Kim's Lemma for NTP_2 theories: a simpler proof of a result by Chernikov and Kaplan

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When Saharon Shelah first introduced the notion of forking, he did so via the more straightforward notion of dividing, and in the general context of an arbitrary complete first-order theory. At first, it was only in stable theories that forking and dividing were known to coincide and to be a very useful notion of independence. Shelah had looked at forking and dividing in the context of theories without the tree property in his first paper on such theories, for which he coined the term simple theories. But it was Byunghan Kim who much later proved that forking and dividing are almost as well-behaved in simple theories as in stable theories. Kim's key result, usually not named because it is now so fundamental, but sometimes known as Kim's Lemma, says that in a simple theory, a formula forks if and only if it divides, if and only if some or, equivalently, every appropriate Morley sequence witnesses that it divides.

It came as a surprise to most researchers in the field that this is not the end of the story. Artem Chernikov and Itay Kaplan, in an admirable tour de force, strengthened the notion of Morley sequence in a mysterious way due to Shelah, and proved that with this modification, theories without the tree property of the second kind satisfy Kim's Lemma. What makes this exciting is the fact that these theories form a large class including all simple theories but also all theories without the independence property. The latter are also known as NIP or dependent theories, and a theory is simple and dependent if and only if it is stable.

In a stable theory, these so-called strict Morley sequences coincide precisely with the ordinary Morley sequences. In a simple theory, the situation is a bit more complicated, so that the result of Chernikov and Kaplan is not a straightforward generalization of Kim's Lemma. In any case it almost looks as if some odd features of a further generalization of classical stability theory can now be discerned in the mist, of a generalization to a context in which independence is no longer a symmetric relation.

The present notes grew out of my attempt to understand the proof of the Chernikov-Kaplan version of Kim's Lemma. They were the basis of a lecture I gave in the *Mini-Course in Model Theory* in Torino in February 2011 as part of a tutorial coordinated with the one by Enrique Casanovas [2].

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Definition 1 $\varphi(x, y)$ has TP_2, the tree property of the second kind, if the following exists:

$$\begin{array}{cccc} \varphi(x,b_{00}) & \varphi(x,b_{01}) & \varphi(x,b_{02}) & \ldots \\ \varphi(x,b_{10}) & \varphi(x,b_{11}) & \varphi(x,b_{12}) & \ldots \\ \varphi(x,b_{20}) & \varphi(x,b_{21}) & \varphi(x,b_{22}) & \ldots \\ \vdots & \vdots & \vdots & \end{array}$$

where

- Each row is k-inconsistent for some k (always the same k).
- For every function $f\colon \omega \to \omega$, $\{\varphi(x, b_{if(i)}) \mid i < \omega\}$ is consistent.

Remark 2 *1.* TP_2 *implies the tree property: just use the same row repeatedly in every tree level.*

2. TP_2 *implies the independence property: observe that for every subset of formulas in the first column there is a tuple making precisely these true.*[1]

Definition 3 1. $a \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}_C B$ iff $\mathrm{tp}(a/BC)$ does not fork over C.

2. $a \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{i}}_C B$ iff $\mathrm{tp}(a/BC)$ has a global extension that is invariant over C (or, equivalently: that does not split over C).

Definition 4 C is an *invariance base* if for all A, B there is $A' \equiv_C A$ such that $A' \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{i}}_C B$.

Remark 5 *1. All models are invariance bases.*

2. In NIP *theories,* $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{i}} = \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}$.

Proof: *1.* A global coheir of $p(x) \in S(M)$ is M-invariant.

2. See Remark 39 in [1] or Remark 5.3 in [2]. $\square$

Definition 6 1. A global type $\mathfrak{p}(x)$ is *strictly invariant* over C if it is invariant over C and for all $B \supseteq C$, all $a \models \mathfrak{p} \restriction B$: $B \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}_C a$. Note that the first condition says $a \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{i}}_C B$.

2. A *strict Morley sequence* over C is a sequence that is generated by a global type $\mathfrak{p}(x)$ strictly invariant over C. Generated means: $a_0 \models \mathfrak{p} \restriction C$, $a_1 \models \mathfrak{p} \restriction Ca_0$, $a_2 \models \mathfrak{p} \restriction Ca_0a_1, \ldots$

Lemma 7 (NTP_2 **I)** *Assume* NTP_2*. If* $\varphi(x, b)$ *divides over* C *and* $\mathfrak{q}(y) \supseteq \mathrm{tp}(b/C)$ *is a strictly invariant global extension, then every sequence generated by* $\mathfrak{q}$ *over* C *(is a strict Morley sequence over* C *and) witnesses that* $\varphi(x, b)$ *divides over* C*.*

Proof: Choose a sequence $\bar{b}_0 = (b_{0i})_{i<\omega}$ indiscernible over C, which witnesses that $\varphi(x, b)$ divides over C and $b \models \mathfrak{q} \restriction C\bar{b}_0$. Since $\bar{b}_0 \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}_C b$, we may choose now a sequence $\bar{b}_1 = (b_{1i})_{i<\omega}$ such that $\bar{b}_0 \equiv_C \bar{b}_1$ and $\bar{b}_1$ is indiscernible over $C\bar{b}_0$. Moreover, we may choose $\bar{b}_1$ so that $b \models \mathfrak{q} \restriction N\bar{b}_0\bar{b}_1$.

[1] For $k > 2$ the observation need not be true, strictly speaking, but still gives the right idea. We can choose the b_{ij} in such a way that it becomes true, e.g. by ensuring as much indiscernibility as possible.

Continuing in this way, we get a sequence $\bar{b}_0, \bar{b}_1, \bar{b}_2, \ldots$ giving rise to a matrix

$$\begin{array}{cccc} \varphi(x,b_{00}) & \varphi(x,b_{01}) & \varphi(x,b_{02}) & \ldots \\ \varphi(x,b_{10}) & \varphi(x,b_{11}) & \varphi(x,b_{12}) & \ldots \\ \varphi(x,b_{20}) & \varphi(x,b_{21}) & \varphi(x,b_{22}) & \ldots \\ \vdots & \vdots & \vdots & \end{array}$$

with k-inconsistent rows (for some k).

Since $(b_{if(i)})_{i<\omega}$ is generated over C by $\mathfrak{q}$, $\{\varphi(x, b_{if(i)}) \mid i < \omega\}$ is consistent either for all $f\colon \omega \to \omega$ or for none. In the former case the matrix witnesses TP_2, a contradiction. In the latter case we are finished. $\square$

Lemma 8 (NTP_2 **II)** *Assume* NTP_2. *If $\varphi(x, b)$ divides over an invariance base M, then there is an $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{i}}$-Morley sequence over M which witnesses this.*

Proof: For big enough κ, let $\bar{b} = (b_i)_{i<\kappa}$ witness that $\varphi(x, b)$ divides over M. Choose $N \supseteq M$ $(|T| + |M|)^+$-saturated such that $\bar{b} \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{i}}_M N$.

Extract from $(b_i)_{i<\kappa}$ a sequence of indiscernibles over N and replace $\bar{b}$ by this new sequence, $\bar{b} = (b_i)_{i<\omega}$. Note $\bar{b} \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{i}}_M N$ still holds, by finite character of $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{i}}$. $\mathrm{tp}(\bar{b}/N)$ generates over M: $\bar{b}_0, \bar{b}_1, \bar{b}_2, \ldots$ For all n, $\bar{b}_n$ is indiscernible over $M\bar{b}_{<n} \subseteq N$ because this is true for $\bar{b}$. Since $\bar{b}_{>n} \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{i}}_M \bar{b}_{\leq n}$, by base monotonicity $\bar{b}_{>n} \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{i}}_{M\bar{b}_{<n}} \bar{b}_n$ and therefore $\bar{b}_n$ is indiscernible over $M\bar{b}_{\neq n}$.

We get a matrix as in the previous proof, with k-inconsistent rows for some k. Since the rows are mutually indiscernible over M, again $(b_{if(i)})_{i<\omega}$ has the same type over M for all $f\colon \omega \to \omega$. In the same way as before we see that $\{\varphi(x, b_{i0}) \mid i < \omega\}$ must be k'-inconsistent for some k', so the $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{i}}$-Morley sequence over M $(b_{i0})_{i<\omega}$ witnesses that $\varphi(x, b)$ divides over M. $\square$

The following technical result is a less cumbersome derivative of the Broom Lemma of Chernikov and Kaplan.

Lemma 9 (Vacuum cleaner) *Assume* NTP_2. *Let $p(x)$ be a partial type that is invariant over an invariance base M. Suppose $p(x) \vdash \psi(x, b) \vee \bigvee_{i<n} \varphi^i(x, c)$, where $b \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{i}}_M c$ and each $\varphi^i(x, c)$ divides over M. Then $p(x) \vdash \psi(x, b)$.*

Proof: Trivial for $n = 0$. Suppose the lemma holds for n, and $p(x) \vdash \psi(x, b) \vee \bigvee_{i\leq n} \varphi^i(x, c)$, where $b \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{i}}_M c$ and each $\varphi^i(x, c)$ divides over M. Let $(c_i)_{i<\omega}$ be an $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{i}}$-Morley sequence over M which witnesses that $\varphi^n(x, c)$ divides over M. Since $b \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{i}}_M c = c_0$, we may assume $b \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{i}}_M (c_i)_{i<\omega}$, and in particular $(c_i)_{i<\omega}$ is indiscernible over Mb. By invariance of p

$$p(x) \vdash \psi(x, b) \vee \bigwedge_{j<k} \bigvee_{i\leq n} \varphi^i(x, c_j)$$

for any k.

If k is chosen so that $\bigwedge_{j<k} \varphi^n(x, b_i)$ is inconsistent, it follows that

$$p(x) \vdash \psi(x, b) \vee \bigvee_{i<n, j<k} \varphi^i(x, c_j)$$

For each j, $b \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{i}}_M c_{\geq j}$ implies $b \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{i}}_{Mc_{>j}} c_j$; since $c_{>j} \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{i}}_M c_j$ we conclude that $bc_{>j} \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{i}}_M c_j$.

Applying the induction hypothesis k times, we get

$$\begin{array}{lcl} p(x) & \vdash & \psi(x,b) \vee \bigvee_{1\leq j<k} \bigvee_{i<n} \varphi^i(x,c_j) \\ p(x) & \vdash & \psi(x,b) \vee \bigvee_{2\leq j<k} \bigvee_{i<n} \varphi^i(x,c_j) \\ & \vdots & \\ p(x) & \vdash & \psi(x,b) \vee \bigvee_{k-1\leq j<k} \bigvee_{i<n} \varphi^i(x,c_j) \\ p(x) & \vdash & \psi(x,b) \end{array}$$

□

Corollary 10 *Assume* NTP_2. *A consistent partial global type that is invariant over an invariance base M does not fork over M.*

Proof: Set $\psi = \bot$. □

Lemma 11 (Existence) *Assume* NTP_2. *Every type over an invariance base M has a strictly invariant global extension.*

Proof: Given a complete type $p(x) = \mathrm{tp}(a/M)$, consider the partial global type

$$p(x) \cup \{\neg\varphi(x,b) \mid \varphi(a,y) \text{ forks over } M\} \cup \{\psi(x,c) \leftrightarrow \psi(x,c') \mid c \equiv_M c'\}$$

We need to show that this partial type is consistent. If not, then

$$p(x) \vdash \varphi(x,b) \vee \bigvee_{i<n} \neg(\psi_i(x,c_i) \leftrightarrow \psi_i(x,c_i'))$$

where $\varphi(a,y)$ forks over M and $c_i \equiv_M c_i'$. Since $\varphi(a,y)$ forks over M, the partial type $q(y) = \{\varphi(a',y) \mid a' \equiv_M a\}$ also forks over M. As it is invariant over M, by the (corollary to the) Vacuum Cleaner Lemma, $q(y)$ is inconsistent.

Let $a_0, a_1, \ldots, a_{m-1} \models \mathrm{tp}(a/M)$ be such that $\{\varphi(a_i,y) \mid i < m\}$ is inconsistent. Since M is an invariance base, $\mathrm{tp}(a_0,\ldots,a_{m-1}/M)$ has a global extension $\mathfrak{p}(x_0,\ldots,x_{m-1})$ that is invariant over M. Each $\mathfrak{p} \restriction x_j$ is invariant over M and

$$\mathfrak{p} \restriction x_j \supseteq p(x_j) \vdash \varphi(x_j,b) \vee \bigvee_{i<n} \neg(\psi_i(x_j,c_i) \leftrightarrow \psi_i(x_j,c_i'))$$

It follows that

$$\mathfrak{p}(x_0,\ldots,x_{m-1}) \vdash \varphi(x_0,b) \wedge \ldots \wedge \varphi(x_{m-1},b),$$

a contradiction. □

Theorem 12 (Kim's Lemma for NTP_2 theories) *In an NTP_2 theory, for any formula $\varphi(x,b)$ and any invariance base M the following are equivalent:*

1. *Every strict Morley sequence in $\mathrm{tp}(b/M)$ witnesses that $\varphi(x,b)$ divides over M.*
2. *Some strict Morley sequence in $\mathrm{tp}(b/M)$ witnesses that $\varphi(x,b)$ divides over M.*
3. *$\varphi(x,b)$ divides over M.*
4. *$\varphi(x,b)$ forks over M.*

Proof: Use the Existence Lemma for *1* $\Rightarrow$ *2* and the NTP_2 I Lemma for *3* $\Rightarrow$ *1*. We prove *4* $\Rightarrow$ *3*. Assume $\varphi(x,b) \vdash \psi_1(x,a_1) \vee \ldots \vee \psi_n(x,a_n)$, where every $\psi_i(x,a_i)$ divides over M. Let $(b_i a_{1i} \ldots a_{ni})_{i<\omega}$ be a strict Morley sequence in $\mathrm{tp}(ba_1 \ldots a_n/M)$. If $\varphi(x,b)$ does not divide over M, then $\{\varphi(x,b_i) \mid i<\omega\}$ is consistent. Let c realize this set of formulas. Then for each $i<\omega$ there is some $j \leq n$ such that $\models \psi_j(c,a_{ji})$. For some $j \leq n$ there are infinitely many $i<\omega$ such that $\models \psi_j(c,a_{ji})$. By indiscernibility, $\{\psi_j(x,a_{ji}) \mid i<\omega\}$ is consistent. Then $\psi_j(x,a_j)$ does not divide over M, a contradiction. $\square$

References

[1] H. Adler. Introduction to theories without the independence property. Preprint. `http://www.logic.univie.ac.at/ adler/docs/nip.pdf`, January 2007.

[2] E. Casanovas. Lascar strong types and forking in NIP theories. Preprint, March 2014.

[3] A. Chernikov and I. Kaplan. Forking and dividing in NTP_2 theories. *The Journal of Symbolic Logic*, 77(1):1–20, 2012.

[4] B. Kim. Forking in simple unstable theories. *Journal of the London Mathematical Society*, 57:257–267, 1998.

[5] B. Kim and A. Pillay. Simple theories. *Annals of Pure and Applied Logic*, 88:149–164, 1997.

[6] S. Shelah. Simple unstable theories. *Annals of Mathematical Logic*, 19:177–203, 1980.

[7] S. Shelah. *Classification Theory.* North Holland P.C., Amsterdam, second edition, 1990.