1)}{=} \text{[diagram]} + [2-a]\text{[diagram]} + \begin{bmatrix} 2-a \\ 2 \end{bmatrix}\text{[diagram]} \overset{(2.3),(\bullet 1)}{\underset{(\bullet 2)}{=}} \text{[diagram]} + [2-a][a][\mathsf{z};-1]_2 \\ + \begin{bmatrix} 2-a \\ 2 \end{bmatrix}[\mathsf{z};0]_2 \;\Big|$$

$$\overset{(\bullet 3)}{=} \text{[diagram]} + [\mathsf{z};-2]_2 \text{[diagram]} \begin{array}{l} + [2-a][a][\mathsf{z};-1]_2 \\ + \begin{bmatrix} 2-a \\ 2 \end{bmatrix}[\mathsf{z};0]_2 \end{array}\Big| \overset{(2.3)}{=} \text{[diagram]} + [\mathsf{z};-a]_2 \;\Big|$$

A tedious but straightforward computation gives the claimed coefficients. ∎

Lemma 4.9. *For all a,b we have*

$$(4.4) \qquad \text{[diagram]} = \text{[diagram]}$$

Proof. Clear by associativity (**A**1). ∎

Lemma 4.10. *For all a,b we have*

$$(4.5) \qquad \text{[diagram]} + [3]\,\text{[diagram]} = \text{[diagram]} + [3]\,\text{[diagram]}$$

Proof. First, we note that (4.5) is equivalent to

$$\text{[diagram]} + [3]\,\text{[diagram]} = \text{[diagram]} + [3]\,\text{[diagram]}$$

Next, we can apply the **A**-web Serre relations (cf. Remark 2.5) to the marked part (for $s = 2$, $t = 3$) and we are done. ∎

Lemma 4.11. *We have*

$$(4.6)$$

Proof. The first equation is equivalent to the merge-split sliding relation ($\downarrow$4) through the chain of equalities

Using (2.3), the second equality is an immediate consequence of the first one. ∎

Lemma 4.12. *We have*

$$(4.7)$$

Proof. We compute, using (4.6), that

Noting that $[3] - 1 = [2]_2$, we are done. ∎

Lemma 4.13. *For all a, b we have*

$$(4.8)$$

Proof. The proof is a repeated application of the **A**-web Serre relations (cf. Remark 2.5). We always indicate where we apply these and for what values of s, t.

We start by applying these for $s = 1$, $t = 4$ as follows.

Similarly, but for $s = 1$, $t = 3$, we can rewrite the second term as

Combining these gives

Next and as before, this time with $s = 1$, $t = 1$, we get for the second term

Again, by combining this with the above we get

We can rewrite this as

On the other side, using now the $s = 1$, $t = 1$ case, we have

Putting everything together, we get the claimed equality. ∎

5. REPRESENTATION THEORETICAL BACKGROUND

In this section we fix our conventions for the quantum enveloping algebras and recall the definition of the coideal subalgebras $\mathbf{U}'_q(\mathfrak{so}_n)$ and $\mathbf{U}'_q(\mathfrak{sp}_n)$. We will also consider their vector representations, the associated exterior and symmetric powers, and construct some intertwiners.

Quantum enveloping algebras. Let $\mathfrak{g}$ be a reductive Lie algebra with simple roots $(\alpha_i)_{i \in I}$, simple coroots $(h_i)_{i \in I}$ and weight lattice X. Denote by $a_{ij} = \langle h_i, \alpha_j \rangle$ the entries of the Cartan matrix, and by $d_i \in \mathbb{Z}_{\geq 0}$ the minimal values such that the matrix $(d_i a_{ij})_{i,j \in I}$ is symmetric and positive definite, see also the appendix.

Throughout, all indices are always from the evident sets, e.g. if we write E_i, then we always assume that $i \in \mathrm{I}$.

Definition 5.1. The *quantum enveloping algebra* $\mathbf{U}_q(\mathfrak{g})$ *of* $\mathfrak{g}$ is the associative, unital $\mathbb{C}(q)$-algebra generated by $\mathbf{q}^h$ for $h \in X^*$, and by $\mathrm{E}_i, \mathrm{F}_i$ for $i \in \mathrm{I}$, subject to:

$$(5.1) \qquad \mathbf{q}^0 = 1, \quad \mathbf{q}^h \mathbf{q}^{h'} = \mathbf{q}^{h+h'}, \quad \mathbf{q}^h \mathrm{E}_i = q^{\langle h, \alpha_i \rangle} \mathrm{E}_i \mathbf{q}^h, \quad \mathbf{q}^h \mathrm{F}_i = q^{-\langle h, \alpha_i \rangle} \mathrm{F}_i \mathbf{q}^h,$$

$$(5.2) \qquad \mathrm{E}_i \mathrm{F}_j - \mathrm{F}_j \mathrm{E}_i = \delta_{ij} \frac{\mathrm{K}_i - \mathrm{K}_i^{-1}}{q_i - q_i^{-1}},$$

$$(5.3) \qquad \sum_{v=0}^{1-a_{ij}} (-1)^v \begin{bmatrix} 1-a_{ij} \\ v \end{bmatrix}_i \mathrm{E}_i^{1-a_{ij}-s} \mathrm{E}_j \mathrm{E}_i^v = 0, \qquad \text{for } i \neq j,$$

$$(5.4) \qquad \sum_{v=0}^{1-a_{ij}} (-1)^v \begin{bmatrix} 1-a_{ij} \\ v \end{bmatrix}_i \mathrm{F}_i^{1-a_{ij}-s} \mathrm{F}_j \mathrm{F}_i^v = 0, \qquad \text{for } i \neq j.$$

The latter two relations are the so-called *Serre relations*. Here, $\mathrm{K}_i = \mathbf{q}^{d_i h_i}$ and the quantum binomials are as in (1.9).

EXAMPLE 5.2. Besides $\mathfrak{gl}_n$, we will consider the cases $\mathfrak{g} = \mathfrak{sp}_{2k}$ and $\mathfrak{g} = \mathfrak{so}_{2k}$ with conventions fixed in the appendix. The corresponding Serre relations for the E_i's are

$$(5.5) \qquad \mathrm{E}_{k-1}^3 \mathrm{E}_k + [3]\mathrm{E}_{k-1} \mathrm{E}_k \mathrm{E}_{k-1}^2 = \mathrm{E}_k \mathrm{E}_{k-1}^3 + [3]\mathrm{E}_{k-1}^2 \mathrm{E}_k \mathrm{E}_{k-1},$$

$$(5.6) \qquad [2]_2 \mathrm{E}_k \mathrm{E}_{k-1} \mathrm{E}_k = \mathrm{E}_{k-1} \mathrm{E}_k^2 + \mathrm{E}_k^2 \mathrm{E}_{k-1}$$

in case $\mathfrak{g} = \mathfrak{sp}_{2k}$, and for $\mathfrak{g} = \mathfrak{so}_{2k}$ they are

$$(5.7) \qquad \qquad \mathrm{E}_{k-1}\mathrm{E}_k = \mathrm{E}_k\mathrm{E}_{k-1},$$

$$(5.8) \qquad \qquad [2]\mathrm{E}_{k-2}\mathrm{E}_k\mathrm{E}_{k-2} = \mathrm{E}_{k-2}^2\mathrm{E}_k + \mathrm{E}_k\mathrm{E}_{k-2}^2,$$

$$(5.9) \qquad \qquad [2]\mathrm{E}_k\mathrm{E}_{k-2}\mathrm{E}_k = \mathrm{E}_{k-2}\mathrm{E}_k^2 + \mathrm{E}_k^2\mathrm{E}_{k-2}.$$

Additionally, there are versions involving F_k's, and the type $\mathbf{A}$ Serre relations.

As usual, we define the *divided powers*

$$\mathrm{E}_i^{(s)} = \tfrac{1}{[s]_i!}\mathrm{E}_i^s \quad \text{and} \quad \mathrm{F}_i^{(s)} = \tfrac{1}{[s]_i!}\mathrm{F}_i^s, \qquad s \in \mathbb{Z}_{\geq 0}.$$

One can then show that the *higher order Serre relations*

$$(5.10) \qquad \begin{aligned} &\sum_{u+v=t}(-1)^v q_i^{\varepsilon u(-a_{ij}s-t+1)}\mathrm{E}_i^{(u)}\mathrm{E}_j^{(s)}\mathrm{E}_i^{(v)} = 0, \qquad \text{for } i \neq j, \\ &\sum_{u+v=t}(-1)^v q_i^{\varepsilon u(-a_{ij}s-t+1)}\mathrm{F}_i^{(u)}\mathrm{F}_j^{(s)}\mathrm{F}_i^{(v)} = 0, \qquad \text{for } i \neq j, \end{aligned}$$

hold for $\varepsilon = \pm 1$, for all $s, t \in \mathbb{Z}$ with $s \geq 1$ and $t > -a_{ij}$ (see e.g. [Lus10, Chapter 7] and in particular Proposition 7.1.5 therein).

Moreover, recall that $\mathbf{U}_q(\mathfrak{g})$ has the structure of a Hopf algebra. We use the following conventions for the comultiplication $\Delta\colon \mathbf{U}_q(\mathfrak{g}) \to \mathbf{U}_q(\mathfrak{g}) \otimes \mathbf{U}_q(\mathfrak{g})$, the counit $\varepsilon\colon \mathbf{U}_q(\mathfrak{g}) \to \mathbb{C}(q)$ and the antipode $S\colon \mathbf{U}_q(\mathfrak{g}) \to \mathbf{U}_q(\mathfrak{g})$:

$$(5.11) \qquad \begin{aligned} \Delta(\mathbf{q}^h) = \mathbf{q}^h \otimes \mathbf{q}^h, \ \ \Delta(\mathrm{E}_i) &= \mathrm{E}_i \otimes \mathrm{K}_i + 1 \otimes \mathrm{E}_i, \ \ \Delta(\mathrm{F}_i) = \mathrm{F}_i \otimes 1 + \mathrm{K}_i^{-1} \otimes \mathrm{F}_i, \\ \varepsilon(\mathbf{q}^h) &= 1, \quad \varepsilon(\mathrm{E}_i) = 0, \quad \varepsilon(\mathrm{F}_i) = 0, \\ S(\mathbf{q}^h) = \mathbf{q}^{-h}, \quad S(\mathrm{E}_i) &= -\mathrm{E}_i\mathrm{K}_i^{-1}, \quad S(\mathrm{F}_i) = -\mathrm{K}_i\mathrm{F}_i. \end{aligned}$$

The idempotented versions. Next, following [Lus10, Chapter 23], we define:

Definition 5.3. The *idempotented quantum enveloping algebra* $\dot{\mathbf{U}}_q(\mathfrak{g})$ is the additive closure of the $\mathbb{C}(q)$-linear category with:

     ▷ objects $\mathbf{1}_\lambda$ for $\lambda \in X$, and
     ▷ morphisms $\mathrm{Hom}_{\dot{\mathbf{U}}_q(\mathfrak{g})}(\mathbf{1}_\lambda, \mathbf{1}_\mu) = \mathbf{U}_q(\mathfrak{g})/I_{\lambda,\mu}$, where

$$I_{\lambda,\mu} = \sum_{h\in X^*}\mathbf{U}_q(\mathfrak{g})(\mathbf{q}^h - q^{\langle h,\lambda\rangle}) + \sum_{h\in X^*}(\mathbf{q}^h - q^{\langle h,\mu\rangle})\mathbf{U}_q(\mathfrak{g}).$$

The reader unfamiliar with the idempotented version of $\mathbf{U}_q(\mathfrak{g})$ in its categorical disguise is referred to [CKM14, §4.1], whose type $\mathbf{A}$ treatment immediately generalizes to a general $\mathfrak{g}$. Sometimes it is also convenient to regard $\dot{\mathbf{U}}_q(\mathfrak{g})$ as an algebra, and we use both viewpoints interchangeably.

We denoted the morphism of $\dot{\mathbf{U}}_q(\mathfrak{g})$ by $\mathsf{X}\mathbf{1}_\lambda = \mathbf{1}_\mu\mathsf{X}\mathbf{1}_\lambda \in \mathrm{Hom}_{\dot{\mathbf{U}}_q(\mathfrak{g})}(\mathbf{1}_\lambda, \mathbf{1}_\mu)$ for X being some product of E_i's and F_i's, and appropriate λ and μ. In particular,

$$\mathsf{E}_i\mathbf{1}_\lambda \in \mathrm{Hom}_{\dot{\mathbf{U}}_q(\mathfrak{g})}(\mathbf{1}_\lambda, \mathbf{1}_{\lambda+\alpha_i}) \quad \text{and} \quad \mathsf{F}_i\mathbf{1}_\lambda \in \mathrm{Hom}_{\dot{\mathbf{U}}_q(\mathfrak{g})}(\mathbf{1}_\lambda, \mathbf{1}_{\lambda-\alpha_i}).$$

(Note that we write E_i etc. for elements of $\mathbf{U}_q(\mathfrak{g})$, and $\mathsf{E}_i\mathbf{1}_\lambda$ etc. for $\dot{\mathbf{U}}_q(\mathfrak{g})$.)

The quantum enveloping algebra $\mathbf{U}_q(\mathfrak{gl}_n)$. We denote by $\boldsymbol{\mathcal{R}}\mathbf{ep}_q(\mathfrak{gl}_n)$ the braided monoidal category of finite-dimensional representations of $\mathbf{U}_q(\mathfrak{gl}_n)$. Let us recall some basic facts about some representations of $\mathbf{U}_q(\mathfrak{gl}_n)$.

We denote by $\mathbb{C}_q = \mathbb{C}(q)$ the trivial and by V_q the (quantum analog of the) vector representation of $\mathbf{U}_q(\mathfrak{gl}_n)$. On the standard basis $v_1, \ldots, v_n$ of V_q, the action of the generators is explicitly given by

$$\mathrm{K}_i^{\pm 1} v_j = \begin{cases} q^{\pm 1} v_j, & \text{if } i = j, \\ q^{\mp 1} v_j, & \text{if } i = j-1, \\ v_j, & \text{else}, \end{cases}$$

$$\mathrm{E}_i v_j = \begin{cases} v_{j-1}, & \text{if } i = j-1, \\ 0, & \text{else}, \end{cases} \qquad \mathrm{F}_i v_j = \begin{cases} v_{j+1}, & \text{if } i = j, \\ 0, & \text{else}. \end{cases}$$

As usual, we define the *(q-)exterior algebra* of V_q as

$$\textstyle\bigwedge_q^\bullet \mathrm{V}_q = \mathrm{TV}_q / \langle \mathrm{S}_q^2 \mathrm{V}_q \rangle,$$

where TV_q denotes the tensor algebra of V_q and $\mathrm{S}_q^2 \mathrm{V}_q \subset \mathrm{V}_q \otimes \mathrm{V}_q$ is the $\mathbb{C}(q)$-linear subspace spanned by

(5.12) $\qquad v_i \otimes v_i, \text{ for all } i = 1, \ldots, n, \quad \text{and} \quad q^{-1} v_i \otimes v_j + v_j \otimes v_i, \text{ for all } i < j.$

Since $\mathrm{T}V$ is naturally graded and the ideal $\langle \mathrm{S}_q^2 \mathrm{V}_q \rangle$ is homogeneous, $\bigwedge_q^\bullet \mathrm{V}_q$ is also graded and decomposes as a $\mathbf{U}_q(\mathfrak{gl}_n)$-module as $\bigoplus_{a \in \mathbb{Z}_{\geq 0}} \bigwedge_q^a \mathrm{V}_q$, with $\bigwedge_q^0 \mathrm{V}_q \cong \mathbb{C}_q$ and $\bigwedge_q^1 \mathrm{V}_q \cong \mathrm{V}_q$. We call $\bigwedge_q^a \mathrm{V}_q$ the ath *exterior power* (of V_q), and we write $v_{i_1} \wedge \cdots \wedge v_{i_a}$ for the image of $v_{i_1} \otimes \cdots \otimes v_{i_a}$ in the quotient $\bigwedge_q^a \mathrm{V}_q$.

Similarly, we define the *(q-)symmetric algebra* as

$$\mathrm{Sym}_q^\bullet \mathrm{V}_q = \mathrm{TV}_q / \langle \mathrm{E}_q^2 \mathrm{V}_q \rangle,$$

where $\mathrm{E}_q^2 \mathrm{V}_q \subset \mathrm{V}_q \otimes \mathrm{V}_q$ is spanned by

(5.13) $\qquad\qquad\qquad q v_i \otimes v_j - v_j \otimes v_i, \text{ for all } i < j.$

As before, we have a $\mathbf{U}_q(\mathfrak{gl}_n)$-module decomposition $\mathrm{Sym}_q^\bullet \mathrm{V}_q = \bigoplus_{a \in \mathbb{Z}_{\geq 0}} \mathrm{Sym}_q^a \mathrm{V}_q$, with $\mathrm{Sym}_q^0 \mathrm{V}_q \cong \mathbb{C}_q$ and $\mathrm{Sym}_q^1 \mathrm{V}_q \cong \mathrm{V}_q$. We call $\mathrm{Sym}_q^a \mathrm{V}_q$ the ath *symmetric power* (of V_q). We write $v_{i_1} \cdots v_{i_a}$ for the corresponding element of $\mathrm{Sym}_q^a \mathrm{V}_q$.

Clearly, $\bigwedge_q^a \mathrm{V}_q$ and $\mathrm{Sym}_q^a \mathrm{V}_q$ are $\mathbb{C}(q)$-linearly spanned by elements of the form

$$v_{i_1} \wedge \cdots \wedge v_{i_a}, \quad i_1 < \cdots < i_a, \quad \text{and} \quad v_{i_1} \cdots v_{i_a}, \quad i_1 \leq \cdots \leq i_a.$$

Henceforth, we will always assume that the indices are (strictly) increasing.

The multiplication of the tensor algebra TV_q is clearly $\mathbf{U}_q(\mathfrak{gl}_n)$-equivariant, and therefore induces $\mathbf{U}_q(\mathfrak{gl}_n)$-equivariant multiplications on $\bigwedge_q^\bullet \mathrm{V}_q$ and $\mathrm{Sym}_q^\bullet \mathrm{V}_q$. Moreover, both $\bigwedge_q^\bullet \mathrm{V}_q$ and $\mathrm{Sym}_q^\bullet \mathrm{V}_q$ are coalgebras, with $\mathbf{U}_q(\mathfrak{gl}_n)$-equivariant comultiplications. (This follows from Howe duality in type **A**, see [CKM14, Lemma 3.1.2] for $\bigwedge_q^\bullet \mathrm{V}_q$ and [RT16, Lemma 2.21] for $\mathrm{Sym}_q^\bullet \mathrm{V}_q$.) Thus, we can define $\mathbf{U}_q(\mathfrak{gl}_n)$-equivariant maps

$$\textstyle\lambda_{a,b}^{a+b} \colon \bigwedge_q^a \mathrm{V}_q \otimes \bigwedge_q^b \mathrm{V}_q \to \bigwedge_q^{a+b} \mathrm{V}_q \quad \text{and} \quad \lambda_{a,b}^{a+b} \colon \mathrm{Sym}_q^a \mathrm{V}_q \otimes \mathrm{Sym}_q^b \mathrm{V}_q \to \mathrm{Sym}_q^{a+b} \mathrm{V}_q,$$

$$\textstyle\mathsf{Y}_{a+b}^{a,b} \colon \bigwedge_q^{a+b} \mathrm{V}_q \to \bigwedge_q^a \mathrm{V}_q \otimes \bigwedge_q^b \mathrm{V}_q \quad \text{and} \quad \mathsf{Y}_{a+b}^{a,b} \colon \mathrm{Sym}_q^{a+b} \mathrm{V}_q \to \mathrm{Sym}_q^a \mathrm{V}_q \otimes \mathrm{Sym}_q^b \mathrm{V}_q,$$

to be the corresponding (co)multiplications.

REMARK 5.4. In order to facilitate the distinction between the exterior and the symmetric power, we use the color code from [TVW15], i.e. "exterior=red" and "symmetric=green" (with "black=$\bigwedge_q^1 V_q = V_q = \mathrm{Sym}_q^1 V_q$"). However, our web categories are "red and green at the same time" (cf. Figure 2), so we do not color their webs.

EXAMPLE 5.5. The base cases of the $\mathbf{U}_q(\mathfrak{gl}_n)$-intertwiners from above are the ones with $a = b = 1$. In these cases we omit the sub- and superscripts and we have

$$\curlyvee : \bigwedge_q^2 V_q \to V_q \otimes V_q, \quad v_i \wedge v_j \mapsto q v_i \otimes v_j - v_j \otimes v_i,$$

$$\curlyvee : \mathrm{Sym}_q^2 V_q \to V_q \otimes V_q, \quad v_i v_j \mapsto \begin{cases} q^{-1} v_i \otimes v_j + v_j \otimes v_i, & \text{for } i < j, \\ [2]\, v_i \otimes v_i, & \text{for } i = j. \end{cases}$$

The coideal subalgebra $\mathbf{U}_q'(\mathfrak{so}_n)$. Next, we recall the definition of the coideal subalgebra $\mathbf{U}_q'(\mathfrak{so}_n)$ of $\mathbf{U}_q(\mathfrak{gl}_n)$, following [KP11, Section 3].

Definition 5.6. Let $\mathbf{U}_q'(\mathfrak{so}_n)$ be the $\mathbb{C}(q)$-subalgebra of $\mathbf{U}_q(\mathfrak{gl}_n)$ generated by

$$(5.14) \qquad\qquad B_i = F_i - K_i^{-1} E_i, \qquad \text{for } i = 1, \ldots, n.$$

REMARK 5.7. Despite the similar notation, $\mathbf{U}_q(\mathfrak{so}_n)$ and $\mathbf{U}_q'(\mathfrak{so}_n)$ are different algebras. In fact, the standard embedding $\mathbf{U}(\mathfrak{so}_n) \hookrightarrow \mathbf{U}(\mathfrak{gl}_n)$ does not lift to the quantum level as an embedding of $\mathbf{U}_q(\mathfrak{so}_n)$ into $\mathbf{U}_q(\mathfrak{gl}_n)$. In contrast, $\mathbf{U}_q'(\mathfrak{so}_n)$ is, by definition, a subalgebra of $\mathbf{U}_q(\mathfrak{gl}_n)$. Both of them are, however, quantizations of the $\mathbb{C}$-algebra $\mathbf{U}(\mathfrak{so}_n)$, cf. [Let99, Section 4, especially Theorem 4.8].

The algebra $\mathbf{U}_q'(\mathfrak{so}_n)$ is not a Hopf subalgebra of $\mathbf{U}_q(\mathfrak{gl}_n)$ (in particular, it is not closed under the comultiplication). Indeed, using (5.11), we get

$$(5.15) \qquad\qquad \Delta(B_i) = B_i \otimes 1 + K_i^{-1} \otimes B_i \in \mathbf{U}_q(\mathfrak{gl}_n) \otimes \mathbf{U}_q'(\mathfrak{so}_n).$$

However, (5.15) shows that $\mathbf{U}_q'(\mathfrak{so}_n)$ is a so-called *left coideal subalgebra*.

The representation category of $\mathbf{U}_q'(\mathfrak{so}_n)$. We denote the category of finite-dimensional representations of $\mathbf{U}_q'(\mathfrak{so}_n)$ by $\mathcal{R}\mathbf{ep}_q'(\mathfrak{so}_n)$. Via restriction, we see that the objects and morphisms from $\mathcal{R}\mathbf{ep}_q(\mathfrak{gl}_n)$ descend to $\mathcal{R}\mathbf{ep}_q'(\mathfrak{so}_n)$. In particular, the $\mathbf{U}_q(\mathfrak{gl}_n)$-intertwiners $\curlywedge_{a,b}^{a+b}$, $\curlywedge_{a,b}^{a+b}$, $\curlyvee_{a+b}^{a,b}$ and $\curlyvee_{a+b}^{a,b}$ are $\mathbf{U}_q'(\mathfrak{so}_n)$-equivariant as well.

Moreover, as recalled above, $\mathbf{U}_q'(\mathfrak{so}_n)$ is not closed under comultiplication. Hence, $\mathcal{R}\mathbf{ep}_q'(\mathfrak{so}_n)$ does not inherit the structure of a monoidal category from $\mathcal{R}\mathbf{ep}_q(\mathfrak{gl}_n)$. However, since $\mathbf{U}_q'(\mathfrak{so}_n)$ is a coideal subalgebra, $\mathcal{R}\mathbf{ep}_q'(\mathfrak{so}_n)$ is a $\mathcal{R}\mathbf{ep}_q(\mathfrak{gl}_n)$-category.

Some intertwiners. We define $\mathbb{C}(q)$-linear maps

$$(5.16) \quad \begin{aligned} \cup &: \mathbb{C}_q \to V_q \otimes V_q, \quad 1 \mapsto \textstyle\sum_{i=1}^n v_i \otimes v_i, \\ \cap &: V_q \otimes V_q \to \mathbb{C}_q, \quad v_i \otimes v_j \mapsto \begin{cases} q^{n+1-2i}, & \text{if } i = j, \\ 0, & \text{else,} \end{cases} \end{aligned}$$

$$(5.17) \quad \begin{aligned} \downarrow &: \mathbb{C}_q \to \mathrm{Sym}_q^2 V_q, \quad 1 \mapsto \tfrac{1}{[2]} \left(\textstyle\sum_{i=1}^n v_i v_i \right), \\ \uparrow &: \mathrm{Sym}_q^2 V_q \to \mathbb{C}_q, \quad v_i v_j \mapsto \begin{cases} q^{n+1-2i}, & i = j, \\ 0, & \text{else.} \end{cases} \end{aligned}$$

Lemma 5.8. *The $\mathbb{C}(q)$-linear maps $\cup$, $\cap$, $\downarrow$ and $\uparrow$ intertwine the $\mathbf{U}_q'(\mathfrak{so}_n)$-actions.*

Proof. First we note that

$$\cup = \curlyvee \circ \downarrow \quad \text{and} \quad \cap = \uparrow \circ \curlywedge.$$

We already know that $\curlyvee$ and $\curlywedge$ intertwine the action of $\mathbf{U}_q(\mathfrak{gl}_n)$. Thus, via restriction, they intertwine the action of $\mathbf{U}_q'(\mathfrak{so}_n)$ as well. So it remains to show that $\downarrow$ and $\uparrow$ intertwine the action of $\mathbf{U}_q'(\mathfrak{so}_n)$.

The $\downarrow$ case: One just has to show that $B_j\left(\sum_{i=1}^n v_i v_i\right) = 0$ for all $j = 1, \ldots, n$, which follows via direct and straightforward computation.

The $\uparrow$ case: The computation boils down to check that

$$\uparrow(B_i(v_i \otimes v_{i+1})) = \uparrow(v_{i+1} \otimes v_{i+1} - q^{-2} v_i \otimes v_i) = 0,$$

and the claim follows. ∎

The coideal subalgebra $\mathbf{U}_q'(\mathfrak{sp}_n)$. Similar to the orthogonal case, we define now the coideal subalgebra $\mathbf{U}_q'(\mathfrak{sp}_n)$, following [KP11, Section 5].

Definition 5.9. Let $\mathbf{U}_q'(\mathfrak{sp}_n)$ be the $\mathbb{C}(q)$-subalgebra of $\mathbf{U}_q(\mathfrak{gl}_n)$ generated by

$$E_i, F_i, K_i^{\pm 1}, \qquad \qquad \text{for } i = 1, 3, \ldots, n-1,$$

$$(5.18) \quad \begin{aligned} B_i = {} & F_i - K_i^{-1} E_{i-1} E_{i+1} E_i + q^{-1} K_i^{-1} E_{i-1} E_i E_{i+1} \\ & + q^{-1} K_i^{-1} E_{i+1} E_i E_{i-1} - q^{-2} K_i^{-1} E_i E_{i-1} E_{i+1}, \end{aligned} \quad \text{for } i = 2, 4, \ldots, n.$$

REMARK 5.10. As before, $\mathbf{U}_q'(\mathfrak{sp}_n)$ should not be confused with $\mathbf{U}_q(\mathfrak{sp}_n)$, although they both de-quantize to $\mathbf{U}(\mathfrak{sp}_n)$ (cf. [Let99, Section 4]).

One again checks that $\mathbf{U}_q'(\mathfrak{sp}_n)$ is a left coideal subalgebra of $\mathbf{U}_q(\mathfrak{gl}_n)$.

The representation category of $\mathbf{U}_q'(\mathfrak{sp}_n)$. We denote by $\mathcal{R}ep_q'(\mathfrak{sp}_n)$ the category of finite-dimensional representations of $\mathbf{U}_q'(\mathfrak{sp}_n)$. Again, the category $\mathcal{R}ep_q'(\mathfrak{sp}_n)$ is a $\mathcal{R}ep_q(\mathfrak{gl}_n)$-category since $\mathbf{U}_q'(\mathfrak{sp}_n)$ is a coideal subalgebra of $\mathbf{U}_q(\mathfrak{gl}_n)$, and, via restriction, the objects and morphisms from $\mathcal{R}ep_q(\mathfrak{gl}_n)$ descend to $\mathcal{R}ep_q'(\mathfrak{sp}_n)$.

Some more intertwiners. We define $\mathbb{C}(q)$-linear maps

$$(5.19) \quad \begin{aligned} & \downarrow : \mathbb{C}_q \to \textstyle\bigwedge_q^2 V_q, \quad 1 \mapsto \sum_{i=1}^{n/2} q^{1-i} v_{2i-1} \wedge v_{2i}, \\ & \uparrow : \textstyle\bigwedge_q^2 V_q \to \mathbb{C}_q, \quad v_i \wedge v_j \mapsto \begin{cases} q^{n-1/2(3i+1)}, & \text{if } i \text{ is odd and } j = i+1, \\ 0, & \text{else.} \end{cases} \end{aligned}$$

$$(5.20) \quad \begin{aligned} & \cup : \mathbb{C}_q \to V_q \otimes V_q, \quad 1 \mapsto \sum_{i=1}^{n/2} q^{1-i}(q v_{2i-1} \otimes v_{2i} - v_{2i} \otimes v_{2i-1}), \\ & \cap : V_q \otimes V_q \to \mathbb{C}_q, \quad v_i \otimes v_j \mapsto \begin{cases} q^{n-1/2(3i+1)}, & \text{if } i \text{ is odd and } j = i+1, \\ -q^{n-1/2(3i)}, & \text{if } i \text{ is even and } j = i-1, \\ 0, & \text{else.} \end{cases} \end{aligned}$$

Lemma 5.11. *The $\mathbb{C}(q)$-linear maps $\downarrow$, $\uparrow$, $\cup$ and $\cap$ intertwine the $\mathbf{U}_q'(\mathfrak{sp}_n)$-actions.*

Proof. As in the proof of Lemma 5.8 we have

$$\cup = \curlyvee \circ \downarrow \quad \text{and} \quad \cap = \uparrow \circ \curlywedge.$$

Hence, as before, we only need to check that $\downarrow$ and $\uparrow$ are $\mathbf{U}_q'(\mathfrak{sp}_n)$-equivariant.

The ↓ case: We need to show for i odd that $\mathrm{K}_i^{\pm 1}$ acts on ↓(1) as the identity and E_i, F_i as zero, and for i even that $\mathrm{B}_i(\downarrow(1)) = 0$. The former is clear, while the latter computation essentially boils down to

$$\mathrm{B}_i(v_{i-1} \wedge v_i + q^{-1} v_{i+1} \wedge v_{i+2}) = \mathrm{F}_i(v_{i-1} \wedge v_i) - q^{-1} \mathrm{K}_i^{-1} \mathrm{E}_{i-1} \mathrm{E}_{i+1} \mathrm{E}_i (v_{i+1} \wedge v_{i+2})$$
$$= v_{i-1} \wedge v_{i+1} - q^{-1} q v_{i-1} \wedge v_{i+1} = 0,$$

since $\mathrm{E}_{i-1}(v_{i-1} \wedge v_i) = v_{i-1} \wedge v_{i-1} = 0$ and $\mathrm{E}_{i+1}(v_{i+1} \wedge v_{i+2}) = v_{i+1} \wedge v_{i+1} = 0$.

The ↑ case: We have to show that

$$\uparrow(\mathrm{X}(v_i \wedge v_j)) = \mathrm{X}(\uparrow(v_i \wedge v_j)), \quad \text{for all X as in } (5.18).$$

This is clear for $\mathrm{X} = \mathrm{K}_l^{\pm 1}$ with l odd, so let us assume that X is either an E, an F or a B. Of course, we can also assume that $i < j$. Still, we have a few cases to check, where we only need to verify $\uparrow(\mathrm{X}(v_i \wedge v_j)) = 0$, since the other side is always zero:

▶ If $j > i+2$, then it is easily shown that $\uparrow(\mathrm{X}(v_i \wedge v_j)) = 0$. Indeed, the only thing to observe hereby is

$$\mathrm{E}_i \mathrm{E}_{i+1} \mathrm{E}_{i+2}(v_i \wedge v_{i+3}) = v_i \wedge v_i = 0,$$

which shows that $\uparrow(\mathrm{B}_{i+1}(v_i \wedge v_{i+3})) = 0$ for i odd.

▶ If $j = i+1$ and i is odd, then $\mathrm{E}_i(v_i \wedge v_{i+1}) = \mathrm{F}_i(v_i \wedge v_{i+1}) = 0$. Moreover,

$$\uparrow(\mathrm{B}_{i-1}(v_i \wedge v_{i+1})) = \uparrow(-q v_{i-2} \wedge v_i) = 0 \quad \text{and} \quad \uparrow(\mathrm{B}_{i+1}(v_i \wedge v_{i+1})) = \uparrow(v_i \wedge v_{i+2}) = 0.$$

▶ If $j = i+1$ and i is even, then clearly $\uparrow(\mathrm{X}(v_i \wedge v_{i+1})) = 0$ for X being either of $\mathrm{E}_{i-1}, \mathrm{E}_{i+1}, \mathrm{F}_{i-1}, \mathrm{F}_{i+1}$. Moreover, one also directly sees that $\mathrm{B}_i(v_i \wedge v_{i+1}) = 0$.

▶ If $j = i+2$ and i is odd, then clearly $\mathrm{E}_l(v_i \wedge v_{i+2}) = 0$ for all l odd. We also see directly that $\uparrow(\mathrm{F}_{i+2}(v_i \wedge v_{i+2})) = 0$ and $\mathrm{B}_{i+1}(v_i \wedge v_{i+2}) = 0$. Moreover, noting that $i+1$ is even, we get

$$\uparrow(\mathrm{F}_i(v_i \wedge v_{i+2})) = \uparrow(v_{i+1} \wedge v_{i+2}) = 0.$$

▶ Finally, if $j = i+2$ and i is even, then $\mathrm{F}_l(v_i \wedge v_{i+2}) = 0$ for all l odd. We also directly see that $\uparrow(\mathrm{E}_{i-1}(v_i \wedge v_{i+2})) = 0$. Further, because i is even, we have

$$\uparrow(\mathrm{E}_{i+1}(v_i \wedge v_{i+2})) = \uparrow(v_i \wedge v_{i+1}) = 0.$$

Moreover, noting that $i-1$ and $i+1$ are odd, we get

$$\uparrow(\mathrm{B}_i(v_i \wedge v_{i+2})) = \uparrow(v_{i+1} \wedge v_{i+2} - q^{-3} v_{i-1} \wedge v_i)$$
$$= q^{n-3i+4/2} - q^{-3} q^{n-3i-2/2} = 0,$$

and $\uparrow(\mathrm{B}_{i+2}(v_i \wedge v_{i+2})) = 0$ follows again because $\uparrow(v_i \wedge v_{i+1}) = 0$. ∎

An integral form. For the purpose of later specialization, we need a version of Lusztig's integral form for $\mathbf{U}_q'(\mathfrak{so}_n)$ and $\mathbf{U}_q'(\mathfrak{sp}_n)$. To this end, we let $\mathscr{A} = \mathbb{C}[q, q^{-1}, \frac{1}{[n]}]$. We denote by $^{\mathscr{A}}\mathbf{U}_q(\mathfrak{g})$ the $\mathscr{A}$-*form of* $\mathbf{U}_q(\mathfrak{g})$, which is the $\mathscr{A}$-subalgebra generated by the E_i's, F_i's and $\mathbf{q}^h$'s. Note that we clearly have $^{\mathscr{A}}\mathbf{U}_q(\mathfrak{g}) \otimes_{\mathscr{A}} \mathbb{C}(q) = \mathbf{U}_q(\mathfrak{g})$.

Definition 5.12. We let $^{\mathscr{A}}\mathbf{U}_q'(\mathfrak{so}_n) \subset {}^{\mathscr{A}}\mathbf{U}_q(\mathfrak{gl}_n)$ be the $\mathscr{A}$-*form of* $\mathbf{U}_q'(\mathfrak{so}_n)$, which is defined to be the $\mathscr{A}$-subalgebra generated by the B_i's from (5.14). Similarly, we define the $\mathscr{A}$-*form of* $\mathbf{U}_q'(\mathfrak{sp}_n)$ using the B_i's from (5.18).

Again, we clearly have that

$$^{\mathscr{A}}\mathbf{U}_q'(\mathfrak{so}_n) \otimes_{\mathscr{A}} \mathbb{C}(q) = \mathbf{U}_q'(\mathfrak{so}_n) \quad \text{and} \quad {}^{\mathscr{A}}\mathbf{U}_q'(\mathfrak{sp}_n) \otimes_{\mathscr{A}} \mathbb{C}(q) = \mathbf{U}_q'(\mathfrak{sp}_n).$$

6. Connecting webs and representation categories

We are now going to define the functors from Figure 2.

Actions on representations in types BCD. We will now define actions of our diagrammatic web categories on representations of $\mathbf{U}_q'(\mathfrak{so}_n)$ and $\mathbf{U}_q'(\mathfrak{sp}_n)$.

The presentation functors for $\mathbf{U}_q(\mathfrak{gl}_n)$. First, we recall that in type **A** we can define functors $\Gamma_{\mathbf{A}}^{\mathrm{ext}}\colon \mathcal{W}\mathrm{eb}_q^{\lambda} \to \mathcal{R}\mathbf{ep}_q(\mathfrak{gl}_n)$ and $\Gamma_{\mathbf{A}}^{\mathrm{sym}}\colon \mathcal{W}\mathrm{eb}_q^{\lambda} \to \mathcal{R}\mathbf{ep}_q(\mathfrak{gl}_n)$ (sending the object a to $\bigwedge_q^a V_q$ and $\mathrm{Sym}_q^a V_q$, respectively) using the $\mathbf{U}_q(\mathfrak{gl}_n)$-intertwiners $\lambda_{a,b}^{a+b}$, $\curlyvee_{a+b}^{a,b}$ and $\lambda_{a+b}^{a,b}$, $\curlyvee_{a+b}^{a,b}$ from Section 5. By Example 5.5, we get

(6.1)
$$\Gamma_{\mathbf{A}}^{\mathrm{ext}}\left(\begin{matrix}1 \quad 1 \\ \diagdown\diagup \\ \diagup\diagdown \\ 1 \quad 1\end{matrix}\right) = \mathord{\times}\colon v_i \otimes v_j \mapsto \begin{cases} q v_i \otimes v_j - v_j \otimes v_i, & \text{if } i < j, \\ q^{-1} v_i \otimes v_j - v_j \otimes v_i, & \text{if } i > j, \\ 0, & \text{if } i = j, \end{cases}$$

$$\Gamma_{\mathbf{A}}^{\mathrm{sym}}\left(\begin{matrix}1 \quad 1 \\ \diagdown\diagup \\ \diagup\diagdown \\ 1 \quad 1\end{matrix}\right) = \mathord{\times}\colon v_i \otimes v_j \mapsto \begin{cases} q^{-1} v_i \otimes v_j + v_j \otimes v_i, & \text{if } i < j, \\ q v_i \otimes v_j + v_j \otimes v_i, & \text{if } i > j, \\ [2]\, v_i \otimes v_i, & \text{if } i = j. \end{cases}$$

We will use (6.1) frequently below.

REMARK 6.1. Note that $\Gamma_{\mathbf{A}}^{\mathrm{ext}}$ is the functor from [CKM14, §3.2], while $\Gamma_{\mathbf{A}}^{\mathrm{sym}}$ is its cousin as in [RT16, Definition 2.18] or [TVW15, Definition 3.17].

One can check that both $\Gamma_{\mathbf{A}}^{\mathrm{ext}}$ and $\Gamma_{\mathbf{A}}^{\mathrm{sym}}$ are functors of braided monoidal categories (see e.g. [TVW15, Theorem 3.20]) – a fact that we use silently below.

The presentation functors for $\mathbf{U}_q'(\mathfrak{so}_n)$. We now specialize $\mathsf{z} = q^n \in \mathbb{C}(q)$ in the exterior and $\mathsf{z} = -q^{-n} \in \mathbb{C}(q)$ in the symmetric case. (Note that in both cases $[\mathsf{z}; a]$ specializes to $[n + a]$ and $[\mathsf{z}; a]_2$ specializes to $[n + a]_2$.)

We define $\Gamma_{\mathbf{BD}}^{\mathrm{ext}}\colon \mathcal{W}\mathrm{eb}_{q,q^n}^{\cup} \to \mathcal{R}\mathbf{ep}_q'(\mathfrak{so}_n)$ on objects by $a \mapsto \bigwedge_q^a V_q$ and on the generating morphisms by the assignment

(6.2)
$$\overset{1 \quad 1}{\smile} \mapsto \left(\cup\colon \mathbb{C}_q \to V_q \otimes V_q\right) \quad \text{and} \quad \underset{1 \quad 1}{\frown} \mapsto \left(\cap\colon V_q \otimes V_q \to \mathbb{C}_q\right),$$

and to be $\Gamma_{\mathbf{A}}^{\mathrm{ext}}$ on the **A**-web generators (**A**gen). Similarly, we define its symmetric counterpart $\Gamma_{\mathbf{BD}}^{\mathrm{sym}}\colon \mathcal{W}\mathrm{eb}_{q,q^n}^{\bullet} \to \mathcal{R}\mathbf{ep}_q'(\mathfrak{so}_n)$ on objects by $a \mapsto \mathrm{Sym}_q^a V_q$ and on the generating morphisms by the assignment

(6.3)
$$\overset{2}{\underset{\bullet}{\big|}} \mapsto \left(\mathord{\downarrow}\colon \mathbb{C}_q \to \mathrm{Sym}_q^2 V_q\right) \quad \text{and} \quad \underset{2}{\overset{\bullet}{\big|}} \mapsto \left(\mathord{\uparrow}\colon \mathrm{Sym}_q^2 V_q \to \mathbb{C}_q\right),$$

and to be $\Gamma_{\mathbf{A}}^{\mathrm{sym}}$ on the **A**-web generators (**A**gen). The $\mathbf{U}_q'(\mathfrak{so}_n)$-intertwiners in (6.2) and (6.3) are defined in (5.16) and (5.17).

In order to prove that $\Gamma_{\mathbf{BD}}^{\mathrm{ext}}$ and $\Gamma_{\mathbf{BD}}^{\mathrm{sym}}$ are well-defined, we need to show that the defining relations of $\mathcal{W}\mathrm{eb}_{q,q^n}^{\cup}$ are satisfied in the image. For $\Gamma_{\mathbf{BD}}^{\mathrm{ext}}$, we do this in detail in the following lemmas, where we denote by $\mathrm{id}_a = \mathrm{id}_{\bigwedge_q^a V_q}$ the identity morphisms (we write $\mathrm{id} = \mathrm{id}_1$ for short) and all indexes are from $\{1, \dots, n\}$. Further, we abbreviate $v_{j_1 \cdots j_\ell} = v_{j_1} \otimes \cdots \otimes v_{j_l}$.

Lemma 6.2 (Circle removal). *We have* $\cap \circ \cup = [n]\,\mathrm{id}_0$.

Proof. By definition, $\cap \circ \cup (1) = \cap\left(\sum_{i=1}^n v_{ii}\right) = \sum_{i=1}^n q^{n+1-2i} = [n]$. ∎

Lemma 6.3 (Bubble removal). *We have* $(\mathrm{id} \otimes \cap)(\mathord{\times} \otimes \mathrm{id})(\mathrm{id} \otimes \cup) = [n-1]\,\mathrm{id}$.

Proof. We compute

$$(\mathrm{id} \otimes \cap)(\mathord{\times} \otimes \mathrm{id})(\mathrm{id} \otimes \cup)(v_x)$$
$$= (\mathrm{id} \otimes \cap)(\mathord{\times} \otimes \mathrm{id})(\textstyle\sum_{i=1}^n v_{xii}) = (\mathrm{id} \otimes \cap)(\textstyle\sum_{i<x} q^{-1}v_{xii} + \sum_{i>x} q v_{xii})$$
$$= \textstyle\sum_{i<x} q^{-1} q^{n+1-2i} v_x + \sum_{i>x} q q^{n+1-2i} v_x = \sum_{i=1}^{n-1} q^{n-2i} v_x = [n-1]v_x. \qquad ∎$$

Lemma 6.4 (Lasso move). *We have*

$$(\mathrm{id} \otimes \mathrm{id} \otimes \cap)(\mathrm{id} \otimes \mathord{\times} \otimes \mathrm{id})(\mathord{\times} \otimes \mathord{\times})(\mathrm{id} \otimes \mathord{\times} \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes \cup) - \mathord{\cup\!\!\cap} = [n-2]\,\mathrm{id} \otimes \mathrm{id}.$$

Proof. We compute

$$(\mathrm{id} \otimes \mathrm{id} \otimes \cap)(\mathrm{id} \otimes \mathord{\times} \otimes \mathrm{id})(\mathord{\times} \otimes \mathord{\times})(\mathrm{id} \otimes \mathord{\times} \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes \cup)(v_{xy})$$
$$= (\mathrm{id} \otimes \mathrm{id} \otimes \cap)(\mathrm{id} \otimes \mathord{\times} \otimes \mathrm{id})(\mathord{\times} \otimes \mathord{\times})(\mathrm{id} \otimes \mathord{\times} \otimes \mathrm{id})(\textstyle\sum_{i=1}^n v_{xyii})$$
$$= (\mathrm{id} \otimes \mathrm{id} \otimes \cap)(\mathrm{id} \otimes \mathord{\times} \otimes \mathrm{id})(\mathord{\times} \otimes \mathord{\times})(-\textstyle\sum_{i\neq b} v_{xiyi}).$$

Now, if $x < y$, then we get

$$= (\mathrm{id} \otimes \mathrm{id} \otimes \cap)(\mathrm{id} \otimes \mathord{\times} \otimes \mathrm{id})(-\textstyle\sum_{i<x} q^{-2}v_{xiyi} - q^{-1}v_{xiiy} - q^{-1}v_{ixyi} + v_{ixiy})$$
$$+ (\mathrm{id} \otimes \mathrm{id} \otimes \cap)(\mathrm{id} \otimes \mathord{\times} \otimes \mathrm{id})(-\textstyle\sum_{x<i<y} v_{xiyi} - q^{-1}v_{xiiy} - q v_{ixyi} + v_{ixiy})$$
$$+ (\mathrm{id} \otimes \mathrm{id} \otimes \cap)(\mathrm{id} \otimes \mathord{\times} \otimes \mathrm{id})(-\textstyle\sum_{i>y} q^2 v_{xiyi} - q v_{xiiy} - v_{ixyi} + v_{ixiy})$$
$$= (\mathrm{id} \otimes \mathrm{id} \otimes \cap)(\textstyle\sum_{i<y} q^{-2}v_{xyii} + \sum_{x<i<y} v_{xyii} + \sum_{i>y} q^2 v_{xyii})$$
$$= \textstyle\sum_{i=1}^{n-2} q^{n-2i-1} v_{xy} = [n-2]v_{xy}.$$

Similarly, if $x > y$, then we get

$$= (\mathrm{id} \otimes \mathrm{id} \otimes \cap)(\mathrm{id} \otimes \mathord{\times} \otimes \mathrm{id})(-\textstyle\sum_{i<y} q^{-2}v_{xiyi} - q^{-1}v_{xiiy} - q^{-1}v_{ixyi} + v_{ixiy})$$
$$+ (\mathrm{id} \otimes \mathrm{id} \otimes \cap)(\mathrm{id} \otimes \mathord{\times} \otimes \mathrm{id})(-\textstyle\sum_{y<i<x} v_{xiyi} - q v_{xiiy} - q^{-1}v_{ixyi} + v_{ixiy})$$
$$+ (\mathrm{id} \otimes \mathrm{id} \otimes \cap)(\mathrm{id} \otimes \mathord{\times} \otimes \mathrm{id})(-\textstyle\sum_{i>x} q^2 v_{xiyi} - q v_{xiiy} - v_{ixyi} + v_{ixiy})$$
$$= (\mathrm{id} \otimes \mathrm{id} \otimes \cap)(\textstyle\sum_{i<x} q^{-2}v_{xyii} + \sum_{y<i<x} v_{xyii} + \sum_{i>x} q^2 v_{xyii})$$
$$= \textstyle\sum_{i=1}^{n-2} q^{n-2i-1} v_{xy} = [n-2]v_{xy}.$$

So the statement is proved on $v_x \otimes v_y$ if $x \neq y$. Finally, if $x = y$, then we get

$$= (\mathrm{id} \otimes \mathrm{id} \otimes \cap)(\mathrm{id} \otimes \mathord{\times} \otimes \mathrm{id})(-\textstyle\sum_{i<x} q^{-2}v_{xixi} - q^{-1}v_{xiix} - q^{-1}v_{ixxi} + v_{ixix})$$
$$+ (\mathrm{id} \otimes \mathrm{id} \otimes \cap)(\mathrm{id} \otimes \mathord{\times} \otimes \mathrm{id})(-\textstyle\sum_{i>x} q^2 v_{xixi} - q v_{xiix} - v_{ixxi} + v_{ixix})$$
$$= (\mathrm{id} \otimes \mathrm{id} \otimes \cap)(\textstyle\sum_{i<x} q^{-2}v_{xxii} + v_{iixx}) + (\mathrm{id} \otimes \mathrm{id} \otimes \cap)(\textstyle\sum_{i>x} q^2 v_{xxii} + v_{iixx})$$
$$= \textstyle\sum_{i<x}(q^{n-2i-1}v_{xx} + q^{n-2x+1}v_{ii}) + \sum_{i>x}(q^{n-2i+3}v_{xx} + q^{n-2x+1}v_{ii})$$
$$= \textstyle\sum_{i=1}^{n-2}(q^{n-2i-1}v_{xx}) + \sum_{i=1}^n (q^{n-2x+1}v_{xx}) = [n-2]v_{xx} + \sum_{i=1}^n (q^{n-2x+1}v_{ii})$$
$$= ([n-2]\mathrm{id} + \mathord{\cup\!\!\cap})(v_{xx}),$$

and we are done. ∎

Lemma 6.5 (Lollipop relation). *We have* $\mathord{\times} \circ \cup = 0$ *and* $\cap \circ \mathord{\times} = 0$.

Proof. First, if $x < y$, then $(\cap \circ \,\chi)(v_{xy}) = \cap(qv_{xy} - v_{yx}) = 0$ while, if $x > y$, then $(\cap \circ \,\chi)(v_x \otimes v_y) = \cap(q^{-1}v_{xy} - v_{yx}) = 0$. Next, $(\chi \circ \,\cup)(1) = \chi(\sum_{i=1}^n v_{ii}) = 0$. ∎

Lemma 6.6 (Merge-split sliding relations). *We have*

$$(\cap \otimes \cap)(\mathrm{id} \otimes \times \otimes \mathrm{id})(\chi \otimes \mathrm{id} \otimes \mathrm{id}) = (\cap \otimes \cap)(\mathrm{id} \otimes \times \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes \chi),$$

$$(\chi \otimes \mathrm{id} \otimes \mathrm{id})(\mathrm{id} \otimes \times \otimes \mathrm{id})(\cup \otimes \cup) = (\mathrm{id} \otimes \mathrm{id} \otimes \chi)(\mathrm{id} \otimes \times \otimes \mathrm{id})(\cup \otimes \cup).$$

Proof. First, we compute

$$(\cap \otimes \cap)(\mathrm{id} \otimes \times \otimes \mathrm{id})(v_{wxyz}) = \begin{cases} -q^{2(n+1-2w)-1}, & \text{if } w = x = y = z, \\ q^{2(n+1)-w-y}(q - q^{-1}), & \text{if } w = x < y = z, \\ -q^{2(n+1)-w-x}, & \text{if } w = y \neq x = z, \\ 0, & \text{else.} \end{cases}$$

Now, it is easy to see that both $(\cap \otimes \cap)(\mathrm{id} \otimes \times \otimes \mathrm{id})(\chi \otimes \mathrm{id} \otimes \mathrm{id})(v_{wxyz})$ and $(\cap \otimes \cap)(\mathrm{id} \otimes \times \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes \chi)(v_{wxyz})$ can only be non-zero if $w = y$ and $x = z$, and that they are equal in this case. This shows the first equation.

For the second equation, we compute

$$(6.4) \qquad \begin{aligned} (\mathrm{id} \otimes \times \otimes \mathrm{id})(\cup \otimes \cup)(1) &= (\mathrm{id} \otimes \times \otimes \mathrm{id})(\textstyle\sum_{i,j=1}^n v_{iijj}) \\ &= -\textstyle\sum_{i \neq j} v_{ijij} + (q - q^{-1})\textstyle\sum_{i<j} v_{iijj} - q^{-1}\textstyle\sum_{i=j} v_{iiii}. \end{aligned}$$

Next, applying both $\chi \otimes \mathrm{id} \otimes \mathrm{id}$ or $\mathrm{id} \otimes \mathrm{id} \otimes \chi$ to (6.4) yields

$$\textstyle\sum_{i<j}\left(v_{jiij} - qv_{ijij}\right) + \textstyle\sum_{i>j}\left(v_{jiij} - q^{-1}v_{ijij}\right). \qquad ∎$$

The proof that (6.3) is well-defined works very similar. Namely, it follows basically by the above, by comparison of the topological version of the relations in $\boldsymbol{\mathcal{W}}\mathrm{eb}_{q,q^n}^{\cup}$ and $\boldsymbol{\mathcal{W}}\mathrm{eb}_{q,-q^{-n}}^{\bullet}$, and by comparison of (5.16) and (5.17). We omit the details for brevity. Hence, we get:

Proposition 6.7. *The two functors* $\Gamma_{\mathbf{BD}}^{\mathrm{ext}}$ *and* $\Gamma_{\mathbf{BD}}^{\mathrm{sym}}$ *are well-defined. Moreover, we have commuting diagrams*

$$\begin{array}{ccc}
\boldsymbol{\mathcal{W}}\mathrm{eb}_q^{\wedge} & \xrightarrow{\Gamma_{\mathbf{A}}^{\mathrm{ext}}} & \boldsymbol{\mathcal{R}}\mathrm{ep}_q(\mathfrak{gl}_n) \\
\circlearrowleft & & \circlearrowleft \\
\boldsymbol{\mathcal{W}}\mathrm{eb}_{q,q^n}^{\cup} & \xrightarrow[\Gamma_{\mathbf{BD}}^{\mathrm{ext}}]{} & \boldsymbol{\mathcal{R}}\mathrm{ep}_q'(\mathfrak{so}_n)
\end{array} \quad \text{and} \quad \begin{array}{ccc}
\boldsymbol{\mathcal{W}}\mathrm{eb}_q^{\wedge} & \xrightarrow{\Gamma_{\mathbf{A}}^{\mathrm{sym}}} & \boldsymbol{\mathcal{R}}\mathrm{ep}_q(\mathfrak{gl}_n) \\
\circlearrowleft & & \circlearrowleft \\
\boldsymbol{\mathcal{W}}\mathrm{eb}_{q,-q^{-n}}^{\bullet} & \xrightarrow[\Gamma_{\mathbf{BD}}^{\mathrm{sym}}]{} & \boldsymbol{\mathcal{R}}\mathrm{ep}_q'(\mathfrak{so}_n).
\end{array} \quad ∎$$

The presentation functors for $\mathbf{U}_q'(\mathfrak{sp}_n)$. Again, we specialize $\mathsf{z} = q^n \in \mathbb{C}(q)$ in the exterior and $\mathsf{z} = -q^{-n} \in \mathbb{C}(q)$ in the symmetric case.

We define $\Gamma_{\mathbf{C}}^{\mathrm{ext}} \colon \boldsymbol{\mathcal{W}}\mathrm{eb}_{q,q^n}^{\bullet} \to \boldsymbol{\mathcal{R}}\mathrm{ep}_q'(\mathfrak{sp}_n)$ on generators by the assignment

$$(6.5) \qquad \overset{\bullet}{\underset{2}{\vert}} \mapsto \left(\uparrow \colon \textstyle\bigwedge_q^2 \mathrm{V}_q \to \mathbb{C}_q\right) \quad \text{and} \quad \overset{2}{\underset{\bullet}{\vert}} \mapsto \left(\downarrow \colon \mathbb{C}_q \to \textstyle\bigwedge_q^2 \mathrm{V}_q\right),$$

and, as before, to be $\Gamma_{\mathbf{A}}^{\mathrm{ext}}$ on $\mathbf{A}$-web generators. Analogously, we define its symmetric counterpart $\Gamma_{\mathbf{C}}^{\mathrm{sym}} \colon \boldsymbol{\mathcal{W}}\mathbf{eb}_{q,-q^{-n}}^{\cup} \to \boldsymbol{\mathcal{R}}\mathbf{ep}_q'(\mathfrak{sp}_n)$ on generators via

$$(6.6) \qquad \overset{1\ \ 1}{\cup} \mapsto \big(\cup \colon \mathrm{V}_q \otimes \mathrm{V}_q \to \mathbb{C}_q\big) \quad \text{and} \quad \underset{1\ \ 1}{\cap} \mapsto \big(\cap \colon \mathbb{C}_q \to \mathrm{V}_q \otimes \mathrm{V}_q\big),$$

and, as before, to be $\Gamma_{\mathbf{A}}^{\mathrm{sym}}$ on $\mathbf{A}$-web generators.

Again, in order to prove that (6.5) is well-defined, we need to show that the defining relations of $\boldsymbol{\mathcal{W}}\mathbf{eb}_{q,q^n}^{\bullet}$ are satisfied in the image. This boils down to prove the following lemmas, which can be verified, similarly as in type $\mathbf{BD}$, via involved and lengthy computations. In order to keep the number of (boring) computations in this paper in reasonable boundaries, we omit their proofs.

Lemma 6.8 (Barbell removal). *We have* $\uparrow \circ \downarrow = [\frac{n}{2}]_2 \mathrm{id}_0$. $\blacksquare$

Lemma 6.9 (Thin K removal). *We have*

$$(\mathrm{id} \otimes \uparrow) \circ \ \chi\ \circ (\mathrm{id} \otimes \downarrow) = [\tfrac{n}{2} - 1]_2 \mathrm{id}. \qquad\qquad \blacksquare$$

Lemma 6.10 (Thick K opening). *We have*

$$(\mathrm{id}_2 \otimes \uparrow) \circ \ \chi\ \circ (\mathrm{id}_2 \otimes \downarrow) = \ \overset{\bullet}{\underset{\bullet}{|}}\ + [\tfrac{n}{2} - 2]_2 \mathrm{id}_2. \qquad\quad \blacksquare$$

Lemma 6.11 (Merge-split sliding relations). *We have*

$$(\cap \otimes \cap)(\mathrm{id} \otimes \chi \otimes \mathrm{id})(\chi \otimes \mathrm{id} \otimes \mathrm{id}) = (\cap \otimes \cap)(\mathrm{id} \otimes \chi \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes \chi),$$

$$(\chi \otimes \mathrm{id} \otimes \mathrm{id})(\mathrm{id} \otimes \chi \otimes \mathrm{id})(\cup \otimes \cup) = (\mathrm{id} \otimes \mathrm{id} \otimes \chi)(\mathrm{id} \otimes \chi \otimes \mathrm{id})(\cup \otimes \cup). \qquad \blacksquare$$

Again, the proof that (6.6) is well-defined goes similarly, and we immediately obtain:

Proposition 6.12. *The two functors* $\Gamma_{\mathbf{C}}^{\mathrm{ext}}$ *and* $\Gamma_{\mathbf{C}}^{\mathrm{sym}}$ *are well-defined. Moreover, we have commuting diagrams*

$$
\begin{array}{ccc}
\boldsymbol{\mathcal{W}}\mathbf{eb}_q^{\curlywedge} & \xrightarrow{\ \Gamma_{\mathbf{A}}^{\mathrm{ext}}\ } & \boldsymbol{\mathcal{R}}\mathbf{ep}_q(\mathfrak{gl}_n) \\
\circlearrowright & & \circlearrowright \\
\boldsymbol{\mathcal{W}}\mathbf{eb}_{q,q^n}^{\bullet} & \xrightarrow[\ \Gamma_{\mathbf{C}}^{\mathrm{ext}}\ ]{} & \boldsymbol{\mathcal{R}}\mathbf{ep}_q'(\mathfrak{sp}_n).
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
\boldsymbol{\mathcal{W}}\mathbf{eb}_q^{\curlywedge} & \xrightarrow{\ \Gamma_{\mathbf{A}}^{\mathrm{sym}}\ } & \boldsymbol{\mathcal{R}}\mathbf{ep}_q(\mathfrak{gl}_n) \\
\circlearrowright & & \circlearrowright \\
\boldsymbol{\mathcal{W}}\mathbf{eb}_{q,-q^{-n}}^{\cup} & \xrightarrow[\ \Gamma_{\mathbf{C}}^{\mathrm{sym}}\ ]{} & \boldsymbol{\mathcal{R}}\mathbf{ep}_q'(\mathfrak{sp}_n).
\end{array}
\qquad \blacksquare
$$

The ladderfication functor in types BCD. We define now the *ladderfication functors* $\Upsilon_{\mathfrak{so}}$ and $\Upsilon_{\mathfrak{sp}}$, which relate our web categories to the quantum groups $\mathbf{U}_q(\mathfrak{so}_{2k})$ and $\mathbf{U}_q(\mathfrak{sp}_{2k})$. We stress that the definition of the ladderfication functors does not depend whether we are in the exterior or the symmetric case.

The ladderfication functor for $\cup$-webs. We write $\overline{\lambda} = \lambda + \frac{n}{2}$ and we define the *ladderfication functor* $\Upsilon_{\mathfrak{so}} \colon \dot{\mathbf{U}}_q(\mathfrak{so}_{2k}) \to \boldsymbol{\mathcal{W}}\mathbf{eb}_{q,q^n}^{\cup}$ via

$$1_\lambda \longmapsto (\overline{\lambda}_1 = \lambda_1 + \tfrac{n}{2}, \ldots, \overline{\lambda}_k = \lambda_k + \tfrac{n}{2}),$$

Here, we silently assume that λ, as an object of $\boldsymbol{\mathcal{W}}\mathbf{eb}_{q,q^n}^{\cup}$, is the zero object if $\lambda \notin \mathbb{Z}_{\geq 0}^k$.

Lemma 6.13. *The ladderfication functor $\Upsilon_{\mathfrak{so}}$ is well-defined.*

Proof. We need to check that the relations of $\dot{\mathbf{U}}_q(\mathfrak{so}_{2k})$ are satisfied in the image.

Assignment of the generators: Recall that

$$\mathsf{E}_i 1_\lambda \in \mathrm{Hom}_{\dot{\mathbf{U}}_q(\mathfrak{so}_{2k})}(1_\lambda, 1_{\lambda+\alpha_i}) \quad \text{and} \quad \mathsf{F}_i 1_\lambda \in \mathrm{Hom}_{\dot{\mathbf{U}}_q(\mathfrak{so}_{2k})}(1_\lambda, 1_{\lambda-\alpha_i}),$$

where α_i are the simple roots. By our conventions for types **A** and **D** (cf. the appendix), we see that (6.7) lands in the correct morphisms spaces.

The $\dot{\mathbf{U}}_q(\mathfrak{gl}_{2k})$ relations: The relations involving only E_i's and F_i's with $i \neq k-1$ are clearly satisfied by the web calculus in type **A**, i.e. by [CKM14, Proposition 5.2.1].

The $\dot{\mathbf{U}}_q(\mathfrak{so}_{2k})$ relations: We just have to check case by case that the defining relations of $\dot{\mathbf{U}}_q(\mathfrak{so}_{2k})$ which involve E_k's and F_k's hold in the web calculus (for this purpose, recall the anti-involution ω from Remark 3.3):

▶ The commutator relation (5.2) between E_k and F_k holds in $\boldsymbol{\mathcal{W}}\mathbf{eb}_{q,q^n}^{\cup}$ by Lemma 3.9.
▶ The images of F_{k-1} and E_k commute thanks to Lemma 3.10. Applying ω shows that the images of E_{k-1} and F_k commute as well.
▶ The Serre relation (5.7) holds in $\boldsymbol{\mathcal{W}}\mathbf{eb}_{q,q^n}^{\cup}$ by Lemma 3.12. The F version of it holds by applying ω.
▶ The Serre relation (5.8) holds in $\boldsymbol{\mathcal{W}}\mathbf{eb}_{q,q^n}^{\cup}$ by Lemma 3.13. The versions involving F's hold by applying ω.
▶ The Serre relation (5.9) holds in $\boldsymbol{\mathcal{W}}\mathbf{eb}_{q,q^n}^{\cup}$ by Lemma 3.14. Again, the versions involving F's hold by applying ω.

Note here that the quantum numbers work out thanks to the shift by $\frac{n}{2}$ in (6.7). All other relations, e.g. far-commutativity, are clearly satisfied. ∎

The ladderfication functor for $\downarrow$-webs. Using the same notation as above, we define the *ladderfication functor* $\Upsilon_{\mathfrak{sp}} \colon \dot{\mathbf{U}}_q(\mathfrak{sp}_{2k}) \to \boldsymbol{\mathcal{W}}\mathbf{eb}_{q,q^n}^{\downarrow}$ via

$$1_\lambda \longmapsto (\overline{\lambda}_1 = \lambda_1 + \tfrac{n}{2}, \ldots, \overline{\lambda}_k = \lambda_k + \tfrac{n}{2}),$$

(6.8)

$$\mathsf{E}_i 1_\lambda \longmapsto \begin{array}{c} \overline{\lambda}_1 \cdots \overline{\lambda}_{i+1}\ \overline{\lambda}_{i+1}-1\cdots\ \overline{\lambda}_k \\ \Big| \ \ldots\ \Big\diagdown\ \Big|\ \ldots\ \Big| \\ \overline{\lambda}_1 \cdots\ \overline{\lambda}_i\ \ \overline{\lambda}_{i+1}\ \cdots\ \overline{\lambda}_k \end{array}, \quad \text{for all } i = 1, \ldots, k-1,$$

$$\mathsf{F}_i 1_\lambda \longmapsto \begin{array}{c} \overline{\lambda}_1 \cdots \overline{\lambda}_{i-1}\ \overline{\lambda}_{i+1}+1\cdots\ \overline{\lambda}_k \\ \Big| \ \ldots\ \Big\diagup\ \Big|\ \ldots\ \Big| \\ \overline{\lambda}_1 \cdots\ \overline{\lambda}_i\ \ \overline{\lambda}_{i+1}\ \cdots\ \overline{\lambda}_k \end{array}, \quad \text{for all } i = 1, \ldots, k-1,$$

$$\mathsf{E}_k 1_\lambda \longmapsto \begin{array}{c} \overline{\lambda}_1 \cdots\ \overline{\lambda}_{k-2}\ \overline{\lambda}_{k-1}\ \overline{\lambda}_k+2 \\ \Big| \ \ldots\ \Big|\ \ \ \Big|\ \ \succ\!\!\bullet \\ \overline{\lambda}_1 \cdots\ \overline{\lambda}_{k-2}\ \overline{\lambda}_{k-1}\ \ \overline{\lambda}_k \end{array}$$

$$\mathsf{F}_k 1_\lambda \longmapsto \begin{array}{c} \overline{\lambda}_1 \cdots\ \overline{\lambda}_{k-2}\ \overline{\lambda}_{k-1}\ \overline{\lambda}_k-2 \\ \Big| \ \ldots\ \Big|\ \ \ \Big|\ \ \bullet\!\!\!\prec \\ \overline{\lambda}_1 \cdots\ \overline{\lambda}_{k-2}\ \overline{\lambda}_{k-1}\ \ \overline{\lambda}_k \end{array}$$

Again, we assume that λ, as an object of $\boldsymbol{\mathcal{W}}\mathbf{eb}_{q,q^n}^{\downarrow}$, is the zero object if $\lambda \notin \mathbb{Z}_{\geq 0}^k$.

Lemma 6.14. *The ladderfication functor $\Upsilon_{\mathfrak{sp}}$ is well-defined.*

Proof. The proof is, mutatis mutandis, as the proof of Lemma 6.13. In particular:

- ▶ The E_k-F_k commutator relation holds in $\boldsymbol{\mathcal{W}}\mathbf{eb}_{q,q^n}^{\downarrow}$ by Lemma 4.8.
- ▶ The images of F_{k-1} and E_k commute by Lemma 4.9. That the images of E_{k-1} and F_k commute follows by applying ω.
- ▶ The Serre relation (5.5) holds in $\boldsymbol{\mathcal{W}}\mathbf{eb}_{q,q^n}^{\downarrow}$ by Lemma 4.10. As before, the versions involving F's follow then applying ω.
- ▶ The Serre relation (5.6) holds in $\boldsymbol{\mathcal{W}}\mathbf{eb}_{q,q^n}^{\downarrow}$ by Lemma 4.13. As usual, the versions involving F's follow then applying ω. ∎

The Howe functors. Notice that we never used that z was specialized to q^n in the definition of the ladderfication functors, and we actually get ladderfication functors $\dot{\mathbf{U}}_q(\mathfrak{so}_{2k}) \to \boldsymbol{\mathcal{W}}\mathbf{eb}_{q,z}^{\vee}$ and $\dot{\mathbf{U}}_q(\mathfrak{sp}_{2k}) \to \boldsymbol{\mathcal{W}}\mathbf{eb}_{q,z}^{\downarrow}$ for any $z \in \mathbb{C}(q)$. In particular, we also get ladderfication functors $\dot{\mathbf{U}}_q(\mathfrak{so}_{2k}) \to \boldsymbol{\mathcal{W}}\mathbf{eb}_{q,-q^{-n}}^{\vee}$ and $\dot{\mathbf{U}}_q(\mathfrak{sp}_{2k}) \to \boldsymbol{\mathcal{W}}\mathbf{eb}_{q,-q^{-n}}^{\downarrow}$, which, by slight abuse of notation, we still denote by $\Upsilon_{\mathfrak{so}}$ and $\Upsilon_{\mathfrak{sp}}$.

Composition the presentation and the ladderfication functors, we finally obtain the *Howe functors*:

$$
(6.9)
\begin{aligned}
\Phi_{\mathbf{BD}}^{\mathrm{ext}} &: \dot{\mathbf{U}}_q(\mathfrak{so}_{2k}) \xrightarrow{\Upsilon_{\mathfrak{so}}} \mathcal{W}\mathrm{eb}_{q,q^n}^{\vee} \xrightarrow{\Gamma_{\mathbf{BD}}^{\mathrm{ext}}} \mathcal{R}\mathrm{ep}_q'(\mathfrak{so}_n), \\
\Phi_{\mathbf{C}}^{\mathrm{sym}} &: \dot{\mathbf{U}}_q(\mathfrak{so}_{2k}) \xrightarrow{\Upsilon_{\mathfrak{so}}} \mathcal{W}\mathrm{eb}_{q,-q^{-n}}^{\vee} \xrightarrow{\Gamma_{\mathbf{C}}^{\mathrm{sym}}} \mathcal{R}\mathrm{ep}_q'(\mathfrak{sp}_n), \\
\Phi_{\mathbf{BD}}^{\mathrm{sym}} &: \dot{\mathbf{U}}_q(\mathfrak{sp}_{2k}) \xrightarrow{\Upsilon_{\mathfrak{sp}}} \mathcal{W}\mathrm{eb}_{q,-q^{-n}}^{\blacklozenge} \xrightarrow{\Gamma_{\mathbf{BD}}^{\mathrm{sym}}} \mathcal{R}\mathrm{ep}_q'(\mathfrak{so}_n), \\
\Phi_{\mathbf{C}}^{\mathrm{ext}} &: \dot{\mathbf{U}}_q(\mathfrak{sp}_{2k}) \xrightarrow{\Upsilon_{\mathfrak{sp}}} \mathcal{W}\mathrm{eb}_{q,q^n}^{\blacklozenge} \xrightarrow{\Gamma_{\mathbf{C}}^{\mathrm{ext}}} \mathcal{R}\mathrm{ep}_q'(\mathfrak{sp}_n).
\end{aligned}
$$

7. Main results

We are finally ready to state and prove our main results.

Quantizing Howe dualities in types BCD.

A brief reminder on (quantum) highest weight theory. The finite-dimensional representation theory of $\mathbf{U}_q(\mathfrak{g})$ at generic q is fairly well-understood. In particular, all such representations are semisimple, and, if we restrict to so-called *type 1 representations* (where $\mathbf{q}^h$ acts by powers of q, cf. [Jan96, Section 5.2]), then the simple modules are in bijection with dominant integral weights $\lambda \in X^+$. We denote by $\mathrm{L}_q(\mathfrak{g}, \lambda)$ the corresponding simple $\mathbf{U}_q(\mathfrak{g})$-module.

The situation for the coideal subalgebras, on the contrary, is more difficult and less understood. For $\mathbf{U}_q'(\mathfrak{so}_n)$ and $\mathbf{U}_q'(\mathfrak{sp}_n)$, in particular, one cannot consider weights and weight representations, since there is no natural analog of a Cartan subalgebra. Still, we will encounter some of their representations through Howe duality.

Before we can start, we need some more notation. Let $\mathfrak{P}$ be the set of partitions (or Young diagrams). Given a partition $\lambda = (\lambda_1, \ldots, \lambda_s) \in \mathfrak{P}$ (with $\lambda_s \neq 0$), we write $\ell(\lambda) = s$ for its length, and we denote by $\lambda^{\mathrm{T}} = (\lambda_1^{\mathrm{T}}, \ldots, \lambda_t^{\mathrm{T}}) \in \mathfrak{P}$ its transpose. For the rest, we keep the notation from Section 6.

We start with the sympletic case since it is easier to state (cf. Remark 1.2).

Skew quantum Howe duality for the pair $(\mathbf{U}_q'(\mathfrak{sp}_n), \mathbf{U}_q(\mathfrak{sp}_{2k}))$.

Theorem 7.1. *There are commuting actions*

$$
(7.1) \qquad \mathbf{U}_q'(\mathfrak{sp}_n) \;\circlearrowright\; \underbrace{\textstyle\bigwedge_q^{\bullet} \mathrm{V}_q \otimes \cdots \otimes \bigwedge_q^{\bullet} \mathrm{V}_q}_{k \ times} \;\circlearrowleft\; \mathbf{U}_q(\mathfrak{sp}_{2k})
$$

generating each other's centralizer. Hence, the $\mathbf{U}_q'(\mathfrak{sp}_n)$-$\mathbf{U}_q(\mathfrak{sp}_{2k})$-bimodule (7.1) is multiplicity-free. The $\mathbf{U}_q(\mathfrak{sp}_{2k})$-modules appearing in its decomposition are

$$
(7.2) \qquad \mathrm{L}_q(\mathfrak{sp}_{2k}, \textstyle\sum_{j=1}^{k}(\lambda_j^{\mathrm{T}} - \frac{n}{2})\varepsilon_j), \quad \text{for } \lambda \in \mathfrak{P} \text{ with } \ell(\lambda^{\mathrm{T}}) \leq k, \ell(\lambda) \leq \tfrac{n}{2}.
$$

Proof. We denote the space in (7.1) by M_q. All λ's appearing below will always satisfy the conditions in (7.2).

By construction, M_q has an action of $\mathbf{U}_q'(\mathfrak{sp}_n)$ via restricting the action of $\mathbf{U}_q(\mathfrak{gl}_n)$. Using $\Phi_{\mathbf{C}}^{\mathrm{ext}}$ from (6.9), we see that there is a commuting action of $\mathbf{U}_q(\mathfrak{sp}_{2k})$. (In fact, we get an action of $\dot{\mathbf{U}}_q(\mathfrak{sp}_{2k})$ which then gives an action of $\mathbf{U}_q(\mathfrak{sp}_{2k})$ since M_q is finite-dimensional, cf. [Lus10, Section 23.1.4].)

Next, we want to use the analog result in the non-quantized setting (see [How95] and [CW12, Corollary 5.33]), but beware that the roles of k and n are swapped

in [CW12]). It states that there is an action of $\mathbf{U}(\mathfrak{sp}_{2k})$ on $\mathrm{M} = \bigwedge^\bullet(\mathbb{C}^n \otimes \mathbb{C}^k)$ commuting with the natural action of $\mathbf{U}(\mathfrak{sp}_n)$ and that these two actions generate each others centralizer. Moreover, [CW12, Corollary 5.33] gives the bimodule decomposition of M, similar to (7.2).

Now, we can easily compare the action of $\mathbf{U}_q(\mathfrak{sp}_{2k})$ on M_q and the action of $\mathbf{U}(\mathfrak{sp}_{2k})$ on M, and see that the weights and their multiplicities are the same. Hence, we can deduce that the decomposition of M_q as a $\mathbf{U}_q(\mathfrak{sp}_{2k})$-module is the quantum analog of the one in [CW12, Corollary 5.33]. It follows that the $\mathbf{U}'_q(\mathfrak{sp}_n)$-$\mathbf{U}_q(\mathfrak{sp}_{2k})$-bimodule M_q decomposes as

$$\mathrm{M}_q \cong \bigoplus_\lambda \mathrm{L}'_q(\mathfrak{sp}_n, \lambda) \otimes \mathrm{L}_q(\mathfrak{sp}_{2k}, \textstyle\sum_{j=1}^k (\lambda_j^{\mathrm{T}} - \frac{n}{2})\varepsilon_j),$$

with λ as in (7.2) and where the $\mathrm{L}'_q(\mathfrak{sp}_n, \lambda)$'s denote just some $\mathbf{U}'_q(\mathfrak{sp}_n)$-modules (which are indexed by the λ's).

We want to show that all appearing $\mathrm{L}'_q(\mathfrak{sp}_n, \lambda)$ are irreducible, or, equivalently, that the action gives a surjection

$$(7.3) \qquad \mathbf{U}'_q(\mathfrak{sp}_n) \twoheadrightarrow \mathrm{End}_{\mathbf{U}_q(\mathfrak{sp}_{2k})}(\mathrm{M}_q) \cong \mathrm{End}_{\mathbb{C}(q)}\left(\bigoplus_\lambda \mathrm{L}'_q(\mathfrak{sp}_n, \lambda)\right).$$

To this end, consider the integral version $^{\mathscr{A}}\mathrm{M}_q$ of the representation M_q, defined as the $\mathscr{A}$-span of tensor products of wedges of the standard basis vectors v_i inside M_q. Note that $^{\mathscr{A}}\mathrm{M}_q$ is a free $\mathscr{A}$-module, and this will be important for what follows.

It can be easily checked that $^{\mathscr{A}}\mathrm{M}_q$ is stable under the actions of $^{\mathscr{A}}\mathbf{U}'_q(\mathfrak{sp}_n)$ and $^{\mathscr{A}}\mathbf{U}_q(\mathfrak{sp}_{2k})$. Moreover, setting $q = 1$, we can identify $^{\mathscr{A}}\mathrm{M}_q \otimes_{\mathscr{A}} \mathscr{A}/(q-1)$ with M, and it is then clear that the action of $^{\mathscr{A}}\mathbf{U}'_q(\mathfrak{sp}_n) \otimes_{\mathscr{A}} \mathscr{A}/(q-1)$ matches the natural action of $\mathbf{U}(\mathfrak{sp}_n)$, i.e.

$$
\begin{array}{ccc}
^{\mathscr{A}}\mathbf{U}'_q(\mathfrak{sp}_n) \otimes_{\mathscr{A}} \mathscr{A}/(q-1) & \longrightarrow & \mathrm{End}_{\mathscr{A}/(q-1)}(^{\mathscr{A}}\mathrm{M}_q \otimes_{\mathscr{A}} \mathscr{A}/(q-1)) \\
 & & \| \\
\mathbf{U}(\mathfrak{sp}_n) & \longrightarrow & \mathrm{End}_{\mathbb{C}}(\mathrm{M}).
\end{array}
$$

(One could actually show that $^{\mathscr{A}}\mathbf{U}'_q(\mathfrak{sp}_n) \otimes_{\mathscr{A}} \mathscr{A}/(q-1)$ and $\mathbf{U}(\mathfrak{sp}_n)$ are isomorphic, but since we do not need it, we prefer to avoid this additional complication.) In particular, the images of these two actions agree, and their dimensions are both equal to

$$\textstyle\sum_\lambda \dim_{\mathbb{C}} \mathrm{L}(\mathfrak{sp}_n, \lambda) = d = \sum_\lambda \dim_{\mathbb{C}(q)} \mathrm{L}'_q(\mathfrak{sp}_n, \lambda).$$

It follows that the dimension of the image for generic q cannot be strictly smaller, and in particular the dimension of the image of (7.3) has to be greater or equal than d. Hence, the map in (7.3) is surjective, and we are done. $\blacksquare$

Symmetric quantum Howe duality for the pair $(\mathbf{U}'_q(\mathfrak{sp}_n), \dot{\mathbf{U}}_q(\mathfrak{so}_{2k}))$.

Theorem 7.2. *There are commuting actions*

$$(7.4) \qquad \mathbf{U}'_q(\mathfrak{sp}_n) \;\circlearrowright\; \underbrace{\mathrm{Sym}_q^\bullet \mathrm{V}_q \otimes \cdots \otimes \mathrm{Sym}_q^\bullet \mathrm{V}_q}_{k\ times} \;\circlearrowleft\; \dot{\mathbf{U}}_q(\mathfrak{so}_{2k})$$

generating each other's centralizer. Hence, the $\mathbf{U}'_q(\mathfrak{sp}_n)$-$\dot{\mathbf{U}}_q(\mathfrak{so}_{2k})$-*bimodule* (7.4) *is multiplicity-free. The* $\dot{\mathbf{U}}_q(\mathfrak{so}_{2k})$-*modules appearing in its decomposition are*

$$(7.5) \qquad \mathrm{L}_q(\mathfrak{so}_{2k}, \textstyle\sum_{j=1}^k (\lambda_j + \frac{n}{2})\varepsilon_j), \quad for\ \lambda \in \mathfrak{P}\ with\ \ell(\lambda) \leq \min\{\tfrac{n}{2}, k\}.$$

Proof. The proof is similar to the proof of Theorem 7.3, but using the functor $\Phi_{\mathbf{C}}^{\mathrm{sym}}$ and the non-quantized Howe duality from [CW12, Corollary 5.32]. (Note hereby that we cannot easily pass from $\dot{\mathbf{U}}_q(\mathfrak{so}_{2k})$ to $\mathbf{U}_q(\mathfrak{so}_{2k})$ since the $\mathbb{C}(q)$-vector space in (7.4) is infinite-dimensional.) ∎

Skew quantum Howe duality for the pair $(\mathbf{U}'_q(\mathfrak{so}_n), \mathbf{U}_q(\mathfrak{so}_{2k}))$.

Theorem 7.3. *There are commuting actions*

$$(7.6) \qquad \mathbf{U}'_q(\mathfrak{so}_n) \; \circlearrowright \; \underbrace{\textstyle\bigwedge_q^\bullet V_q \otimes \cdots \otimes \bigwedge_q^\bullet V_q}_{k \text{ times}} \; \circlearrowleft \; \mathbf{U}_q(\mathfrak{so}_{2k}).$$

In case n is odd they generate each other's centralizer. In any case, the $\mathbf{U}_q(\mathfrak{so}_{2k})$-modules appearing in the decomposition of (7.6) *are*

$$(7.7) \qquad \mathrm{L}_q(\mathfrak{so}_{2k}, \textstyle\sum_{j=1}^k (\lambda_j^{\mathrm{T}} - \tfrac{n}{2})\varepsilon_j), \quad \text{for } \lambda \in \mathfrak{P} \text{ with } \ell(\lambda^{\mathrm{T}}) \le k, \lambda_1^{\mathrm{T}} + \lambda_2^{\mathrm{T}} \le n.$$

Proof. Mutatis mutandis as in the proof of Theorem 7.1, but using the functor $\Phi_{\mathbf{BD}}^{\mathrm{ext}}$ and the non-quantized Howe duality from [CW12, Corollary 5.41]. Note that one has $\mathrm{O}(n) \cong \mathrm{SO}(n) \times \mathbb{Z}/2\mathbb{Z}$ in type **B**. As explained in [CW12, above Proposition 5.35] or [LZ06, § 5.1.3], the extra generator in $\mathrm{O}(n) - \mathrm{SO}(n)$ acts trivially on the de-quantized analog of (7.6). It follows that [CW12, Corollary 5.41] works in this case for $\mathrm{SO}(n)$ instead of $\mathrm{O}(n)$, and hence also for $\mathfrak{so}_n$, cf. also Remark 1.2. ∎

Symmetric quantum Howe duality for the pair $(\mathbf{U}'_q(\mathfrak{so}_n), \dot{\mathbf{U}}_q(\mathfrak{sp}_{2k}))$.

Theorem 7.4. *There are commuting actions*

$$(7.8) \qquad \mathbf{U}'_q(\mathfrak{so}_n) \; \circlearrowright \; \underbrace{\mathrm{Sym}_q^\bullet V_q \otimes \cdots \otimes \mathrm{Sym}_q^\bullet V_q}_{k \text{ times}} \; \circlearrowleft \; \dot{\mathbf{U}}_q(\mathfrak{sp}_{2k}).$$

In case n is odd they generate each other's centralizer. In any case, the $\dot{\mathbf{U}}_q(\mathfrak{sp}_{2k})$-modules appearing in the decomposition of (7.8) *are*

$$(7.9) \qquad \mathrm{L}_q(\mathfrak{sp}_{2k}, \textstyle\sum_{j=1}^k (\lambda_j + \tfrac{n}{2})\varepsilon_j), \quad \text{for } \lambda \in \mathfrak{P} \text{ with } \ell(\lambda) \le k, \lambda_1^{\mathrm{T}} + \lambda_2^{\mathrm{T}} \le n.$$

Proof. Mutatis mutandis as in the proof of Theorem 7.1, but using the functor $\Phi_{\mathbf{BD}}^{\mathrm{sym}}$ and the non-quantized Howe duality from [CW12, Corollary 5.40]. (Keeping the same remarks as in the proofs of Theorems 7.2 and 7.3 in mind.) ∎

Some concluding remarks.

REMARK 7.5. We stress again that Theorems 7.3 and 7.4 can be strengthened to include the double centralizer property for type **D** as well, cf. Remark 1.2.

REMARK 7.6. In the spirit of [TVW15], one could use the Howe dualities involving the orthosymplectic Lie superalgebra $\mathfrak{osp}$, as in [How89], [CZ04] or [CW12], to give a unified treatment of the exterior and the symmetric story. Since quantization in our setup is already quite involved, we decided to not pursue this further.

REMARK 7.7. One feature of web categories is that they are "amenable for categorification". For example, one can use *foams* in the sense of [Kho04], see e.g. [Bla10], [LQR15], [EST15] and [EST16] for categorifying webs. Or category $\mathcal{O}$ as e.g. in [Sar16a] or [Sar16b]. Categorifications of Howe dualities involving coideal subalgebras (of different kinds) have already been obtained in [ES13] (which also connects to foams, cf. [ETW16]), and there are good reasons to hope that our story categorifies as well.

Relation of the web categories to the (quantum) Brauer algebra. In ground-breaking work, Brauer [Bra37] has introduced the so-called *Brauer algebra*, which arises naturally when studying the centralizer of the action of the orthogonal group $O(n)$ and of the symplectic group $Sp(n)$ acting on the k-fold tensor product $V^{\otimes k}$ of their vector representations. Comparing this to the de-quantized versions of Theorems 7.1, 7.2, 7.3 and 7.4 suggests that there should be a connection to our web categories. We make this more precise in the following.

Various quantizations of the Brauer algebra. The first quantization of the Brauer algebra, called *BMW algebra*, was introduced by Birman-Wenzl [BW89] and Murakami [Mur87]. The BMW algebra plays the role of Brauer's algebra with respect to the actions of $\mathbf{U}_q(\mathfrak{so}_n)$ and $\mathbf{U}_q(\mathfrak{sp}_n)$ on their quantum tensor spaces. However, since we are looking at the centralizers of actions of $\mathbf{U}'_q(\mathfrak{so}_n)$ and $\mathbf{U}'_q(\mathfrak{sp}_n)$, and not of $\mathbf{U}_q(\mathfrak{so}_n)$ and $\mathbf{U}_q(\mathfrak{sp}_n)$, the BMW algebra does not fit into our picture.

In contrast, Molev [Mol03] defined a new quantization $\mathrm{Br}^k_{q,z}$ of the Brauer algebra, called *quantum or q-Brauer algebra*. This $\mathbb{C}(q)[z^{\pm 1}]$-algebra is related by a version of q-Schur-Weyl duality to $\mathbf{U}'_q(\mathfrak{so}_n)$ and $\mathbf{U}'_q(\mathfrak{sp}_n)$. Thus, $\mathrm{Br}^k_{q,z}$ is the natural candidate to be connected to our web categories.

A quantized Brauer category. First, let us quickly recall the situation in type **A**:

Definition 7.8. The *Hecke category* $\boldsymbol{\mathcal{H}}_q$ is the additive closure of the (strict) monoidal, $\mathbb{C}(q)$-linear category generated by one object 1 and by one morphism $T\colon 1 \otimes 1 \to 1 \otimes 1$ modulo the relations

$$T^2 = (q - q^{-1})T + \mathrm{id}_{1 \otimes 1},$$
$$(T \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes T)(T \otimes \mathrm{id}_1) = (\mathrm{id}_1 \otimes T)(T \otimes \mathrm{id}_1)(\mathrm{id}_1 \otimes T).$$

(The second relation is known as the *braid relation*.)

We depict the generator T by an overcrossing, cf. (2.4). Then, by sending T in the evident way to the braiding of $\boldsymbol{\mathcal{W}}\mathrm{eb}_q^{\boldsymbol{\lambda}}$, we get a functor

$$\beta_{\mathbf{A}}\colon \boldsymbol{\mathcal{H}}_q \to \boldsymbol{\mathcal{W}}\mathrm{eb}_q^{\boldsymbol{\lambda}}, \quad T \mapsto \quad \raisebox{-1.2em}{\rule{0pt}{3em}}$$

which is fully faithful, see e.g. [QS15, Proposition 5.9] or [TVW15, Lemma 2.25]. Note, in particular, that crossings span $\mathrm{End}_{\boldsymbol{\mathcal{W}}\mathrm{eb}_q^{\boldsymbol{\lambda}}}(1^{\otimes k})$.

Our next goal is to extend this to types **BCD**.

Definition 7.9. The *quantum or q-Bauer category* $\boldsymbol{\mathcal{B}}\mathrm{r}_{q,z}$ is the additive closure of the $\mathbb{C}(q)[z^{\pm 1}]$-linear $\boldsymbol{\mathcal{H}}_q$-category generated by $\varnothing$ and by the cup and cap morphisms (depicted as in ($\cup$gen)) modulo the relations ($\cup$1), ($\cup$a), ($\cup$b), ($\cup$c) and ($\cup$d).

Recall that the relations ($\cup$a), ($\cup$b), ($\cup$c) and ($\cup$d) are the topological analogs of the relations in Definitions 3.2 and 4.1 (for $\downarrow$-webs with slightly different parameters), and that ($\cup$1) is equivalent to ($\downarrow$1) in case of $\downarrow$-webs. Hence, the functor $\beta_{\mathbf{A}}$ extends to two functors

$$\beta_{\cup}\colon \boldsymbol{\mathcal{B}}\mathrm{r}_{q,z} \to \boldsymbol{\mathcal{W}}\mathrm{eb}_{q,z}^{\cup} \quad \text{and} \quad \beta_{\downarrow}\colon \boldsymbol{\mathcal{B}}\mathrm{r}_{-q^{-1},z} \to \boldsymbol{\mathcal{W}}\mathrm{eb}_{q,z}^{\downarrow}.$$

Connection with the quantum Brauer algebra. Let us now denote by $\mathrm{Br}^k_{q,\mathbf{z}}$ the q-Brauer algebra as defined by Molev in [Mol03, Definition 2.3]. This is a $\mathbb{C}(q)[\mathbf{z}^{\pm 1}]$-algebra with generators T_i for $i = 1, \ldots, k-1$ and additionally e_{k-1}. (Note that Molev uses the notation σ_i instead of T_i.)

Lemma 7.10. *The assignment*

$$T_i \mapsto \; \cdots \; \quad and \quad e_{k-1} \mapsto \; \cdots$$

defines an algebra homomorphism $\psi_k \colon \mathrm{Br}^k_{-q^{-1},-\mathbf{z}^{-1}} \to \mathrm{End}_{\boldsymbol{\mathcal{B}r}_{q,z}}(1^{\otimes k})$.

Proof. This is immediate up to the last relation in [Mol03, Definition 2.3]. Verifying the last relation in [Mol03, Definition 2.3] is a lengthy, but straightforward computation, which can be done by using $(\cup\mathrm{b})$ and $(\cup\mathrm{d})$ repeatedly. ∎

In particular, the composition $\Gamma^{\mathrm{ext}}_{\mathbf{BD}} \circ \beta_\cup \circ \psi_k$ defines an action of the q-Brauer algebra which commutes which the natural action of $\mathbf{U}'_q(\mathfrak{so}_n)$:

$$(7.10) \qquad \mathbf{U}'_q(\mathfrak{so}_n) \; \circlearrowright \; \underbrace{V_q \otimes \cdots \otimes V_q}_{k \text{ times}} \; \circlearrowleft \; \mathrm{Br}^k_{-q,-q^{-n}}.$$

Up to scaling conventions, this is the action defined in [Mol03, Theorem 4.2]. Similarly, the composition $\Gamma^{\mathrm{ext}}_{\mathbf{C}} \circ \beta_\bullet \circ \psi_k$ provides commuting actions

$$(7.11) \qquad \mathbf{U}'_q(\mathfrak{sp}_n) \; \circlearrowright \; \underbrace{V_q \otimes \cdots \otimes V_q}_{k \text{ times}} \; \circlearrowleft \; \mathrm{Br}^k_{-q,q^n}.$$

(Clearly, we could have also chosen $\Gamma^{\mathrm{sym}}_{\mathbf{BD}}$ and $\Gamma^{\mathrm{sym}}_{\mathbf{C}}$ instead of $\Gamma^{\mathrm{ext}}_{\mathbf{BD}}$ and $\Gamma^{\mathrm{ext}}_{\mathbf{C}}$.)

We show now that Molev's q-Brauer algebra can be identified with the endomorphism algebra of $1^{\otimes k}$ in our q-Brauer category:

Proposition 7.11. *The map ψ_k is an algebra isomorphism, and the functors $\beta_\cup$ and $\beta_\bullet$ are fully faithful.*

Proof. **Surjectivity of ψ_k:** Because crossings span $\mathrm{End}_{\boldsymbol{\mathcal{W}}\mathbf{eb}_q^{\boldsymbol{\lambda}}}(1^{\otimes k})$, it is enough to show that $\mathrm{End}_{\boldsymbol{\mathcal{B}r}_{q,z}}(1^{\otimes k})$ is spanned by diagrams of the form $w_{\mathrm{top}} e^{(l)} w_{\mathrm{bot}}$, with $w_{\mathrm{bot}}, w_{\mathrm{top}} \in \mathrm{End}_{\boldsymbol{\mathcal{H}}_q}(1^{\otimes k})$ and

$$e^{(l)} = \; \cdots$$

This can be easily seen by induction on the number of crossings of some fixed diagram.

Injectivity of ψ_k: This follows because the representations in (7.10) and (7.11) are faithful for $n \gg k$ (the precise bound is irrelevant for us). Indeed, the proof that they are faithful for $n \gg k$ can be done, as in the proof of [Wen12, Theorem 3.8], by the same results in the non-quantized setting (see e.g. [AST15, Theorem 3.17], but the statement therein can implicitly already found in the work of Brauer [Bra37]).

Fully faithfulness of $\beta_\cup$ and $\beta_\bullet$: Very similar arguments as for the proof of bijectivity of ψ_k imply that the functors $\beta_\cup$ and $\beta_\bullet$ are fully faithful. ∎

REMARK 7.12. Because of Proposition 7.11, our web categories can be seen as (vast) generalizations of the (quantum) Brauer calculus.

APPENDIX: ROOT AND WEIGHT CONVENTIONS

Fix $m \in \mathbb{Z}_{\geq 1}$ (the *rank*). Let $\mathfrak{g}$ be either $\mathfrak{g} = \mathfrak{sp}_{2m}$ or $\mathfrak{g} = \mathfrak{so}_{2m}$, and we denote by Φ and Π the sets of *roots* and *simple roots*, which we choose accordingly to Table 1. Here $\{\varepsilon_i \mid i \in \mathrm{I}\}$ for $\mathrm{I} = \{1, \ldots, m\}$ denotes a chosen basis of the dual of the *Cartan* $\mathfrak{h}$, which is orthonormal with respect to the *Killing form* $(\cdot, \cdot)$. Correspondingly, we have a *weight lattice* X and *dominant integral weights* X^+. We let also $\{h_i \in \mathfrak{h} \mid i \in \mathrm{I}\}$ be the basis of $\mathfrak{h}$ determined by $\langle h_i, \lambda \rangle = 2\frac{(\alpha_i, \lambda)}{(\alpha_i, \alpha_i)}$ for $\lambda \in X$. Moreover, recall that the *Cartan matrix* $A = (a_{ij})_{i,j \in \mathrm{I}}$ is defined via $a_{ij} = \langle h_i, \alpha_j \rangle$. The sequence $(d_1, \ldots, d_m)$ is chosen with $d_i \in \mathbb{Z}_{\geq 0}$ for $i = 1, \ldots, m$ minimal such that the matrix $(d_i a_{ij})_{i,j \in \mathrm{I}}$ is symmetric and positive definite. (The Cartan datum can also be read off from the corresponding *Dynkin diagram D*.)

We do not need to fix a Cartan datum for type **B**, since in this paper we only encounter the type **B** Lie algebra $\mathfrak{so}_n$ (for n odd) in the coideal $\mathbf{U}'_q(\mathfrak{so}_n)$, and never in the quantum enveloping algebra $\mathbf{U}_q(\mathfrak{so}_n)$.

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	$\mathfrak{sp}_{2m}$	$\mathfrak{so}_{2m}$
D		
A	$\begin{pmatrix} 2 & -1 & 0 & \cdots & \big| & 0 \\ -1 & 2 & \ddots & \ddots & \big| & \vdots \\ 0 & \ddots & \ddots & -1 & \big| & 0 \\ \vdots & \ddots & -1 & 2 & \big| & -2 \\ 0 & \cdots & 0 & -1 & & 2 \end{pmatrix}$	$\begin{pmatrix} 2 & -1 & 0 & \cdots & \big| & 0 \\ -1 & 2 & \ddots & \ddots & \big| & \vdots \\ 0 & \ddots & \ddots & -1 & \big| & -1 \\ \vdots & \ddots & -1 & 2 & \big| & 0 \\ 0 & \cdots & -1 & 0 & & 2 \end{pmatrix}$
$\vec{d}$	$(1,\ldots,1,2)$	$(1,\ldots,1)$
Π	$\begin{aligned} \alpha_1 &= \varepsilon_1 - \varepsilon_2 \\ &\vdots \\ \alpha_{m-1} &= \varepsilon_{m-1} - \varepsilon_m \\ \alpha_m &= 2\varepsilon_m \end{aligned}$	$\begin{aligned} \alpha_1 &= \varepsilon_1 - \varepsilon_2 \\ &\vdots \\ \alpha_{k-1} &= \varepsilon_{m-1} - \varepsilon_m \\ \alpha_m &= \varepsilon_{m-1} + \varepsilon_m \end{aligned}$
Φ	$\{\pm 2\varepsilon_i, \pm\varepsilon_i \pm \varepsilon_j \mid i \neq j \in \mathrm{I}\}$	$\{\pm\varepsilon_i \pm \varepsilon_j \mid i \neq j \in \mathrm{I}\}$
X	$\mathbb{Z}^m$	$\mathbb{Z}^m \oplus \left(\frac{1}{2} + \mathbb{Z}\right)^m$
X^+	$\{(\lambda_1,\ldots,\lambda_m) \in X \mid$ $\lambda_1 \geq \cdots \geq \lambda_m \geq 0\}$	$\{(\lambda_1,\ldots,\lambda_m) \in X \mid$ $\lambda_1 \geq \cdots \geq \lambda_{m-1} \geq |\lambda_m|\}$

TABLE 1. Our conventions for types $\mathbf{C}_m$ and $\mathbf{D}_m$. Here we also specify the type $\mathbf{A}_{m-1}$ Cartan datum by considering the subgraphs of D and the submatrix of A as indicated (for both types).

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