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Sur les théories sans la propriété de l'arbre du second type

(On theories without the tree property of the second kind)

Thèse de doctorat en mathématiques

par

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2012

Acknowledgments

I am grateful to my advisor Itai Ben Yaacov who didn't mind that there was nothing continuous in my logic and who was always eager to discuss mathematics. I am very grateful to my collaborators, friends and people with whom I have spent infinite amount of time discussing model theory and all other sorts of things: Martin Hils, Itay Kaplan and Pierre Simon (and for their great tolerance of my e-mail responding style). I am grateful to the MALOA grant which had allowed me to spend a lot of time in many wonderful places around the world and to make very beneficial visits to the logic groups of Paris, Muenster, Bogota and Jerusalem. I am grateful to the Lyon logic group for the stimulating environment and to the Berlin logic group where it all had started. Finally, I am grateful to Ale for keeping my spirits high.

Résumé / Abstract

Cette thèse en théorie des modèles pure présente la première étude systématique de la classe des théories NTP_2 introduites par Shelah, avec un accent particulière sur le cas NIP. Dans les premier et deuxième chapitres, nous développons la théorie de la bifurcation sur des bases d'extension (par exemple, nous prouvons l'existence de suites de Morley universelles, l'égalité de la bifurcation avec la division, un théorème d'indépendance et d'égalité du type Lascar avec le type compact). Ceci rend possible de considérer les résultats de Kim et Pillay sur des théories simples comme un cas particulier, tout en fournissant une contrepartie manquante pour le cas des théories NIP. Cela répond à des questions de Adler, Hrushovski et Pillay. Dans le troisième chapitre, nous développons les rudiments de la théorie du fardeau (une généralisation du calcul du poids), en particulier, nous montrons qu'il est sous-multiplicatif, répondant à une question de Shelah. Nous étudions ensuite les types simples et NIP en théories NTP_2: nous montrons que les types simples sont co-simples, caractérisés par le théorème de coindépendance, et que la bifurcation entre les réalisations d'un type simple et des éléments arbitraires satisfait la symétrie complète; nous montrons qu'un type est NIP si et seulement si toutes ses extensions ont un nombre borné d'extensions globales non-bifurquantes. Nous prouvons aussi une préservation de type d'Ax-Kochen pour NTP_2, montrant que, par exemple, tout ultraproduit de p-adics est NTP_2. Nous continuons à étudier le cas particulier des théories NIP. Dans le chapitre 4, nous introduisons les définitions honnêtes et les utilisons pour donner une nouvelle preuve du théorème de l'expansion de Shelah et un critère général pour la dépendance d'une paire élémentaire. Comme une application, nous montrons que le fait de nommer une petite suite indiscernable préserve NIP. Dans le chapitre 5, nous combinons les définitions honnêtes avec des résultats combinatoires plus profonds de la théorie de Vapnik-Chervonenkis pour déduire que, dans théories NIP, des types sur ensembles finis sont uniformément définissables. Cela confirme une conjecture de Laskowski pour les théories NIP. Par ailleurs, nous donnons une nouvelle condition suffisante pour une théorie d'une paire d'éliminer les quantificateurs en des quantificateurs sur le prédicat et quelques exemples concernant la définissabilité de 1-types vs la définissabilité de n-types sur les modèles. Le dernier chapitre concernes la classification des taux de croissance du nombre des extensions non-bifurquantes. Nous avançons vers la conjecture qu'il existe un nombre fini de possibilités différentes et développons une technique générale pour la construction de théories avec un nombre prescrit d'extensions non- bifurquantes que nous appelons la circularisation. En particulier, nous répondons par la négative à une question d'Adler en donnant un exemple d'une théorie qui a IP où le nombre des extensions non- bifurquantes de chaque type est bornée. Par ailleurs, nous résolvons une question de Keisler sur le nombre de coupures de Dedekind dans les ordres linéaires: il est compatible avec ZFC que $\mathrm{dded}\,\kappa < (\mathrm{ded}\,\kappa)^{\omega}$.

This thesis in pure model theory presents the first systematic study of the class of NTP_2 theories introduced by Shelah, with a special accent on the NIP case.

In the first and second chapters we develop the theory of forking over extension bases (e.g. we prove existence of universal Morley sequences, equality of forking and dividing, an independence theorem and equality of Lascar type and compact type) thus making it possible to view the results of Kim and Pillay on simple theories as a special case, but also providing a missing counterpart for the case of NIP theories. This answers questions of Adler, Hrushovski and Pillay.

In the third chapter we develop the basics of the theory of burden (a generalization of the weight calculus), in particular we show that it is submultiplicative, answering a question of Shelah. We then study simple and NIP types in NTP_2 theories: we prove that simple types are co-simple, characterized by the co-independence theorem, and forking between realizations of a simple type and arbitrary elements satisfies full symmetry; we show that a type is NIP if and only if all of its extensions have only boundedly many global non-forking extensions. We also prove an Ax-Kochen type preservation of NTP_2, thus showing that e.g. any ultraproduct of p-adics is NTP_2.

We go on to study the special case of NIP theories. In Chapter 4 we introduce honest definitions and using them give a new proof of the Shelah expansion theorem and a general criterion for dependence of an elementary pair. As an application we show that naming a small indiscernible sequence preserves NIP. In Chapter 5, we combine honest definitions with some deeper combinatorial results from the Vapnik-Chervonenkis theory to deduce that in NIP theories, types over finite sets are uniformly definable. This confirms a conjecture of Laskowski for NIP theories. Besides, we give a new sufficient condition for a theory of a pair to eliminate quantifiers down to the predicate (in particular answering a question of Baldwin and Benedikt about naming an indiscernible sequence) and some examples concerning definability of 1-types vs definability of n-types over models.

The last chapter is devoted to the study of non-forking spectra. To a countable first-order theory we associate its non-forking spectrum — a function of two cardinals kappa and lambda giving the supremum of the possible number of types over a model of size lambda that do not fork over a sub-model of size kappa. This is a natural generalization of the stability function of a theory. We make progress towards classifying the non-forking spectra. Besides, we answer a question of Keisler regarding the number of cuts a linear order may have. Namely, we show that it is possible that $\mathrm{ded}\,\kappa < (\mathrm{ded}\,\kappa)^{\omega}$.

Contents

Introduction

0.1. Introduction (français)

La théorie des modèles est une branche de la logique mathématique qui étudie les structures, les algèbres de Boole des parties définissables par des formules du premier ordre, et les espaces de types correspondants (c'est à dire les espaces d'ultrafiltres d'ensembles définissables donnés par la dualité de Stone). L'objet d'étude initial de la théorie des modèles était la logique du premier ordre elle-même, mais elle a finalement évolué pour devenir l'étude des théories du premier ordre complètes et leur classification (poétiquement la théorie des modèles est parfois appelée "la géographie des mathématiques apprivoisées"). Ces dernières années, la théorie des modèles a trouvé de nombreuses applications profondes en algèbre, géométrie algébrique, géométrie algébrique réelle, théorie des nombres et analyse combinatoire.

Des recherches approfondies de Shelah [**She90**] et d'autres sur le programme de classification des théories du premier ordre ont produit un vaste corpus de résultats et de techniques pour analyser les types et les modèles dans les théories stables (par exemple calcul de bifurcation, poids, orthogonalité, définissabilité, multiplicité, etc). Cependant, ce n'est que relativement récemment qu'il est devenu évident que beaucoup de ces outils pourraient être généralisés à des classes beaucoup plus grandes de théories considérées par les théoriciens des modèles, ou même plus généralement, pourraient être faites localement par rapport à un certain type dans une théorie arbitraire (et donc le notion d'apprivoisé devient peu à peu sauvage). Cette ligne de recherche, motivé à la fois par de nouveaux exemples algébriques importants et développements de théorie des modèles pure, constitue la "théorie de néo-stabilité", et c'est le domaine dans lequel cette thèse contribue.

0.1.1. Histoire de le sujet. Habituellement, le théorème fondamental suivant de Morley de 1965 est considéré comme le début de la théorie des modèles moderne.

Fact 0.1.1. *Soit T une théorie de premier ordre dans un langage dénombrable. Supposons que T a un modèle unique de taille κ (à isomorphisme près) pour un certain $\kappa > \aleph_0$. Alors T a un modèle unique de taille κ pour* ***tous*** $\kappa > \aleph_0$.

La preuve de Morley a introduit un certain nombre d'idées essentielles pour les développements ultérieurs : la méthode fondamentale de l'analyse de l'espace de types au moyen de le rang de Cantor-Bendixon et l'utilisation de la ω-stabilité. Dans le même article Morley a posé l'hypothèse selon laquelle la fonction $f_T : \kappa \to |\{M : M \models T, |M| = \kappa\}|$ est non décroissante.

Dans un corps incroyable de travail [**She90**], Saharon Shelah a adopté une approche radicale vers cette conjecture en visant à décrire toutes les possibilités pour la fonction f_T. L'idée philosophique principale était celle de lignes de démarcation : on isole certaines configurations combinatoires de telle sorte que toute théorie qui les "code" est mauvaise (on peut lui prouver un théorème de non-structure, par exemple démontrer que f_T est maximale), tandis que pour les théories qui ne les codent pas on développe une théorie de structure avec une compréhension plus fine des types.

Dans l'un des premiers résultats de ce programme Shelah démontré qu'on peut limiter le domaine de consideration à des théories stables (théories avec les " petits " espaces de types, ou de façon équivalente les théories qui ne sont pas capable de "coder" ordres linéaires, voir la section suivante). Le programme a culminé essentiellement à isoler les conditions pour que les modèles puissent être classifiées par des invariants cardinaux (généralisant la dimension des espaces vectoriels ou de degré de transcendance des corps algébriquement clos) et le calcul du nombre de modèles dans ces cas. Ces techniques ont permis à Shelah d'affirmer conjecture de Morley, et les travaux suivant [**HHL00**] conduit à une description complète des possibilités pour f_T.

0.1.2. Le paradis stable.

Soit T une complète théorie du premier ordre, et nous fixons un modèle monstre $\mathbb{M}$, très grand et saturée (un "domaine universel"). Pour un modèle $M \models T$, soit $S(M)$ l'espace des types sur M, c'est à dire le dual de Stone de l'algèbre booléenne des parties définissables de M (i.e., l'ensemble d'ultrafiltres sur cette algèbre), avec la base ouvert-fermé constitué d'ensembles de la forme $[\varphi] = \{p \in S(M) : \varphi \in p\}$. C'est un espace compact et totalement discontinu.

Soit $s_T(\kappa) = \sup\{|S(M)| : M \models T, |M| = \kappa\}$. Notez que toujours $s_T(\kappa) \geq \kappa$.

Definition 0.1.2. T est stable si elle satisfait l'une des propriétés équivalentes suivantes:

(1) Pour chaque cardinal κ, $s_T(\kappa) \leq \kappa^{\aleph_0}$.
(2) Il existe un cardinal κ de telle sorte que $s_T(\kappa) = \kappa$.
(3) Il n'existe pas de formule $\varphi(x, y)$ et $(a_i)_{i\in\omega}$ (dans un certain modèle) telle que $\varphi(a_i, a_j) \Leftrightarrow i < j$.

Des exemples de théories stables sont les suivants:

- modules,
- Les corps algébriquement clos,
- Les corps séparablement clos,
- Les corps différentiellement clos,
- Les groupes libres (Z. Sela [**Sel**]),
- Les graphes planaires (K. Podewski and M. Ziegler [**PZ78**]).

Shelah a développé un certain nombre de techniques d'analyse des types et des modèles de théories stables (modèles prime, le poids, les types réguliers, ...). Une notion clé est *bifurcation.*

Definition 0.1.3. (1) Une formule$\varphi(x, a)$ *divise sur* A s'il y a une suite $(a_i)_{i\in\omega}$ et $k \in \omega$ telle que:
- $\mathrm{tp}(a_i/A) = \mathrm{tp}(a/A)$,
- $\{\varphi(x, a_i)\}_{i\in\omega}$ et k-incompatible (c'est à dire l'intersection de tout k éléments distincts est vide).

(2) Une formule$\varphi(x, a)$ *bifurque* sur A si elle appartient à l'idéal engendré par les formules divisant sur A, i.e. il ya $\varphi_i(x, a_i)$ pour $i < n \in \omega$ telle que
- $\varphi(x, a) \vdash \bigvee_{i<n} \varphi_i(x, a_i)$,
- $\varphi_i(x, a_i)$ divise sur A pour chaque $i < n$.

Le but de l'introduction de bifurcation en plus de la division, c'est que chaque type partiel non-bifurquant s'étend à un type non-bifurquant global, sur chaque ensemble de paramètres (par le théorème de l'Idéal Premier). L'idée est que une extension non-bifurquante capture une "extension générique" d'un type (qui est une généralisation profonde de la notion d'un point générique d'une variété). En général la bifurcation n'est pas la même que la division.

Example 0.1.4. Considérons la théorie d'un ordre dense circulaire, c'est à dire d'une relation ternaire $R(x, y, z)$ qui contient chaque x, y, z qui sont des points sur un cercle unité et y est entre x et z, dans le sens des aiguilles d'une montre. La formule "$x = x$" ne divise pas sur $\emptyset$ (et en fait aucune formule ne divise sur ses paramètres). D'autre part, $x = x \vdash \bigvee_{i<3} R(a_i, x, b_i)$ pour certains choix de $(a_i b_i)_{i<3}$, et il est facile de voir que $R(a_i, x, b_i)$ divise pour chaque $i < 3$.

Dans les théories stables, bifurcation bénéficie d'un certain nombre de propriétés merveilleuses qui peuvent être disposés dans les trois groupes suivants:

F_1 Belle structure combinatoire de l'idéal de bifurcation : bifurcation est égal à division, l'existence de suites de Morley universelles, la condition de la chaîne, etc...

F_2 Disons que $a \mathop{\smile\!\!\!\!\!\!|}_c b$ lorsque $\mathrm{tp}(a/bc)$ ne bifurque pas sur c. Alors $\mathop{\smile\!\!\!\!\!\!|}$ est une relation d'indépendance agréable : invariante par automorphismes de $\mathbb{M}$, symétrique, transitive, ayant le caractère local, le caractère fini, ...

F_3 Multiplicité : chaque type admet une extension non-bifurquent unique, les types sont définissables, le théorème de relation d'équivalence finie, ...

Ces trois groupes de propriétés ont été quelque peu entrelacés dans le développement initial de la stabilité. Les travaux sur les théories simples (voir la section suivante), tout en ne distinguant pas entre les F_1 and F_2, a précisé leur indépendance à partir de F_3. Une grande partie de cette thèse est de démontrer que, en fait F_1 peut être développé de manière indépendante dans une classe beaucoup plus vaste de théories.

En utilisant le combinaison de F_1–F_3, Shelah a développé des outils puissants pour l'analyse des types et des modèles dans les théories stables, accomplissant son objectif initial : compter le nombre de modèles d'une théorie de premier ordre.

D'autres travaux, notamment par Hrushovski (et en grande partie basés sur des idées de Zilber autour des théories fortement minimales), ont conduit à l'analyse raffinée et le développement de la théorie de la stabilité géométrique, et ont rendu précise l'idée que la complexité de la bifurcation doit être en corrélation avec la complexité des structures algébriques interprétables dans la théorie : trichotomie, la configuration du groupe, etc. Ces développements constituent un pont technique majeur reliant la théorie de modèles pur avec ses applications à la géométrie algébrique et la théorie des nombres. Malheureusement, la plupart des théories ne sont pas stables.

0.1.3. Théories simples.

La classe des théories simples a été introduit par Shelah dans [**She80**] dans le cadre de la caractérisation du spectre de saturation. Mais le renouveau d'intérêt réel avait eu lieu 15 ans plus tard, provenant des travaux de Hrushovski sur corps pseudo-finis et d'autres exemples de rang fini [**Hru02**], et d'un travail dans la théorie de modèles pure de Kim et Pillay [**Kim98, KP97, Kim01, Kim96**].

Une théorie est simple si tous les types ne dévie pas sur un sous-ensemble petit son domaine. De manière équivalente, si ce n'est pas possible d'encoder un arbre d'une manière définissable (voir le chapitre 3 pour les définitions précises). Exemples de théories simples sont:

- chaque théorie stable est simple,
- la théorie de la graphe aléatoire de Rado,
- les corps pseudo-finis,
- la théorie des corps algébriquement clos augmenté par un automorphisme générique, ACFA.

Dans sa thèse [**Kim96**], Kim avait prouvé que bifurcation est égal à division, et qu'elles donnent lieu à une relation d'indépendance transitive et symétrique, récupérant ainsi complètement les propriétés de F_1 et F_2 dans le théories simples.

Concernant F_3, bifurcation n'est plus décrit par définissabilité des types, et de stationnarité échoue. Mais, dans le travail de Hrushovski sur le cas de rang fini, il est devenu évident que dans la plupart des situations, on pourrait remplacer le caractère unique de l'extensions non-bifurquent par la capacité de amalgamer deux extensions dans une position suffisamment générale. Cela a conduit au théorème suivant importante de Kim et Pillay.

FACT 0.1.5. [**KP97**] *La théorème d'indépendance. Soit* T *une théorie simple et* $M \models T$*. Soit* $p_0(x)$ *un type complet sur* M*,* $p_1 \in S(A)$ *et* $p_2 \in S(B)$ *sont extensions non-bifurquantes de* p_0*, et* $A \mathop{\smile\!\!\!\!\!\mid}_M B$*. Alors il y a un certain type global* $p(x)$ *non-bifurquant sur* M *et telle que* $p_1, p_2 \subseteq p$*.*

Nous avions formulé le théorème de l'indépendance sur un modèle, alors qu'en fait, une analyse plus poussée montre que le seul obstacle de l'amalgamation est caractérisée par l'action du groupe des automorphismes forts de Lascar. En fait, l'existence d'une relation satisfaisant F_2 et le théorème d'indépendance implique que la théorie est simple et que cette relation est donnée par non-bifurcation.

Les travaux ultérieurs de nombreux de chercheurs a conduit à un développement rapide du champ, parmi les résultats notables sont l'existence de bases canoniques et la théorie de hyperimaginaries (et leur élimination dans les théories supersimple), résultats sur la configuration de groupe, travail de Chatzidakis, Hrushovski et Peterzil sur ACFA — culminant dans la théorie de simplicité géométrique, trichotomie pour les ensembles de rang 1 au ACFA et de la preuve de Mordell-Lang par Hrushovski.

0.1.4. NIP.

La classe des théories NIP (non propriété d'indépendance) a été introduit par Shelah dans l'un des premiers articles sur le programme de la classification. Une théorie est NIP si elle ne peut pas "coder le graphe aléatoire bipartite par une formule". Plus précisément:

DEFINITION 0.1.6. Une formule$\varphi(x,y)$ a NIP si pour un certain $n<\omega$ il n'y a pas $(a_i)_{i<n}$ et $(b_s)_{s\subseteq n}$ de telle sorte que $\varphi(a_i,b_s)\Leftrightarrow i\in s$. Une théorie est NIP si elle implique que toute les formules sont NIP.

Il a été observé très tôt par Laskowski [**Las92**] que NIP est équivalente à la finitude de dimension de Vapnik-Chervonenkis des familles φ-définissables pour tous φ. Nous remarquons que, si une théorie est à la fois simple et NIP, alors elle est stable.

Des exemples de théories NIP sont:

- les théories stables,
- les ordres linéaires et les arbres,
- les groupes abéliens ordonnée (Gurevich-Schmitt),
- les théories o-minimales,
- les corps valués algébriquement clos (et en fait tout les théories c-minimales),
- $\mathbb{Q}_p$.

Il existait bien certaines résultats sur NIP dans les années 80, et elles connaissent actuellement un renouveau d'intérêt. La motivation est double : le travail sur l'exemple particulier de corps valués algébriquement clos (élimination des imaginaires et la domination stable dans $ACVF$ par Haskell, Hrushovski et Macpherson [**HHM08**], Hrushovski-Loeser sur les types génériquement stables et des espaces de Berkovich, Hrushovski-Peterzil-Pillay sur la conjecture de Pillay pour groupes o-minimal [**HPP08**]), et les développements de caractère purement modèle théorique (travail de Shelah: théorème sur les ensembles extérieurement définissables [**She04, ?**], la conjecture de paire générique et le comptage de types à automorphisme près [**Sheb, Shea, Shec**], les travaux sur le dp-rang et notions de dp-minimalité, les mesures, ...).

Les théories NIP ont de nombreuses propriétés combinatoires caractéristiques aux théories stables, mais il s'manifeste est un phénomène essentiellement nouveau — la présence des ensembles extérieurement définissables qui ne sont pas intérieurement définissables. Il semble inévitable pour le développement futur de saisir un certain contrôle sur leur structure. Et qu'est peut-on dire de la bifurcation dans les théories NIP ? D'une part, F_2 échoue complètement — la non-bifurcation n'est pas symétrique, ni transitive, déjà dans un ordre dense. Cependant, il s'avère qu'un type global ne dévie pas sur un modèle si et seulement si il est invariant par tous les automorphismes fixant ce modèle. Cela implique que chaque type a un nombre bornée d'extensions non-bifurquantes et laisse d'espoir pour de meilleurs résultats à l'égard F_3. Effectivement, nous faisons du progrès dans ces deux directions dans les chapitres 4,5. En ce qui concerne F_1, nous en discutons dans la section suivante.

0.1.5. NTP$_2$**.** Enfin, nous arrivons à la question centrale de cette thèse — la classe des théories sans la propriété d'arbre du deuxième type, ou théories NTP$_2$. Il a été introduit par Shelah implicitement dans [**She90**] et explicitement dans [**She80**], comme une généralisation de la simplicité.

DEFINITION 0.1.7. On dit que $\varphi(x,y)$ a TP$_2$ s'il ya $(a_{ij})_{i,j\in\omega}$ et $k\in\omega$ de telle sorte que:

(1) $\{\varphi(x,a_{ij})\}_{j\in\omega}$ est k-incompatible pour chaque $i\in\omega$.
(2) $\left\{\varphi\left(x,a_{if(i)}\right)\right\}_{i\in\omega}$ est compatible pour tous $f:\omega\to\omega$.

Une théorie est NTP_2 si aucune formule a TP_2.

La classe de théories NTP_2 est une généralisation naturelle des théories simples et des théories NIP. D'autres exemples de théories NTP_2 sont les suivantes:

- Expansion d'une théorie NTP_2 géométrique par un prédicat générique reste NTP_2. "Géométrique" signifie que la clôture algébrique satisfait échange et que le quantificateur $\exists^\infty$ est éliminé. "Générique" est dans le sens de [**CP98**]. Par exemple, l'expansion d'une théorie o-minimale par l'ajout d'un graphe aléatoire est NTP_2 (voir le chapitre 3).
- Les ultraproduits de $\mathfrak{p}$-adics sont NTP_2. Plus généralement, un corp valués hensélien de caractéristique 0 est NTP_2 si et seulement si son corp residuel est NTP_2 (voir le chapitre 3).
- Certains corps valués augmentés d'un automorphism σ-hensélien. E.g. automorphisme de Frobenius non-standard sur un corp valué algébriquement clos de caractéristique 0 ([**CH**]).

Cette thèse contient la première étude systématique de la classe de théories NTP_2. Une grande partie de cette étude est consacrée au développement du calcul de bifurcation dans le cadre de théories NTP_2 (nous parvenons à démonstrer F_1 complètement et offrir une théorème d'indépendance faible pour F_3), à la compréhension des types particuliers sur théories NTP_2 (avec un accent sur les types simples et NIP) et à la fourniture de nouveaux exemples. Des résultats supplémentaires sur les groupes et les corps (type-)définissables dans des structures avec théories NTP_2 qui n'ont pas trouvé leur place dans ce texte seront disponible en [**CH**] et [**CKS**].

0.1.6. Résumé des résultats. Les chapitres 1 et 2 sont consacrés au développement de la théorie de la bifurcation dans les théories NTP_2 : nous démontrons qu'une grande partie du calcul de la bifurcation peut être développée dans le contexte général des théories NTP_2 sur des bases d'extension (la coïncidence de la bifurcation et de la division, l'existence d'extensions strictement invariantes, la condition de chaîne, le théorème d'indépendance, etc), généralisant le travail de Kim et Pillay sur les théories simples et répondant à une question de Pillay, qui a été ouverte même dans le cas des théories NIP, ainsi qu'à des questions d'Adler et de Hrushovski au sujet du nombre d'extensions non-bifurquantes et la condition de chaîne de la non-bifurcation. Le chapitre 1 est un travail en commun avec Itay Kaplan (et est publié comme "Forking and dividing in NTP_2 theories" dans le Journal of Symbolic Logic [**CK12**]) et le chapitre 2 est un travail conjoint avec Itaï Ben Yaacov (et est en circulation comme un preprint "An independence theorem for NTP_2 theories").

Chapitre 3 (soumis à les Annals of Pure and Applies Logic comme "Theories without the tree property of the second kind") développe les bases de la théorie du fardeau, une notion généralisée de poids (par exemple, nous démontrons qu'il est sous-multiplicative, répondant à une question de Shelah [**She90**]). Par ailleurs, nous étudions les types simples et NIP dans les théories NTP_2 et les effets que ces hypothèses ont pour le calcul du fardeau.

Pour les types simples nous établissons une symétrie complète de la bifurcation entre les réalisations du type et des éléments arbitraires, répondant ainsi à une question de Casanovas dans le cas de théories NTP_2. Pour les types NIP, nous démontrons que leur dp-rang (de façon équivalente, le fardeau) est toujours témoigné par des suites mutuellement indiscernables de réalisations du type. Enfin, nous donnons des exemples nouveaux de théories NTP_2 : toute expansion d'une théorie géométrique NTP_2 par un prédicat générique est NTP_2 ; tous les corps hensélien value de caractéristique 0 est NTP_2 en supposant que le corps résiduel est NTP_2.

Le chapitre 4 (à paraître dans le Israel Journal of Mathematics comme "Externally definable sets and dependent pairs") et le chapitre 5 (soumis à Transactions of AMS) sont un travail en commun avec Pierre Simon, et sont dédiés à l'étude de l'ensembles extérieurement définissables dans les théories NIP.

Dans le chapitre 4, nous introduisons les définitions honnêtes et les utilisons pour donner une nouvelle preuve du théorème de l'expansion Shelah et un critère général de la dépendance d'une paire élémentaire. Comme application nous répondons à une question de Baldwin et de Benedikt [**BB00**] sur le nommage d'une suite indiscernable, et montrons que le résultat recouvre la grande majorité des résultats existants sur les paires dépendantes. Nous montrons aussi que les ensembles extérieurement définissables dans les théories NIP qui sont suffisamment grands ont des sous-ensembles intérieurement définissables.

Dans le chapitre 5, nous combinons les définitions honnêtes avec des résultats combinatoires plus profonds de la théorie de Vapnik-Chervonenkis pour déduire que, dans théories NIP, des types sur ensembles finis sont uniformément définissable. Cela confirme une conjecture de Laskowski pour les théories NIP. Par ailleurs, nous donnons une nouvelle condition suffisante pour une théorie d'une paire d'éliminer les quantificateurs en des quantificateurs sur le prédicat et quelques exemples concernant la définissabilité de 1-types vs la définissabilité de n-types sur les modèles. Nous montrons aussi des résultats sur la couverture des familles non-bifurquent par types invariants.

Le dernier chapitre (travail en commune avec Itay Kaplan et Saharon Shelah, soumis à Transactions of AMS comme "On non-forking spectra") concernes la classification des taux de croissance du nombre des extensions non-bifurquantes. Nous avançons vers la conjecture que il existe nombre fini de possibilités différentes et développons une technique générale pour la construction de théories avec un nombre prescrit d'extensions non-bifurquantes que nous appelons la *circularisation*. En particulier, nous répondons par la négative à une question d'Adler en donnant un exemple d'une théorie qui a IP où le nombre des extensions non-bifurquantes de chaque type est bornée. Par ailleurs, nous résolvons une question de Keisler sur le nombre de coupures de Dedekind dans les ordres linéaires: il est compatible avec ZFC que $\operatorname{ded}\kappa < (\operatorname{ded}\kappa)^{\aleph_0}$.

0.2. Introduction (English)

Model theory is a branch of mathematical logic studying structures, Boolean algebras of subsets definable by means of first order formulas, and the corresponding spaces of types (that is, the spaces of ultrafilters of definable sets given by the Stone duality). While the early focus of model theory was on the first-order logic itself, it had eventually moved on to become the study of complete first-order theories and their classification (somewhat poetically model theory is sometimes called "the geography of tame mathematics"). In recent years model theory had found numerous (and deep) applications to algebra, algebraic geometry and real algebraic geometry, number theory and combinatorics.

Extensive research of Shelah [**She90**] and others on the classification program for first-order theories had produced a large and coherent body of results and techniques for analyzing types and models in stable theories (e.g. forking-calculus, weight and orthogonality, definability and multiplicity, etc). However, only relatively recently it became apparent that many of these tools could be generalized to considerably larger classes of theories considered by model theorists, or even more generally, could be done locally with respect to a certain type in an arbitrary theory (thus the model theoretic notion of "tame" is gradually becoming wilder). This line of research, motivated both by new important algebraic examples and purely model theoretic developments, constitutes the so-called "neo-stability theory", and it is the field to which this thesis contributes.

0.2.1. History of the subject. Usually the following fundamental theorem of Morley from 1965 is considered as the beginning of modern model theory.

FACT 0.2.1. *Let T be a first-order theory in a countable language. Assume that T has a unique model of size κ (up to isomorphism) for some $\kappa > \aleph_0$. Then T has a unique model of size κ for* ***all*** $\kappa > \aleph_0$.

Morley's proof had introduced a number of ideas essential for the later developments: the fundamental method of analyzing the space of types by means of the Cantor-Bendixon rank and the use of ω-stability. In the same paper Morley posed the conjecture that the function $f_T : \kappa \to |\{M : M \models T, |M| = \kappa\}|$ is non-decreasing.

In an amazing body of work [**She90**], Saharon Shelah took a radical approach to this conjecture by aiming to describe all the possibilities for the function f_T. The main philosophical idea was that of dividing lines, namely one isolates certain combinatorial pattern such that any theory "encoding" it is bad (namely one can prove a strong non-structure theorem e.g. demonstrating that f_T is maximal), while for theories not able to encode it one develops a structure theory with a finer understanding of types. In one of the early results of this program Shelah demonstrated that the domain of consideration can be restricted to stable theories (theories with "small" spaces of types, or equivalently theories which are not able to "encode" linear orders, see the next section). The programme essentially culminated in isolating the conditions for models to be classifiable by cardinal invariants (generalizing the dimension of vector spaces or transcendence degree of algebraically closed fields) and computing the number of models in these cases. These techniques allowed Shelah to affirm Morley's conjecture, and further work [**HHL00**] led to a complete description of possibilities for f_T.

0.2.2. Stable paradise.

Let T be a complete first-order theory, and we fix a very large saturated monster model $\mathbb{M}$ (a "universal domain"). For a model $M \models T$, let $S(M)$, the space of types over M, be the Stone dual of the Boolean algebra of definable subsets of M. I.e. the set of ultrafilters on it, with the clopen basis consisting of sets of the form $[\varphi] = \{p \in S(M) : \varphi \in p\}$. It is a totally disconnected compact Hausdorff space.

Let $s_T(\kappa) = \sup\{|S(M)| : M \models T, |M| = \kappa\}$. Note that always $s_T(\kappa) \geq \kappa$.

DEFINITION 0.2.2. T is called stable if it satisfies any of the following equivalent properties:

(1) For every cardinal κ, $s_T(\kappa) \leq \kappa^{\aleph_0}$.
(2) There is some cardinal κ such that $s_T(\kappa) = \kappa$.
(3) There is no formula $\varphi(x, y)$ and $(a_i)_{i\in\omega}$ (in some model) such that $\varphi(a_i, a_j) \Leftrightarrow i < j$.

Examples of stable theories are:

- modules,
- algebraically closed fields,
- separably closed fields,
- differentially closed fields,
- free groups (a deep result of Z. Sela [**Sel**]),
- planar graphs (K. Podewski and M. Ziegler [**PZ78**]).

Shelah had developed a number of techniques for analyzing types and models of stable theories (prime models, weight, regular types, ...). A key notion introduced was that of *forking*.

Definition 0.2.3. (1) A formula $\varphi(x, a)$ *divides* over A if there is a sequence $(a_i)_{i\in\omega}$ and $k \in \omega$ such that:
   - $\mathrm{tp}(a_i/A) = \mathrm{tp}(a/A)$,
   - $\{\varphi(x, a_i)\}_{i\in\omega}$ is k-inconsistent (i.e. the intersection of any k distinct elements is empty).

(2) A formula $\varphi(x, a)$ *forks* over A if it belong to the ideal generated by the formulas dividing over A, i.e. there are $\varphi_i(x, a_i)$ for $i < n \in \omega$ such that
   - $\varphi(x, a) \vdash \bigvee_{i<n} \varphi_i(x, a_i)$,
   - $\varphi_i(x, a_i)$ divides over A for each $i < n$.

The purpose of introducing forking in addition to dividing is that every partial non-forking type extends to a complete non-forking type over possibly a larger set of parameters (by the Prime Ideal Theorem). The idea is that a non-forking extension captures a "generic extension" of a type (which is a far-reaching generalization of the concept of a generic point of a variety). In general forking is not the same as dividing.

Example 0.2.4. Consider the theory of a dense circular order, i.e. of a ternary relation $R(x, y, z)$ which holds whenever x, y, z are points on a unit circle and y is between x and z taken clock-wise. The formula "$x = x$" does not divide over $\emptyset$ (and in fact no formula divides over its parameters). On the other hand, $x = x \vdash \bigvee_{i<3} R(a_i, x, b_i)$ for some choice of $(a_i b_i)_{i<3}$ and it is easy to see that $R(a_i, x, b_i)$ divides for each $i < 3$.

In stable theories, forking enjoys a number of wonderful properties which can be arranged into the following three groups:

F_1 Nice combinatorial structure of the forking ideal: forking equals dividing, existence of universal Morley sequences, chain condition, ...

F_2 Let $a \mathop{\smile\!\!\!\!\!\mid}_c b$ denote that $\mathrm{tp}(a/bc)$ does not fork over c. Then $\mathop{\smile\!\!\!\!\!\mid}$ is a nice independence relation: invariant under automorphisms of $\mathbb{M}$, symmetric, transitive, finite character, ...

F_3 Multiplicity: every type has a unique non-forking extension, types are definable, finite equivalence relation theorem, ...

This three groups of properties were somewhat interwined in the original development of stability. Work on simple theories (see the next section), while still not distinguishing between F_1 and F_2, clarified their independence from F_3. A large part of this thesis is devoted to demonstrating that in fact F_1 can be developed independently in a much larger class of theories.

Using the combination of F_1–F_3, Shelah had developed powerful tools for analyzing types and models in stable theories, fulfilling his original purpose: to count the number of models a first-order theory may have.

Further work, notably by Hrushovski (and largely based on Zilber's ideas around strongly minimal theories), led to the refined analysis and development of the so-called geometric stability theory, making precise the idea that the complexity of forking should be interrelated with the complexity of algebraic structures interpretable in the theory: trichotomy, group configuration, etc. These developments form a major technical bridge connecting pure model theory with its applications to algebraic geometry and number theory. Unfortunately, most theories are not stable.

0.2.3. Simple theories.

The class of simple theories was introduced by Shelah in [**She80**] in connection to characterizing the saturation spectrum. However, the real revival of interest had occurred 15 years later stemming from the Hrushovski's work on pseudo-finite fields and other finite rank examples [**Hru02**] and a purely model-theoretical work of Kim and Pillay [**Kim98, KP97, Kim01, Kim96**].

A theory is simple if every type does not fork over some small subset of its domain. Equivalently if it is not possible to encode a tree in a definable way (see Chapter 3 for precise definitions). Examples of simple theories are:

- every stable theory is simple,
- the theory of the random Rado graph,
- pseudo-finite fields,
- the theory of algebraically closed fields expanded by a generic automorphism, ACFA.

In his thesis [**Kim96**], Kim had proved that forking equals dividing, and that it gives rise to a symmetric transitive independence relation, thus recovering completely the properties in F_1 and F_2 in the context of simple theories.

Concerning F_3, forking is no longer described by definability of types, and stationarity fails badly. However, in the work of Hrushovski on the finite rank case it became apparent that in most situations one could replace the uniqueness of non-forking extensions by the ability to amalgamate any two of them in a sufficiently general position. This led to the following important theorem of Kim and Pillay.

Fact 0.2.5. [**KP97**] *The independence theorem. Let* T *be a simple theory and* $M \models T$. *Let* $p_0(x)$ *be a complete type over* M, $p_1 \in S(A)$ *and* $p_2 \in S(B)$ *be non-forking extensions of* p_0, *and* $A \mathop{\smile\!\!\!\!\!\downarrow}_M B$. *Then there is some global type* $p(x)$ *non-forking over* M *and such that* $p_1, p_2 \subseteq p$.

We had stated the independence theorem over a model, while in fact further analysis demonstrates that the only obstacle to amalgamation is characterized by the action of the Lascar group of strong automorphisms. In fact, existence of a relation satisfying F_2 and the independence theorem implies that the theory is simple and that this relation is given by non-forking.

Subsequent work of numerous researchers led to a rapid development of the field, among notable results are existence of canonical bases and the theory of hyperimaginaries (and their elimination in supersimple theories), results on group configuration, work of Chatzidakis, Hrushovski and Peterzil on ACFA — culminating in geometric simplicity theory, trichotomy for sets of rank 1 in ACFA and the proof of Mordell-Lang by Hrushovski.

0.2.4. NIP.

The class of NIP theories (No Independence Property, also called dependent) was introduced by Shelah in one of the earliest papers on classification programme. A theory is NIP if it cannot "encode the random bipartite graph by a formula". More precisely:

DEFINITION 0.2.6. A formula $\varphi(x, y)$ has NIP if for some $n < \omega$ there are no $(a_i)_{i<n}$ and $(b_s)_{s \subseteq n}$ such that $\varphi(a_i, b_s) \Leftrightarrow i \in s$. A theory is NIP if it implies that every formula is NIP.

It was observed early on by Laskowski [**Las92**] that NIP is equivalent to the finite Vapnik-Chervonenkis dimension of families of φ-definable sets for all φ. We remark that if a theory is both simple and NIP, then it is stable.

Examples of NIP theories are:

- stable theories,
- linear orders and trees,
- ordered abelian groups (Gurevich-Schmitt),
- any o-minimal theory,
- algebraically closed valued fields (and in fact any c-minimal theory),
- $\mathbb{Q}_p$.

While there were some results on NIP in the 80's, it is currently experiencing a revival of interest. The motivation is again two-fold and stems both from the work on particular example of algebraically closed valued fields (elimination of imaginaries and stable domination in $ACVF$ by Haskell, Hrushovski and Macpherson [**HHM08**], Hrushovski-Loeser on generically stable types and Berkovich spaces , Hrushovski-Peterzil-Pillay on Pillay's o-minimal group conjecture [**HPP08**]) and the purely model theoretic developments (Shelah's work: theorem on externally definable sets [**She04, ?**], the generic pair conjecture and the recounting of types up to automorphism [**Sheb, Shea, Shec**], work on dp-rank and related notions of dp-minimality, measures...).

NIP theories have many of the combinatorial properties characteristic to stable theories, however there is an essentially new phenomenon — presence of externally definable sets which are not internally definable. It seems unavoidable for the further development to grasp some control over their structure. What about forking in NIP theories? On the one hand, F_2 fails badly — forking is neither symmetric nor transitive, already in a dense linear order. However it turns out that a global type does not fork over a model if and only if it is invariant under all automorphisms fixing this model. It follows that every type has boundedly many non-forking extensions and leaves some hope for better results towards F_3. Indeed, we make some progress towards both of these directions in Chapters 4,5. As for F_1, we discuss it in the next section.

0.2.5. NTP$_2$**.** Finally, we arrive to the central topic of this thesis — the class of theories without the tree property of the second kind, or NTP$_2$ theories. It was introduced by Shelah implicitly in [**She90**] and explicitly in [**She80**], as a generalization of simplicity.

DEFINITION 0.2.7. We say that $\varphi(x, y)$ has TP$_2$ if there are $(a_{ij})_{i,j\in\omega}$ and $k \in \omega$ such that:

(1) $\{\varphi(x, a_{ij})\}_{j\in\omega}$ is k-inconsistent for every $i \in \omega$.
(2) $\{\varphi(x, a_{if(i)})\}_{i\in\omega}$ is consistent for every $f : \omega \to \omega$.

A theory is called NTP$_2$ if no formula has TP$_2$.

The class of NTP$_2$ theories is a natural generalization of both simple and NIP theories. Further examples of NTP$_2$ theories are:

- Expansion of a geometric NTP$_2$ theory by a generic predicate remains NTP$_2$. Geometric means that algebraic closure satisfies exchange and that the quantifier $\exists^\infty$ is eliminated. Generic is in the sense of [**CP98**]. For example, an expansion of an o-minimal theory by adding a random graph is NTP$_2$ (see Chapter 3).
- Ultraproducts of p-adics are NTP$_2$, and more generally henselian valued fields of characteristic 0 with NTP$_2$ residue fields (see Chapter 3).
- Certain σ-henselian valued difference fields, e.g. non-standard Frobenius automorphism on an algebraically closed field of characteristic 0 ([**CH**]).

Further results on groups and fields (type-) definable in structures with NTP$_2$ theories which have not found their place in this text will be available in [**CH**] and [**CKS**].

This thesis contains the first systematic study of the class of NTP$_2$ theories. Large part of it is devoted to developing forking calculus in the setting of NTP$_2$ theories (we succeed with recovering F$_1$ fully and provide a weak independence theorem for F$_3$), understanding special kinds of types in NTP$_2$ theories (with focus on simple and NIP types) and providing new examples.

0.2.6. Overview of results. First a very quick overview of the thesis.

Chapters 1 and 2 are devoted to developing the theory of forking in NTP$_2$ theories: we demonstrate that a large part of the forking calculus can be developed in the general context of NTP$_2$ theories (e.g. forking=dividing, existence of strictly invariant extensions, chain condition, weak independence theorem, etc) thus generalizing the work of Kim and Pillay on simple theories and answering a question of Pillay which was open even in the case of NIP theories, along with questions of Adler and Hrushovski around the number of non-forking extensions and the chain condition of non-forking. Chapter 1 is a joint work with Itay Kaplan (and is published as "Forking and dividing in NTP$_2$ theories" in the Journal of Symbolic Logic [**CK12**]) and Chapter 2 is a joint work with Itai Ben Yaacov (and is in circulation as a preprint "A weak independence theorem for NTP$_2$ theories").

Chapter 3 (submitted to the Annals of Pure and Applies Logic as "Theories without the tree property of the second kind") develops the basics of the theory of burden, a generalized notion of weight (e.g. we demonstrate that it is submultiplicative, answering a question of Shelah from [**She90**]). Besides, we study simple and NIP types in NTP$_2$ theories and the effect these assumptions have for burden calculus. For simple types we establish full symmetry of forking between realizations of the type and arbitrary elements, thus answering a question of Casanovas in the case of NTP_2 theories. For NIP types, we demonstrate that their dp-rank (equivalently, burden) is always witnessed by mutually indiscernible sequences of realizations of the type. Finally, we give new examples of NTP_2 theories: any expansion of a geometric NTP_2 theory by a generic predicate is NTP_2; any henselian valued field of characteristic 0 is NTP_2 assuming that the residue field is NTP_2. So in particular any ultraproduct of p-adics is NTP_2.

Chapters 4 (to appear in the Israel Journal of Mathematics as “Externally definable sets and dependent pairs”) and Chapter 5 (submitted to the Transactions of AMS) are a joint work with Pierre Simon and are devoted to the study of externally definable sets in NIP theories. In Chapter 4 we introduce *honest definitions* and using them give a new proof of the Shelah expansion theorem and a general criterion for dependence of an elementary pair. As an application we answer a question of Baldwin and Benedikt [**BB00**] about naming an indiscernible sequence. In Chapter 5, we combine honest definitions with some deeper combinatorial results from the Vapnik-Chervonenkis theory to deduce that in NIP theories, types over finite sets are uniformly definable. This confirms a conjecture of Laskowski for NIP theories. Besides, we give a new sufficient condition for a theory of a pair to eliminate quantifiers down to the predicate and some examples concerning definability of 1-types vs definability of n-types over models.

The final chapter (joint work with Itay Kaplan and Saharon Shelah, submitted as “On non-forking spectra” to the Transactions of AMS) is devoted to the classification of possible growth rates of the number of non-forking extensions. We make progress towards the conjecture that there could be only finitely many different possibilities for it and develop a general technique for constructing theories with a prescribed number of non-forking extension which we call *circularization*. In particular we answer negatively a question of Adler by giving an example of a theory which has IP yet every type has only boundedly many non-forking extensions. Besides, we resolve a question of Keisler on the number of Dedekind cuts in linear orders: it is consistent with ZFC that $\operatorname{ded}\kappa < (\operatorname{ded}\kappa)^{\aleph_0}$.

In the following sections we give a more detailed overview of each chapter, along with the statements of main theorems.

0.2.7. Forking and dividing in NTP_2 theories (joint work with Itay Kaplan). In this chapter we develop the basics of forking and dividing in NTP_2 theories. It is easy to see that the theory in Example 0.2.4 is NIP. Thus, forking is not the same as dividing in general.

PROBLEM 0.2.8. (Pillay) Is forking = dividing over models in NIP theories?

Working on this question, to our own surprise it eventually became clear that going to a larger class of NTP_2 theories clarifies the situation.

DEFINITION 0.2.9. We say that a set A is an *extension base* if every $p(x) \in S(A)$ does not fork over A.

E.g. every model in every theory is an extension base. In simple, o-minimal or c-minimal theories, every set is an extension base.

THEOREM 0.2.10. *Let* T *be* NTP_2 *and* A *an extension base. Then* $\varphi(x,a)$ *divides over* A *if and only if it forks over* A.

While it is not true that every indiscernible sequence witnesses dividing, in a simple theory every Morley sequence does, and in fact this property characterizes simplicity [**Kim01**].

DEFINITION 0.2.11.
(1) A global type $p(x)$ is *strictly invariant* over A if it is invariant over A and for every $B \supseteq A$ and $a \models p|_B$, $\mathrm{tp}(B/aA)$ does not fork over A.
(2) We say that $\bar{a} = (a_i)_{i\in\omega}$ is a *strict Morley sequence* over A if $\mathrm{tp}(a_i/a_{<i}A)$ extends to a global strictly invariant type, for each $i \in \omega$. In particular $\bar{a}$ is indiscernible.

It turns out that the notion of strict Morley sequence is the right one for generalizing Kim's lemma to NTP_2.

THEOREM 0.2.12. *Assume that* $\varphi(x,a)$ *divides over* M *and that* $(a_i)_{i\in\omega}$ *is a strict Morley sequence in* $\mathrm{tp}(a/M)$. *Then* $\{\varphi(x,a_i)\}_{i\in\omega}$ *is inconsistent.*

The only remaining (and the main technical) difficulty is to establish (using the so-called Broom lemma):

THEOREM 0.2.13. *For every* $M \models T$, *every* $p(x) \in S(M)$ *has a global strictly invariant extension.*

As an application we give a positive answer to a question of Adler in the case of NTP_2 theories:

THEOREM 0.2.14. T *is NIP if and only if it is* NTP_2 *and every type has only boundedly many non-forking extensions.*

In the last section we give examples demonstrating optimality of the results.

0.2.8. A weak independence theorem for NTP_2 theories (joint work with Itai Ben Yaacov). In this chapter we continue the development of the theory of forking calculus in NTP_2.

We begin by considering a multi-dimensional generalization of dividing, the so-called array-dividing.

DEFINITION 0.2.15.
(1) We say that $(a_{ij})_{i,j\in\kappa}$ is an *indiscernible array* over A if both $\left((a_{ij})_{j\in\kappa}\right)_{i\in\kappa}$ and $\left((a_{ij})_{i\in\kappa}\right)_{j\in\kappa}$ are indiscernible sequences.
(2) Let us say that a formula $\varphi(x,a)$ array-divides over A if there is an A-indiscernible array $(a_{ij})_{i,j\in\kappa}$ such that $a_{00} \equiv_A a$ and $\{\varphi(x,a_{ij})\}_{i,j\in\kappa}$ is inconsistent.

THEOREM 0.2.16. *Let* T *be* NTP_2 *and* A *an arbitrary set. Then* $\phi(x,a)$ *divides over* A *if and only if it array-divides over* A.

DEFINITION 0.2.17. We will say that forking in T satisfies the *chain condition* over A if: for any $I = (a_i)_{i\in\omega}$ indiscernible over A, assume that $\phi(x,a_0)$ does not fork over A. Then $\phi(x,a_0) \wedge \phi(x,a_1)$ does not fork over A.

This condition can be understood as saying that the forking ideal (in the Boolean algebra of definable sets) is "generically" prime (or equivalently that there are no anti-chains of non-forking formulas of unbounded size, hence the name).

PROBLEM 0.2.18. Adler / Hrushovski: what is the relation between NTP_2 and the chain condition?

On the one hand, combining the equivalence of dividing and array-dividing with the results of the previous chapter on strict invariance we get:

THEOREM 0.2.19. *Let* T *be* NTP_2 *and* A *an extension base. Then* T *satisfies the chain condition over* A.

On the other hand, we give an example of a theory with TP_2 satisfying the chain condition (in fact, we use one of the examples constructed in the last chapter of the thesis).

In his work on approximate subgroups [**Hru12**], Hrushovski had found a reformulation of the Independence theorem for simple theories with respect to an invariant $S1$-ideal for type with a global invariant extension.

Using the chain condition we prove a version of this theorem for forking over an arbitrary extension base in an NTP_2 theory.

THEOREM 0.2.20. *The Weak Independence Theorem. Let* T *be* NTP_2 *and* A *an extension base. Assume that* $c \mathop{\smile\!\!\!\!\!\!\mid}_A ab$, $a \mathop{\smile\!\!\!\!\!\!\mid}_A bb'$ *and* $b \equiv_A^{\mathrm{Lstp}} b'$. *Then there is* c' *such that* $c' \mathop{\smile\!\!\!\!\!\!\mid}_A ab'$, $c'a \equiv_A ca$, $c'b' \equiv_A cb$.

The usual independence theorem for simple theories easily follows from this one using symmetry of forking. As an application we deduce that Lascar strong type equals Kim-Pillay strong type over an extension base in an NTP_2 theory. It also follows that the stabilizer theorem of Hrushovski holds over models in NTP_2 theories.

In the last part of the chapter we discuss several possible generalizations of the notion of fundamental order to the class of NTP_2 theories, connections to existence of universal Morley sequences and some related conjectures.

The conclusion is that a large part of the forking calculus of simple theories, up to the independence theorem (recovering fully F_1 and the corresponding counterpart of F_3), can be redeveloped in a much larger class of NTP_2 theories when properly formulated (and giving the results for simple theories as easy special cases).

0.2.9. Burden, simple and NIP types, examples. In this chapter we continue investigating the class of NTP_2 theories. We begin by considering the notion of burden introduced by Adler (which is in fact a localization of Shelah's cardinal invariant κ_{inp} with respect to a type). It generalizes both weight in simple theories and dp-rank in NIP theories. A theory is NTP_2 if and only if every type has bounded burden.

We show that burden is sub-multiplicative, in any theory. More precisely,

THEOREM 0.2.21. *If* $\mathrm{bdn}(a) < \kappa$ *and* $\mathrm{bdn}(b/a) < \lambda$, *then* $\mathrm{bdn}(ab) < \kappa \times \lambda$.

This answers a question of Shelah from [**She90**]. In particular, it follows that if a theory has TP_2, then already some formula $\varphi(x, y)$ has TP_2 with x a singleton.

We elaborate on this topic and give an equivalent way of computing burden of a type as the supremum of lengths of strictly invariant sequences such that some realization of the type forks with all of its elements. Using it we show that in fact NTP_2 is characterized by the generalized Kim's lemma from the previous section, and that any theory in which dividing of a type is always witnessed by an instance of a dependent formula has to be NTP_2.

We continue with the analysis of two extremal kinds of types in NTP_2 theories — simple and NIP types.

- NIP types: Combining the results of the previous chapters on forking localized to an NTP_2 type with honest definitions from Chapter 4 we prove that dp-rank of a type in arbitrary theory is always witnessed by mutually indiscernible sequences of its realizations, after adding some parameters. So the dp-rank of a 1-type in any theory is always witnessed by sequences of singletons. We also observe that in an NTP_2 theory, a type is NIP if and only if every extension of it has only boundedly many global non-forking extensions (parallel to the characterization of stable types as those types every completion of which has a unique non-forking extension).
- Simple types are defined as those type for which every completion satisfies the local character. While it is more or less immediate that on the set of realizations of a simple type forking satisfies all the properties of forking in simple theories, the interaction between the realizations of a simple type and arbitrary tuples seems more intricate as a simple type need not be stably embedded. We establish full symmetry of forking between realizations of a simple type and arbitrary elements, answering a question of Casanovas in the case of NTP_2 theories (showing that simple types are co-simple). Then we show that simple types are characterized as those satisfying the co-independence theorem and that co-simple stably embedded types are simple (so in particular a theory is simple if and only if it is NTP_2 and satisfies the independence theorem).

In the final section of this chapter we give some new examples of NTP_2 theories. Most importantly we show:

THEOREM 0.2.22. *Let* $\bar{K} = (K, \Gamma, k)$ *be a henselian valued field in the Denef-Pas language. Assume that* k *is* NTP_2*, then* $\bar{K}$ *is* NTP_2*.*

In particular, any ultraproduct of p-adics is NTP_2 (actually strong of finite burden), while it is neither simple nor NIP even in the pure field language. We also demonstrate that adding a generic predicate to a geometric NTP_2 theory, in the sense of Chatzidakis and Pillay [**CP98**], preserves NTP_2.

0.2.10. Externally definable sets and dependent pairs (joint work with Pierre Simon). In the following two chapters we concentrate on the special case of NIP theories (or often NIP types in an arbitrary theory, without explicitly stressing it), trying to recover some elements of the definability of types from stable theories in this larger context.

Let M be a model of a theory T. An *externally definable* subset of M^k is an $X \subseteq M^k$ that is equal to $\phi(M^k, d)$ for some formula ϕ and d in some $N \succ M$. In a stable theory, by definability of types, any externally definable set coincides with some M-definable set. By contrast, in a random graph for example, any subset in dimension 1 is externally definable.

Assume now that T is NIP. A theorem of Shelah ([**Shed**]), generalizing a result of Poizat and Baisalov in the o-minimal case ([**BP98**]), states that the projection of an externally definable set is again externally definable. His proof does not give any information on the formula defining the projection. A slightly clarified account is given by Pillay in [**Pil07**].

In section 1, we show how this result follows from a stronger one: existence of honest definitions. An *honest definition* of an externally definable set is a formula $\phi(x, d)$ whose trace on M is X and which implies all M-definable subsets containing X. Then the projection of X can be obtained simply by taking the trace of the projection of $\phi(x, d)$.

Combining this notion with an idea from [**Gui11**], we can adapt honest definitions to make sense over any subset A instead of a model M. We obtain a property of *weak stable-embeddedness* of sets in NIP structures. Namely, consider a pair (M, A), where we have added a unary predicate $\mathbf{P}(x)$ for the set A. Take $c \in M$ and $\phi(x, c)$ a formula. We consider $\phi(A, c)$. If A is stably embedded, then this set is A-definable. Guingona shows that in an NIP theory, this set is externally A-definable, *i.e.*, coincides with $\psi(A, d)$ for some $\psi(x, y) \in L$ and $d \in A'$ where $(M', A') \succ (M, A)$. We strengthen this by showing that one can find such a $\phi(x, d)$ with the additional property that $\psi(x, d)$ never lies, namely $(M', A') \models \psi(x, d) \to \phi(x, c)$. In particular, the projection of $\psi(x, d)$ has the same trace on A as the projection of $\phi(x, c)$.

In the second part of the chapter we try to understand when dependence of a theory is preserved after naming a new subset by a predicate. We provide a quite general sufficient condition for dependence of the pair, in terms of the structure induced on the predicate and the restriction of quantification to the named set.

This question was studied for stable theories by a number of people (see [**CZ01**] and [**BB04**] for the most general results). In the last few years there has been a large number of papers proving dependence of some pair-like structures, e.g. [**BDO11**], [**GH11**], [**Box11**], etc. However, our approach differs in an important way from the previous ones, in that we work in a general NIP context and do not make any assumption of minimality of the structure (by asking for example that the algebraic closure controls relations between points). In particular, in the case of pairs of models, we obtain

THEOREM 0.2.23. *If M is NIP, $N \succ M$ and (N, M) is bounded (i.e. every formula is equivalent to one in which quantification is restricted to the predicate), then (N, M) is NIP.*

Those results seem to apply to most, if not all, of the pairs known to be dependent. It also covers some new cases, in particular answering a question of Baldwin and Benedikt [**BB00**] we establish:

THEOREM 0.2.24. *Let M be* NIP *and assume that* I *is a small indiscernible sequence. Then (M, I) is NIP.*

0.2.11. Externally definable sets and dependent pairs II (joint work with Pierre Simon).

In this chapter we continue the investigation of externally definable sets in NIP theories.

As it was established in the previous chapter, every externally definable set $X = \phi(x, b) \cap A$ has an *honest definition*, which can be seen as the existence of a uniform family of internally definable subsets approximating X. Formally, there is $\theta(x, z)$ such that for any *finite* $A_0 \subseteq X$ there is some $c \in A$ satisfying

$A_0 \subseteq \theta(A, c) \subseteq A$. The first section of this chapter is devoted to establishing existence of *uniform* honest definitions. By uniform we mean that $\theta(x, z)$ can be chosen depending just on $\phi(x, y)$ and not on A or b. We achieve this assuming that the whole theory is NIP, combining careful use of compactness with a strong combinatorial result of Alon-Kleitman [**AK92**] and Matousek [**Mat04**]: the (p, k)-theorem.

Recall the following classical fact characterizing stability of a formula.

FACT 0.2.25. *The following are equivalent:*
(1) $\phi(x, y)$ *is stable.*
(2) There is $\theta(x, z)$ *such that for any* A *and* a*, there is* $b \in A$ *satisfying* $\phi(A, a) = \theta(A, b)$.
(3) There are $m, n \in \omega$ *such that* $|S_\phi(A)| \leq m \cdot |A|^n$ *for any set* A.

DEFINITION 0.2.26. We say that $\phi(x, y)$ has UDTFS (Uniform Definability of Types over Finite Sets) if there is $\theta(x, z)$ such that for every finite A and a there is $b \in A$ such that $\phi(A, a) = \theta(A, b)$. We say that T satisfies UDTFS if every formula does.

If $\phi(x, y)$ has UDTFS, then it is NIP, thus naturally leading to the following conjecture

PROBLEM 0.2.27. [Laskowski] Assume that $\phi(x, y)$ is NIP, then it satisfies UDTFS.

It was proved for weakly o-minimal theories in [**JL10**] and for dp-minimal theories in [**Gui10**]. As an immediate corollary of the uniformity of honest definitions we prove the conjecture assuming that the whole theory is NIP,

THEOREM 0.2.28. *Let* T *be* NIP*. Then it satisfies* UDTFS.

In the next section we consider an implication of the (p, k)-theorem for forking in NIP theories. Combined with the results on forking and dividing from the first chapter, we deduce the following

THEOREM 0.2.29. *Working over a model* M*, let* $\{\phi(x, a) : a \models q(y)\}$ *be a family of non-forking instances of* $\phi(x, y)$*, where the parameter* a *ranges over the set of solutions of a partial type* q*. Then there are finitely many global* M*-invariant types such that each* $\phi(x, a)$ *from the family belongs to one of them.*

In Section 3 we return to the question of naming subsets with a new predicate. In the previous section we gave a general condition for the expansion to be NIP: it is enough that the theory of the pair is *bounded*, i.e. eliminates quantifiers down to the predicate, and the induced structure on the predicate is NIP. Here, we try to complement the picture by providing a general sufficient condition for the boundedness of the pair. In the stable case the situation is quite neatly resolved using the notion of nfcp. However nfcp implies stability, so one has to come up with some generalization of it that is useful in unstable NIP theories. Towards this purpose we introduce *dnfcp*, i.e. no finite cover property for definable sets of parameters, and its relative version with respect to a set. We also introduce dnfcp' – a weakening of dnfcp with separated variables. Using it, we succeed in the distal, stably embedded, case: if one names a subset of M which is small, uniformly stably embedded and the induced structure satisfies dnfcp', then the pair is bounded.

In section 4 we look at the special case of naming an indiscernible sequence. On the one hand, we complement the result in the previous chapter by showing that naming a small indiscernible sequence of *arbitrary* order type is bounded and preserves NIP. On the other hand, naming a large indiscernible sequence does not.

In the last section we consider models over which all types are definable. While in general even o-minimal theories may not have such models, many interesting NIP theories do (RCF, ACVF, Th($\mathbb{Q}_p$), Presburger arithmetic...). In practice, it is often much easier to check definability of 1 types, as opposed to n-types, so it is natural to ask whether one implies the other. Unfortunately, this is not true – we give an NIP counter-example. Can anything be said on the positive side? Pillay [**Pil11**] had established: let M be NIP, $A \subseteq M$ be definable with rosy induced structure. Then if it is 1-stably embedded, it is stably embedded. We observe that Pillay's results holds when the definable set A is replaced with a model, assuming that it is *uniformly* 1-stably embedded. This provides a generalization of the classical theorem of Marker and Steinhorn about definability of types over models in o-minimal theories. We also remark that in NIP theories, there are arbitrary large models with "few" types over them (i.e. such that $|S(M)| \leq |M|^{|T|}$).

0.2.12. On the number of non-forking extensions (joint with Itay Kaplan and Saharon Shelah).

The final chapter is devoted to the question of how many non-forking extension can a type have, in an arbitrary theory. More precisely, we consider the following function.

DEFINITION 0.2.30. For a complete countable first-order theory T and cardinals $\kappa \leq \lambda$, we let

$$f_T(\kappa, \lambda) = \sup \left\{ S^{\mathrm{nf}}(N, M) \,|\, M \preceq N \models T, |M| \leq \kappa, |N| \leq \lambda \right\},$$

where $S^{\mathrm{nf}}(A, B) = \{p \in S_1(A) \,|\, p \text{ does not fork over } B\}$.

This is a generalization of the classical question "how many types can a theory have?". Recall that the stability function of a theory is defined as $f_T(\kappa) = \sup\{S(M) \,|\, M \models T, |M| = \kappa\}$. It is easy to see that $f_T(\kappa, \kappa) = f_T(\kappa)$. This function has been studied extensively by Keisler and Shelah, and the following fundamental result was proved:

FACT 0.2.31. *For any complete countable first-order theory* T*,* f_T *is one of the following:* κ*,* $\kappa + 2^{\aleph_0}$*,* $\kappa^{\aleph_0}$*,* $\mathrm{ded}(\kappa)$*,* $\mathrm{ded}(\kappa)^{\aleph_0}$*,* 2^κ*.*

Where $\mathrm{ded}(\kappa)$ is the supremum of the number of cuts that a linear order of size κ may have. While this result is unconditional, in some models of ZFC, some of these functions may coincide. Namely, if GCH holds, $\mathrm{ded}(\kappa) = \mathrm{ded}(\kappa)^{\aleph_0} = 2^\kappa$. By a result of Mitchell [**Mit73**], it was known that for any cardinal κ with $\mathrm{cof}\,\kappa > \aleph_0$ consistently $\mathrm{ded}(\kappa) < 2^\kappa$. In 1976, Keisler [**Kei76**, Problem 2] asked:

PROBLEM 0.2.32. Is $\mathrm{ded}(\kappa) < \mathrm{ded}(\kappa)^{\aleph_0}$ consistent with ZFC?
We give a positive answer.

However, the main aim of this chapter is to classify the possibilities of $f_T(\kappa, \lambda)$. The philosophy of "dividing lines" suggests that the possible non-forking spectra are quite far from being arbitrary, and that there should be finitely many possible functions, distinguished by the lack (or presence) of certain combinatorial configurations. We work towards justifying this philosophy and arrive at the following picture.

THEOREM 0.2.33. *Let T be a countable complete first-order theory. Then for $\lambda \gg \kappa$, $f_T(\kappa, \lambda)$ can be one of the following, in increasing order (meaning that we have an example for each item in the list except for (13), and "???" means that we don't know if there is anything between the previous and the next item, while the lack of "???" means that there is nothing in between):*

(1) κ
(2) $\kappa + 2^{\aleph_0}$
(3) $\kappa^{\aleph_0}$
(4) $\operatorname{ded} \kappa$
(5) *???*
(6) $(\operatorname{ded} \kappa)^{\aleph_0}$
(7) $2^{2^{\kappa}}$
(8) λ
(9) $\lambda^{\aleph_0}$
(10) *???*
(11) $\lambda^{<\beth_{\aleph_1}(\kappa)}$
(12) $\operatorname{ded} \lambda$
(13) *???*
(14) $(\operatorname{ded} \lambda)^{\aleph_0}$
(15) *???*
(16) 2^{λ}

In particular, we note that the existence of an example of $f_T(\kappa, \lambda) = 2^{2^{\kappa}}$ answers negatively a question of Adler [**Adl08**, Section 6] whether NIP is equivalent to bounded non-forking in general (compare with Theorem 0.2.14).

The restriction $\lambda \gg \kappa$ is in order to make the statement clearer. It can be taken to be $\lambda \geq \beth_{\aleph_1}(\kappa)$. In fact we can say more about smaller λ in some cases. In the class of NTP$_2$ theories, we have a much nicer picture, meaning that there is a gap between (6) and (20).

In the first part of the chapter, we prove that the non-forking spectra cannot take values which are not listed in the Main Theorem. The proofs here combine techniques from generalized stability theory (including results on stable and NIP theories, splitting and tree combinatorics) with a two cardinal theorem for $L_{\omega_1, \omega}$.

The second part of the chapter is devoted to examples.

We introduce a general construction which we call *circularization*. Roughly speaking, the idea is the following: modulo some technical assumptions, we start with an arbitrary theory T_0 in a finite relational language and an (essentially) arbitrary prescribed set of formulas F. We expand T by putting a circular order on the set of solutions of each formula in F, iterate the construction and take the limit. The point is that in the limit all the formulas in F are forced to fork, and we have gained some control on the set of non-forking types. This construction turns out to be quite flexible: by choosing the appropriate initial data, we can find a wide range of examples of non-forking spectra previously unknown.

CHAPTER 1

Forking and dividing in NTP_2 theories

This chapter is a joint work with Itay Kaplan and is published as "Forking and dividing in NTP_2 theories", J. Symbolic Logic, 77(1):1–20, 2012 [**CK12**].

We prove that in theories without the tree property of the second kind (which include dependent and simple theories) forking and dividing over models are the same, and in fact over any extension base. As an application we show that dependence is equivalent to bounded non-forking assuming NTP_2.

1.1. Introduction

Background.
The study of forking in the dependent (NIP) setting was initiated by Shelah in full generality [**She09**] and by Dolich in the case of nice $\mathfrak{o}$-minimal theories [**Dol04a**]. Further results appear in [**Adl08**], [**HP11**], [**OU11**] and [**Sta**]. The main trouble is that apparently non-forking independence outside of the simple context no longer corresponds to a notion of dimension in any possible way. Moreover it is neither symmetric nor transitive (at least in the classical sense). However in dependent theories it corresponds to invariance of types, which is undoubtedly a very important concept, and it is a meaningful combinatorial tool.

Main results.
The crucial property of forking in simple theories is that it equals dividing (thus the useful concept – forking – becomes somewhat more understandable in real-life situations). It is known that there are dependent theories in which forking does not equal dividing in general (for example in circular order over the empty set, see section 1.5). However there is a natural restatement of the question due to Anand Pillay: whether forking and dividing are equal over models? After failing to find a counter-example we decided to prove it instead. And so the main theorem of the paper is:

THEOREM 1.1.1. *Let T be an NTP_2 theory (a class which includes dependent and simple theories). Then forking and dividing over models are the same – a formula $\varphi(x, a)$ forks over a model M if and only if it divides over it.*

In fact, a more general result is attained. Namely that:

THEOREM 1.1.2. *Let T be NTP_2. Then for a set A, the following are equivalent:*

(1) A is an extension base for $\mathop{\smile\!\!\!\!\!\!|}^{f}$ (non-forking) (see definition 1.2.7).
(2) $\mathop{\smile\!\!\!\!\!\!|}^{f}$ has left extension over A (see definition 1.2.4).
(3) Forking equals dividing over A (i.e. a formula $\varphi(x, b)$ divides over A iff if forks over A).

So theorem 1.1.1 is a corollary of 1.1.2 (types over models are finitely satisfiable, so (1) is true), and of course:

COROLLARY 1.1.3. *If T is NTP_2 and all sets are extension bases for non-forking, then forking equals dividing. (This class contains simple theories, o-minimal and c-minimal theories).*

The idea of the proof.

The idea is to generalize the proof of the theorem in simple theories. There, "Kim's lemma" was the main tool. The lemma says, that in a simple theory, if $\varphi(x, a)$ divides over A, then *every* Morley Sequence over A (i.e. an indiscernible sequence $\langle a_i \,| i < \omega \rangle$ such that for all $i < \omega$, $\mathrm{tp}(a_i / A a_0 \dots a_{i-1})$ does not fork over A and $a_i \equiv_A a$) witnesses this. As there is no problem to construct Morley sequences over any set, one shows that forking equals dividing by constructing a Morley sequence that starts with the parameters of the formulas witnessing forking.
To prove the parallel result in the NTP_2 context, we find a new notion of independence, $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ist}}$ such that every $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ist}}$-Morley sequence witnesses dividing. Then we show that this notion satisfies "existence over a model", i.e. that for every a, $a \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ist}}_M M$. For this we shall need the so-called "broom lemma". Essentially it says that if a formula is covered by finitely many formulas arranged in a "nice position", then we can throw away the dividing ones, by passing to an intersection of finitely many conjugates.

Applications.

We give some corollaries, among them that in dependent theories forking is type definable, has left extension over models (answering a question of Itai Ben Yaacov), and that if p is a global φ type which is invariant over a model, then it can be extended to a global type invariant over the same model (strengthening a result that appears in [**HP11**]).
Hans Adler asked in [**Adl08**] whether NIP is equivalent to boundedness of non-forking. In section 1.4 we show that assuming NTP_2, this is indeed the case. This generalizes a well-known analogous result describing the subclass of stable theories inside the class of simple theories. Finally in section 1.5 we present 2 examples that show that the NTP_2 assumption is needed, and explain why we work over models. These are variants of an example due to Martin Ziegler of a theory in which forking does not equal dividing over models.

Further remarks.

In Chapter 6, we give an example of a theory with IP, such that forking is bounded (moreover, a global type does not fork over a set iff it is finitely satisfiable in this set). This, together with the result appearing in section 1.4, completely solves Adler's question from [**Adl08**] mentioned above.

Acknowledgments.

We would like to thank Alex Usvyatsov for many helpful discussions and comments, Martin Ziegler for allowing us to include his example, Itai Ben Yaacov, Frank Wagner, the entire Lyon logic group for a very fruitful atmosphere, and the MATHLOGAPS/MODNET networks which made our collaboration possible.
The first author thanks Berlin logic group for organizing the seminar on NIP which triggered his interest for the questions concerned.

Finally, both authors are grateful to Vissarion for leading their thoughts in the right direction.

1.2. Preliminaries

Notation.

Notations are standard.
As usual, T is a first order theory; $\mathfrak{C}$ is the monster model (a big saturated model); all sets are subsets of $\mathfrak{C}$ of size smaller than $|\mathfrak{C}|$ and all models are elementary substructures of $\mathfrak{C}$.
We shall not always distinguish between sets and sequences, i.e. a can be a singleton, a set, an n-tuple or a sequence of any length of members of $\mathfrak{C}$.
The variables x, y are singletons or finite sequences.
For sets A, B we write AB for the union, and for an element (or a sequence) a, we write Aa for $A \cup \{a\}$ (or $A \cup \operatorname{im}(a)$). In some contexts, ab will denote the concatenation of the sequences a and b (for instance when we write $ab \equiv cd$).
For us, I, J denote infinite sequences.
A global type is a type over $\mathfrak{C}$.

Preliminaries on dependent theories.

Let us recall:

Definition 1.2.1. A theory T has the *independence property* if there is a formula $\phi(x, y)$ and tuples $\{a_i \,|\, i < \omega\}$, $\{b_u \,|\, u \subseteq \omega\}$ (in $\mathfrak{C}$) such that $\phi(a_i, b_u)$ if and only if $i \in u$. T is *dependent* iff it does not have the independence property (also known as *NIP*).

Definition 1.2.2. The alternation rank of a formula: $\operatorname{alt}(\varphi(x, y)) =$

$$= \max\{n < \omega \,|\, \exists \langle a_i \,|\, i < \omega\rangle \text{ indiscernible, } \exists b : \varphi(a_i, b) \leftrightarrow \neg\varphi(a_{i+1}, b) \text{ for } i < n-1\}$$

Fact 1.2.3. T *is dependent iff every formula has finite alternation rank.*

To the best of our knowledge, this fact first appeared in [**Poi81**], and is an easy exercise in the definition.

Pre-independence relations, dividing and forking.

To make the presentation clearer, we chose to follow the style of Adler in [**Adl05**], and define an abstract notion of independence. By a pre-independence relation we shall mean a ternary relation $\mathop{\smile\!\!\!\!\!\!|}$ on sets which satisfies one or more of the properties below. For a more general definition of a pre-independence relation see e.g. [**Adl08**, Section 5]. Note that since normally our relation is not symmetric many properties can be formulated both on the left side and on the right side.

Definition 1.2.4. A pre-independence relation $\mathop{\smile\!\!\!\!\!\!|}$ is an invariant ternary relation on sets. We write $a \mathop{\smile\!\!\!\!\!\!|}_A b$ for: a is $\mathop{\smile\!\!\!\!\!\!|}$-independent from b over A. The following are the properties we consider for a pre-independence relation:

(1) Monotonicity: If $aa' \mathop{\smile\!\!\!\!\!\!|}_A bb'$ then $a \mathop{\smile\!\!\!\!\!\!|}_A b$.
(2) Base monotonicity: If $a \mathop{\smile\!\!\!\!\!\!|}_A bc$ then $a \mathop{\smile\!\!\!\!\!\!|}_{Ab} c$.
(3) Transitivity on the left (over A): $a \mathop{\smile\!\!\!\!\!\!|}_{Ab} c$ and $b \mathop{\smile\!\!\!\!\!\!|}_A c$ implies $ab \mathop{\smile\!\!\!\!\!\!|}_A c$.
(4) Right extension (over A): if $a \mathop{\smile\!\!\!\!\!\!|}_A b$ then for all c there is $c' \equiv_{Ab} c$ such that $a \mathop{\smile\!\!\!\!\!\!|}_A bc'$.

(5) Left extension (over A): if $a \mathop{\smile\!\!\!\!\!\!\>|}_A b$ then for all c there is $c' \equiv_{Aa} c$ such that $ac' \mathop{\smile\!\!\!\!\!\!\>|}_A b$.

Remark 1.2.5. We shall not discuss independence relations, but for completeness we mention that an independence relation is a pre-independence relation that satisfies (1) – (3) and symmetry (i.e. $a \mathop{\smile\!\!\!\!\!\!\>|}_A b$ iff $b \mathop{\smile\!\!\!\!\!\!\>|}_A a$).

Definition 1.2.6. We say that a pre-independence relation is *standard* if it satisfies (1) – (4) from definition 1.2.4.

Definition 1.2.7. We say that A is an extension base for a pre-independence relation $\mathop{\smile\!\!\!\!\!\!\>|}$ if for all a, $a \mathop{\smile\!\!\!\!\!\!\>|}_A A$.

Now let us recall the definition of forking and dividing.

Definition 1.2.8. (dividing) Let A be be a set, and a a tuple. We say that the formula $\varphi(x, a)$ *divide*s over A iff there is a number $k < \omega$ and tuples $\{a_i \mid i < \omega\}$ such that

(1) $\operatorname{tp}(a_i/A) = \operatorname{tp}(a/A)$.
(2) The set $\{\varphi(x, a_i) \mid i < \omega\}$ is k-inconsistent (i.e. every subset of size k is not consistent).

In this case, we say that a formula k-divides.

Remark 1.2.9. From Ramsey and compactness it follows that $\varphi(x, a)$ divides over A iff there is an indiscernible sequence over A, $\langle a_i \mid i < \omega \rangle$ such that $a_0 = a$ and $\{\varphi(x, a_i) \mid i < \omega\}$ is inconsistent.

Definition 1.2.10. We say that a type p divides over A iff there is a finite conjunction of formulas from p which divides over A. The notation $a \mathop{\smile\!\!\!\!\!\!\>|}^d_A b$ means $\operatorname{tp}(a/Ab)$ does not divide over A.

Fact 1.2.11. *(see* [**She80**, 1.4]*) The following are equivalent for every* T*:*

(1) $a \mathop{\smile\!\!\!\!\!\!\>|}^d_A b$.
(2) For every indiscernible sequence I *over* A *such that* $b \in I$*, there is an indiscernible sequence* I' *such that* $I' \equiv_{Ab} I$ *and* I' *is indiscernible over* Aa.
(3) For every indiscernible sequence I *over* A *such that* $b \in I$*, there is* a' *such that* $a' \equiv_{Ab} a$ *and* I *is indiscernible over* Aa'.

Definition 1.2.12. (forking) Let A be be a set, and a a tuple.

(1) Say that the formula $\varphi(x, a)$ *forks* over A if there are formulas $\psi_i(x, a_i)$ for $i < n$ such that $\varphi(x, a) \vdash \bigvee_{i<n} \psi_i(x, a_i)$ and $\psi_i(x, a_i)$ divides over A for every $i < n$.
(2) Say that a type p forks over A if there is a finite conjunction of formulas from p which forks over A.
(3) The notation $a \mathop{\smile\!\!\!\!\!\!\>|}^f_A b$ means: $\operatorname{tp}(a/Ab)$ does not fork over A.

Note that:

Remark 1.2.13.

(1) If $\varphi(x, a)$ divides over A then it forks over A.
(2) If $M \supseteq A$ is an $|A|^+$ saturated model and $p \in S(M)$ does not divide over A, then it does not fork over A.

Remark 1.2.14. $\mathop{\smile\!\!\!\!\!\!|}^{f}$ is standard (see, e.g. [**Adl08**, section 5]).

Two other pre-independence relations we shall use are $\mathop{\smile\!\!\!\!\!\!|}^{u}$ (finite satisfiability – the u comes from "ultrafilter"), and $\mathop{\smile\!\!\!\!\!\!|}^{i}$ (invariance).

Definition 1.2.15. We write $a \mathop{\smile\!\!\!\!\!\!|}^{u}_{A} b$ when $\operatorname{tp}(a/Ab)$ is finitely satisfiable in A.

Remark 1.2.16. $\mathop{\smile\!\!\!\!\!\!|}^{u}$ is standard and satisfies left extension over models. Every model is an extension base for $\mathop{\smile\!\!\!\!\!\!|}^{u}$.

Proof. The fact that $\mathop{\smile\!\!\!\!\!\!|}^{u}$ is standard can be seen in e.g. [**Adl08**, section 5]. For left extension over models: Consider inheritance ($\mathop{\smile\!\!\!\!\!\!|}^{h}$) over a model M: $a \mathop{\smile\!\!\!\!\!\!|}^{h}_{M} b$ iff $\operatorname{tp}(a/Mb)$ is an heir over M, iff $b \mathop{\smile\!\!\!\!\!\!|}^{u}_{M} a$. It is well known that $\mathop{\smile\!\!\!\!\!\!|}^{h}$ satisfies right extension over models, so the result follows. The fact that every model is an extension base follows from the fact that filters can be extended to ultrafilters. $\square$

Let us recall the definition of Lascar strong types.

Definition 1.2.17. $\operatorname{Aut} f_L(\mathfrak{C}/A)$ is the subgroup of all automorphisms of $\mathfrak{C}$ generated by the set $\{f \in \operatorname{Aut}(\mathfrak{C}/M) \mid M \supseteq A \text{ is some small model}\}$. We write $a \equiv^L_A b$ (a is Lascar equivalent to b, or a and b have the same Lascar strong type) if there is $\sigma \in \operatorname{Aut} f_L(\mathfrak{C}/A)$ taking a to b.

Fact 1.2.18. *(See e.g. in* [**Ker07**]*) The relation $\equiv^L_A$ is an equivalence relation, and in fact it is the finest invariant equivalence relation with boundedly many classes. It is also defined as the transitive closure of the relation $E(a,b)$ saying that there is an indiscernible sequence over A containing both a and b.*

Now we can define another pre-independence relation:

Definition 1.2.19. We say that $a \mathop{\smile\!\!\!\!\!\!|}^{i}_{A} b$ iff there is is a global type p extending $\operatorname{tp}(a/Ab)$ which is Lascar invariant over A: for every c, d such that $c \equiv^L_A d$ and every formula $\varphi(x,y)$ over A, $\varphi(x,c) \in p$ iff $\varphi(x,d) \in p$.

Remark 1.2.20. In general, by Fact 1.2.18, if I is an indiscernible sequence over A and $a \mathop{\smile\!\!\!\!\!\!|}^{i}_{A} I$ then I is indiscernible over Aa. So $a \mathop{\smile\!\!\!\!\!\!|}^{i}_{A} b$ iff for every finitely many indiscernible sequences over A, $I_1, \ldots, I_n$, there are sequences $I'_1, \ldots, I'_n$ such that $\langle I'_1 \ldots I'_n \rangle \equiv_{Ab} \langle I_1 \ldots I_n \rangle$ and I'_i is indiscernible over Aa. Hence, it is easy to see that $\mathop{\smile\!\!\!\!\!\!|}^{i}$ is standard. For more details, see [**Adl08**, Corollary 35].
In addition, over a model M, $\mathop{\smile\!\!\!\!\!\!|}^{i}_{M}$ is non-splitting (invariance) – $a \mathop{\smile\!\!\!\!\!\!|}^{i}_{M} b$ iff $\operatorname{tp}(a/Mb)$ can be extended to a global invariant type over M.

Definition 1.2.21. We say that $\mathop{\smile\!\!\!\!\!\!|}$ is at least as strong as $\mathop{\smile\!\!\!\!\!\!|}'$ if for every a, b and A, $a \mathop{\smile\!\!\!\!\!\!|}_{A} b \Rightarrow a \mathop{\smile\!\!\!\!\!\!|}'_{A} b$.

Example 1.2.22. $\mathop{\smile\!\!\!\!\!\!|}^{u}$ is at least as strong as $\mathop{\smile\!\!\!\!\!\!|}^{i}$ which is at least as strong as $\mathop{\smile\!\!\!\!\!\!|}^{f}$. See claim below.

By the remark above, when $\mathop{\smile\!\!\!\!\!\!|}$ is at least as strong as $\mathop{\smile\!\!\!\!\!\!|}^{i}$, if I is indiscernible over A and $a \mathop{\smile\!\!\!\!\!\!|}_{A} I$ then I is indiscernible over Aa. In this case, we'll say that

$\mathop{\smile\!\!\!\!\!\!|}$ preserves indiscernibility. In fact, these two are equivalent (i.e. to preserve indiscernibility and to be as strong as $\mathop{\smile\!\!\!\!\!\!|}^{i}$) for standard pre-independence relations: it follows from right extension and the criterion given in 1.2.20.

REMARK 1.2.23. If N is $|A|^{+}$ saturated, and $p \in S(N)$ is an A-invariant type, then p has a unique extension to a global A-invariant type.

CLAIM 1.2.24. $\mathop{\smile\!\!\!\!\!\!|}^{i}$ is at least as strong as $\mathop{\smile\!\!\!\!\!\!|}^{f}$. If T is dependent, then $\mathop{\smile\!\!\!\!\!\!|}^{i} = \mathop{\smile\!\!\!\!\!\!|}^{f}$.

PROOF. The first statement is clear, and the second appears in [**She09**] and also in [**Adl08**]. □

Generating indiscernible sequences.

Recall the following fact:

FACT 1.2.25. *Assume that p is global A-invariant type. Then p generates an indiscernible sequence over A: $a_0 \models p|_A$, $a_{i+1} \models p|_{Aa_0\dots a_i}$. The type of this indiscernible sequence depends only on p, and will be denoted by $p^{(\omega)}|_A \in S^{(\omega)}(A)$. The type we get after n steps is denoted by $p^{(n)}|_A \in S^{n}(A)$.*

DEFINITION 1.2.26.

(1) A type p is $\mathop{\smile\!\!\!\!\!\!|}$-free over A if for any b such that $Ab \subseteq \mathrm{dom}(p)$ and every $a \models p|_{Ab}$, $a \mathop{\smile\!\!\!\!\!\!|}_A b$.

(2) A Morley sequence $\langle a_i \,|i < \omega\rangle$ for $\mathop{\smile\!\!\!\!\!\!|}$ with base A over $B \supseteq A$ is an indiscernible sequence over B, such that for all i, $a_i \mathop{\smile\!\!\!\!\!\!|}_A Ba_0 \dots a_{i-1}$.

Note that if a global type p is $\mathop{\smile\!\!\!\!\!\!|}$-free and invariant over A, then for every $B \supseteq A$, the sequence p generates over B is a Morley sequence with base A over B.

NTP$_2$ **Theories.**

DEFINITION 1.2.27. A theory T has TP$_2$ (the tree property of the second kind) if there exists a formula $\varphi(x, y)$, a number $k < \omega$ and an array of elements $\left\langle a_i^j \,|i,j < \omega \right\rangle$ (in $\mathfrak{C}$) such that:

- Every row is k-inconsistent: for every $i < \omega$ and $j_0, \dots, j_{k-1} < \omega$, $\mathfrak{C} \models \neg\left(\exists x \bigwedge_{l<k} \varphi\left(x, a_i^{j_l}\right)\right)$.
- Every vertical path is consistent: for every function $\eta : \omega \to \omega$, the set $\left\{\varphi\left(x, a_{i,\eta(i)}\right) |i < \omega\right\}$ is consistent.

We say that T is NTP$_2$ when it does not have TP$_2$.

FACT 1.2.28. *Every dependent theory as well as every simple one is* NTP$_2$.

PROOF. The tree property of the second kind implies the tree property (so every simple theory is NTP$_2$) and the Independence property. □

The tree property of the second kind was defined in [**She80**]. There it is proved that a theory is non-simple (has the tree property) iff it has the tree property of the first kind (which we shall not define here) or the the tree property of the second kind.

1.3. Main results

1.3.1. The Broom lemma.

We start with the main technical lemma. Here there are no assumptions on T.

LEMMA 1.3.1. *Suppose that* $\mathop{\smile\!\!\!\!\!\!|}$ *satisfies all properties from 1.2.4 but we demand that it satisfies left extension only over* A*, and in addition that it preserves indiscernibility. Assume that*

$$\alpha(x,e)\vdash\psi(x,c)\vee\bigvee_{i<n}\varphi_i(x,a_i)$$

where

(1) *For* $i<n$*, the formula* $\varphi_i(x,a_i)$ k*-divides over* A*, as witnessed by the indiscernible sequence* $I_i=\langle a_{i,l}\,|l<\omega\rangle$ *where* $a_{i,0}=a_i$*.*
(2) *For each* $i<n$ *and* $1\leq l$*,* $a_{i,l}\mathop{\smile\!\!\!\!\!\!|}_A a_{i,<l}I_{<i}$ *where* $a_{i,<l}=a_{i,0}\dots a_{i,l-1}$*, and* $I_{<i}=I_0\dots I_{i-1}$*.*
(3) $c\mathop{\smile\!\!\!\!\!\!|}_A I_{<n}$*.*

Then for some $m<\omega$ *there is* $\{e_i\,|i<m\}$ *with* $e_i\equiv_A e$ *for* $i<m$ *and* $\bigwedge_{i<m}\alpha(x,e_i)\vdash\psi(x,c)$*. In particular, if* $\psi(x,c)=\bot$ *(i.e.* $\forall x(x\neq x)$*), then* $\{\alpha(x,e_i)\,|i<m\}$ *is inconsistent.*

PROOF. By induction on n. For $n=0$ there is nothing to prove.
Assume that the claim is true for n and we prove it for $n+1$. Let $b_0=a_{n,0}\dots a_{n,k-2}$ and $b_1=a_{n,1}\dots a_{n,k-1}$ (where k is from (1)). Since $\mathop{\smile\!\!\!\!\!\!|}$ preserves indiscernibility, as $c\mathop{\smile\!\!\!\!\!\!|}_A I_n$ we have

$$cb_1\equiv_A cb_0.$$

We build by induction on $j<k$ sequences $\left\langle I^{l,j}_{<n}\,|l\leq j\right\rangle$ (so $I^{l,j}_{<n}=I^{l,j}_0\dots I^{l,j}_{n-1}$) such that:

(1) $I^{l,j}_{<n}=I^{l,j}_0\dots I^{l,j}_{n-1}$ and each $I^{l,j}_i$ is of the same length as I_i,
(2) $I^{0,j}_{<n}=I_{<n}$.
(3) $I^{l,j}_{<n}ca_{n,l}\equiv_A I^{0,j}_{<n}ca_{n,0}$ for all $l\leq j$ and
(4) For all $0\leq l<j$, $cI^{j,j}_{<n}I^{j-1,j}_{<n}\dots I^{l+1,j}_{<n}\mathop{\smile\!\!\!\!\!\!|}_A I^{l,j}_{<n}$ and $c\mathop{\smile\!\!\!\!\!\!|}_A I^{j,j}_{<n}$ (which already follows from the previous clauses).

For $j=0$, use (2): $I^{0,0}_{<n}=I_{<n}$.
So suppose we have this sequence for j and we build it for $j+1<k$.
By (2), let $I^{0,j+1}_{<n}=I_{<n}$.
As $cb_1\equiv_A cb_0$ we can find some $J^{l,j+1}_{<n}$ for $1\leq l\leq j+1$ so that:

$$J^{j+1,j+1}_{<n}J^{j,j+1}_{<n}\dots J^{1,j+1}_{<n}cb_1\equiv_A I^{j,j}_{<n}I^{j-1,j}_{<n}\dots I^{0,j}_{<n}cb_0. \tag{I}$$

By transitivity on the left and base monotonicity (and by (2)) we have $cb_1\mathop{\smile\!\!\!\!\!\!|}_A a_{n,0}I_{<n}$, and by left extension we can find $\left\langle I^{l,j+1}_{<n}\,|1\leq l\leq j+1\right\rangle$ such that

$$I^{j+1,j+1}_{<n}I^{j,j+1}_{<n}\dots I^{1,j+1}_{<n}cb_1\equiv_A J^{j+1,j+1}_{<n}J^{j,j+1}_{<n}\dots J^{1,j+1}_{<n}cb_1 \tag{II}$$

and

$$\left\langle I^{l,j+1}_{<n}\,|1\leq l\leq j+1\right\rangle cb_1\mathop{\smile\!\!\!\!\!\!|}_A a_{n,0}I_{<n}. \tag{III}$$

And so we have constructed $\left\langle I_{<n}^{l,j+1} \mid l \leq j+1 \right\rangle$.
Note that from equations (I) and (II) it follows that

$$\text{(IV)} \qquad I_{<n}^{j+1,j+1} I_{<n}^{j,j+1} \dots I_{<n}^{1,j+1} cb_1 \equiv_A I_{<n}^{j,j} I_{<n}^{j-1,j} \dots I_{<n}^{0,j} cb_0.$$

Now to check that we have our conditions satisfied:
(1) and (2) follows directly from construction.
(3): First of all, $I_{<n} ca_{n,0} \equiv_A I_{<n}^{1,j+1} ca_{n,1}$ by equation (IV). For $1 \leq l \leq j$,

$$I_{<n} ca_{n,0} \equiv_A I_{<n}^{l,j} ca_{n,l}$$

by the hypothesis regarding j. By (IV),

$$I_{<n}^{l,j} ca_{n,l} \equiv_A I_{<n}^{l+1,j+1} ca_{n,l+1}$$

and so we have (3) for $l \leq j+1$.
(4) follows from (III), the invariance of $\downarrow$ and induction.
So, for $j = k-1$ we have $\langle I_{<n}^{l,k-1} \mid l \leq k-1 \rangle$. We shall now use only this last sequence.
There are some $\langle e_l \mid l < k \rangle$ such that $e_0 = e$ and for $0 < l < k$, $e_l I_{<n}^{l,k-1} ca_{n,l} \equiv_A eI_{<n} ca_{n,0}$, so applying some automorphism fixing Ac, we replace $a_{n,0}$ by $a_{n,l}$, e by e_l and $I_{<n}$ by $I_{<n}^{l,k-1}$. So we get

$$\alpha(x, e_l) \vdash \psi(x, c) \vee \bigvee_{i<n} \varphi_i\left(x, a_i^{l,k-1}\right) \vee \varphi_n(x, a_{n,l})$$

where $a_i^{l,k-1}$ starts $I_i^{l,k-1}$. Hence $\alpha^0 = \bigwedge_{l<k} \alpha(x, e_l)$ implies the conjunction of these formulas. But as I_n witnesses that $\varphi_n(x, a_n)$ is k dividing, we have the following:

$$\alpha^0 \vdash \psi(x, c) \vee \bigvee_{i<n, l<k} \varphi_i\left(x, a_i^{l,k-1}\right).$$

Define a new formulas $\psi^r(x, c^r) = \psi(x, c) \vee \bigvee_{i<n, r \leq l<k} \varphi_i\left(x, a_i^{l,k-1}\right)$ for $r \leq k$. By induction on $r \leq k$, we find α^r such that α^r is a conjunction of conjugates over A of $\alpha(x, e)$, and $\alpha^r \vdash \psi^r(x, c^r)$. It will follow of course, that $\alpha^k \vdash \psi(x, c)$ as desired. For $r = 0$, we already found α^0. Assume we found α^r, so we have

$$\alpha^r \vdash \psi^{r+1}\left(x, c^{r+1}\right) \vee \bigvee_{i<n} \varphi_i\left(x, a_i^{r,k-1}\right)$$

One can easily see that the hypothesis of the lemma is true for this implication (where $c = c^{r+1}$, and $I_i = I_i^{r,k-1}$) so by the induction hypothesis (on n), there is some α^{r+1}(which is a conjunction of conjugates of α^r over A, and so also of α) such that $\alpha^{r+1} \vdash \psi^{r+1}\left(x, c^{r+1}\right)$. $\square$

Definition 1.3.2. We say that a formula $\alpha(x, e)$ quasi-divides over A if there are $m < \omega$ and $\{e_i \mid i < m\}$ such that $e_i \equiv_A e$ and $\{\alpha(x, e_i) \mid i < m\}$ is inconsistent.

So this lemma shows that under certain conditions, a forking formula also quasi-divides.

Remark 1.3.3. The name of this lemma is due to its method of proof, which reminded the authors (and also Itai Ben Yaacov who thought of the name) of a sweeping operation.

1.3.2. On pre-independence relations in NTP_2.

Existence of global free co-free types.
The title of this section may seem a bit mysterious, but it will become clearer with the next Proposition. Let T be any theory.

DEFINITION 1.3.4. Let $\downfree$ be a pre-independence relation. We say that $\downfree$ has *finite character* if whenever $a \not\downfree_B b$, there is a formula $\varphi(x)$ over Bb such that $\varphi(a)$ and for all a' if $\varphi(a')$ then $a' \not\downfree_B b$.

REMARK 1.3.5. This definition is taken from [**Adl08**], where it is called strong finite character, but since there is no room for confusion, we decided to omit "strong".

EXAMPLE 1.3.6. All the pre-independence relations we mentioned satisfy this: $\downfree^f$, $\downfree^u$ and $\downfree^i$.

PROPOSITION 1.3.7. *Assume that* $\downfree$ *is a standard pre-independence relation with finite character. Assume that* B *is an extension base for* $\downfree$ *and that if* $\varphi(x,a)$ *forks over* B*, then* $\varphi(x,a)$ *quasi-divides over* B *(see 1.3.2; in this case we say that forking implies quasi dividing over* B*).*
Then: for every type p *over* B*,*

(1) *There exists a global extension* q*,* $\downfree$*-free over* B*, such that for every* $C \supseteq B$ *and every* $c \models q|_C$*,* $C \downfree^f_B c$*.*
(2) *There exists a global extension* q' *that doesn't fork over* B *(i.e.* $\downfree^f$*-free over* B*), such that for every* $C \supseteq B$ *and every* $c \models q'|_C$*,* $C \downfree_B c$*.*

PROOF. (1): Let $a \models p$. By finite character, it is enough to see that the following set is consistent

$$\begin{aligned} p(x) \quad &\cup \quad \{\neg\varphi(x,b)\,|\,\varphi(x,y) \text{ is over } B\,\&\, b \in \mathfrak{C}\,\&\,\varphi(a,y) \text{ forks over } B\} \\ &\cup \quad \left\{\neg\psi(x,d)\,\middle|\,\psi(x,z) \text{ is over } B\,\&\, d \in \mathfrak{C}\,\&\forall c\left[\psi(c,d) \Rightarrow c \underset{B}{\not\downfree} d\right]\right\}. \end{aligned}$$

Since then every global type q that contains this set will suffice.
Indeed: assume not, then we have an implication of the form

$$p \vdash \bigvee_{i<n} \varphi_i(x,b_i) \vee \bigvee_{j<m} \psi_j(x,d_j)$$

where $\varphi_i(x,y_i)$, $\psi_j(x,z_j)$ formulas over B, $\forall c\left[\psi_j(c,d_j) \Rightarrow c \not\downfree_B d_j\right]$ and $\varphi_i(a,y_i)$ forks over B.
Note that $\bigvee_{i<n} \varphi_i(a,y_i)$ forks over B, so we may assume $n=1$.
By assumption, $\varphi_0(a,y)$ quasi-divides over B, so there are $h_0,\ldots,h_{k-1}$ such that $h_i \equiv_B a$ and $\{\varphi_0(h_i,y)\,|\,i<k\}$ is inconsistent. Denote $h = h_0h_1\ldots h_{k-1}$ and $r(x_0,\ldots,x_{k-1}) = \mathrm{tp}(h/B)$. Then

$$r \restriction x_i \vdash \varphi_0(x_i,b) \vee \bigvee_{j<m} \psi_j(x_i,d_j).$$

So

$$r \vdash \bigwedge_{i<k} \left[\varphi_0 (x_i, b) \vee \bigvee_{j<m} \psi_j (x_i, d_j) \right].$$

But

$$r \vdash \neg \exists z \left(\bigwedge_{i<k} \varphi_0 (x_i, z) \right),$$

so $r \vdash \bigvee_{i<k,j<m} \psi_j (x_i, d_j)$.
The set B is an extension base for $\downfree$, so $h \downfree_B B$, and by right extension there is $h' \equiv_B h$ such that $h' \downfree_B \{d_j \,|\, j < m\}$. It follows that there are i, j such that $\psi_j (h'_i, d_j)$. This is a contradiction to the choice of ψ_j.

(2): The proof is very similar. Let $a \models p$. We must show that

$$\begin{aligned} p(x) \quad & \cup \quad \{\neg\varphi (x, b) \,|\, \varphi (x, y) \text{ is over } B \,\&\, b \in \mathfrak{C} \,\&\, \varphi (x, b) \text{ forks over } B\} \\ & \cup \quad \left\{ \neg\psi (x, d) \,\middle|\, \psi (x, z) \text{ is over } B \,\&\, d \in \mathfrak{C} \,\&\forall c \left[\psi (a, c) \Rightarrow c \not\downfree_B a \right] \right\} \end{aligned}$$

is consistent. If not, then $p \vdash \bigvee_{i<n} \varphi_i (x, b_i) \vee \bigvee_{j<m} \psi_j (x, d_j)$ and we may assume $n = 1$. As $\varphi_0 (x, b_0)$ forks over B, it quasi-divides over B, so there are $e_0, \ldots, e_{k-1}$ such that $e_i \equiv_B b_0$ and $\{\varphi (x, e_i) \,|\, i < k\}$ is inconsistent. Let $\bar{d} = \langle d_{i,j} \,|\, j < m \rangle$ be such that $\bar{d}_i e_i \equiv_B \bar{d} b_0$. As p is over B, for every $i < k$,

$$p \vdash \varphi_0 (x, e_i) \vee \bigvee_{j<m} \psi_j (x, d_{i,j}).$$

So it follows that $p \vdash \bigvee_{i,j} \psi_j (x, d_{i,j})$. Denote $\bar{d}' = \langle d_{i,j} \,|\, i < k, j < m \rangle$. As B is an extension base for $\downfree$, $\bar{d}' \downfree_B B$, and by right extension, wlog $\bar{d}' \downfree_B a$. So there are i, j such that $\psi_j \left(a, d'_{i,j} \right)$ which contradicts the choice of ψ_j. $\square$

The following pre-independence relation is instrumental in the proof of the main theorem.

DEFINITION 1.3.8. We say that $\mathrm{tp}(a/Bb)$ is strictly invariant over B (denoted by $a \downfree^{\mathrm{ist}}_B b$) if there is a global extension p, which is Lascar invariant over B (so $a \downfree^{i}_B b$) and for any $C \supseteq Bb$, if $c \models p|_C$ then $C \downfree^{f}_B c$.

REMARK 1.3.9.

(1) $\downfree^{\mathrm{ist}}$ satisfies extension, invariance and monotonicity.
(2) Strictly invariant types are a special case of strictly non-forking types. We say that $\mathrm{tp}(a/Bb)$ strictly does not fork over B (denoted by $a \downfree^{\mathrm{st}}_B b$) if there is a global extension p, which does not fork over B, and for any $C \supseteq B$, if $c \models p|_C$ then $C \downfree^{f}_B c$. They coincide in dependent theories, and in stable theories they are the same as non-forking. The notion originated in [**She09**, 5.6]. More on strict non-forking can be found in [**Usv**] and in [**UK**].

As $\downfree^{i}$ has finite character, we conclude from (1) in Proposition 1.3.7 that:

COROLLARY 1.3.10. *Assume forking implies quasi dividing over* B *and that* B *is an extension base for* $\downfree^{i}$. *Then* B *is an extension base for* $\downfree^{\mathrm{ist}}$.

Working with an abstract pre-independence relation.
Here we shall prove the following theorem:

THEOREM 1.3.11. *Let* T *be* NTP$_2$. *Then (1) implies (2) where:*

(1) *There exists a standard pre-independence relation* $\downfree$ *with left extension over* B*, which preserves indiscernibility over* B *and such that* B *is an extension base for it.*
(2) *Forking equals dividing over* B*.*

In addition, if T *is dependent then (1) and (2) are equivalent.*

(1) implies (2).
So assume T is NTP$_2$, and that $\downfree$ is a pre-independence relation as in (1). We do not need left extension for this next claim:

LEMMA 1.3.12. *Assume* $\varphi(x, a)$ *divides over* B*. Then there is a model* $M \supseteq B$ *and a global* $\downfree$*-free type over* B*,* $p \in S(\mathfrak{C})$*, extending* $\operatorname{tp}(a/M)$*, such that every Morley sequence generated by* p *over* M *(as in 1.2.25) witnesses that* $\varphi(x, a)$ *divides.*

PROOF. Let $I = \langle b_i \,|i < \omega\rangle$ be a B-indiscernible sequence that witnesses k dividing of $\varphi(x, a)$. Let N be a $(|B| + |T|)^+$ saturated model containing B. By compactness we may assume that the length of I is $\left(2^{|N|+|T|}\right)^+$. As B is an extension base, we may assume that $I \downfree_B N$. The number of types over N is bounded by $2^{|N|+|T|}$, so I has infinitely many elements with the same type p over N, and wlog they are the first ω. Replace I with $I \restriction \omega$. Let $B \subseteq M \subseteq N$ be any model such that $|M| \leq |B| + |T|$.
Let $Q(x_0, x_1, \ldots) = \operatorname{tp}(I/N)$. Then Q is an invariant type over M (as M is a model and Q is Lascar invariant over B), and so is $p(x_i) = Q \restriction x_i$. By saturation, we can define a sequence $\langle I_i \,|i < \omega\rangle$ in N as in 1.2.25: $I_0 \models Q|_M$, $I_{i+1} \models Q|_{MI_0 \ldots I_i}$. Then $\langle I_i \,|i < \omega\rangle$ is an indiscernible sequence. Let $I_i = \langle a_{i,j} \,|j < \omega\rangle$. It follows that for every $\eta : \omega \to \omega$, $a_{0,\eta(0)} a_{1,\eta(1)} \ldots \equiv_M a_{0,0} a_{1,0} \ldots$, as both sequences satisfy the type $p^{(\omega)}|_M$.
As T is NTP$_2$, $\{\varphi(x, a_{i,0}) \,|i < \omega\}$ is inconsistent (otherwise $\{\varphi(x, a_{i,j}) \,|i, j < \omega\}$ witnesses that T has the tree property of the second kind because of the choice of I).
By 1.2.23, the type p has a unique extension to a global $\downfree$-free type over B (which we shall also call p).
Let $a' \models p|_M$, then $a' \equiv_B a$, so after applying an automorphism over B (and changing M), we may assume that p extends $\operatorname{tp}(a/M)$, and it is the required type: it is $\downfree$-free (as Q is), and there is a Morley sequence generated by p that witnesses dividing, so every such sequence does so as well. □

COROLLARY 1.3.13. *Forking implies quasi dividing over* B*.*

PROOF. Suppose $\varphi(x, a)$ forks over B, then $\varphi(x, a) \vdash \bigvee_{i<n} \varphi_i(x, a_i)$ where for all $i < n$, $\varphi_i(x, a_i)$ divides over B. By Lemma 1.3.12, for $i < n$, there are models $M_i \supseteq B$ and types p_i which are global $\downfree$-free extension of $\operatorname{tp}(a_i/B)$. Let I_0 be some indiscernible sequence witnessing dividing of $\varphi_0(x, a_0)$. For $0 < i$, let $I_i = \langle a_{i,l} \,|l < \omega\rangle$ be a Morley sequence generated by p_i as follows: $a_{i,0} = a_i \models p_i|_{M_i}$, and for all $j > 0$, $a_{i,l+1} \models p_i|_{M_i I_{<i} a_{i,\leq l}}$. This will set us in the situation of the broom lemma 1.3.1 hence φ quasi-divides over B. □

For the next claims, let A be any set.
The importance of $\mathop{\smile\!\!\!\!\!\!\mid}^{ist}$ lies in the following lemma, which is analogous to "Kim's Lemma" (see [**Kim98**, 2.1]).

Lemma 1.3.14. *If $\varphi(x,a)$ divides over A, and $\langle b_i \mid i<\omega\rangle$ is a sequence satisfying $b_i \equiv_A a$ and $b_i \mathop{\smile\!\!\!\!\!\!\mid}^{ist}_{A} b_{<i}$. Then $\{\varphi(x,a_i) \mid i<\omega\}$ is inconsistent. In particular, if $\langle b_i \mid i<\omega\rangle$ is an indiscernible sequence then it witnesses dividing of $\varphi(x,a)$.*

Proof. Wlog $b_0=a$. Let I be an indiscernible sequence witnessing the dividing of $\varphi(x,a)$ over A. We build by induction on n sequences $I_i=\langle a_{i,j} \mid j<\omega\rangle$ for $i<n$ such that

- Each I_i is indiscernible over $AI_{<i}a_{>i,0}$ (where $a_{>i,0}=a_{i+1,0}\dots a_{n-1,0}$).
- For $i<\omega$, $I_i \equiv_A I$.
- $a_{i,0}=b_i$.

This is enough, because then by compactness we can find an infinite such array and then if $\{\varphi(x,b_i) \mid i<\omega\}$ is consistent, we reach a contradiction to NTP$_2$: In the infinite array $\langle a_{i,j} \mid i,j<\omega\rangle$, for every function $\eta:\omega\to\omega$ and every n, one may show by decreasing induction on $i\le n$ (starting with $i=n$), that

$$a_{0,\eta(0)}\dots a_{n-1,\eta(n-1)} \equiv_A a_{0,\eta(0)}\dots a_{i-1,\eta(i-1)}a_{i,0}\dots a_{n-1,0}.$$

And this shows that every vertical path has the same type, but each row is k-inconsistent for the same k (because $I_i \equiv_A I$).
For $n\le 1$ it is clear. Suppose we have built these sequences up to n and we consider $n+1$. Denote our array of n rows by $I_{<n}$. By right extension, there is $J_{<n} \equiv_{Ab_{<n}} I_{<n}$ such that $b_n \mathop{\smile\!\!\!\!\!\!\mid}^{ist}_{A} J_{<n}$. Hence also $J_{<n} \mathop{\smile\!\!\!\!\!\!\mid}^{f}_{A} b_n$. As $b_n \equiv_A a$, there is an indiscernible sequence $I' \equiv_A I$ starting with b_n. By 1.2.11, there is an A-indiscernible sequence J_n such that $J_n \equiv_{Ab_n} I'$ and J_n is indiscernible over $J_{<n}$. Now it is easy to check that the conditions we demanded are met with this new array. The only non-trivial one is the first condition: J_n is indiscernible over $J_{<n}$ by construction. For every $i<n$, J_i is indiscernible over $AJ_{<i}b_{>i}$ by the induction hypothesis (where $b_{>i}=b_{i+1}\dots b_{n-1}$). As $b_n \mathop{\smile\!\!\!\!\!\!\mid}^{i}_{A} J_{<n}$, by the base monotonicity of $\mathop{\smile\!\!\!\!\!\!\mid}^{i}$ it follows that $b_n \mathop{\smile\!\!\!\!\!\!\mid}^{i}_{AJ_{<i}b_{>i}} J_i$, and as $\mathop{\smile\!\!\!\!\!\!\mid}^{i}$ preserves indiscernibility, it follows that J_i is indiscernible over $AJ_{<i}b_{>i}b_n$. □

Remark 1.3.15. In fact we need less than Lemma 1.3.14. For our needs, it suffices to see that if $\varphi(x,a)$ divides over A, and there exists p, a global $\mathop{\smile\!\!\!\!\!\!\mid}$-free type over A, containing $\mathrm{tp}(a/A)$, then every Morley sequence p generates (over a model $M\supseteq A$) witnesses dividing. The proof of this fact is a bit easier: Assume that I witnesses dividing, and that N is $|M|^+$ saturated. Let $c\models p|_N$. Then $c \mathop{\smile\!\!\!\!\!\!\mid}^{ist}_{A} N$ and in particular $N \mathop{\smile\!\!\!\!\!\!\mid}^{f}_{A} c$, so (by 1.2.11) we may find I' such that $cI' \equiv_A aI$ and I' is indiscernible over N. Now, as in the proof of 1.3.12, we define $I_i \models \mathrm{tp}(I'/N)|_{MI_{<i}}$ in N. Then, every vertical path realizes the type $p^{(\omega)}|_M$ and we get a contradiction.

Corollary 1.3.16. *If A is an extension base for $\mathop{\smile\!\!\!\!\!\!\mid}^{ist}$, then forking equals dividing over A.*

Proof. Suppose $\varphi(x,a)\vdash \bigvee_{i<n}\varphi_i(x,a_i)$, each $\varphi_i(x,a_i)$ divides over A. Let $\bar{a}=aa_0\dots a_{n-1}$ and let $p=\mathrm{tp}(\bar{a}/A)$. As $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}^{ist}_{A} A$, by definition there is q, a global $\mathop{\smile\!\!\!\!\!\!\mid}^{ist}$-free type over A, containing p.

Let $\left\langle \bar{a}^j = a^j a^j_0 \dots a^j_{n-1} \mid j < \omega \right\rangle$ be a Morley sequence generated by q over a model M containing A. It is enough to see that $\left\{ \varphi\left(x, a^j\right) \mid j < \omega \right\}$ is inconsistent (as it is an indiscernible sequence whose elements have the same type as a over A).
If this set is consistent, let c realize it. Then for all $j < \omega$, there is $i_j < n$ such that $\varphi\left(c, a^j_{i_j}\right)$, so there is $\iota < n$ and infinitely many j's such that $\iota = i_j$. Then $\left\{ \varphi_{i_0}\left(x, a^j_\iota\right) \mid i_j = \iota \right\}$ is consistent – a contradiction to 1.3.14. ☐

Lemma 1.3.17. *The set* B *(from our assumptions) is an extension base for* $\downarrow^{ist}$
.

Proof. Forking implies quasi-dividing over B by 1.3.13, and B is an extension base for $\downarrow^{i}$ by our assumption (because $\downarrow$ is at least as strong as $\downarrow^{i}$), so the lemma follows immediately from 1.3.10. ☐

Summing up, we have

Corollary 1.3.18. *Forking equals dividing over* B.

By this we have proved one direction of Theorem 1.3.11.

(2) implies (1).
Here we assume that T is dependent and that forking equals dividing over B. We shall prove that $\downarrow^{f}$ satisfy all the demands that appear in (1) in Theorem 1.3.11. Note that by 1.2.24, $\downarrow^{f} = \downarrow^{i}$, and $\downarrow^{f}$ is standard. We are left with showing that B is an extension base for $\downarrow^{f}$ and that there is left extension over B.
Since no type divides over its domain, we get

Claim 1.3.19. (No need for NIP) B is an extension base for $\downarrow^{f}$.

Claim 1.3.20. (No need for NIP) We have left extension for $\downarrow^{f}$ over B.

Proof. Suppose $a \downarrow^{f}_{B} b$ and we have some c. We want to find some $c' \equiv_{Ba} c$ such that $c'a \downarrow^{f}_{B} b$. Let $p = \mathrm{tp}(c/Ba)$. We need to show that the following set is consistent:

$$p(x) \cup \{\neg\varphi(x, a, b) \mid \varphi \text{ is over } B \text{ and } \varphi(x, y, b) \text{ divides over } B\}.$$

If not, then $p(x) \vdash \bigvee_{i<n} \varphi_i(x, a, b)$ where $\varphi_i(x, y, b)$ divides over B.
So $\psi(x, y, b) := \bigvee_{i<n} \varphi_i(x, y, b)$ forks over B, hence divides over B. Assume that $I = \langle b_i \mid i < \omega \rangle$ is an indiscernible sequence that witnesses dividing (with $b_0 = b$). By 1.2.11, there is $I' \equiv_{Bb} I$ such that I' is indiscernible over Ba and wlog $I' = I$. The type p is over Ba, so $p(x) \vdash \psi(x, a, b_i)$ for all i. But this is a contradiction as p is consistent.
This concludes the proof of 1.3.11. ☐

More conclusion from forking = dividing.
Here there are no assumption on the theory T.

Lemma 1.3.21. *Assume forking equals dividing over* B*. Then we have*
(1) $a \downarrow^{f}_{B} a$ *iff* $a \in \mathrm{acl}(B)$.
(2) $a \downarrow^{f}_{B} b$ *iff* $a \downarrow^{f}_{\mathrm{acl}(B)} b$ *iff* $\mathrm{acl}(Ba) \downarrow^{f}_{B} b$ *iff* $a \downarrow^{f}_{B} \mathrm{acl}(Bb)$.

Proof. (2): Every indiscernible sequence I over B is indiscernible over $\operatorname{acl}(B)$: Every 2 increasing sub-sequences from I have the same Lascar strong type over B. As every model containing B contains $\operatorname{acl}(B)$, they have the same type over $\operatorname{acl}(B)$. It follows that a formula divides over B iff it divides over $\operatorname{acl}(B)$. Hence $a \mathop{\smile\!\!\!\!\!\!\;|}^{f}_{\operatorname{acl}(B)} b$ implies $a \mathop{\smile\!\!\!\!\!\!\;|}^{f}_{B} b$.
Assume that $a \mathop{\smile\!\!\!\!\!\!\;|}^{f}_{B} b$, and assume that I is a B-indiscernible sequence starting with b. Then there is an indiscernible sequence $I' \equiv_{Bb} I$ such that I' is indiscernible over Ba. So it is also indiscernible over $\operatorname{acl}(Ba)$. This shows that $\operatorname{acl}(Ba) \mathop{\smile\!\!\!\!\!\!\;|}^{f}_{B} b$ (by 1.2.11). By right extension, there is $a' \equiv_{Bb} a$ such that $a' \mathop{\smile\!\!\!\!\!\!\;|}^{f}_{B} \operatorname{acl}(Bb)$. But every automorphism fixing Bb pointwise fixes $\operatorname{acl}(Bb)$ setwise, so $a \mathop{\smile\!\!\!\!\!\!\;|}^{f}_{B} \operatorname{acl}(Bb)$. By base monotonicity, we get $a \mathop{\smile\!\!\!\!\!\!\;|}^{f}_{\operatorname{acl}(B)} b$.
The rest follows from monotonicity.
(1): Assume that $a \in \operatorname{acl}(B)$, then since $a \mathop{\smile\!\!\!\!\!\!\;|}^{f}_{B} B$, it follows from (2) that $a \mathop{\smile\!\!\!\!\!\!\;|}_{B} a$. On the other hand, if $a \mathop{\smile\!\!\!\!\!\!\;|}^{f}_{B} a$, then the formula $x = a$ does not divide over B, so there are not infinitely many realizations of $\operatorname{tp}(a/B)$, so this type is algebraic and we are done. $\square$

1.3.3. Applying the previous sections.

Here we assume T is NTP$_2$ unless stated otherwise.

Corollary 1.3.22. *Forking equals dividing over models.*

Proof. We use Theorem 1.3.11 with $\mathop{\smile\!\!\!\!\!\!\;|} = \mathop{\smile\!\!\!\!\!\!\;|}^{u}$. We saw in 1.2.16 that $\mathop{\smile\!\!\!\!\!\!\;|}^{u}$ satisfies all the demands. $\square$

We saw that if the conditions of Theorem 1.3.11 on the existence of $\mathop{\smile\!\!\!\!\!\!\;|}$ and B are met, then forking equals dividing, and moreover B is an extension base for $\mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{ist}}$. So in this case we can use our version of “Kim’s lemma”. It gives more information than just “forking equals dividing”, so naturally we are interested in knowing when this happens.

Lemma 1.3.23. *Suppose $\mathop{\smile\!\!\!\!\!\!\;|}$ is a standard pre-independence relation. Moreover, assume that every set containing B is an extension base for $\mathop{\smile\!\!\!\!\!\!\;|}$. Then $\mathop{\smile\!\!\!\!\!\!\;|}$ has left extension over B.*

Proof. Assume $a \mathop{\smile\!\!\!\!\!\!\;|}_{B} b$ and we are given c. We want to find $c' \equiv_{Ba} b$ such that $ac' \mathop{\smile\!\!\!\!\!\!\;|}_{B} b$. Well, by assumption $c \mathop{\smile\!\!\!\!\!\!\;|}_{Ba} Ba$, so by right extension there is $c' \equiv_{Ba} c$ such that $c' \mathop{\smile\!\!\!\!\!\!\;|}_{Ba} Bab$. This means that $c' \mathop{\smile\!\!\!\!\!\!\;|}_{Ba} b$, so by transitivity we get $c'a \mathop{\smile\!\!\!\!\!\!\;|}_{B} b$ as requested. $\square$

Definition 1.3.24. If B satisfies the condition of the previous lemma, we say that B is a *good extension base*.

Corollary 1.3.25. *If B is a good extension base for a standard pre-independence relation $\mathop{\smile\!\!\!\!\!\!\;|}$, and in addition $\mathop{\smile\!\!\!\!\!\!\;|}$ is at least as strong as $\mathop{\smile\!\!\!\!\!\!\;|}^{i}$, then B is a good extension base for $\mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{ist}}$ as well. In particular, forking equals dividing over B.*

For instance, this corollary is true if B is a good extension base for $\mathop{\smile\!\!\!\!\!\!\;|}^{i}$. In dependent theories, since $\mathop{\smile\!\!\!\!\!\!\;|}^{i} = \mathop{\smile\!\!\!\!\!\!\;|}^{f}$, we have

Corollary 1.3.26. *If* T *is dependent and for every* A *and* $p \in S(A)$, p *does not fork over* A*, then every set is an extension base for* $\mathop{\smile\!\!\!\!\!\!|}^{\text{ist}}$ *and forking equals dividing.*

This corollary is true for o-minimal theories and c-minimal theories (see [**HP11**, 2.14]).

Now we turn to the proof of the main Theorem 1.1.2. We abandon for a moment our desire to find extension basis for $\mathop{\smile\!\!\!\!\!\!|}^{\text{ist}}$ and concentrate on forking and dividing. In the end we shall conclude a corollary which is stronger than both 1.3.22 and 1.3.25.

Claim 1.3.27. (T any theory) Assume that $a \mathop{\smile\!\!\!\!\!\!|}^{f}_{B} b$ and $\varphi(x, b)$ forks over B, then $\varphi(x, b)$ forks over Ba as well.

Proof. Assume $\varphi(x, b)$ forks over B, so there are $n < \omega$, $\varphi_i(x, y_i)$ and b_i for $i < n$ such that $\varphi_i(x, b_i)$ divides over B and $\varphi(x, b) \vdash \bigvee_{i<n} \varphi_i(x, b_i)$. By extension, we may assume $a \mathop{\smile\!\!\!\!\!\!|}^{f}_{A} b \langle b_i | i < n \rangle$. By 1.2.11, $\varphi_i(x, b_i)$ divides over Ba. Hence $\varphi(x, b)$ forks over Ba. □

Theorem 1.3.28. *For a set* B *the following are equivalent:*

(1) Forking equals dividing over B*.*
(2) B *is an extension base for* $\mathop{\smile\!\!\!\!\!\!|}^{f}$ *(i.e. types over* B *do not fork over* B*).*
(3) $\mathop{\smile\!\!\!\!\!\!|}^{f}$ *has left extension over* B*.*

Proof. We saw that (1) implies (2) and (3) in 1.3.19 and 1.3.20. Assume that (2) or (3) are true. Assume that $\varphi(x, a)$ forks over B, and let M be any model containing B.
If (2) is true then $M \mathop{\smile\!\!\!\!\!\!|}^{f}_{B} B$, so by right extension we may assume wlog that $M \mathop{\smile\!\!\!\!\!\!|}^{f}_{B} a$.
If (3) is true, then $B \mathop{\smile\!\!\!\!\!\!|}^{f}_{B} a$ (even $B \mathop{\smile\!\!\!\!\!\!|}^{u}_{B} a$). So by left extension we can assume wlog that $M \mathop{\smile\!\!\!\!\!\!|}^{f}_{B} a$.
So in both cases we are in a situation where we have a model M that satisfies $M \mathop{\smile\!\!\!\!\!\!|}^{f}_{B} a$. Hence, by 1.3.27, $\varphi(x, a)$ forks over M. By 1.3.22, $\varphi(x, a)$ divides over M, so it also divides over B. □

The next corollary is stronger than both 1.3.22 and 1.3.25:

Corollary 1.3.29. *A set* B *is an extension base for* $\mathop{\smile\!\!\!\!\!\!|}^{\text{ist}}$ *iff it is an extension base for* $\mathop{\smile\!\!\!\!\!\!|}^{i}$*. In this case, by the previous theorem, forking equals dividing over* B*.*

Proof. If B is an extension base for $\mathop{\smile\!\!\!\!\!\!|}^{\text{ist}}$, it is an extension base for $\mathop{\smile\!\!\!\!\!\!|}^{i}$ by definition. On the other hand, if B is an extension base for $\mathop{\smile\!\!\!\!\!\!|}^{i}$, then, since $\mathop{\smile\!\!\!\!\!\!|}^{i}$ is at least as strong as $\mathop{\smile\!\!\!\!\!\!|}^{f}$, B is an extension base for $\mathop{\smile\!\!\!\!\!\!|}^{f}$, so forking equals dividing over B by the previous theorem. By corollary 1.3.10, we are done (since if $\varphi(x, a)$ forks over B, it divides over B so it quasi-divides over B). □

1.3.4. Some corollaries for dependent theories.

Assume T is dependent. We shall see some consequences about the behavior of forking.

Theorem 1.3.30. *The following are equivalent for* B*:*

(1) Forking equals dividing over B.
(2) B *is an extension base for* $\mathop{\smile\!\!\!\!\!\!|}^{f}$.
(3) $\mathop{\smile\!\!\!\!\!\!|}^{f}$ *has left extension over* B.
(4) B *is an* $\mathop{\smile\!\!\!\!\!\!|}^{ist}$ *extension base.*

Proof. (1) – (3) are equivalent by 1.3.28. If B is an extension base for $\mathop{\smile\!\!\!\!\!\!|}^{ist}$, then it is an extension base for $\mathop{\smile\!\!\!\!\!\!|}^{f}$, and we are done by the same theorem. Recall that in a dependent theory $\mathop{\smile\!\!\!\!\!\!|}^{f} = \mathop{\smile\!\!\!\!\!\!|}^{i}$, so if B is an extension base for $\mathop{\smile\!\!\!\!\!\!|}^{f}$, it is an extension base for $\mathop{\smile\!\!\!\!\!\!|}^{i}$, so by 1.3.29, also for $\mathop{\smile\!\!\!\!\!\!|}^{ist}$. $\square$

Assume from now on that forking equals dividing over B (for instance, B is a model).

Corollary 1.3.31. *The following are equivalent for a formula* $\varphi(x, a)$*:*

- φ *forks over* B.
- φ *quasi Lascar divides over* B*: there are* $\{e_i \mid i < m\}$ *such that* $e_i \equiv^L_B a$ *and* $\{\varphi(x, e_i)\}$ *is inconsistent.*

Proof. If $\varphi(x, a)$ forks over B, then it quasi Lascar divides because forking equals dividing over B. If $\varphi(x, a)$ does not fork over B, then extend it to p, a global non forking type over B. By dependence, p is Lascar invariant over B. This means that it contains all Lascar conjugates of φ over B, and in particular it is impossible for φ to quasi Lascar divide. $\square$

Definition 1.3.32. We say that dividing over B is type definable when for every formula $\varphi(x, y)$ there is a (partial) type $\pi(x)$ over B such that $\pi(a)$ iff $\varphi(x, a)$ divides over B.

Remark 1.3.33. Dividing is type definable, so in dependent theories all these notions – dividing, forking and quasi Lascar dividing – are type-definable over B (i.e. dependent theories are low, see [**Bue99**])

Proof. (Due to Itai Ben Yaacov) First we shall see that for any set B, if $\varphi(x, a)$ divides over B then it k divides over B, with $k = \mathrm{alt}(\varphi)$. If $\langle a_i \mid i < \omega \rangle$ is an indiscernible sequence witnessing $m > k$ dividing but not k dividing, it means that $\exists x \bigwedge_{i<k} \varphi(x, a_i)$, and by indiscernibility, $\exists x \bigwedge_{i<k} \varphi(x, a_{mi})$. So assume $\varphi(c, a_{mi})$ for $i < k$. But for each i, there must be some $mi < j_i \leq mi + m - 1$ such that $\neg\varphi(c, a_{j_i})$. This is a contradiction to the definition of the alternation rank (see definition 1.2.2).
The remark now follows: The type $\pi(y)$ says that there exists a sequence $\langle y_i \mid i < \omega \rangle$ of elements having the same type as y over B, and that every subset of size k of formulas of the form $\varphi(x, y_i)$ is inconsistent. $\square$

The following is a strengthening of [**HP11**, Lemma 8.10]

Corollary 1.3.34. *Let* r *be a partial type which is Lascar invariant over* B*. Then there exists some global* B*-Lascar invariant extension of* r*.*

Proof. If $\varphi_1, \ldots, \varphi_n \in r$, then $\bigwedge_i \varphi_i$ does not quasi Lascar divide over A (because all the conjugates of φ_i are in r for all i). Hence r does not fork over B, hence there is a global non-forking (hence Lascar invariant) extension. $\square$

1.4. Bounded non-forking + NTP$_2$ = Dependent

It is well-known that stable theories can be characterized as those simple theories in which every type over model has boundedly many non-forking extensions (see e.g. [**Adl08**, theorem 45]). Our aim in this section is to prove a generalization of this fact: if non-forking is bounded, and the theory is NTP$_2$, then the theory is actually dependent. This gives a partial answer to a question of Adler.

Definition 1.4.1. We say that a pre-independence relation $\mathop{\smile\!\!\!\!\!\mid}$ is bounded if there is a function f on cardinals such that for every type $p(x) \in S(C)$ (where x is a finite tuple), and every model $M \supseteq C$, the size of the set

$$\left\{ \operatorname{tp}(a/M) \,\middle|\, a \models p \,\&\, a \mathop{\smile\!\!\!\!\!\mid}_C M \right\}$$

is bounded by $f(|T| + |C|)$.

We quote from [**Adl08**, Corollary 38]:

Fact 1.4.2. *The following are equivalent for a theory* T*:*

1. $\mathop{\smile\!\!\!\!\!\mid}^f$ is bounded.
2. $\mathop{\smile\!\!\!\!\!\mid}^f$ is bounded by the function $f(\kappa) = 2^{2^{\kappa}}$.
3. $\mathop{\smile\!\!\!\!\!\mid}^f = \mathop{\smile\!\!\!\!\!\mid}^i$.

The question Adler asks in [**Adl08**] is whether it is true that T is dependent iff $\mathop{\smile\!\!\!\!\!\mid}^f$ is bounded. The answer in general is no (see Chapter 6), but under the assumption of NTP$_2$ it is true.

Theorem 1.4.3. *Assume* T *is* NTP$_2$*, and that* $\mathop{\smile\!\!\!\!\!\mid}^f$ *is bounded. Then* T *is dependent.*

Proof. Assume $\varphi(x, y)$ has the independence property. This means that there is an infinite set A of tuples, such that for any subset $B \subseteq A$, there is some b such that for all $a \in A$, $\varphi(b, a)$ iff $a \in B$. Let $r(x) = \{x \neq a \,|\, a \in A\}$ be a partial type over A. Since it is finitely satisfiable in A there is a global type p containing r which is finitely satisfied in A. Let $q = p^{(2)}$. Denote $\psi(x, y, z) = \varphi(x, y) \wedge \neg\varphi(x, z)$.
Note that if $M \supseteq A$ is a model and $b \equiv_M c$ then $\psi(x, b, c)$ forks over M (otherwise there is a global non-forking type over M which is not invariant over M in contradiction to our assumption) and hence divides over M.
We build by induction on $\alpha < \omega_1$ a sequence of indiscernible sequences $J_\alpha = \langle I_i \,|\, i < \alpha \rangle$ such that

1. $J_{\alpha'} \subseteq J_\alpha$ for $\alpha' < \alpha$.
2. $I_i = \langle a_{i,j} \,|\, j < \omega \rangle$.
3. For all $i < \alpha$, $j < \omega$, $a_{i,j} \models q|_{AJ_i}$.
4. For all $i < \alpha$, I_i witnesses the dividing of $\psi(x, a_{i,0})$ (over $\emptyset$).

For $\alpha = 0$ there is nothing to do, for α limit we take the union.
For $\alpha + 1$: Let M be a model containing AJ_α. Let $a_{\alpha,0} \models q|_M$. Then $\psi(x, a_{\alpha,0})$ divides over M, and let I_α witness this. It is easy to see that all demands are met.
Since the array is of length ω_1, there is some k such that for infinitely many $i < \omega_1$, I_i witnesses k-dividing . Wlog, these are the first ω. It follows that for every vertical path $\eta : \omega \to \omega$, $\operatorname{tp}\left(\left\langle a_{i,\eta(i)} \,|\, i < \omega \right\rangle / A\right) = q^{(\omega)}|_A$.

Now we shall show that the set $\{\psi(x, a_{i,0}) \mid i < \omega\}$ is consistent and reach a contradiction to NTP$_2$.
Denote $a_i = a_{i,0} = (b_i, c_i)$. Note that by the choice of p and q, for every formula $\phi(x_0, y_0, \ldots, x_{n-1}, y_{n-1})$, if $\phi(a_0, \ldots, a_{n-1})$, then there are pairwise distinct $b'_0, c'_0, \ldots, b'_{n-1}, c'_{n-1} \in A$ such that

$$\phi\left(b'_0, c'_0, \ldots, b'_{n-1}, c'_{n-1}\right).$$

For $n < \omega$, let $\phi = \neg\exists x \bigwedge_{i<n} \psi(x, a_i)$, then there are pairwise distinct $b'_0, c'_0, \ldots, b'_{n-1}, c'_{n-1} \in A$ such that $\neg\exists x \bigwedge_{i<n} \psi(x, b'_i, c'_i)$, which contradicts the choice of ψ, i.e. this set is consistent. □

1.5. Optimality of results

In general, forking is not the same as dividing, and Shelah already gave an example in [**She90**, III,2]. Kim gave another example in his thesis ([**Kim96**, Example 2.11]) – circular ordering. Both examples were over the empty set, and the theory was dependent.
Here we give 2 examples. The first shows that outside the realm of NTP$_2$, our results are not necessarily true, and the second shows that even in dependent theories, forking is not the same as dividing even over sets containing models.
In both examples, we use the notion of a (directed) circular order, so here is the definition:

Definition 1.5.1. A circular order on a finite set is a ternary relation obtained by placing the points on a circle and taking all triples in clockwise order. For an infinite set, a circular order is a ternary relation such that the restriction to any finite set is a circular order.
A first order definition is: a circular order is a ternary relation C such that for every x, $C(x, -, -)$ is a linear order on $\{y \mid y \neq x\}$ and $C(x, y, z) \to C(y, z, x)$ for all x, y, z.

1.5.1. Example 1.

Here we present a variant of an example found by Martin Ziegler, showing that

(1) forking and dividing over models are different in general,
(2) strictly non-forking types need not exist over models (see 1.3.9), so in particular, strictly invariant types and non-forking heirs need not necessarily exist over models.

Let L be a 2 sorted language: one sort P for "points", for which we will use the variables $t, t_0, \ldots$ and another S for "sets", for which we will use the variables $s, s_0, \ldots$. L consists of 1 binary relation $E(t, s)$ to denote "membership" (so a subset of $P \times S$), and two 4-ary relations: $C(t_1, t_2, t_3, s)$ and $D(s_1, s_2, s_3, t)$.
Consider the following universal theory $T^\forall$ saying:

(1) For all s, $C(-, -, -, s)$ is a circular order on the set of all t such that $E(t, s)$, and if $C(t_1, t_2, t_3, s)$ then $E(t_i, s)$ for $i = 1, 2, 3$, and
(2) For all t, $D(-, -, -, t)$ is a circular order on the set of all s such that $\neg E(t, s)$, and if $D(s_1, s_2, s_3, t)$ then $\neg(E(t, s_i))$ for $i = 1, 2, 3$.

This theory has the joint embedding property and the amalgamation property as can easily be verified by the reader. Hence, as the language has no function symbols, by Fraïssé's theorem it has a model completion T, so T eliminates quantifiers (see [**Hod93**, Theorem 7.4.1]).

Let M be a model of T. We choose $t_0, s_0 \in \mathfrak{C}\backslash M$, such that for all $t \in M$, $\neg E(t, s_0)$ and for all $s \in M$, $E(t_0, s)$. Now, $E(x, s_0)$ forks over M, and $\neg E(t_0, y)$ forks over M, but none of them (quasi) divides.
Why? Non quasi dividing is straightforward from the construction of T.
We show that $\neg E(t_0, y)$ forks (for $E(x, s_0)$ use the same argument): choose some circular order on P^M, and choose s'_i for $i < \omega$ such that:

- $\neg E(t_0, s'_i)$ for $i < \omega$.
- $D(s'_i, s'_j, s'_k, t_0)$ whenever $i < j < k$.
- For all $i < \omega$ and for all $t \in M$ we have $E(t, s'_i)$, and $C(-,-,-,s'_i)$ orders P^M using the pre-chosen circular order.

Now,

$$\neg E(t_0, y) \vdash D(s'_0, y, s'_1, t_0) \vee D(s'_1, y, s'_0, t_0) \vee y = s'_0 \vee y = s'_1$$

and $D(s'_0, y, s'_1, t_0)$ divides over Mt_0 as witnessed by $\langle s'_i s'_{i+1} | i < \omega \rangle$, and so does $D(s'_1, y, s'_0, t_0)$, because for all n, $s'_1 s'_0 \equiv_{Mt_0} s'_{n+1} s'_n$.
Let $p(t)$ be $\mathrm{tp}(t_0/M)$. We show that p is not a strictly non-forking type over M: suppose q is a global strictly non-forking extension, and let $t'_0 \models q|_{s_0}$. Then $t'_0 \mathop{\smile\!\!\!\!\!\!\mid}^{f}_{M} s_0$ and $s_0 \mathop{\smile\!\!\!\!\!\!\mid}^{f}_{M} t'_0$. So surely $\neg E(t, s_0) \in q$, so $\neg E(t'_0, s_0)$ holds. But $t'_0 \equiv_M t_0$ so $s_0 \not\mathop{\smile\!\!\!\!\!\!\mid}^{f}_{M} t'_0$ – a contradiction.
Note that T has the tree property of the second kind: Let s_i for $i < \omega$ be such that they are all different, and for each i, let t^i_j for $j < \omega$, be such that for $j < k < l$, $C(t^i_j, t^i_k, t^i_l, s_i)$. The array $\{C(t^i_j, x, t^i_{j+1}, s_i) | i, j < \omega\}$ witnesses TP_2.

1.5.2. Example 2. We give an example showing that even if T is dependent, and S contains a model, forking is not necessarily the same as dividing over S. Hence models are not good extension bases for non-forking in dependent theories in general (see 1.3.24).
Let L the language $\{C, E\}$ where E is a binary relation and C is a ternary relation. Let $T^\forall$ be the universal theory saying that E is an equivalence relation and that C induces a circular order on every equivalence class, and that in addition $\forall x, y, z (C(x, y, z) \to E(x, y) \wedge E(y, z))$.
This theory has the JEP and AP so it has a model completion (as in Example 1). Moreover, T is dependent: To show this, it's enough to show that all formulas $\varphi(x, y)$ where x is one variable have finite alternation rank. As T eliminates quantifiers, it's enough to consider atomic formulas (see e.g. [**Adl08**, Section 1]), and this is straightforward and left to the reader.
Consider T^{eq}. It is also dependent.
Let M be a model. Let $c \in \mathfrak{C}\backslash M$ be a code of an E-equivalence class without any M-points. Then for every $a_1 \neq a_2$ in this class, both $C(a_2, x, a_1)$ and $C(a_1, x, a_2)$ divide over Mc (like in Example 1). So we have

$$\pi_E(x) = c \vdash C(a_1, x, a_2) \vee C(a_2, x, a_1) \vee x = a_1 \vee x = a_0$$

forks but does not divide over Mc (where π_E is the canonical projection into the sort of codes of E-classes).

1.6. Further remarks

Our understanding of forking in dependent theories was highly influenced by Section 5 (Non-forking) in [**She09**]. This section contains the definition of strict

non-forking, that we generalized to $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{ist}}$ (in dependent theories they are equal). Essentially, the ideas of the proof of Lemma 1.3.14 ("Kim's Lemma") appears there. Alex Usvyatsov also noticed a variant of that lemma independently.
The claim and proof of 1.3.12, with some modifications and generalizations is due to Usvyatsov and Onshuus in [**OU11**]. It should be noted that H. Adler and A. Pillay were the first to realize that NTP_2 is all the assumption one needs there.
Alex Usvyatsov noticed that one can use the broom lemma to prove that types over models can be extended to global non-forking heirs (see [**Usv**]). In fact, this follows directly from 1.3.7.

1.7. Questions and remarks

(1) Are simple theories $\mathop{\smile\!\!\!\!\!\mid}^{i}$-extensible NTP_2 theories?
(2) Can similar results be proved for NSOP theories? Or at least NTP_1 theories?
(3) It would be nice to find some purely semantic characterization of theories in which forking equals dividing over models. For example we know that all NTP_2 theories are such, however the opposite is not true: there is a theory with TP_2 in which forking equals dividing (essentially the example from section 1.5, but with dense linear orders instead of circular ones).

CHAPTER 2

A weak independence theorem for NTP_2 theories

This chapter is a joint work with Itai Ben Yaacov and is in circulation as a preprint "A weak independence theorem for NTP_2 theories" [**BC12**]. We establish new results about dividing and forking in NTP_2 theories. We show that dividing is the same as array-dividing. Combining it with existence of strictly invariant sequences we deduce that forking satisfies the chain condition over extension bases (i.e. the forking ideal is $S1$, in Hrushovski's terminology). Using it we prove a weak independence theorem over an extension base (which, in the case of simple theories, specializes to the ordinary independence theorem). As an application we show that Lascar strong type and compact strong type coincide over an extension base in an NTP_2 theory. After that we define the dividing order of a theory — a generalization of Poizat's fundamental order from stable theories — and give some equivalent characterizations under the assumption of NTP_2. The last section is devoted to a refinement of the class of strong theories and its place in the classification hierarchy.

2.1. Introduction

The class of NTP_2 theories, namely theories without the tree property of the second kind, was introduced by Shelah [**She80**] and is a natural generalization of both simple and NIP theories containing new important examples (e.g. any ultraproduct of p-adics is NTP_2, see Chapter 3).

The realization that it is possible to develop a good theory of forking in the NTP_2 context came from the paper [**CK12**], where it was demonstrated that the basic theory can be carried out as long as one is working over an extension base (a set is called an extension base if every complete type over it has a global non-forking extension, e.g. any model or any set in a simple, o-minimal or C-minimal theory is an extension base).

Here we establish further important properties of forking, thus demonstrating that a large part of simplicity theory can be seen as a special case of the theory forking in NTP_2 theories.

In Section 2.2 we consider the notion of *array dividing*, which is a multi-dimensional generalization of dividing. We show that in an NTP_2 theory, dividing coincides with array dividing over an arbitrary set (thus generalizing a corresponding result of Kim for the class of simple theories).

Section 2.3 is devoted to a property of forking called the *chain condition.* We say that forking in T satisfies the chain condition over a set A if for any A-indiscernible sequence $(a_i)_{i\in\omega}$ and any formula $\varphi(x,y)$, if $\varphi(x,a_0)$ does not fork over A, then $\varphi(x,a_0)\wedge\varphi(x,a_1)$ does not fork over A. This property is equivalent to requiring that there are no anti-chains of unbounded size in the partial order of formulas non-forking over A ordered by implication (hence the name, see Section 2.3 for more equivalences and the history of the notion). The following question had been raised by Adler and by Hrushovski:

PROBLEM 2.1.1. What are the implications between NTP$_2$ and the chain condition?

We resolve it by showing that:

(1) Forking in NTP$_2$ theories satisfies the chain condition over extension bases (Theorem 2.3.9, our proof combines the equality of dividing and array-dividing with the existence of universal Morley sequences from Chapter 1).
(2) There is a theory with TP$_2$ in which forking satisfies the chain condition (Section 2.3.3).

In his work on approximate subgroups, Hrushovski [**Hru12**] reformulated the independence theorem for simple theories with respect to an arbitrary invariant $S1$-ideal. In Section 2.4 we observe that the chain condition means that the forking ideal is $S1$. Using it we prove a weak independence theorem for forking over an arbitrary extension base in an NTP$_2$ theory (Theorem 2.4.3), which is a natural generalization of the independence theorem of Kim and Pillay for simple theories. As an application we show that Lascar type coincides with compact strong type over an extension base in an NTP$_2$ theory.

In Section 2.5 we discuss a possible generalization of the fundamental order of Poizat which we call the *dividing order*. We prove some equivalent characterizations and connections to the existence of universal Morley sequences in the case of NTP$_2$ theories, and make some conjectures.

In the final section we define burden^2 and strong^2 theories (which coincide with strongly^2 dependent theories under the assumption of NIP, just as Adler's strong theories specialize to strongly dependent theories). We establish some basic properties of burden^2 and prove that NTP$_2$ is characterized by the boundedness of burden^2.

Preliminaries. We assume some familiarity with the basics of forking and dividing (e.g. [**CK12**, Section 2]), simple theories (e.g. [**Cas07**]) and NIP theories (e.g. [**Adl08**]).

As usual, T is a complete first-order theory, $\mathbb{M} \models T$ is a monster model. We write $a \mathop{\smile\!\!\!\!\!\!\mid}_C b$ when $\mathrm{tp}(a/bC)$ does not fork over C and $a \mathop{\smile\!\!\!\!\!\!\mid}^{d}_C b$ when $\mathrm{tp}(a/bC)$ does not divide over C. In general these relations are not symmetric. We say that a global type $p(x) \in S(\mathbb{M})$ is *invariant* (*Lascar-invariant*) over A if whenever $\varphi(x, a) \in p$ and $b \equiv_A a$ (resp. $b \equiv^L_A a$, see Definition 2.4.1), then $\varphi(x, b) \in p$. We use the plus sign to denote concatenation of sequences, as in $I + J$, or $a_0 + I + b_1$ and so on.

DEFINITION 2.1.2. Recall that a formula $\varphi(x, y)$ is TP$_2$ if there are $(a_{ij})_{i,j \in \omega}$ and $k \in \omega$ such that:

- $\{\varphi(x, a_{ij})\}_{j \in \omega}$ is k-inconsistent for each $i \in \omega$,
- $\{\varphi(x, a_{if(i)})\}_{i \in \omega}$ is consistent for each $f : \omega \to \omega$.

A formula is NTP$_2$ if it is not TP$_2$, and a theory T is NTP$_2$ if it implies that every formula is NTP$_2$.

2.2. Array dividing

For the clarity of exposition (and since this is all that we will need) we only deal in this section with 2-dimensional arrays. All our results generalize to n-dimensional arrays by an easy induction (or even to λ-dimensional arrays for an arbitrary ordinal λ, by compactness; see [**Ben03**, Section 1]).

Definition 2.2.1. (1) We say that $(a_{ij})_{i,j\in\kappa}$ is an *indiscernible array* over A if both $\left((a_{ij})_{j\in\kappa}\right)_{i\in\kappa}$ and $\left((a_{ij})_{i\in\kappa}\right)_{j\in\kappa}$ are indiscernible sequences. Equivalently, all $n\times n$ sub-arrays have the same type over A, for all $n<\omega$. Equivalently, $\mathrm{tp}(a_{i_0 j_0} a_{i_0 j_1} ... a_{i_n j_n}/A)$ depends just on the quantifier-free order types of $\{i_0,...,i_n\}$ and $\{j_0,...,j_n\}$. Notice that, in particular, $\left(a_{if(i)}\right)_{i\in\kappa}$ is an A-indiscernible sequence of the same type for any strictly increasing function $f:\kappa\to\kappa$.

(2) We say that an array $(a_{ij})_{i,j\in\kappa}$ is strongly indiscernible over A if it is an indiscernible array over A, and in addition its rows are mutually indiscernible over A, i.e. $(a_{ij})_{j\in\kappa}$ is indiscernible over $(a_{i'j})_{i'\in\kappa\setminus\{i\},j\in\kappa}$ for each $i\in\kappa$.

Definition 2.2.2. We say that $\varphi(x,a)$ *array-divides* over A if there is an A-indiscernible array $(a_{ij})_{i,j\in\omega}$ such that $a_{00}=a$ and $\{\varphi(x,a_{ij})\}_{i,j\in\omega}$ is inconsistent.

Definition 2.2.3. (1) Given an array $\mathbf{A}=(a_{ij})_{i,j\in\omega}$ and $k\in\omega$, we define:

(a) $\mathbf{A}^k=\left(a'_{ij}\right)_{i,j\in\omega}$ with $a'_{ij}=a_{(ik)j}a_{(ik+1)j}\dots a_{(ik+i-1)j}$.

(b) $\mathbf{A}^{\mathrm{T}}=(a_{ji})_{i,j\in\omega}$, namely the transposed array.

(2) Given a formula $\varphi(x,y)$, we let $\varphi^k(x,y_0\dots y_{k-1})=\bigwedge_{i<k}\varphi(x,y_i)$.

(3) Notice that with this notation $\left(\mathbf{A}^k\right)^l=\mathbf{A}^{kl}$ and $\left(\varphi^k\right)^l=\varphi^{kl}$.

Lemma 2.2.4. *(1) If $\mathbf{A}$ is a B-indiscernible array, then $\mathbf{A}^k$ (for any $k\in\omega$) and $\mathbf{A}^{\mathrm{T}}$ are B-indiscernible arrays.*

(2) If $\mathbf{A}$ is a strongly indiscernible array over B, then $\mathbf{A}^k$ is a strongly indiscernible array over B (for any $k\in\omega$).

Lemma 2.2.5. *Assume that T is NTP_2 and let $(a_{ij})_{i,j\in\omega}$ be a strongly indiscernible array. Assume that the first column $\{\varphi(x,a_{i0})\}_{i\in\omega}$ is consistent . Then the whole array $\{\varphi(x,a_{ij})\}_{i,j\in\omega}$ is consistent.*

Proof. Let $\varphi(x,y)$ and a strongly indiscernible array $\mathbf{A}=(a_{ij})_{i,j\in\omega}$ be given. By compactness, it is enough to prove that $\{\varphi(x,a_{ij})\}_{i<k,j\in\omega}$ is consistent for every $k\in\omega$. So fix some k, and let $\mathbf{A}^k=(b_{ij})_{i,j\in\omega}$ — it is still a strongly indiscernible array by Lemma 2.2.4. Besides $\left\{\varphi^k(x,b_{i0})\right\}_{i\in\omega}$ is consistent. But then $\left\{\varphi^k(x,b_{ij})\right\}_{j\in\omega}$ is consistent for some $i\in\omega$ (as otherwise φ^k would have TP_2 by the mutual indiscernibility of rows), thus for $i=0$ (as the sequence of rows is indiscernible). Unwinding, we conclude that $\{\varphi(x,a_{ij})\}_{i<k,j\in\omega}$ is consistent. $\square$

Lemma 2.2.6. *Let $\mathbf{A}=(a_{ij})_{i,j\in\omega}$ be an indiscernible array and assume that the diagonal $\{\varphi(x,a_{ii})\}_{i\in\omega}$ is consistent. Then for any $k\in\omega$, if $\mathbf{A}^k=(b_{ij})_{i,j\in\omega}$ then the diagonal $\left\{\varphi^k(x,b_{ii})\right\}_{i\in\omega}$ is consistent.*

PROOF. By compactness we can extend our array $\mathbf{A}$ to $(a_{ij})_{i\in\omega\times\omega, j\in\omega}$ and let $b_{ij} = a_{i\times\omega+j,i}$.

It then follows that $(b_{ij})_{i,j\in\omega}$ is a strongly indiscernible array and that $\{\varphi(x, b_{i0})\}_{i\in\omega}$ is consistent. But then $\{\varphi(x, b_{ij})\}_{i,j\in\omega}$ is consistent by Lemma 2.2.5 , and we can conclude by indiscernibility of $\mathbf{A}$.

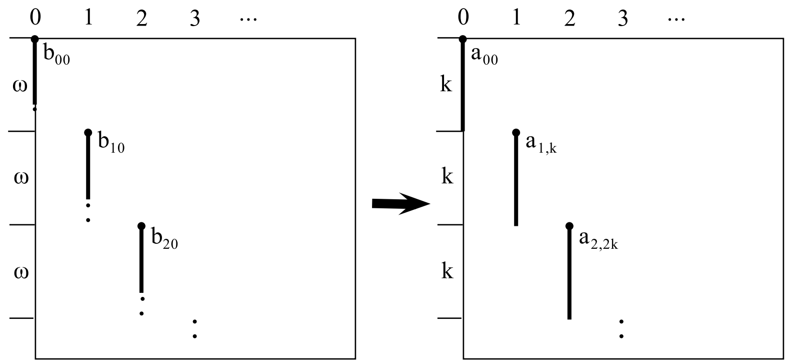

FIGURE 2.2.1.

□

PROPOSITION 2.2.7. *Assume* T *is* NTP_2. *If* $(a_{ij})_{i,j\in\omega}$ *is an indiscernible array and the diagonal* $\{\varphi(x, a_{ii})\}_{i\in\omega}$ *is consistent, then the whole array* $\{\varphi(x, a_{ij})\}_{i,j\in\omega}$ *is consistent. Moreover, this property characterizes* NTP_2.

PROOF. Let $\kappa \in \omega$ be arbitrary. Let $\mathbf{A}^k = (b_{ij})_{i,j\in\omega}$, then its diagonal $\left\{\varphi^k(x, b_{ii})\right\}_{i\in\omega}$ is consistent by Lemma 2.2.6. As $\mathbf{B} = \left(\mathbf{A}^k\right)^T$ has the same diagonal, using Lemma 2.2.6 again we conclude that if $\mathbf{B}^k = (c_{ij})_{i,j\in\omega}$, then its diagonal $\left\{\varphi^{k^2}(x, c_{ii})\right\}_{i\in\omega}$ is consistent. In particular $\{\varphi(x, a_{ij})\}_{i,j<k}$ is consistent. Conclude by compactness.

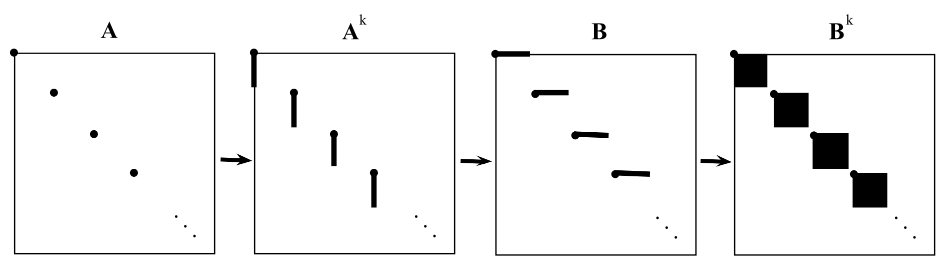

FIGURE 2.2.2.

"Moreover" follows from the fact that if T has TP_2, then there is a strongly indiscernible array witnessing this. □

COROLLARY 2.2.8. *Let* T *be* NTP_2. *Then* $\varphi(x, a)$ *divides over* A *if and only if it array-divides over* A.

PROOF. If $(a_{ij})_{i,j\in\omega}$ is an A-indiscernible array with $a_{00} = a$, then $\{\varphi(x, a_{ii})\}_{i\in\omega}$ is consistent since $(a_{ii})_{i\in\omega}$ is indiscernible over A and $\varphi(x, a)$ does not divide over A, apply Proposition 2.2.7. $\square$

REMARK 2.2.9. Array dividing was apparently first considered for the purposes of classification of Zariski geometries in [**HZ96**]. Kim [**Kim96**] proved that in simple theories dividing equals array dividing. Later the first author used it to develop the basics of simplicity theory in the context of compact abstract theories [**Ben03**], and Adler used it in his presentation of thorn-forking in [**Adl09**].

2.3. The chain condition

2.3.1. The chain condition.

DEFINITION 2.3.1. We say that forking in T satisfies the *chain condition* over A if whenever $I = (a_i)_{i\in\omega}$ is an indiscernible sequence over A and $\varphi(x, a_0)$ does not fork over A, then $\varphi(x, a_0) \wedge \varphi(x, a_1)$ does not fork over A. It then follows that $\{\varphi(x, a_i)\}_{i\in\omega}$ does not fork over A.

LEMMA 2.3.2. *The following are equivalent for any theory T and a set A:*

(1) Forking in T satisfies the chain condition over A.

(2) For every $p(x) \in S(A)$, whenever $(p(x) \cup \{\varphi_i(x, a_i)\})_{i\in(2^{|T|+|A|})^+}$ is a family of partial types non-forking over A, there are $i < j \in \kappa$ such that $p(x) \cup \{\varphi_i(x, a_i)\} \cup \{\varphi_j(x, a_j)\}$ does not fork over A.

(3) There are no anti-chains of unbounded size in the partial order of non-forking types of a fixed size over A: there is κ such that given $p(x) \in S(A)$, whenever $(p(x) \cup \{\varphi_i(x, a_i)\})_{i\in\lambda}$ is a family of partial types non-forking over A, there are $i < j \in \kappa$ such that $p(x) \cup \{\varphi_i(x, a_i)\} \cup \{\varphi_j(x, a_j)\}$ does not fork over A.

(4) If $b \mathop{\smile\!\!\!\!\!\!\;|}_A a_0$ and $I = (a_i)_{i\in\omega}$ is indiscernible over A, then there is $I' \equiv_{Aa_0} I$, indiscernible over Ab and such that $b \mathop{\smile\!\!\!\!\!\!\;|}_A I'$.

PROOF. (1) implies (2): Follows from the fact that in every set S with elements of size λ, if $|S| > 2^{\lambda+|T|}$ then some two different elements appear in an indiscernible sequence (see e.g. [**Cas03**, Proposition 3.3]).

(2) implies (3) is obvious.

(3) implies (4): We may assume that I is of length κ, long enough. Let $p(x, a_0) = \mathrm{tp}(b/a_0A)$. It follows from (3) by compactness that $\bigcup_{i<\kappa} p(x, a_i)$ does not fork over A. Then there is b' realizing it, such that in addition $b' \mathop{\smile\!\!\!\!\!\!\;|}_A I$. By Ramsey, automorphism and compactness we find an I' as wanted.

(4) implies (1): Assume that (1) fails, let I and $\varphi(x, y)$ witness this, so $\varphi(x, a_0) \wedge \varphi(x, a_1)$ forks over A. Let $b \models \varphi(x, a_0) \wedge \varphi(x, a_1)$. It is clearly not possible to find I' as in (4). $\square$

REMARK 2.3.3. The term "chain condition" refers to Lemma 2.3.2(3) interpreted as saying that there are no antichains of unbounded size in the partial order of non-forking formulas (ordered by implication). The chain condition was introduced and proved by Shelah with respect to weak dividing, rather than dividing, for simple theories in the form of (2) in [**She80**]. Later [**GIL02**, Theorem 4.9] presented a proof due to Shelah of the chain condition with respect to dividing for simple theories using the independence theorem, again in the form of (2). The chain condition in the form of (1) was proved for simple theories by Kim [**Kim96**]. It was further studied by Dolich [**Dol04b**], Lessman [**Les00**], Casanovas [**Cas03**] and Adler [**Adl06**] establishing the equivalence of (1), (2) and (3). In the case of NIP theories, the chain condition follows immediately from the fact that non-forking is equivalent to Lascar-invariance (see Lemma 2.3.11).

Of course, the chain condition need not hold in general.

EXAMPLE 2.3.4. Let T be the model completion of the theory of triangle-free graphs. It eliminates quantifiers. Let $M \models T$ and let $(a_i)_{i\in\omega}$ be an M-indiscernible sequence such that $\models \neg R a_i b$ for any i and $b \in M$. Notice that by indiscernibility $\models \neg R a_i a_j$ for $i \neq j$. It is easy to see that Rxa_0 does not divide over M. On the other hand, $Rxa_0 \wedge Rxa_1$ divides over M.

2.3.2. NTP$_2$ implies the chain condition.

We will need some facts about forking and dividing in NTP$_2$ theories established in Chapter 1. Recall that a set C is an *extension base* if every type in $S(C)$ does not fork over C.

DEFINITION 2.3.5. We say that $(a_i)_{i\in\kappa}$ is a *universal Morley sequence* in $p(x) \in S(A)$ when:

- it is indiscernible over A with $a_i \models p(x)$
- for any $\varphi(x,y) \in L(A)$, if $\varphi(x,a_0)$ divides over , then $\{\varphi(x,a_i)\}_{i\in\kappa}$ is inconsistent.

FACT 2.3.6. *[Chapter 1] Assume that* T *is* NTP$_2$.

(1) Let M *be a model. Then for every* $p(x) \in S(M)$*, there is a universal Morley sequence in it.*
(2) Let C *be an extension base. Then* $\varphi(x,a)$ *divides over* C *if and only if* $\varphi(x,a)$ *forks over* C.

First we observe that the chain condition always implies equality of dividing and array dividing:

PROPOSITION 2.3.7. *If* T *satisfies the chain condition over* C*, then* $\varphi(x,a)$ *divides over* C *if and only if it array-divides over* C.

PROOF. Assume that $\varphi(x,a)$ does not divide over C. Let $(a_{ij})_{i,j\in\omega}$ be a C-indiscernible array and $a_{00} = a$. It follows by the chain condition and compactness that $\{\varphi(x,a_{i0})\}_{i\in\omega}$ does not divide over C. But as $\left((a_{ij})_{i\in\omega}\right)_{j\in\omega}$ is also a C-indiscernible sequence, applying the chain condition and compactness again we conclude that $\{\varphi(x,a_{ij})\}_{i,j\in\omega}$ does not divide over C, so in particular it is consistent. □

And in the presence of universal Morley sequences witnessing dividing, the converse holds:

PROPOSITION 2.3.8. *Let* T *be* NTP$_2$ *and* $M \models T$*. Then forking satisfies the chain condition over* M.

PROOF. Let κ be very large compared to $|M|$, assume that $\bar{a}_0 = (a_{0i})_{i\in\kappa}$ is indiscernible over M, $\varphi(x,a_{00})$ does not divide over M, but $\varphi(x,a_{00}) \wedge \varphi(x,a_{01})$ does. By Fact 2.3.6, let $(\bar{a}_i)_{i\in\omega}$ be a universal Morley sequence in $\mathrm{tp}(\bar{a}_0/M)$. By the universality and indiscernibility of $\bar{a}_0$, $\{\varphi(x, a_{ij_1}) \wedge \varphi(x, a_{ij_2})\}_{i\in\omega}$ is inconsistent for any $j_1 \neq j_2$. We can extract an M-indiscernible sequence $\left(\left(a'_{ij}\right)_{i\in\omega}\right)_{j\in\omega}$ from $\left((a_{ij})_{i\in\omega}\right)_{j\in\kappa}$, such that type of every finite subsequence over M is already present in the original sequence. It follows that $\left(a'_{ij}\right)_{i,j\in\omega}$ is an M-indiscernible array and that $\left\{\varphi(x, a'_{ij})\right\}_{i,j\in\omega}$ is inconsistent, thus $\varphi(x, a_{00})$ array-divides over M, thus divides over M by Corollary 2.2.8 — a contradiction. $\square$

Theorem 2.3.9. *If T is NTP$_2$, then it satisfies the chain condition over extension bases.*

Proof. Let C be an extension base and $\bar{a} = (a_i)_{i\in\omega}$ be an A-indiscernible sequence. As C is an extension base, we can find $M \supseteq C$ such that $M \mathop{\smile\!\!\!\!\!\!\;|}_C \bar{a}$. It follows that for any $n \in \omega$, $\bigwedge_{i<n} \varphi(x, a_i)$ divides over C if and only if it divides over M. It follows from Proposition 2.3.8 that if $\varphi(x, a_0)$ does not divide over C, then $\{\varphi(x, a_i)\}_{i\in\omega}$ does not divide over C. $\square$

Corollary 2.3.10. *If T is NTP$_2$, A is an extension base, $(a_{ij})_{i,j\in\omega}$ is an A-indiscernible array, and $\varphi(x, a_{00})$ does not divide over A, then $\{\varphi(x, a_{ij})\}_{i,j\in\omega}$ does not divide over A.*

2.3.3. The chain condition does not imply NTP$_2$.

Lemma 2.3.11. *Let T be a theory satisfying:*

- *For every set A and a global type $p(x)$, it does not fork over A if and only if it is Lascar-invariant over A.*

Then T satisfies the chain condition.

Proof. Let $\bar{a} = (a_i)_{i\in\omega}$ be an A-indiscernible sequence and assume that $\varphi(x, a_0)$ does not fork over A. Then there is a global type $p(x)$ containing $\varphi(x, a_0)$ and non-forking over A, thus Lascar-invariant over A. Taking $c \models p|_{\bar{a}A}$, it follows by Lascar-invariance that $c \models \{\varphi(x, a_i)\}_{i\in\omega}$. $\square$

In Chapter 6, Section 5.3 the following example is constructed:

Fact 2.3.12. *There is a theory T such that:*

(1) T has TP$_2$.
(2) A global type does not fork over a small set A if and only if it is finitely satisfiable in A (therefore, if and only if it is Lascar-invariant over A).

It follows from Lemma 2.3.11 that this T satisfies the chain condition.

2.4. The weak independence theorem and Lascar types

Definition 2.4.1. As usual, we write $a \equiv^L_C b$ to denote that a and b have the same *Lascar type* over C. That is, if any of the following equivalent properties holds:

(1) a and b are equivalent under every C-invariant equivalence relation with a bounded number of classes.
(2) There are $n \in \omega$ and $a = a_0, ..., a_n = b$ such that a_i, a_{i+1} start a C-indiscernible sequence for each $i < n$.

We let $d_C(a,b)$ be the *Lascar distance*, that is the smallest n as in (2) or ∞ if it does not exist.

Now we will use the chain condition in order to deduce a weak independence theorem over an extension base.

LEMMA 2.4.2. *Assume that* $d_A(b,b')=1$ *and* $a \mathop{\smile\!\!\!\!\!\!\mid}_{Ab} b'$. *Then there exists a sequence* $(a_i b_i)_{i\in\omega}$ *indiscernible over* A *and such that* $a_0 b_0 b_1 = abb'$.

PROOF. Standard. $\square$

THEOREM 2.4.3. *Let* T *be* NTP$_2$ *and* A *an extension base. Assume that* $c \mathop{\smile\!\!\!\!\!\!\mid}_A ab$, $a \mathop{\smile\!\!\!\!\!\!\mid}_A bb'$ *and* $b \equiv_A^L b'$. *Then there is* c' *such that* $c' \mathop{\smile\!\!\!\!\!\!\mid}_A ab'$, $c'a \equiv_A ca$, $c'b' \equiv_A cb$.

PROOF. Let us first consider the case $d_A(b,b')=1$. Since $a \mathop{\smile\!\!\!\!\!\!\mid}_{Ab} b'$, by Lemma 2.4.2 we can find $(a_i b_i)_{i\in\omega}$ indiscernible over A and such that $a_0 b_0 b_1 = abb'$. As $c \mathop{\smile\!\!\!\!\!\!\mid}_A a_0 b_0$, it follows by the chain condition that there exists $c' \equiv_{A a_0 b_0} c$ such that $c' \mathop{\smile\!\!\!\!\!\!\mid}_A (a_i b_i)_{i\in\omega}$ and $(a_i b_i)_{i\in\omega}$ is indiscernible over $c'A$. In particular $c' \mathop{\smile\!\!\!\!\!\!\mid}_A ab'$, $c'a \equiv_A ca$ and $c'b' \equiv_A c'b \equiv_A cb$, as desired.

For the general case, assume that $d_A(b,b') \leq n$, namely that there are $b_0,...,b_n$ be such that $b_i b_{i+1}$ start an A-indiscernible sequence for all $i<n$ and $b_0=b$, $b_n=b'$. We may assume that $a \mathop{\smile\!\!\!\!\!\!\mid}_A b_0 ... b_n$.

By induction on $i \leq n$ we choose c_i such that:

(1) $c_i \mathop{\smile\!\!\!\!\!\!\mid}_A ab_i$,
(2) $c_i a \equiv_A ca$,
(3) $c_i b_i \equiv_A cb_0$.

Let $c_0=c$, it satisfies (1)–(3) by hypothesis. Given c_i, by the Lascar distance 1 case there is some $c_{i+1} \mathop{\smile\!\!\!\!\!\!\mid}_A ab_{i+1}$ such that $c_{i+1}a \equiv_A c_i a \equiv_A ca$ and $c_{i+1}b_{i+1} \equiv_A c_i b_i \equiv_A cb_0$ (by the inductive assumption).

It follows that $c'=c_n$ is as wanted. $\square$

REMARK 2.4.4. For simplicity of notation, let us work over $A=\emptyset$.

(1) It is easy to see that the usual independence theorem implies the weak one. Indeed, let c_1 be such that $c_1 b' \equiv^L cb$. Then $c_1 \mathop{\smile\!\!\!\!\!\!\mid} b'$, $c \mathop{\smile\!\!\!\!\!\!\mid} a$, $a \mathop{\smile\!\!\!\!\!\!\mid} b'$ and $c_1 \equiv^L c$. By the independence theorem we find c' such that $c' \mathop{\smile\!\!\!\!\!\!\mid} ab'$, $c'a \equiv ca$ and $c'b' \equiv c_1 b' \equiv cb$.

(2) In a simple theory, the usual independence theorem follows from the weak one by a direct forking calculus argument. Indeed, assume that we are given $d_1 \mathop{\smile\!\!\!\!\!\!\mid} e_1$, $d_2 \mathop{\smile\!\!\!\!\!\!\mid} e_2$, $d_1 \equiv^L d_2$ and $e_1 \mathop{\smile\!\!\!\!\!\!\mid} e_2$. Using symmetry and Lemma 2.4.10 we find $e_1' d_2'$ such that $e_1' d_2' \mathop{\smile\!\!\!\!\!\!\mid} e_1 e_2$ and $e_1' d_2' \equiv^L e_1 d_1$. It is easy to check that all the assumptions of the weak independence theorem are satisfied with $c=d_2'$, $b=e_1'$, $a=e_2$ and $b'=e_1$. Applying it we find some d such that $d \mathop{\smile\!\!\!\!\!\!\mid} e_1 e_2$, $de_1 \equiv d_2' e_1' \equiv d_1 e_1$ and $de_2 \equiv d_2 e_2$.

We observe that the chain condition means precisely that the ideal of forking formulas is S1, in the terminology of Hrushovski [**Hru12**]. Combining Proposition 2.3.7 with [**Hru12**, Theorem 2.18] we can slightly relax the assumption on the independence between the elements, at the price of assuming that some type has a global invariant extension:

PROPOSITION 2.4.5. *Let* T *be* NTP$_2$ *and* A *an extension base. Assume that* $c \mathop{\smile\!\!\!\!\!\mid}_A ab$*,* $b \mathop{\smile\!\!\!\!\!\mid}_A a$*,* $b' \mathop{\smile\!\!\!\!\!\mid}_A a$*,* $b \equiv_A b'$ *and* $\mathrm{tp}(a/A)$ *extends to a global* A*-invariant type. Then there exists* $c' \mathop{\smile\!\!\!\!\!\mid}_A ab'$ *and* $c'b' \equiv_A cb$*,* $c'a \equiv_A ca$*.*

Using the weak independence theorem, we can show that in NTP$_2$ theories Lascar types coincide with Kim-Pillay strong types over extension bases.

COROLLARY 2.4.6. *Assume that* T *is* NTP$_2$ *and* A *is an extension base. Then* $d \equiv_A^L e$ *if and only if* $d_A(d, e) \leq 3$*.*

PROOF. Let $d \equiv_A^L e$ and let $(d_i)_{i\in\omega}$ be a Morley sequence over A starting with $d = d_0$. As $d_{\geq 1} \mathop{\smile\!\!\!\!\!\mid}_A d_0$, we may assume that $d_{\geq 1} \mathop{\smile\!\!\!\!\!\mid}_A d_0 e$.

We have:

- $d_{>1} \mathop{\smile\!\!\!\!\!\mid}_A d_0 d_1$
- $d_1 \mathop{\smile\!\!\!\!\!\mid}_A d_0 e$
- $d_0 \equiv_A^L e$

Applying the weak independence theorem (with $a = d_1$, $b = d_0$, $b' = e$ and $c = d_{>1}$) we get some $d'_{>1}$ such that $d_1 d'_{>1} \equiv_A d_1 d_{>1}$ (thus $d_1 + d'_{>1}$ is an A-indiscernible sequence) and $ed'_{>1} \equiv_A d_0 d_{>1}$ (thus $e + d'_{>1}$ is an A-indiscernible sequence). It follows that $d_A(d, e) \leq 3$ along the sequence d, d_1, d'_2, e. □

REMARK 2.4.7. Consider the standard example [**CLPZ01**, Section 4] showing that the Lascar distance can be exactly n for any $n \in \omega$. It is easy to see that this theory is NIP, as it is interpretable in the real closed field. However, $\emptyset$ is not an extension base.

It is known that both in simple theories (for arbitrary A) and in NIP theories (for A an extension base), $a \equiv_A b$ implies that $d_A(a, b) \leq 2$ ([**HP11**, Corollary 2.10(i)]), while our argument only gives an upper bound of 3. Thus it is natural to ask:

PROBLEM 2.4.8. Is there an NTP$_2$ theory T, an extension base A and tuples a, b such that $d_A(a, b) = 3$?

DEFINITION 2.4.9. Let $a \equiv'_A b$ be the transitive closure of the relation "a, b start a Morley sequence over A, or b, a starts a Morley sequence over A". This is an A-invariant equivalence relation refining $\equiv_A^L$.

The proof of Corollary 2.4.6 demonstrates in particular that if A is an extension base in an NTP$_2$ theory, then $a \equiv_A^L b$ if and only if $a \equiv'_A b$. We show that in fact this holds in a much more general setting.

Let T be an arbitrary theory. We call a type $p(x) \in S(A)$ *extensible* if it has a global extension non-forking over A, equivalently if it does not fork over A (thus A is an extension base if and only if every type over it is extensible).

LEMMA 2.4.10. *Let* $\mathrm{tp}(a/A)$ *be extensible. Then for any* b *there is some* a' *such that* $a' \equiv'_A a$ *and* $a' \mathop{\smile\!\!\!\!\!\mid}_A b$*.*

PROOF. Let $(a_i)_{i\in\omega}$ be a Morley sequence over A starting with a_0. It follows that $a_{\geq 1} \mathop{\smile\!\!\!\!\!\mid}_A a_0$. Then there is $a'_{\geq 1} \mathop{\smile\!\!\!\!\!\mid}_A a_0 b$ and such that $a_{\geq 1} \equiv_{a_0 A} a'_{\geq 1}$. In particular $a_0 + a'_{\geq 1}$ is still a Morley sequence over A, thus $a'_1 \equiv'_A a_0$, and $a'_1 \mathop{\smile\!\!\!\!\!\mid}_A b$ as wanted. □

PROPOSITION 2.4.11. *Let* p *be an extensible type. Then* $a \equiv^L_A b$ *if and only if* $a \equiv'_A b$*, for any* $a, b \models p(x)$.

PROOF. By Definition 2.4.1(1) it is enough to show that $\equiv'_A$ has boundedly many classes on the set of realizations of p.

Assume not, and let κ be large enough. We will choose $\equiv'$-inequivalent $(a_i)_{i\in\kappa}$ such that in addition $a_i \mathop{\smile\!\!\!\!\!\!\mid}\,_A a_{<i}$. Suppose we have chosen $a_{<j}$ and let us choose a_j. Let $b \models p$ be $\equiv'_A$-inequivalent to a_i for all $i < j$. By Lemma 2.4.10, there exists $a_j \equiv'_A b$ such that $a_j \mathop{\smile\!\!\!\!\!\!\mid}\,_A a_{<j}$. In particular $a_j \not\equiv'_A a_i$ for all $i < j$ as desired.

With κ sufficiently large, we may extract an A-indiscernible sequence $\bar{b} = (b_i)_{i\in\omega}$ from $(a_i)_{i\in\kappa}$ — a contradiction, as then $\bar{b}$ is a Morley sequence over A but $b_i \not\equiv'_A b_j$ for any $i \neq j$. □

2.5. The dividing order

In this section we suggest a generalization of the fundamental order of Poizat [**Poi85**] in the context of NTP$_2$ theories. For simplicity of notation, we only consider 1-types, but everything we do holds for n-types just as well.

Given a partial type $r(x)$ over A, we let $S^{EM,r}(A)$ be the set of Ehrenfeucht-Mostowski types of A-indiscernible sequences in $r(x)$. We will omit A when $A = \emptyset$ and omit r when it is "$x = x$".

DEFINITION 2.5.1. Given $p \in S^{EM}(A)$, let $\mathrm{cl}^{div}(p)$ be the set of all $\varphi(x, y) \in L(A)$ such that for some (any) infinite indiscernible sequences $\bar{a} \models p$, the set $\{\varphi(a_i, y)\}_{i\in\omega}$ is consistent. For $p, q \in S^{EM}(A)$, we say that $p \sim^{div}_A q$ (respectively, $p \leq^{div}_A q$) if $\mathrm{cl}^{div}(p) = \mathrm{cl}^{div}(q)$ (respectively, $\mathrm{cl}^{div}(p) \supseteq \mathrm{cl}^{div}(q)$). We obtain a partial order $\left(S^{EM}_A / \sim^{div}_A, \leq^{div}_A\right)$.

PROPOSITION 2.5.2. *Let* T *be stable. Then* $p \sim^{div} q$ *if and only if* $p = q$*, and* $\left(S^{EM}, \leq^{div}\right)$ *is isomorphic to the fundamental order of* T.

PROOF. For a type p over a model M we let $\mathrm{cl}(p)$ denote its fundamental class, namely the set of formulas $\varphi(x, y)$ such that there exists an instance $\varphi(x, b) \in p(x)$. We denote the fundamental order of T by $\left(S/\sim^{fund}, \leq^{fund}\right)$ where S is the set of all types over all models of T, $p \leq^{fund} q$ if $\mathrm{cl}(p) \supseteq \mathrm{cl}(q)$ and $\sim^{fund}$ is the corresponding equivalence relation. Given $p \in S(M)$, let $p^{(\omega)} \in S_\omega(M)$ be the type of its Morley sequence over M. By stability $p^{(\omega)}$ is determined by p. Let p^{EM} be the Ehrenfeucht-Mostowski type over the empty set of $\bar{a} \models p^{(\omega)}|_M$. Let $f : S \to S^{EM}$, $f : p \mapsto p^{EM}$.

(1) Given $p \in S(M)$, let $\bar{a} \models p^{(\omega)}$, and let us show that $\varphi(x, y) \in \mathrm{cl}(p)$ if and only if $\{\varphi(a_i, y)\}_{i\in\omega}$ is consistent. Indeed, by stability, either condition is equivalent to: $\varphi(a_0, y)$ does not divide over M. In other words, $\mathrm{cl}(p) = \mathrm{cl}^{div}(f(p))$, so $p \leq^{fund} q \Leftrightarrow f(p) \leq^{div} f(q)$.
(2) We show that f is onto. Let $P \in S^{EM}$ be arbitrary, and let $(a_i)_{i\in 2\omega}$ be an indiscernible sequence with P as its EM type. Let M be a model containing $I = (a_i)_{i\in\omega}$, such that $J = (a_{\omega+i})_{i\in\omega}$ is indiscernible over M. Then J is a Morley sequence in $p(x) = \mathrm{tp}(a_\omega/M)$, and $f(p) = P$, as wanted.
(3) To conclude, let $P, Q \in S^{EM}$, $P \sim^{div} Q$, and let us show that they are equal. Let $p \in S(M)$ and $q \in S(N)$ be sent by f to P and Q, respectively.

Since $\mathrm{Th}(M) \subseteq \mathrm{cl}^{\mathrm{div}}(P)$ and similarly for N, Q, we have $M \equiv N$. Taking non-forking extensions of p, q, we may therefore assume that $M = N$ is a monster model. Since $\mathrm{cl}(p) = \mathrm{cl}(q)$, the types of (the parameters of) their definitions are the same, so there exists an automorphism sending one definition to the other, and therefore sending $p \mapsto q$. Since $f(p)$ does not involve any parameters, it follows that $P = f(p) = f(q) = Q$.

$\square$

REMARK 2.5.3. A couple of remarks on the existence of the greatest element in the dividing order in NTP_2 theories.

(1) Given a type $r(x_1, x_2) \in S(A)$, assume that $p\left((x_{1j}, x_{2j})_{j\in\omega}\right)$ is the greatest element in $S^{\mathrm{EM},r}(A)$ (modulo $\sim_A^{\mathrm{div}}$). Then for $i = 1, 2$, $p_i\left((x_{ij})_{j\in\omega}\right) = p|_{(x_{ij})_{j\in\omega}}$ is the greatest element in $S^{\mathrm{EM},r_i}(A)$ with $r_i = r|_{x_i}$.
(2) If for every $r \in S(A)$ there is a $\leq^{\mathrm{div}}$-greatest element in $S^{\mathrm{EM},r}(A)$, then a formula $\varphi(x, a)$ forks over A if and only if it divides over A.
(3) If T is NTP_2 then for every extension base A and $r \in S(A)$ there is a $\leq^{\mathrm{div}}$-greatest element in $S^{\mathrm{EM},r}(M)$.

PROOF. (1) Clear as e.g. given an A-indiscernible sequence $(a_{1j})_{j\in\omega}$ in $r_1(x_1)$, by compactness and Ramsey we can find $(a_{2j})_{j\in\omega}$ such that $(a_{1j}a_{2j})_{j\in\omega}$ is an A-indiscernible sequence in $r(x_1, x_2)$.
(2) Assume that $\varphi(x, a) \vdash \bigvee_{i<k} \varphi_i(x, a_i)$ and $\varphi_i(x, a_i)$ divides over A for each $i < k$. Let $r(xx_0 \dots x_{k-1}) = \mathrm{tp}(aa_0 \dots a_{k-1}/A)$, let $p(\bar{x}\bar{x}_0 \dots \bar{x}_{k-1})$ be the greatest element in $S^{\mathrm{EM},r}(A)$ and let $\left(a_j a_{0j} \dots a_{(k-1)j}\right)_{j\in\omega}$ realize it. As $\{\varphi(x, a_j)\}_{j\in\omega}$ is consistent, it follows that $\{\varphi_i(x, a_{ij})\}_{j\in\omega}$ is consistent for some $i < k$ — contradicting the assumption that $\varphi_i(x, a_i)$ divides by (1).
(3) Let $a \models r$. As A is an extension base, let $M \supseteq A$ be a model such that $M \mathop{\smile\!\!\!\!\!\!\mid}_A a$. Let $I = (a_i)_{i\in\omega}$ be a universal Morley sequence in $\mathrm{tp}(a/M)$ which exists by Fact 2.3.6. Then $\mathrm{tp}(I/A)$ is the greatest element in $S^{\mathrm{EM},r}(A)$. Indeed, $\varphi(x, a)$ divides over $A \Leftrightarrow \varphi(x, a)$ divides over $M \Leftrightarrow \{\varphi(x, a_i)\}_{i\in\omega}$ is inconsistent.

$\square$

DEFINITION 2.5.4. For $p, q \in S^{\mathrm{EM}}$, we write $p \leq^{\#} q$ if there is an array $(a_{ij})_{i,j\in\omega}$ such that:

- $(a_{ij})_{j\in\omega} \models p$ for each $i \in \omega$,
- $\left(a_{if(i)}\right)_{i\in\omega} \models q$ for each $f : \omega \to \omega$.

PROPOSITION 2.5.5. *Let* $p, q \in S^{\mathrm{EM}}$.

(1) If $p \leq^{\mathrm{div}} q$, *then* $p \leq^{\#} q$.
(2) If T *is* NTP_2 *and* $p \leq^{\#} q$, *then* $p \leq^{\mathrm{div}} q$.

PROOF. (1): We show by induction that for each $n \in \omega$ we can find $(\bar{a}_i)_{i\in n}$ and $\bar{b}$ such that: $\bar{a}_i \models p$ and $a_{0j_1} + \dots + a_{(n-1)j_{n-1}} + \bar{b} \models q$ for any $j_1, \dots, j_{n-1} \in \omega$. Assume we have found $(\bar{a}_i)_{i<n}$ and $\bar{b}$, without loss of generality $\bar{b} = \bar{b}' + \bar{b}'' = (b'_i)_{i\in\omega} + (b''_i)_{i\in\omega}$. Consider the type

$$r(\bar{x}_0 ... \bar{x}_{n-1}, y, \bar{z}) = \bigcup_{i \leq n} p(\bar{x}_i) \cup q(\bar{z}) \cup$$
$$\cup \bigcup_{j_0,...,j_n \in \omega} \text{"}x_{0j_0} + x_{1j_1} + ... + x_{nj_n} + y + \bar{z} \text{is indiscernible"}$$

For every finite $r' \subset r$, $\{r'(\bar{x}_0 ... \bar{x}_{n-1}, y_i, \bar{z})\}_{i\in\omega} \cup q(\bar{y})$ is consistent — since by the inductive assumption $\models r'(\bar{a}_0 ... \bar{a}_{n-1}, b'_i, \bar{b}'')$ for all $i \in \omega$. Together with $p \leq^{\text{div}} q$ this implies that $\{r'(\bar{x}_0 ... \bar{x}_{n-1}, y_i, \bar{z})\}_{i\in\omega} \cup p(\bar{y})$ is consistent. By compactness we find $\bar{a}_0, ..., \bar{a}_{n-1}, \bar{a}_n, \bar{b}$ realizing it, and they are what we were looking for.

(2): Follows from the definition of TP$_2$. □

Definition 2.5.6. We write $p \leq^{+} q$[1] if there is $\bar{a} = (a_i)_{i\in\mathbb{Z}} \models q$ and $\bar{b} = (b_i)_{i\in\mathbb{Z}} \models p$ such that $a_0 = b_0$ and $\bar{b}$ is indiscernible over $(a_i)_{i\neq 0}$.

Remark 2.5.7. In any theory, $p \leq^{\#} q$ implies $p \leq^{+} q$ (and so $p \leq^{\text{div}} q$ implies $p \leq^{+} q$).

Proof. If $p \leq^{\#} q$, then by compactness and Ramsey we can find an array $(c_{ij})_{i,j\in\mathbb{Z}}$ such that:

- $\bar{c}_i$ is indiscernible over $\bar{c}_{\neq i}$,
- $(\bar{c}_i)_{i\in\mathbb{Z}}$ is an indiscernible sequence,
- $\bar{c}_i \models p$ for all $i \in \omega$,
- $\left(c_{if(i)}\right)_{i\in\omega} \models q$ for all $f : \omega \to \omega$.

Then take $\bar{a} = (c_{0j})_{j\in\mathbb{Z}}$ and $\bar{b} = (c_{i0})_{i\in\mathbb{Z}}$. □

It is much less clear, however, if the converse implication holds.

Definition 2.5.8. We say that T is *resilient*[2] if we cannot find indiscernible sequences $\bar{a} = (a_i)_{i\in\mathbb{Z}}$, $\bar{b} = (b_i)_{i\in\mathbb{Z}}$ and a formula $\varphi(x, y)$ such that:

- $a_0 = b_0$,
- $\bar{b}$ is indiscernible over $(a_i)_{i\neq 0}$,
- $\{\varphi(x, a_i)\}_{i\in\omega}$ is consistent,
- $\{\varphi(x, b_i)\}_{i\in\omega}$ is inconsistent.

Remark 2.5.9. It follows by compactness that we get an equivalent definition replacing $\mathbb{Z}$ by $\mathbb{Q}$.

Lemma 2.5.10. *The following are equivalent:*

(1) T *is resilient.*
(2) For every $p, q \in S^{\text{EM}}$, $p \leq^{+} q$ *implies* $p \leq^{\text{div}} q$.
(3) For any indiscernible sequence $\bar{a} = (a_i)_{i\in\mathbb{Z}}$ *and* $\varphi(x, y) \in L$, *if* $\varphi(x, a_0)$ *divides over* $(a_i)_{i\neq 0}$, *then* $\{\varphi(x, a_i)\}_{i\in\mathbb{Z}}$ *is inconsistent.*
(4) There is no array $(a_{ij})_{i,j\in\omega}$ *and* $\varphi(x, y) \in L$ *such that* $\{\varphi(x, a_{i0})\}_{i\in\omega}$ *is consistent,* $\{\varphi(x, a_{ij})\}_{j\in\omega}$ *is inconsistent for each* $i \in \omega$ *and* $\bar{a}_i = (a_{ij})_{j\in\omega}$ *is indiscernible over* $(a_{j0})_{j\neq i}$ *for each* $i \in \omega$.

[1] Note that "#" and "+" are supposed to graphically represent the combinatorial configuration which we are using in the definition of the order.

[2] The term was suggested by Hans Adler as a replacement for "NTP$_2$" but we prefered to use it for a (possibly) smaller class of theories.

(5) *There is a cardinal* κ *such that for any* $(a_i)_{i\in\kappa}$ *and* b *with* a_i, b *finite,* $b \mathop{\smile\!\!\!\!\!\!\,|}^{d}_{a_{\neq i}} a_i$ *for some* $i \in \kappa$.

PROOF. (1) is equivalent to (2) Assume that $p \leq^{+} q$, i.e. there is $\bar{a} = (a_i)_{i\in\mathbb{Z}} \models q$ and $\bar{b} = (b_i)_{i\in\mathbb{Z}} \models p$ such that $a_0 = b_0$ and $\bar{b}$ is indiscernible over $(a_i)_{i\neq 0}$. For any $\varphi(x, y)$, if $\{\varphi(x, b_i)\}_{i\in\omega}$ is inconsistent, then $\{\varphi(x, a_i)\}_{i\in\omega}$ is inconsistent by resilience, which means precisely that $p \leq^{\mathrm{div}} q$. The converse is clear.

(1) is equivalent to (3) If $\varphi(x, a_0)$ divides over $a_{\neq 0}$, then there is a sequence $(b_i)_{i\in\mathbb{Z}}$ indiscernible over $a_{\neq 0}$ and such that $b_0 = a_0$ and $\{\varphi(x, b_i)\}_{i\in\mathbb{Z}}$ is inconsistent. It follows by resilience that $\{\varphi(x, a_i)\}_{i\in\mathbb{Z}}$ is inconsistent. On the other hand, assume that $\{\varphi(x, a_i)\}_{i\in\mathbb{Z}}$ is inconsistent. By compactness we can extend our indiscernible sequence to $\bar{a}' + \bar{a} + \bar{a}'' = (a'_i)_{i\in\omega^*} + (a_i)_{i\in\mathbb{Z}} + (a''_i)_{i\in\omega}$. But then $\bar{a}$ witnesses that $\varphi(x, a_0)$ divides over $\bar{a}'\bar{a}''$. Sending $\bar{a}'$ to $a_{\leq -1}$ and $\bar{a}''$ to $a_{\geq 1}$ by an automorphism fixing a_0 we conclude that $\varphi(x, a_0)$ divides over $a_{\neq 0}$.

(1) is equivalent to (4) Let $\bar{a}$, $\bar{b}$ and $\varphi(x, y)$ witness that T is not resilient. Then we let $\bar{a}_0 = \bar{b}$ and we let $\bar{a}_i$ be an image of $\bar{b}$ under some automorphism sending b_0 to a_i by indiscernibility. It follows that $(a_{ij})_{i,j\in\omega}$ is an array as wanted.

Conversely, if we have an array as in (4), by compactness we may assume that it is of the form $(a_{ij})_{i,j\in\mathbb{Z}}$ and and that in addition $(a_{i0})_{i\in\mathbb{Z}}$ is indiscernible. Then $\bar{a} = (a_{i0})_{i\in\mathbb{Z}}$, $\bar{b} = (a_{0j})_{j\in\omega}$ and $\varphi(x, y)$ contradicts resilience.

(5) is equivalent to (4) Let κ be arbitrary. By compactness we may assume that we have an array $(a_{ij})_{i\in\kappa, j\in\omega}$ as in (4). Let $b \models \{\varphi(x, a_{i0})\}_{i\in\kappa}$. It then follows that $b \not\mathop{\smile\!\!\!\!\!\!\,|}^{d}_{a_{\neq i0}} a_{i0}$ (as $\varphi(x, a_{i0})$ divides over $a_{\neq i0}$, witnessed by $\bar{a}_i$) — contradicting (5).

(3) implies (5): Assume that we have $(a_i)_{i\in\kappa}$ and b with a_i, b finite, $b \not\mathop{\smile\!\!\!\!\!\!\,|}^{d}_{a_{\neq i}} a_i$ for all $i \in \kappa$. If κ is large enough then by Erdős-Rado and compactness we can extract a b-indiscernible sequence $(a_i)_{i\in\mathbb{Z}}$ such that still $b \not\mathop{\smile\!\!\!\!\!\!\,|}^{d}_{a\neq i} a_i$. Then some $\varphi(x, a_0) \in \mathrm{tp}(b/a_0)$ divides over $a_{\neq 0}$, while $b \models \{\varphi(x, a_i)\}_{i\in\mathbb{Z}}$ by indiscernibility over b. □

PROPOSITION 2.5.11. *(1) If* T *is* NIP*, then it is resilient.*
(2) If T *is simple, then it is resilient.*
(3) If T *is resilient, then it is* NTP$_2$.

PROOF. (1): Fix $\varphi(x, y)$ and assume that $\{\varphi(x, a_i)\}_{i\in\mathbb{Q}}$ is consistent. Then by NIP there is a maximal $k \in \omega$ such that $\{\neg\varphi(x, a_i)\}_{i\in s} \cup \{\varphi(x, a_i)\}_{i\notin s}$ is consistent, for $s = \{1, 2, ..., k\} \subseteq \mathbb{Q}$. Let d realize it. If $\{\varphi(x, b_i)\}$ was inconsistent, then we would have $\neg\varphi(d, b_i)$ for some $i \in \omega$, and thus $\{\neg\varphi(x, a_i)\}_{i\in s\cup\{k+1\}} \cup \{\varphi(x, a_i)\}_{i\notin s\cup\{k+1\}}$ would be consistent, but by all the indiscernibility around — a contradiction to the maximality of k. Thus, $\{\varphi(x, b_i)\}_{i\in\mathbb{Q}}$ is consistent.

(2): It is easy to see that $(a_i)_{i>0}$ is a Morley sequence over $A = (a_i)_{i<0}$ by finite satisfiability. If $\varphi(x, a_0)$ divides over $a_{\neq 0}$, then by Kim's lemma $\{\varphi(x, a_i)\}_{i\in\mathbb{Q}}$ is inconsistent.

(3): By Erdős-Rado and compactness we can find a strongly indiscernible array $(c_{ij})_{i,j\in\mathbb{Z}}$ witnessing TP$_2$ for $\varphi(x, y)$. Set $a_i = c_{i0}$ for $i \in \omega$ and $b_j = b_{0j}$ for $j \in \omega$. Then $\bar{a}$, $\bar{b}$ and $\varphi(x, y)$ witness that T is not resilient. □

CLAIM 2.5.12. Let T be resilient, A an extension base, and let $\bar{a} = (a_i)_{i \in \mathbb{Z}}$ be indiscernible over A, say in and $r = \mathrm{tp}(a_0/A) \in S(A)$. Then the following are equivalent:

(1) The EM type $\mathrm{tp}^{\mathrm{EM}}(\bar{a}/A)$ is $\leq_A^{\mathrm{div}}$-greatest in $S^{\mathrm{EM},r}(A)$.
(2) $\mathrm{tp}(a_{\neq 0}/a_0 A)$ does not divide over A.

PROOF. We may assume that $A = \emptyset$.

(1) implies (2) in any theory: Let $\models \varphi(a_{\neq 0}, a_0)$. By indiscernibility and compactness $\{\varphi(x, a_i)\}_{i \in \mathbb{Z}}$ is consistent, so by (1) $\varphi(x, a_0)$ does not divide.

(2) implies (1): Assume that $\varphi(x, a_0)$ divides. As $\mathrm{tp}(a_{\neq 0}/a_0)$ does not divide, it follows that $\varphi(x, a_0)$ divides over $a_{\neq 0}$. But then by Lemma 2.5.10(3) we have that $\{\varphi(x, a_i)\}_{i \in \mathbb{Z}}$ is inconsistent, hence (1). □

REMARK 2.5.13. Similar observation in the context of NIP theories based on [**She09**] is made in [**KU**].

Recall that a theory is called *low* if for every formula $\varphi(x, y)$ there is $k \in \omega$ such that for any indiscernible sequence $(a_i)_{i \in \omega}$, $\{\varphi(x, a_i)\}_{i \in \omega}$ is consistent if and only if it is k-consistent. The following is a generalization of [**BPV03**, Lemma 2.3].

PROPOSITION 2.5.14. *Let T be resilient. Then the following are equivalent:*

(1) $\varphi(x, y)$ is low.
(2) The set $\{(c, d) : \varphi(x, c)$ divides over $d\}$ is type-definable (where d is allowed to be of infinite length).

PROOF. (1) implies (2) holds in any theory, and we show that (2) implies (1).

Assume that $\varphi(x, y)$ is not low. Then for every $i \in \omega$ we have a sequence $\bar{a}_i = (a_{ij})_{j \in \mathbb{Z}}$ such that $\{\varphi(x, a_{ij})\}_{j \in \mathbb{Z}}$ is i-consistent, but inconsistent. In particular $\varphi(x, a_{i0})$ divides over $(a_{ij})_{j \neq 0}$ for each i.

If (2) holds, then by compactness we can find a sequence $\bar{a} = (a_j)_{j \in \omega}$ such that $\{\varphi(x, a_j)\}_{j \in \omega}$ is consistent and $\varphi(x, a_0)$ still divides over $a_{\neq 0}$. But this is a contradiction to resilience by 2.5.10(3). □

PROBLEM 2.5.15. (1) Does NTP_2 imply resilience?
(2) Is resilience preserved under reducts?
(3) Does type-definability of dividing imply lowness in NTP_2 theories?

2.6. On a strengthening of strong theories

Recently several attempts have been made to define weight outside of the familiar context of simple theories. First Shelah had defined strongly dependent theories and several notions of dp-rank in [**She09, Shed**]. The study of dp-rank was continued in [**OU11**]. After that Adler [**Adl07**] had introduced *burden*, a notion based on the invariant κ_{inp} of Shelah [**She90**] which generalizes simultaneously dp-rank in NIP theories and weight in simple theories. In this section we are going to add yet another version of measuring weight. First we recall the notions mentioned above.

For notational convenience we consider an extension Card^* of the linear order on cardinals by adding a new maximal element ∞ and replacing every limit cardinal κ by two new elements κ_- and κ_+. The standard embedding of cardinals into Card^* identifies κ with κ_+. In the following, whenever we take a supremum of a set of cardinals, we will be computing it in Card^*.

DEFINITION 2.6.1. [**Adl07**] Let $p(x)$ be a (partial) type.

(1) An inp-pattern of depth κ in $p(x)$ consists of $(\bar{a}_i, \varphi_i(x, y_i), k_i)_{i \in \kappa}$ with $\bar{a}_i = (a_{ij})_{j \in \omega}$ and $k_i \in \omega$ such that:
   - $\{\varphi_i(x, a_{ij})\}_{j \in \omega}$ is k_i-inconsistent for every $i \in \kappa$,
   - $p(x) \cup \{\varphi_i(x, a_{if(i)})\}_{i \in \kappa}$ is consistent for every $f : \kappa \to \omega$.

(2) The *burden* of a partial type $p(x)$ is the supremum (in Card*) of the depths of inp-patterns in it. We denote the burden of p as $\mathrm{bdn}(p)$ and we write $\mathrm{bdn}(a/A)$ for $\mathrm{bdn}(\mathrm{tp}(a/A))$.

(3) We get an equivalent definition by taking supremum only over inp-patterns with mutually indiscernible rows.

(4) It is easy to see by compactness that T is NTP$_2$ if and only if $\mathrm{bdn}(\text{"}x = x\text{"}) < \infty$, if and only if $\mathrm{bdn}(\text{"}x = x\text{"}) < |T|^+$.

(5) A theory T is called *strong* if $\mathrm{bdn}(p) \leq (\aleph_0)_-$ for every finitary type p (equivalently, there is no inp-pattern of infinite depth). Of course, if T is strong then it is NTP$_2$.

FACT 2.6.2. [**Adl07**]

(1) Let T be NIP. *Then* $\mathrm{bdn}(p) = \text{dp-rk}(p)$ *for any p.*

(2) Let T be simple. Then the burden of p is the supremum of weights of its complete extensions.

Some basics of the theory of burden are developed in Chapter 3:

FACT 2.6.3. *Let T be an arbitrary theory.*

(1) The following are equivalent:
   (a) $\mathrm{bdn}(p) < \kappa$.
   (b) For any $(\bar{a}_i)_{i \in \kappa}$ mutually indiscernible over A and $b \models p$, there is some $i \in \kappa$ and $\bar{a}'_i$ such that $\bar{a}'_i$ is indiscernible over bA and $\bar{a}'_i \equiv_{A a_{i0}} \bar{a}_i$.

(2) Assume that $\mathrm{bdn}(a/A) < \kappa$ *and* $\mathrm{bdn}(b/aA) < \lambda$, *with κ and λ finite or infinite cardinals. Then* $\mathrm{bdn}(ab/A) < \kappa \times \lambda$.

(3) In particular, in the definition of strong (or NTP$_2$*) it is enough to look at types in one variable.*

In [**KOU11**] it is proved that dp-rank is sub-additive, so burden in NIP theories is sub-additive as well. The sub-additivity of burden in simple theories follows from Fact 2.6.2 and the sub-additivity of weight in simple theories. It thus becomes natural to wonder if burden is sub-additive in general, or at least in NTP$_2$ theories.

Now we are going to define a refinement of the class of strong theories.

DEFINITION 2.6.4. Let $p(x)$ be a partial type.

(1) An inp^2-pattern of depth κ in $p(x)$ consists of formulas $(\varphi_i(x, y_i, z_i))_{i \in \kappa}$, mutually indiscernible sequences $(\bar{a}_i)_{i \in \kappa}$ and $b_i \subseteq \bigcup_{j<i} \bar{a}_j$ such that:
   (a) $\{\varphi_i(x, a_{i0}, b_i)\}_{i \in \omega} \cup p(x)$ is consistent,
   (b) $\{\varphi_i(x, a_{ij}, b_i)\}_{j \in \omega}$ is inconsistent for every $i \in \omega$.

(2) An inp^3-pattern of depth κ in $p(x)$ is defined exactly as an inp^2-pattern of depth κ, but allowing $b_i \subseteq \bigcup_{j \in \kappa, j \neq i} \bar{a}_j$. It is then clear that every inp^2-pattern is an inp^3-pattern of the same depth, but the opposite is not true.

(3) The *burden*^{2} (*burden*^{3}) of a partial type $p(x)$ is the supremum (in Card*) of the depths of inp^2-patterns (resp. inp^3-patterns) in it. We denote the burden^2 of p as $\mathrm{bdn}^2(p)$ and we write $\mathrm{bdn}^2(a/A)$ for $\mathrm{bdn}^2(\mathrm{tp}(a/A))$ (and similarly for bdn^3).

(4) A theory T is called *strong*^{2} if $\mathrm{bdn}^2(p) \leq (\aleph_0)_-$ for every finitary type p (that is, there is no inp^2-pattern of infinite depth). Similarly for *strong*^{3}.

In the following proposition we sum up some of the properties of bdn^2 and bdn^3.

Proposition 2.6.5. *(1) For any partial type* $p(x)$, $\mathrm{bdn}(p) \leq \mathrm{bdn}^2(p) \leq \mathrm{bdn}^3(p)$.

(2) Strong^3 *implies strong*^{2} *implies strong.*

(3) In fact, T *is strong*^{2} *if and only if it is strong*^{3}.

(4) T *is strongly*^{2} *dependent if and only if it is NIP and strong*^{2} *(we recall from* [**KS12a**, Definition 2.2] *that* T *is called strongly*^{2} *dependent when there are no* $\left(\varphi_i(x, y_i, z_i), \bar{a}_i = (a_{ij})_{j\in\omega}, b_i \subseteq \bigcup_{j<i} \bar{a}_j\right)_{i\in\omega}$ *such that* $(\bar{a}_i)_{i\in\omega}$ *are mutually indiscernible and the set* $\{\varphi_i(x, a_{i0}, b_i) \wedge \neg\varphi_i(x, a_{i1}, b_i)\}_{i\in\omega}$ *is consistent.).*

(5) If T *is supersimple, then it is strong*^{2}.

(6) There are strong^2 *stable theories which are not superstable.*

(7) There are strong stable theories which are not strong^2.

(8) We still have that T *is* NTP$_2$ *if and only if every finitary type has bounded burden*^{3}.

Proof. (1) is immediate by comparing the definitions, and (2) follows from (1).

(3) Assume that T is not strong^3, witnessed by $(\varphi_i(x, y_i, z_i), \bar{a}_i, b_i)_{i\in\omega}$. For $i \in \omega$, let $f(i)$ be the smallest $j \in \omega$ such that $b_i \in \bar{a}_{<j}$. Now for $i \in \omega$ we define inductively:

- $\alpha_0 = 0$, $\alpha_{i+1} = f(\alpha_i)$,
- $b'_i = b_{\alpha_i} \cap \bar{a}_{\in\{\alpha_0,\alpha_1,\dots,\alpha_{i-1}\}}$ and $b''_i = b_{\alpha_i} \cap \bar{a}_{\in\{0,1,\dots,\alpha_{i+1}-1\}\setminus\{\alpha_0,\alpha_1,\dots,\alpha_i\}}$, so we may assume that $b_{\alpha_i} = b''_i b'_i$.
- $a'_{ij} = a_{\alpha_i j} b''_i$ for $j \in \omega$,
- $\varphi'_i\left(x, a'_{ij}, b'_i\right) = \varphi_i(x, a_{ij}, b_i)$.

It is now easy to check that $(\bar{a}'_i)_{i\in\omega}$ are mutually indiscernible, $b'_i \in \bar{a}'_{<i}$, $\{\varphi'_i(x, a'_{i0}, b'_i)\}_{i\in\omega}$ is consistent and $\left\{\varphi'_i\left(x, a'_{ij}, b'_i\right)\right\}_{j\in\omega}$ is inconsistent for every $i \in \omega$. This gives us an inp^2-pattern of infinite depth, witnessing that T is not strong^2.

(4) Let $(\varphi_i(x, y_i, z_i), \bar{a}_i, b_i)_{i\in\omega}$ witness that T is not strong^2 and let $c \models \{\varphi_i(x, a_{i0}, b_i)\}_{i\in\omega}$, it follows from the inconsistency of $\{\varphi(x, a_{ij}, b_i)\}_{j\in\omega}$'s that for each $i \in \omega$ there is some $k_i \in \omega$ such that $c \models \{\varphi_i(x, a_{i0}, b_i) \wedge \neg\varphi_i(x, a_{ik_i}, b_i)\}_{i\in\omega}$. Define $a'_{ij} = a_{i,k_i\times j} a_{i,k_i\times j+1} \dots a_{i,k_i\times(j+1)-1}$ and $\varphi'\left(x, a'_{ij}, b_i\right) = \varphi(x, a_{i,k_i\times j}, b_i)$. Then $(\bar{a}'_i)_{i\in\omega}$ are mutually indiscernible, $b_i \in \bigcup_{j<i} \bar{a}'_j$ and $c \models \{\varphi_i(x, a'_{i0}, b_i) \wedge \neg\varphi_i(x, a'_{i1}, b_i)\}_{i\in\omega}$ — witnessing that T is not strongly^2 dependent.

On the other hand, let $(\varphi_i(x, y_i, z_i), \bar{a}_i, b_i)_{i\in\omega}$ witness that T is not strongly^2 dependent and assume that T is NIP. Let $\varphi'_i(x, y'_i, z_i) = \varphi_i\left(x, y^0_i, z_i\right) \wedge \neg\varphi_i\left(x, y^1_i, z_i\right)$, $a'_{ij} = a_{i(2j)} a_{i(2j+1)}$ for all $i, j \in \omega$. We then have that $(\bar{a}'_i)_{i\in\omega}$ are still mutually indiscernible and $b_i \in \bigcup_{j<i} \bar{a}'$, $\{\varphi'_i(x, a'_{i0}, b_i)\}_{i\in\omega}$ is consistent and $\left\{\varphi'_i\left(x, a'_{ij}, b_i\right)\right\}_{j\in\omega}$

is inconsistent (otherwise let c realize it, it follows that $\varphi_i(c, a_{ij}, b_i)$ holds if and only if j is even, contradicting NIP). But this shows that T is not strong^2.

(5) Let T be supersimple, and assume that T is not strong^2, witnessed by $(\varphi_i(x, y_i, z_i), \bar{a}_i, b_i)_{i\in\omega}$ and let $A = \bigcup_{i,j\in\omega} a_{ij}$. Let $c \models \{\varphi_i(x, a_{i0}, b_i)\}_{i\in\omega}$. By supersimplicity, there has to be some finite $A_0 \subset A$ such that $\mathrm{tp}(c/A)$ does not divide over A_0. It follows that there is some $i' \in \omega$ such that $A_0 \subset \bigcup_{i<i', j\in\omega} a_{ij}$. But then $c \models \varphi_{i'}(x, a_{i'0}, b_{i'})$, $(a_{i'j}b_{i'})_{j\in\omega}$ is indiscernible over A_0 and $\{\varphi(x, a_{i'j}, b_{i'})\}_{j\in\omega}$ is inconsistent, so $\mathrm{tp}(c/A)$ divides over A_0 — a contradiction.

(6) It is easy to see that the theory of an infinite family of refining equivalence relations with infinitely many infinite classes satisfies the requirement.

(7) In [**Shed**, Example 2.5] Shelah gives an example of a strongly stable theory which is not strongly^2 stable. In view of (3) this is sufficient. Besides, there are examples of NIP theories of burden 1 which are not strongly^2 dependent (e.g. $(\mathbb{Q}_p, +, \cdot, 0, 1)$ or $(\mathbb{R}, <, +, \cdot, 0, 1)$).

(8) We remind the statement of Fodor's lemma.

Fact (Fodor's lemma). If κ is a regular, uncountable cardinal and $f : \kappa \to \kappa$ is such that $f(\alpha) < \alpha$ for any $\alpha \neq 0$, then there is some γ and some stationary $S \subseteq \kappa$ such that $f(\alpha) = \gamma$ for any $\alpha \in S$.

If T has TP_2, then clearly $\mathrm{bdn}^3(T) = \infty$, and we prove the converse. Assume that $\mathrm{bdn}^3(T) \geq |T|^+$ and let $\kappa = |T|^+$. Then we can find $(\varphi_i(x, y_i, z_i), \bar{a}_i, b_i)_{i\in\kappa}$ with $(\bar{a}_i)_{i\in\kappa}$ mutually indiscernible, finite $b_i \in \bigcup_{j\in\kappa, j\neq i} \bar{a}_j$ such that $\{\varphi_i(x, a_{i0}, b_i)\}_{i\in\kappa}$ is consistent and $\{\varphi_i(x, a_{ij}, b_i)\}_{j\in\omega}$ is inconsistent for every $i \in \kappa$. For each $i \in \kappa$, let $f(i)$ be the largest $j < i$ such that $\bar{a}_j \cap b_i \neq \emptyset$ and let $g(i)$ be the largest $j \in \kappa$ such that $\bar{a}_j \cap b_i \neq \emptyset$. By Fodor's lemma there is some stationary $S \subseteq \kappa$ and $\gamma \in \kappa$ such that $f(i) = \gamma$ for all $i \in S$.

By induction we choose an increasing sequence $(i_\alpha)_{\alpha\in\kappa}$ from S such that $i_0 > \gamma$ and $i_\alpha > g(i_\beta)$ for $\beta < \alpha$. Now let $a'_{\alpha j} = a_{i_\alpha j} b_{i_\alpha}$ and $\varphi'_\alpha(x, y'_\alpha) = \varphi_{i_\alpha}(x, y_{i_\alpha}, z_{i_\alpha})$. It follows by the choice of i_α's that $(\bar{a}'_\alpha)_{\alpha\in\kappa}$ are mutually indiscernible, $\{\varphi'_\alpha(x, a'_{\alpha 0})\}_{\alpha\in\kappa}$ is consistent and $\left\{\varphi'_\alpha\left(x, a'_{\alpha j}\right)\right\}_{j\in\omega}$ is inconsistent for each $\alpha \in \kappa$. It follows that we had found an inp-pattern of depth $\kappa = |T|^+$ — so T has TP_2. ☐

We are going to give an analogue of Fact 2.6.3(1) for burden^{2,3}, but first a standard lemma.

LEMMA 2.6.6. *Let $\bar{a} = (a_i)_{i\in\omega}$ be indiscernible over A and let $p(x, a_0) = \mathrm{tp}(c/a_0A)$. Assume that $\{p(x, a_i)\}_{i\in\omega}$ is consistent. Then there is $\bar{a}' \equiv_{a_0A} \bar{a}$ which is indiscernible over cA.*

LEMMA 2.6.7. *Let $p(x)$ be a partial type over A:*

(1) The following are equivalent:
  - *(a)* $\mathrm{bdn}^3(p) < \kappa$.
  - *(b) For any $(\bar{a}_i)_{i\in\kappa}$ mutually indiscernible over A and $c \models p(x)$ there is some $i \in \kappa$ and $\bar{a}'_i$ such that:*
    - $\bar{a}'_i \equiv_{a_{i0}\bar{a}_{\neq i}A} \bar{a}_i$,
    - *$\bar{a}'_i$ is indiscernible over $c\bar{a}_{\neq i}A$.*

(2) The following are equivalent:
  - *(a)* $\mathrm{bdn}^2(p) < \kappa$.

(b) *For any* $(\bar{a}_i)_{i\in\kappa}$ *mutually indiscernible over* A *and* $c \models p(x)$ *there is some* $i \in \kappa$ *and* $\bar{a}_i'$ *such that:*
- $\bar{a}_i' \equiv_{a_{i0}\bar{a}_{<i}A} \bar{a}_i$,
- $\bar{a}_i'$ *is indiscernible over* $c\bar{a}_{<i}A$.

Proof. (1): (a) implies (b): Let $(\bar{a}_i)_{i\in\kappa}$ mutually indiscernible over A and $c \models p(x)$ be given. Define $p_i(x, a_{i0}) = \mathrm{tp}(c/a_{i0}\bar{a}_{\neq i}A)$. By Lemma 2.6.6 it is enough to show that $\bigcup_{j\in\omega} p_i(x, a_{ij})$ is consistent for some $i \in \kappa$.

Assume not, but then by compactness for each $i \in \kappa$ we have some $\varphi_i(x, a_{i0}, b_i d_i) \in p_i(x, a_{i0})$ with $b_i \in \bar{a}_{\neq i}$ and $d_i \in A$ such that $\{\varphi_i(x, a_{ij}, b_i d_i)\}_{j\in\omega}$ is inconsistent. Let $\varphi_i'(x, a_{ij}', b_i') = \varphi_i(x, a_{ij}, b_i d_i)$ with $a_{ij}' = a_{ij} d_i$ and $b_i' = b_i$. It follows that $(\bar{a}_i')_{i\in\kappa}$ are mutually indiscernible, $c \models \{\varphi_i'(x, a_{i0}', b_i')\}_{i\in\kappa} \cup p(x)$ and $\{\varphi_i'(x, a_{ij}', b_i')\}_{j\in\omega}$ is inconsistent for each $i \in \kappa$, thus witnessing that $\mathrm{bdn}^3(p) \geq \kappa$ — a contradiction.

(b) implies (a): Assume that $\mathrm{bdn}^3(p) \geq \kappa$, witnessed by an inp^3-pattern $(\varphi_i(x, y_i, z_i), \bar{a}_i, b_i)_{i\in\kappa}$ in $p(x)$. Let $c \models \{\varphi_i(x, a_{i0}, b_i)\}_{i\in\kappa}$ and take $A = \emptyset$. It is then easy to check that (2) fails.

(2): Similar. $\square$

CHAPTER 3

Theories without the tree property of the second kind

This chapter is submitted to the Anals of Pure and Appllied Logic as "Theories without the tree property of the second kind" [**Che12**]. We initiate a systematic study of the class of theories without the tree property of the second kind — NTP_2. Most importantly, we show: the burden is "sub-multiplicative" in arbitrary theories (in particular, if a theory has TP_2 then there is a formula with a single variable witnessing this); NTP_2 is equivalent to the generalized Kim's lemma; the dp-rank of a type in an arbitrary theory is witnessed by mutually indiscernible sequences of realizations of the type, after adding some parameters — so the dp-rank of a 1-type in any theory is always witnessed by sequences of singletons; in NTP_2 theories, simple types are co-simple, characterized by the co-independence theorem, and forking between the realizations of a simple type and arbitrary elements satisfies full symmetry; a Henselian valued field of characteristic $(0,0)$ is NTP_2 (strong, of finite burden) if and only if the residue field is NTP_2 (the residue field and the value group are strong, of finite burden respectively); adding a generic predicate to a geometric NTP_2 theory preserves NTP_2.

3.1. Introduction

The aim of this chapter is to initiate a systematic study of theories without the tree property of the second kind, or NTP_2 theories. This class was defined by Shelah implicitly in [**She90**] in terms of a certain cardinal invariant κ_{inp} (see Section 3.3) and explicitly in [**She80**], and it contains both simple and NIP theories. There was no active research on the subject until the recent interest in generalizing methods and results of stability theory to larger contexts, necessitated for example by the developments in the model theory of important algebraic examples such as algebraically closed valued fields [**HHM08**].

We give a short overview of related results in the literature. The invariant κ_{inp}, an upper bound for the number of independent partitions, was considered by Tsuboi in [**Tsu85**] for the case of stable theories. In [**Adl08**] Adler defines burden, by relativizing κ_{inp} to a fixed partial type, makes the connection to weight in simple theories and defines strong theories. Burden in the context of NIP theories, where it is called dp-rank, was already introduced by Shelah in [**Shed**] and developed further in [**OU11**]. Results about forking and dividing in NTP_2 theories were established in [**CK12**]. In particular, it was proved that a formula forks over a model if and only if it divides over it (see Section 3.5). Some facts about ordered inp-minimal theories and groups (that is with $\kappa^1_{\mathrm{inp}} = 1$) are proved in [**Goo10, Sim11b**]. In [**Ben11**, Theorem 4.13] Ben Yaacov shows that if a structure has IP, then its randomization (in the sense of continuous logic) has TP_2. Malliaris

[Mal12] considers TP_2 in relation to the saturation of ultra-powers and the Keisler order. In **[Cha08]** Chatzidakis observes that ω-free PAC fields have TP_2.

A brief description of the results in this paper.

In Section 3.3 we introduce inp-patterns, burden, establish some of their basic properties and demonstrate that burden is sub-multiplicative: that is, if $\mathrm{bdn}(a/C) < \kappa$ and $\mathrm{bdn}(b/aC) < \lambda$, then $\mathrm{bdn}(ab/C) < \kappa \times \lambda$. As an application we show that the value of the invariant of a theory $\kappa_{\mathrm{inp}}(T)$ does not depend on the number of variables used in the computation. This answers a question of Shelah from **[She90]** and shows in particular that if T has TP_2, then some formula $\phi(x, y)$ with x a singleton has TP_2.

In Section 3.4 we describe the place of NTP_2 in the classification hierarchy of first-order theories and the relationship of burden to dp-rank in NIP theories and to weight in simple theories. We also recall some combinatorial "structure / non-structure" dichotomy due to Shelah.

Section 3.5 is devoted to forking (and dividing) in NTP_2 theories. After discussing strictly invariant types, we give a characterization of NTP_2 in terms of the appropriate variants of Kim's lemma, local character and bounded weight relatively to strict non-forking. As an application we consider theories with dependent dividing (i.e. whenever $p \in S(N)$ divides over $M \prec N$, there some $\phi(x, a) \in p$ dividing over M and such that $\phi(x, y)$ is NIP) and show that any theory with dependent dividing is NTP_2. Finally we observe that the the analysis from Chapter 1 generalizes to a situation when one is working inside an NTP_2 type in an arbitrary theory.

A famous equation of Shelah "NIP = stability + dense linear order" turned out to be a powerful ideological principle, at least at the early stages of the development of NIP theories. In this paper the equation "NTP_2 = simplicity + NIP" plays an important role. In particular, it seems very natural to consider two extremal kinds of types in NTP_2 theories (and in general) — simple types and NIP types. While it is perfectly possible for an NTP_2 theory to have neither, they form important special cases and are not entirely understood.

In section 3.6 we look at NIP types. In particular we show that the results of the previous section on forking localized to a type combined with honest definitions from Chapter 4 allow to omit the global NTP_2 assumption in the theorem of **[KS12b]**, thus proving that dp-rank of a type in arbitrary theory is always witnessed by mutually indiscernible sequences of its realizations, after adding some parameters (see Theorem 3.6.3). We also observe that in an NTP_2 theory, a type is NIP if and only if every extension of it has only boundedly many global non-forking extensions.

In Section 3.7 we consider simple types (defined as those type for which every completion satisfies the local character), first in arbitrary theories and then in NTP_2. While it is more or less immediate that on the set of realizations of a simple type forking satisfies all the properties of forking in simple theories, the interaction between the realizations of a simple type and arbitrary tuples seems more intricate. We establish full symmetry between realizations of a simple type and arbitrary elements, answering a question of Casanovas in the case of NTP_2 theories (showing that simple types are co-simple, see Definition 3.7.7). Then we show that simple types are characterized as those satisfying the co-independence theorem and that co-simple stably embedded types are simple (so in particular a theory is simple if and only if it is NTP_2 and satisfies the independence theorem).

Section 3.8 is devoted to examples. We give an Ax-Kochen-Ershov type statement: a Henselian valued field of characteristic $(0,0)$ is NTP_2 (strong, of finite burden) if and only if the residue field is NTP_2 (the residue field and the value group are strong, of finite burden respectively). This is parallel to the result of Delon for NIP [**Del81**], and generalizes a result of Shelah for strong dependence [**Shed**]. It follows that the valued fields of Hahn series over pseudo-finite fields are NTP_2. In particular, the theory of the ultra-product of p-adics is NTP_2 (and in fact strong, of finite burden). We also show that expanding a geometric NTP_2 theory by a generic predicate (Chatzidakis-Pillay style [**CP98**]) preserves NTP_2.

Acknowledgments. I am grateful to Itaï Ben Yaacov, Itay Kaplan and Martin Hils for multiple discussions around the topics of the paper. I would also like to thank Hans Adler and Enrique Casanovas for their interest in this work and for suggesting nice questions.

3.2. Preliminaries

As usual, we will be working in a monster model $\mathbb{M}$ of a complete first-order theory T. We will not be distinguishing between elements and tuples unless explicitly stated.

3.2.1. Mutually indiscernible sequences and arrays.

DEFINITION 3.2.1. We will often be considering collections of sequences $(\bar{a}_\alpha)_{\alpha<\kappa}$ with $\bar{a}_\alpha = (a_{\alpha,i})_{i<\lambda}$ (where each $a_{\alpha,i}$ is a tuple, maybe infinite). We say that they are *mutually indiscernible* over a set C if $\bar{a}_\alpha$ is indiscernible over $C\bar{a}_{\neq\alpha}$ for all $i<\kappa$. We will say that they are *almost mutually indiscernible* over C if $\bar{a}_\alpha$ is indiscernible over $C\bar{a}_{<\alpha}(a_{\beta,0})_{\beta>\alpha}$. Sometimes we call $(a_{\alpha,i})_{\alpha<\kappa,i<\lambda}$ an *array.* We say that $\left(\bar{b}_\alpha\right)_{\alpha<\kappa'}$ is a *sub-array* of $(\bar{a}_\alpha)_{\alpha<\kappa}$ if for each $\alpha<\kappa'$ there is $\beta_\alpha<\kappa$ such that $\bar{b}_\alpha$ is a sub-sequence of $\bar{a}_{\beta_\alpha}$. We say that an array is mutually indiscernible (almost mutually indiscernible) if rows are mutually indiscernible (resp. almost mutually indiscernible). Finally, an array is strongly indiscernible if it is mutually indiscernible and in addition the sequence of rows $(\bar{a}_\alpha)_{\alpha<\kappa}$ is an indiscernible sequence.

The following lemma follows easily by a repeated use of the usual "Erdös-Rado" and Ramsey theorems, and will be constantly used for finding indiscernible arrays.

LEMMA 3.2.2.
(1) *For any small set C and cardinal κ there is λ such that: If $A=(a_{\alpha,i})_{\alpha<n,i<\lambda}$ is an array, $n<\omega$ and $|a_{\alpha,i}|\leq\kappa$, then there is an array $B=(b_{\alpha,i})_{\alpha<n,i<\omega}$ with rows mutually indiscernible over C and such that every finite sub-array of B has the same type over C as some sub-array of A.*
(2) *Let C be small set and $A=(a_{\alpha,i})_{\alpha<n,i<\omega}$ be an array with $n<\omega$. Then for any finite $\Delta\in L(C)$ and $N<\omega$ we can find Δ-mutually indiscernible sequences $(a_{\alpha,i_{\alpha,0}},\dots,a_{\alpha,i_{\alpha,N}})\subset\bar{a}_\alpha$, $\alpha<n$.*

LEMMA 3.2.3. *Let $(\bar{a}_\alpha)_{\alpha<\kappa}$ be almost mutually indiscernible over C. Then there are $(\bar{a}'_\alpha)_{\alpha<\kappa}$, mutually indiscernible over C and such that $\bar{a}'_\alpha\equiv_{a_{\alpha,0}}\bar{a}_\alpha$ for all $\alpha<\kappa$.*

PROOF. By Lemma 3.2.2, taking an automorphism, and compactness. $\square$

DEFINITION 3.2.4. Given a set of formulas Δ, let $R(\kappa, \Delta)$ be the minimal length of a sequence sufficient for the existence of a Δ-indiscernible sub-sequence of length κ. For example, for finite Δ, $R(\kappa, \Delta) = \kappa$ for any infinite κ and and $R(n, \Delta)$ is finite for any $n \in \omega$.

REMARK 3.2.5. Let $(\bar{a}_i)$ be a mutually indiscernible array over A. Then it is still a mutually indiscernible over acl(A).

3.2.2. Invariant types.

We recall that

FACT 3.2.6. *(see e.g.* [**HP11**]*) Let* $p(x)$ *be a global type invariant over a set* C *(that is* $\phi(x, a) \in p$ *if and only if* $\phi(x, \sigma(a)) \in p$ *for any* $\sigma \in \mathrm{Aut}(\mathbb{M}/C)$*). For any set* $D \supseteq C$*, and an ordinal* α*, let the sequence* $\bar{c} = \langle c_i \mid i < \alpha \rangle$ *be such that* $c_i \models p|_{Dc_{<i}}$*. Then* $\bar{c}$ *is indiscernible over* D *and its type over* D *does not depend on the choice of* $\bar{c}$*. Call this type* $p^{(\alpha)}|_D$*, and let* $p^{(\alpha)} = \bigcup_{D \supseteq C} p^{(\alpha)}|_D$*. Then* $p^{(\alpha)}$ *also does not split over* C*.*

Finally, we assume some acquaintance with the basics of simple (e.g. [**Cas07**]) and NIP (e.g. [**Adl08**]) theories.

3.3. Burden and κ_{inp}

Let $p(x)$ be a (partial) type.

DEFINITION 3.3.1. An inp-*pattern* in $p(x)$ of depth κ consists of $(a_{\alpha,i})_{\alpha<\kappa, i<\omega}$, $\phi_\alpha(x, y_\alpha)$ and $k_\alpha < \omega$ such that

- $\{\phi_\alpha(x, a_{\alpha,i})\}_{i<\omega}$ is k_α-inconsistent, for each $\alpha < \kappa$
- $\left\{\phi_\alpha(x, a_{\alpha,f(\alpha)})\right\}_{\alpha<\kappa} \cup p(x)$ is consistent, for any $f : \kappa \to \omega$.

The *burden* of $p(x)$, denoted bdn(p), is the supremum of the depths of all inp-patterns in $p(x)$. By bdn(a/C) we mean bdn$(\mathrm{tp}(a/C))$.

Obviously, $p(x) \subseteq q(x)$ implies bdn$(p) \geq$ bdn(q) and bdn$(p) = 0$ if and only if p is algebraic. Also notice that bdn$(p) < \infty \Leftrightarrow$ bdn$(p) < |T|^+$ by compactness.

First we observe that it is sufficient to look at mutually indiscernible inp-patterns.

LEMMA 3.3.2. *For* $p(x)$ *a (partial) type over* C*, the following are equivalent:*

1. *There is an* inp-*pattern of depth* κ *in* $p(x)$*.*
2. *There is an array* $(\bar{a}_\alpha)_{\alpha<\kappa}$ *with rows mutually indiscernible over* C *and* $\phi_\alpha(x, y_\alpha)$ *for* $\alpha < \kappa$ *such that:*
   - $\{\phi_\alpha(x, a_{\alpha,i})\}_{i<\omega}$ *is inconsistent for every* $\alpha < \kappa$
   - $p(x) \cup \{\phi_\alpha(x, a_{\alpha,0})\}_{\alpha<\kappa}$ *is consistent.*
3. *There is an array* $(\bar{a}_\alpha)_{\alpha<\kappa}$ *with rows almost mutually indiscernible over* C *with the same properties.*

PROOF. (1)$\Rightarrow$(2) is a standard argument using Lemma 3.2.2 and compactness, (2)$\Rightarrow$(3) is clear and (3)$\Rightarrow$(1) is an easy reverse induction plus compactness. $\square$

We will need the following technical lemma.

LEMMA 3.3.3. *Let $(\bar{a}_\alpha)_{\alpha<\kappa}$ be a mutually indiscernible array over C and b given. Let $p_\alpha(x, a_{\alpha,0}) = \mathrm{tp}(b/a_{\alpha,0}C)$, and assume that $p^\infty(x) = \bigcup_{\alpha<\kappa, i<\omega} p_\alpha(x, a_{\alpha,i})$ is consistent. Then there are $(\bar{a}'_\alpha)_{\alpha<\kappa}$ such that:*

(1) $\bar{a}'_\alpha \equiv_{a_{\alpha,0}C} \bar{a}_\alpha$ for all $\alpha < \kappa$
(2) $(\bar{a}'_\alpha)_{\alpha<\kappa}$ is a mutually indiscernible array over Cb.

PROOF. It is sufficient to find b' such that $b' \equiv_{a_{\alpha,0}C} b$ for all $\alpha < \kappa$ and $(\bar{a}_\alpha)_{\alpha<\kappa}$ is mutually indiscernible over $b'C$ (then applying an automorphism over C to conclude). Let $b^\infty \models p^\infty(x)$. By Lemma 3.2.2, for any finite $\Delta \in L(C)$, $S \subseteq \kappa$ and $n < \omega$, there is a $\Delta(b^\infty)$-mutually indiscernible sub-array $(a'_{\alpha,i})_{\alpha\in S, i<n}$ of $(\bar{a}_\alpha)_{\alpha\in S}$. Let σ be an automorphism over C sending $(a'_{\alpha,i})_{\alpha\in S,i<n}$ to $(a_{\alpha,i})_{\alpha\in S,i<n}$ and $b' = \sigma(b^\infty)$. Then $(a_{\alpha,i})_{\alpha\in S,i<n}$ is $\Delta(b')$-mutually indiscernible and $b' \models \bigcup_{\alpha\in S} p_\alpha(x, a_{\alpha,0})$, so $b' \equiv_{a_{\alpha,0}C} b$. Conclude by compactness. $\square$

Next lemma provides a useful equivalent way to compute the burden of a type.

LEMMA 3.3.4. *The following are equivalent for a partial type $p(x)$ over C:*

(1) There is no inp*-pattern of depth κ in p.*
(2) For any $b \models p(x)$ and $(\bar{a}_\alpha)_{\alpha<\kappa}$, an almost mutually indiscernible array over C, there is $\beta < \kappa$ and $\bar{a}'$ indiscernible over bC and such that $\bar{a}' \equiv_{a_{\beta,0}C} \bar{a}_\beta$.
(3) For any $b \models p(x)$ and $(\bar{a}_\alpha)_{\alpha<\kappa}$, a mutually indiscernible array over C, there is $\beta < \kappa$ and $\bar{a}'$ indiscernible over bC and such that $\bar{a}' \equiv_{a_{\beta,0}C} \bar{a}_\beta$.

PROOF. (1)⇒(2): So let $(\bar{a}_\alpha)_{\alpha<\kappa}$ be almost mutually indiscernible over C and $b \models p(x)$ given. Let $p_\alpha(x, a_{\alpha,0}) = \mathrm{tp}(b/a_{\alpha,0}C)$ and let $p_\alpha(x) = \bigcup_{i<\omega} p_\alpha(x, a_{\alpha,i})$.

Assume that p_α is inconsistent for each α, by compactness and indiscernibility of $\bar{a}_\alpha$ over C there is some $\phi_\alpha(x, a_{\alpha,0}c_\alpha) \in p_\alpha(x, a_{\alpha,0})$ with $c_\alpha \in C$ such that $\{\phi_\alpha(x, a_{\alpha,i}c_\alpha)\}_{i<\omega}$ is k_α-inconsistent. As $b \models \{\phi_\alpha(x, a_{\alpha,0}c_\alpha)\}_{\alpha<\kappa}$, by almost indiscernibility of $(\bar{a}_\alpha)_{\alpha<\kappa}$ over C and Lemma 3.3.2 we find an inp-pattern of depth κ in p – a contradiction.

Thus $p_\beta(x)$ is consistent for some $\beta < \kappa$. Then we can find $\bar{a}'$ which is indiscernible over bC and such that $\bar{a}' \equiv_{a_{\beta,0}C} \bar{a}_\beta$ by Lemma 3.3.3.

(2)⇒(3) is clear.

(3)⇒(1): Assume that there is an inp-pattern of depth κ in $p(x)$. By Lemma 3.3.2 there is an inp-pattern $(\bar{a}_\alpha, \phi_\alpha, k_\alpha)_{\alpha<\kappa}$ in $p(x)$ with $(\bar{a}_\alpha)_{\alpha<\kappa}$ a mutually indiscernible array over C. Let $b \models p(x) \cup \{\phi_\alpha(x, a_{\alpha,0})\}_{\alpha<\kappa}$. On the one hand $\models \phi_\alpha(b, a_{\alpha,0})$, while on the other $\{\phi_\alpha(x, a_{\alpha,i})\}_{i<\omega}$ is inconsistent, thus it is impossible to find an $\bar{a}'_\alpha$ as required for any $\alpha < \kappa$. $\square$

THEOREM 3.3.5. *If there is an* inp*-pattern of depth $\kappa_1 \times \kappa_2$ in $\mathrm{tp}(b_1b_2/C)$, then either there is an* inp*-pattern of depth κ_1 in $\mathrm{tp}(b_1/C)$ or there is an* inp*-pattern of depth κ_2 in $\mathrm{tp}(b_2/b_1C)$.*

PROOF. Assume not. Without loss of generality $C = \emptyset$, and let $(\bar{a}_\alpha)_{\alpha\in\kappa_1\times\kappa_2}$ be a mutually indiscernible array. By induction on $\alpha < \kappa_1$ we choose $\bar{a}'_\alpha$ and $\beta_\alpha \in \kappa_2$ such that:

(1) $\bar{a}'_\alpha$ is indiscernible over $b_2\bar{a}'_{<\alpha}\bar{a}_{\geq(\alpha+1,0)}$.
(2) $\mathrm{tp}(\bar{a}'_\alpha/a_{(\alpha,\beta_\alpha),0}\bar{a}'_{<\alpha}\bar{a}_{\geq(\alpha+1,0)}) = \mathrm{tp}(\bar{a}_{(\alpha,\beta_\alpha)}/a_{(\alpha,\beta_\alpha),0}\bar{a}'_{<\alpha}\bar{a}_{\geq(\alpha+1,0)})$.
(3) $\bar{a}'_{\leq\alpha} \cup \bar{a}_{\geq(\alpha+1,0)}$ is a mutually indiscernible array.

For $\alpha = -1$, (1) and (2) are empty conditions and (3) is the assumption. Now assume we have managed up to α, and we need to choose $\bar{a}'_\alpha$ and β_α. Let $D = \bar{a}'_{<\alpha}\bar{a}_{\geq(\alpha+1,0)}$. As $\left(\bar{a}_{(\alpha,\delta)}\right)_{\delta\in\kappa_2}$ is a mutually indiscernible array over D by (3) and there is no inp-pattern of depth κ_2 in $\mathrm{tp}(b_2/D)$, by Lemma 3.3.4(3) there is some $\beta_\alpha < \kappa_2$ and $\bar{a}'_\alpha$ indiscernible over b_2D (which gives us (1)) and such that $\mathrm{tp}(\bar{a}'_\alpha/a_{(\alpha,\beta_\alpha),0}D) = \mathrm{tp}(\bar{a}_{(\alpha,\beta_\alpha)}/a_{(\alpha,\beta_\alpha),0}D)$ (which together with the inductive assumption gives us (2) and (3)).

So we have carried out the induction. Now it is easy to see by (1), noticing that the first elements of $\bar{a}'_\alpha$ and $\bar{a}_{(\alpha,\beta_\alpha)}$ are the same by (2), that $(\bar{a}'_\alpha)_{\alpha<\kappa_1}$ is an almost mutually indiscernible array over b_2. By Lemma 3.2.3, we may assume that in fact $(\bar{a}'_\alpha)_{\alpha<\kappa_1}$ is a mutually indiscernible array over b_2.

As there is no inp-pattern of depth κ_1 in $\mathrm{tp}(b_1/b_2)$, by Lemma 3.3.4 there is some $\gamma < \kappa_1$ and $\bar{a}$ indiscernible over b_1b_2 and such that $\bar{a} \equiv_{a'_{\gamma,0}} \bar{a}'_\gamma \equiv_{a_{(\gamma,\beta_\gamma),0}} \bar{a}_{(\gamma,\beta_\gamma)}$. As $(\bar{a}_\alpha)_{\alpha\in\kappa_1\times\kappa_2}$ was arbitrary, by Lemma 3.3.4(3) this implies that there is no inp-pattern of depth $\kappa_1 \times \kappa_2$ in $\mathrm{tp}(b_1b_2)$. □

Corollary 3.3.6. *"Sub-multiplicativity" of burden: If* $\mathrm{bdn}(a_i) < k_i$ *for* $i < n$ *with* $k_i \in \omega$*, then* $\mathrm{bdn}(a_0...a_{n-1}) < \prod_{i<n} k_i$.

We note that in the case of NIP theories it is known that burden is not only sub-multiplicative, but actually sub-additive [**KOU11**].

Definition 3.3.7. For $n < \omega$, we let $\kappa^n_{\mathrm{inp}(T)}$ be the first cardinal κ such that there is no inp-pattern $(\bar{a}_\alpha, \phi_\alpha(x, y_\alpha), k_\alpha)$ of depth κ with $|x| \leq n$. And let $\kappa_{\mathrm{inp}}(T) = \sup_{n<\omega} \kappa^n_{\mathrm{inp}}(T)$. Notice that $\kappa^m_{\mathrm{inp}} \geq \kappa^n_{\mathrm{inp}}(T) \geq n$ for all $n < m$, just because of having the equality in the language, and thus $\kappa_{\mathrm{inp}(T)} \geq \aleph_0$.

We can use the previous theorem to answer a question of Shelah [**She90**, Ch. III, Question 7.5].

Corollary 3.3.8. $\kappa_{\mathrm{inp}}(T) = \kappa^n_{\mathrm{inp}}(T) = \kappa^1_{\mathrm{inp}}(T)$*, as long as* κ^n_{inp} *is infinite for some* $n < \omega$.

3.4. NTP$_2$ and its place in the classification hierarchy

The aim of this section is to (finally) define NTP$_2$, describe its place in the classification hierarchy of first-order theories and what burden amounts to in the more familiar situations.

Definition 3.4.1. A formula $\phi(x, y)$ has TP$_2$ if there is an array $(a_{\alpha,i})_{\alpha,i<\omega}$ such that $\{\phi(x, a_{\alpha,i})\}_{i<\omega}$ is 2-inconsistent for every $\alpha < \omega$ and $\left\{\phi(x, a_{\alpha,f(\alpha)})\right\}_{\alpha<\omega}$ is consistent for any $f : \omega \to \omega$. Otherwise we say that $\phi(x, y)$ is NTP$_2$, and T is NTP$_2$ if every formula is.

Lemma 3.4.2. *The following are equivalent for* T*:*

(1) Every formula $\phi(x, y)$ *with* $|x| \leq n$ *is* NTP$_2$.
(2) $\kappa^n_{\mathrm{inp}}(T) \leq |T|^+$.
(3) $\kappa^n_{\mathrm{inp}}(T) < \infty$.
(4) $\mathrm{bdn}(b/C) < |T|^+$ *for all* b *and* C*, with* $|b| = n$.

Proof. (1)⇒(2): Assume we have a mutually indiscernible inp-pattern $(\bar{a}_\alpha, \phi_\alpha(x, y_\alpha), k_\alpha)_{\alpha<|T|^+}$ of depth $|T|^+$. By pigeon-hole we may assume that $\phi_\alpha(x, y_\alpha) = \phi(x, y)$ and $k_\alpha = k$.

Then by Ramsey and compactness we may assume in addition that $(\bar{a}_\alpha)$ is a strongly indiscernible array. If $\{\phi(x, a_{\alpha,0}) \wedge \phi(x, a_{\alpha,1})\}_{\alpha<n}$ is inconsistent for some $n < \omega$, then taking $b_{\alpha,i} = a_{n\alpha,i}a_{n\alpha+1,i}...a_{n\alpha+n-1,i}$, $\left(\bigwedge_{i<n} \phi(x, y_i), \bar{b}_\alpha, 2\right)_{\alpha<\omega}$ is an inp-pattern. Otherwise $\{\phi(x, a_{\alpha,0}) \wedge \phi(x, a_{\alpha,1})\}_{\alpha<\omega}$ is consistent, then taking $b_{\alpha,i} = a_{\alpha,2i}a_{\alpha,2i+1}$ we conclude that $\left(\phi(x, y_1) \wedge \phi(x, y_2), \bar{b}_\alpha, \left[\frac{k}{2}\right]\right)_{\alpha<\omega}$ is an inp-pattern. Repeat if necessary.

The other implications are clear by compactness. $\square$

REMARK 3.4.3. (1) implies (2) is from [**Adl08**].

It follows from the lemma and Theorem 3.3.8 that if T has TP_2, then some formula $\phi(x, y)$ with $|x| = 1$ has TP_2. From Lemma 3.8.1 it follows that if $\phi_1(x, y_1)$ and $\phi_2(x, y_2)$ are NTP_2, then $\phi_1(x, y_1) \vee \phi_2(x, y_2)$ is NTP_2. This, however, is the only Boolean operation preserving NTP_2.

DEFINITION 3.4.4. [Adler] T is called *strong* if there is no inp-pattern of infinite depth in it. It is clearly a subclass of NTP_2 theories.

PROPOSITION 3.4.5. *If* $\phi(x, y)$ *is* NIP*, then it is* NTP_2*.*

PROOF. Let $(a_{\alpha,j})_{\alpha,j<\omega}$ be an array witnessing that $\phi(x, y)$ has TP_2. But then for any $s \subseteq \omega$, let $f(\alpha) = 0$ if $\alpha \in s$, and $f(\alpha) = 1$ otherwise. Let $d \models \left\{\phi(x, a_{\alpha,f(\alpha)})\right\}$. It follows that $\phi(d, a_{\alpha,0}) \Leftrightarrow \alpha \in s$. $\square$

We recall the definition of dp-rank (e.g. [**KOU11**]):

DEFINITION 3.4.6. We let the dp-rank of p, denoted $\mathrm{dprk}(p)$, be the supremum of κ for which there are $b \models p$ and mutually indiscernible over C (a set containing the domain of p) sequences $(\bar{a}_\alpha)_{\alpha<\kappa}$ such that none of them is indiscernible over bC.

FACT 3.4.7. *The following are equivalent for a partial type* $p(x)$ *(by Ramsey and compactness):*

(1) $\mathrm{dprk}(p) > \kappa$.
(2) There is an ict-pattern of depth κ *in* $p(x)$*, that is* $(\bar{a}_i, \varphi_i(x, y_i), k_i)_{i<\kappa}$ *such that* $p(x) \cup \left\{\varphi_i\left(x, a_{i,s(i)}\right)\right\}_{i<\kappa} \cup \{\neg\varphi_i(x, a_{i,j})\}_{s(i)\neq j<\kappa}$ *is consistent for every* $s : \kappa \to \omega$*.*

It is easy to see that every inp-pattern with mutually indiscernible rows gives an ict-pattern of the same depth. On the other hand, if T is NIP then every ict-pattern gives an inp-pattern of the same depth (see [**Adl07**, Section 3]). Thus we have:

FACT 3.4.8. *(1) For a partial type* $p(x)$*,* $\mathrm{bdn}(p) \geq \mathrm{dprk}(p)$*. And if* $p(x)$ *is an* NIP *type, then* $\mathrm{bdn}(p) = \mathrm{dprk}(p)$
(2) T *is strongly dependent* $\Leftrightarrow$ T *is* NIP *and strong.*

PROPOSITION 3.4.9. *If* T *is simple, then it is* NTP_2*.*

PROOF. Of course, inp-pattern of the form $(\bar{a}_\alpha, \phi(x, y), k)_{\alpha<\omega}$ witnesses the tree property. $\square$

Moreover,

FACT 3.4.10. [**Adl07**, Proposition 8] *Let* T *be simple. Then the burden of a partial type is the supremum of the weights of its complete extensions. And* T *is strong if and only if every type has finite burden.*

DEFINITION 3.4.11. [Shelah] $\phi(x, y)$ is said to have TP_1 if there are $(a_\eta)_{\eta \in \omega^{<\omega}}$ and $k \in \omega$ such that:

- $\left\{\phi(x, a_{\eta|i})\right\}_{i \in \omega}$ is consistent for any $\eta \in \omega^\omega$
- $\{\phi(x, a_{\eta_i})\}_{i<k}$ is inconsistent for any mutually incomparable $\eta_0, ..., \eta_{k-1} \in \omega^{<\omega}$.

FACT 3.4.12. [**She90**, III.7.7, III.7.11] *Let* T *be* NTP_2*,* $q(y)$ *a partial type and* $\phi(x, y)$ *has* TP *witnessed by* $(a_\eta)_{\eta \in \omega^{<\omega}}$ *with* $a_\eta \models q$*, and such that in addition* $\left\{\phi(x, a_{\eta|i})\right\}_{i \in \omega} \cup p(x)$ *is consistent for any* $\eta \in \omega^\omega$*. Then some formula* $\psi(x, \bar{y}) = \bigwedge_{i<k} \phi(x, y_i) \wedge \chi(x)$ *(where* $\chi(x) \in p(x)$*) has* TP_1*, witnessed by* (b_η) *with* $b_\eta \subseteq q(\mathbb{M})$ *and such that* $\left\{\phi(x, b_{\eta|i})\right\}_{i \in \omega} \cup p(x)$ *is consistent.*

It is not stated in exactly the same form there, but immediately follows from the proof. See [**Adl07**, Section 4] and [**KKS12**, Theorem 6.6] for a more detailed account of the argument. See [**KK11**] for more details on NTP_1.

EXAMPLE 3.4.13.
(1) Triangle free random graph (i.e. the model companion of the theory of graphs without triangles) has TP_2.
(2) The theories of free roots of the random graph (as defined and studied in [**CW04**]) have TP_2. In particular, the rational Urysohn space has TP_2.

PROOF. (1): We can find $(a_{ij} b_{ij})_{ij<\omega}$ such that $R(a_{ij}, b_{ik})$ for every i and $j \neq k$, and this are the only edges around. But then $\{xRa_{ij} \wedge xRb_{ij}\}_{j<\omega}$ is 2-inconsistent for every i as otherwise it would have created a triangle, while $\left\{xRa_{if(i)} \wedge xRb_{if(i)}\right\}_{i<\omega}$ is consistent for any $f : \omega \to \omega$.

(2): Let $(a_{i,j})_{i,j<\omega}$ be such that $d(a_{i,j}, a_{i,j'}) = 3$ for all $i, j \neq j' < \omega$ and $d(a_{i,j}, a_{i',j'}) = 2$ for all $i \neq i', j, j' < \omega$ - possible to find by model completeness as the triangular inequality is not violated. But then $\{xR_1 a_{i,j}\}_{j<\omega}$ is inconsistent for every i, while $\left\{xR_1 a_{i,f(i)}\right\}_{i<\omega}$ is consistent for any $f : \omega \to \omega$. □

In fact it is known that the triangle-free random graph is rosy and 2-dependent (in the sense of [**She07**]), thus there is no implication between rosiness and NTP_2, and between k-dependence and NTP_2 for $k > 1$. We also remark that in [**She90**, Exercise III.7.12] Shelah suggests an example of a theory satisfying $\mathrm{NTP}_2 + \mathrm{NSOP}$ which is not simple.

3.5. Forking in NTP_2

In [**Kim01**, Theorem 2.4] Kim gives several equivalents to the simplicity of a theory in terms of the behavior of forking and dividing.

FACT 3.5.1. *The following are equivalent:*
(1) T *is simple.*
(2) $\phi(x, a)$ *divides over* A *if and only if* $\{\phi(x, a_i)\}_{i<\omega}$ *is inconsistent for every Morley sequence* $(a_i)_{i<\omega}$ *over* A*.*
(3) Dividing in T *satisfies local character.*

In this section we show an analogous characterization of NTP$_2$. But first we recall some facts about forking and dividing in NTP$_2$ theories and introduce some terminology.

DEFINITION 3.5.2. (1) A type $p(x) \in S(C)$ is *strictly invariant* over A if it is Lascar invariant over A and for any small $B \subseteq C$ and $a \models p|_B$, we have that $\mathrm{tp}(B/aA)$ does not fork over A. For example, a definable type or a global type which is both an heir and a coheir over M, are strictly invariant over M.

(2) We will write $a \mathop{\smile\!\!\!\!\!\!\;\mid}^{\mathrm{ist}}_c b$ when $\mathrm{tp}(a/bc)$ can be extended to a global type $p(x)$ strictly invariant over A.

(3) We say that $(a_i)_{<\omega}$ is a strict Morley sequence over A if it is indiscernible over A and $a_i \mathop{\smile\!\!\!\!\!\!\;\mid}^{\mathrm{ist}}_A a_{<i}$ for all $i < \omega$.

(4) As usual, we will write $a \mathop{\smile\!\!\!\!\!\!\;\mid}^{u}_c b$ if $\mathrm{tp}(a/bc)$ is finitely satisfiable in c, $a \mathop{\smile\!\!\!\!\!\!\;\mid}^{d}_c b$ ($a \mathop{\smile\!\!\!\!\!\!\;\mid}^{f}_c b$) if $\mathrm{tp}(a/bc)$ does not divide (resp. does not fork) over c.

(5) We write $a \mathop{\smile\!\!\!\!\!\!\;\mid}^{i}_c b$ if $\mathrm{tp}(a/bc)$ can be extended to a global type $p(x)$ Lascar invariant over c. We point out that if $a \mathop{\smile\!\!\!\!\!\!\;\mid}^{i}_c b$ and $(b_i)_{i<\omega}$ is a c-indiscernible sequence with $b_0 = b$, then it is actually indiscernible over a.

(6) If T is simple, then $\mathop{\smile\!\!\!\!\!\!\;\mid}^{i} = \mathop{\smile\!\!\!\!\!\!\;\mid}^{\mathrm{ist}}$. And if T is NIP, then $\mathop{\smile\!\!\!\!\!\!\;\mid}^{i} = \mathop{\smile\!\!\!\!\!\!\;\mid}^{f}$.

(7) We say that a set A is an *extension base* if every type over A has a global non-forking extension. Every model is an extension base (because every type has a global coheir). A theory in which every set is an extension base is called extensible.

Strictly invariant types exist in any theory (but it is not true that every type over a model has a global extension which is strictly invariant over the same model). In fact, there are theories in which over any set there is some type without a global strictly invariant extension (see Chapter 6).

LEMMA 3.5.3. *Let $p(x)$ be a global type invariant over A, and let $M \supset A$ be $|A|^+$-saturated. Then p is strictly invariant over M.*

PROOF. It is enough to show that p is an heir over M. Let $\phi(x, c) \in p$. By saturation of M, $\mathrm{tp}(c/A)$ is realized by some $c' \in M$. But as p is invariant over A, $\phi(x, c') \in p$ as wanted. □

One of the main uses of strict invariance is the following criterion for making indiscernible sequences mutually indiscernible without changing their type over the first elements.

LEMMA 3.5.4. *Let $(\bar{a}_i)_{i<\kappa}$ and C be given, with $\bar{a}_i$ indiscernible over C and starting with a_i. If $a_i \mathop{\smile\!\!\!\!\!\!\;\mid}^{\mathrm{ist}}_C a_{<i}$, then there are mutually C-indiscernible $\left(\bar{b}_i\right)_{i<\kappa}$ such that $\bar{b}_i \equiv_{a_i C} \bar{a}_i$.*

PROOF. (1): Enough to show for finite κ by compactness. So assume we have chosen $\bar{a}'_0, ..., \bar{a}'_{n-1}$, and lets choose $\bar{a}'_n$. As $a_n \mathop{\smile\!\!\!\!\!\!\;\mid}^{\mathrm{ist}}_C a_{<n}$, there are $\bar{a}''_0 ... \bar{a}''_{n-1} \equiv_{C a_0 ... a_{n-1}} \bar{a}'_0 ... \bar{a}'_{n-1}$ and such that $a_n \mathop{\smile\!\!\!\!\!\!\;\mid}^{\mathrm{ist}}_C \bar{a}''_{<n}$. As $a_n \mathop{\smile\!\!\!\!\!\!\;\mid}^{i}_{C \bar{a}''_{<n, \neq j}} \bar{a}''_j$ for $j < n$, it follows by the inductive assumption and Definition 3.5.2(5) that $\bar{a}''_j$ is indiscernible over

$a_n \bar{a}''_{\neq j}$. On the other hand $\bar{a}''_0 ... \bar{a}''_{n-1} \mathop{\smile\!\!\!\!\!|}^{f}_{C} a_n$, and so by basic properties of forking there is some $\bar{a}'_n \equiv_{Ca_n} \bar{a}_n$ indiscernible over $\bar{a}''_0, ..., \bar{a}''_{n-1}$. Conclude by Lemma 3.2.3. □

REMARK 3.5.5. This argument is essentially from [**She09**, Section 5].

We recall a result about forking and dividing in NTP$_2$ theories from Chapter 1:

FACT 3.5.6. *Let* T *be* NTP$_2$ *and* $M \models T$*.*

(1) Every $p \in S(M)$ *has a global strictly invariant extension.*

(2) For any a*,* $\phi(x, a)$ *divides over* M *if and only if* $\phi(x, a)$ *forks over* M*, if and only if for every* $(a_i)_{i<\omega}$*, a strict Morley sequence in* $\mathrm{tp}(a/M)$*,* $\{\phi(x, a_i)\}_{i<\omega}$ *is inconsistent.*

(3) In fact, just assuming that A *is an extension base, we still have that* $\phi(x, a)$ *does not divide over* A *if and only if* $\phi(x, a)$ *does not fork over* A*.*

3.5.1. Characterization of NTP$_2$**.** Now we can give a method for computing the burden of a type in terms of dividing with each member of an $\mathop{\smile\!\!\!\!\!|}^{\mathrm{ist}}$-independent sequence.

LEMMA 3.5.7. *Let* $p(x)$ *be a partial type over* C*. The following are equivalent:*

(1) There is an inp*-pattern of depth* κ *in* $p(x)$*.*

(2) There is $d \models p(x)$*,* $D \supseteq C$ *and* $(a_\alpha)_{\alpha<\kappa}$ *such that* $a_\alpha \mathop{\smile\!\!\!\!\!|}^{\mathrm{ist}}_{D} a_{<\alpha}$ *and* $d \not\mathop{\smile\!\!\!\!\!|}^{d}_{D} a_\alpha$ *for all* $\alpha < \kappa$*.*

PROOF. (1)⇒(2): Let $(\bar{a}_\alpha, \phi_\alpha(x, y_\alpha), k_\alpha)_{\alpha<\kappa}$ be an inp-pattern in $p(x)$ with $(\bar{a}_\alpha)$ mutually indiscernible over C. Let $q_\alpha(\bar{y}_\alpha)$ be a non-algebraic type finitely satisfiable in $\bar{a}_\alpha$ and extending $\mathrm{tp}(a_{\alpha 0}/C)$. Let $M \supseteq C(\bar{a}_\alpha)_{\alpha<\kappa}$ be $(|C| + \kappa)^+$-saturated. Then q_α is strictly invariant over M by Lemma 3.5.3. For $\alpha, i < \kappa$ let $b_{\alpha,i} \models q_\alpha \restriction_{M(b_{\alpha,j})_{\alpha<\kappa, j<i}(b_{\beta,i})_{\beta<\alpha}}$. Let $e_\alpha = b_{\alpha,\alpha}$. Now we have:

- $e_\alpha \mathop{\smile\!\!\!\!\!|}^{\mathrm{ist}}_{M} e_{<\alpha}$: as $e_\alpha \models q_\alpha \restriction_{e_{<\alpha}M}$.
- there is $d \models p(x) \cup \{\phi_\alpha(x, e_\alpha)\}_{\alpha<\kappa}$: it is easy to see by construction that for any $\Delta \in L(C)$ and $\alpha_0 < ... < \alpha_{n-1} < \kappa$, if $\models \Delta(e_{\alpha_0}, ..., e_{\alpha_{n-1}})$, then $\models \Delta(a_{\alpha_0,i_0}, ..., a_{\alpha_{n-1},i_{n-1}})$ for some $i_0, ..., i_{n-1} < \omega$. By assumption on $(\bar{a}_\alpha)_{\alpha<\kappa}$ and compactness it follows that $p(x) \cup \{\phi_\alpha(x, e_\alpha)\}_{\alpha<\kappa}$ is consistent.
- $\phi_\alpha(x, e_\alpha)$ divides over M: notice that $(b_{\alpha,\alpha+i})_{i<\omega}$ is an M-indiscernible sequence starting with e_α, as $b_{\alpha,\alpha+i} \models q_\alpha \restriction_{M(b_{\alpha,\alpha+j})_{j<i}}$ and q_α is finitely satisfiable in M. As $\mathrm{tp}(\bar{b}_\alpha)$ is finitely satisfiable in $\bar{a}_\alpha$, we conclude that $\{\phi_\alpha(x, b_{\alpha,\alpha+i})\}_{i<\omega}$ is k_α-inconsistent.

(2)⇒(1): Let $d \models p(x)$, $D \supseteq C$ and $(a_\alpha)_{\alpha<\kappa}$ such that $a_\alpha \mathop{\smile\!\!\!\!\!|}^{\mathrm{ist}}_{D} a_{<\alpha}$ and $d \not\mathop{\smile\!\!\!\!\!|}^{f}_{D} a_\alpha$ for all $\alpha < \kappa$ be given. Let $\phi_\alpha(x, a_\alpha) \in \mathrm{tp}(d/a_\alpha D)$ be a formula dividing over D, and let $\bar{a}_\alpha$ indiscernible over D and starting with a_α witness it. By Lemma 3.3.2 we can find a $(\bar{a}'_\alpha)_{\alpha<\kappa}$, mutually indiscernible over D and such that $\bar{a}'_\alpha \equiv_{a_\alpha D} \bar{a}_\alpha$. It follows that $\{\phi_\alpha(x, y_\alpha), \bar{a}'_\alpha\}_{\alpha<\kappa}$ is an inp-pattern of depth κ in $p(x)$. □

DEFINITION 3.5.8. We say that dividing satisfies *generic local character* if for every $A \subseteq B$ and $p(x) \in S(B)$ there is some $A' \subseteq B$ with $|A'| \leq |T|^+$ and such that: for any $\phi(x, b) \in p$, if $b \mathop{\smile\!\!\!\!\!|}^{\mathrm{ist}}_{A} A'$, then $\phi(x, b)$ does not divide over AA'.

Of course, the local character of dividing implies the generic local character. We are ready to prove the main theorem of this section.

THEOREM 3.5.9. *The following are equivalent:*

(1) T *is* NTP$_2$.
(2) T *has absolutely bounded* $\mathop{\smile\!\!\!\!\!\!\!\;|}^{\mathrm{ist}}$*-weight: for every* M*,* b *and* $(a_i)_{i<|T|^+}$ *with* $a_i \mathop{\smile\!\!\!\!\!\!\!\;|}^{\mathrm{ist}}_M a_{<i}$*,* $b \mathop{\smile\!\!\!\!\!\!\!\;|}^{d}_M a_i$ *for some* $i < |T|^+$.
(3) T *has bounded* $\mathop{\smile\!\!\!\!\!\!\!\;|}^{\mathrm{ist}}$*-weight: for every* M *there is some* κ_M *such that given* b *and* $(a_i)_{i<\kappa_M}$ *with* $a_i \mathop{\smile\!\!\!\!\!\!\!\;|}^{\mathrm{ist}}_M a_{<i}$*,* $b \mathop{\smile\!\!\!\!\!\!\!\;|}^{d}_M a_i$ *for some* $i < \kappa_M$.
(4) T *satisfies "Kim's lemma": for any* $M \models T$*,* $\phi(x, a)$ *divides over* M *if and only if* $\{\phi(x, a_i)\}_{i<\omega}$ *is inconsistent for every strict Morley sequence over* M.
(5) Dividing in T *satisfies generic local character.*

PROOF. (1) implies (2): Assume that there are M, b and $(a_i)_{i<|T|^+}$ with $a_i \mathop{\smile\!\!\!\!\!\!\!\;|}^{\mathrm{ist}}_M a_{<i}$ and $b \not\mathop{\smile\!\!\!\!\!\!\!\;|}^{d}_M a_i$ for all i. But then by Lemma 3.5.7 $\mathrm{bdn}(b/M) \geq |T|^+$, thus T has TP$_2$ by Lemma 3.4.2.

(2) implies (3) is clear.

(1) implies (4): by Fact 3.5.6(1)+(2).

(4) implies (3): assume that we have M, b and $(a_i)_{i<\kappa}$ such that, letting $\kappa = \beth_{(2^{|M|})^+}$, $a_i \mathop{\smile\!\!\!\!\!\!\!\;|}^{\mathrm{ist}}_M a_{<i}$ and $b \not\mathop{\smile\!\!\!\!\!\!\!\;|}^{d}_M a_i$ for all $i < \kappa$. We may assume that dividing is always witnessed by the same formula $\phi(x, y)$. Extracting an M-indiscernible sequence $(a'_i)_{i<\omega}$ from $(a_i)_{i<\kappa}$ by Erdös-Rado, we get a contradiction to (4) as $\{\phi(x, a'_i)\}_{i<\omega}$ is still consistent, (a'_i) is a strict Morley sequence over M and $\phi(x, a'_0)$ divides over M.

(3) implies (1): Assume that $\varphi(x, y)$ has TP$_2$, let $A = (\bar{a}_\alpha)_{\alpha<\omega}$ with $\bar{a}_\alpha = (a_{\alpha i})_{i<\omega}$ be a strongly indiscernible array witnessing it (so rows are mutually indiscernible and the sequence of rows is indiscernible). Let $M \supset A$ be some $|A|^+$-saturated model, and assume that κ_M is as required by (3). Let $\lambda = \beth_{(2^{|M|})^+}$ and $\mu = \left(2^{2^\lambda}\right)^+$. Adding new elements and rows by compactness, extend our strongly indiscernible array to one of the form $(\bar{a}_\alpha)_{\alpha\in\omega+\mu^*}$ with $\bar{a}_\alpha = (a_{\alpha i})_{i\in\lambda}$. By all the indiscernibility around it follows that $\bar{a}_\alpha \mathop{\smile\!\!\!\!\!\!\!\;|}^{u}_A \bar{a}_{<\alpha}$ for all $\alpha < \mu$. As there can be at most 2^{2^λ} global types from $S_\lambda(\mathbb{M})$ that are finitely satisfiable in A, without loss of generality there is some $p(\bar{x}) \in S_\lambda(\mathbb{M})$ finitely satisfiable in A and such that $\bar{a}_\alpha \models p(\bar{x})|_{A\bar{a}_{<\alpha}}$.

By Lemma 3.5.3, $p(\bar{x})$ is strictly invariant over M. We choose $\left(\bar{b}_\alpha\right)_{\alpha<\kappa_M}$ such that $\bar{b}_\alpha \models p|_{M\bar{b}_{<\alpha}}$.

By the choice of λ and Erdös-Rado, for each $\alpha < \kappa_M$ there is $i_\alpha < \lambda$ and $\bar{d}_\alpha$ such that $\bar{d}_\alpha$ is an M-indiscernible sequence starting with $b_{\alpha i_\alpha}$ and such that type of every finite subsequence of it is realized by some subsequence of $\bar{b}_\alpha$. Now we have:

- $d_{\alpha 0} \mathop{\smile\!\!\!\!\!\!\!\;|}^{\mathrm{ist}}_M d_{<\alpha 0}$ (as $d_{\alpha 0} = b_{\alpha i_\alpha}$ and $\bar{b}_\alpha \mathop{\smile\!\!\!\!\!\!\!\;|}^{\mathrm{ist}}_M \bar{b}_{<\alpha}$),
- $\varphi(x, d_{\alpha 0})$ divides over M (as $\bar{d}_\alpha$ is M-indiscernible and $\{\varphi(x, d_{\alpha i})\}_{i\in\omega}$ is inconsistent by construction),
- $\{\varphi(x, d_{\alpha 0})\}_{\alpha<\kappa_M}$ is consistent (follows by construction).

Taking some $c \models \{\varphi(x, d_{\alpha 0})\}_{\alpha<\kappa_M}$ we get a contradiction to (3).

(5) implies (2): Let $p(x) = \mathrm{tp}(b/B)$ with $B = M \cup \bigcup_{i<|T|^+} a_i$. Letting $A = M$, it follows by generic local character that there is some $A' \subseteq B$ with $|A'| \leq |T|$, such that $b \mathop{\smile\!\!\!\!\!|}^{d}_{MA'} a$ for any $a \in B$ with $a \mathop{\smile\!\!\!\!\!|}_{M} A'$. Let $i \in |T|$ be such that $i > \{j : a_j \in A'\}$. Then $a_i \mathop{\smile\!\!\!\!\!|}^{ist}_{M} A$, but also $b \not\mathop{\smile\!\!\!\!\!|}^{d}_{MA'} a_i$ (by left transitivity as $A' \mathop{\smile\!\!\!\!\!|}^{d}_{M} a_i$ and $b \not\mathop{\smile\!\!\!\!\!|}^{d}_{M} a_i$) — a contradiction.

(1) implies (5): Let $p(x) \in S(B)$ and $A \subseteq B$ be given. By induction on $i < |T|^+$ we try to choose $a_i \in B$ and $\varphi_i(x, a_i) \in p$ such that $a_i \mathop{\smile\!\!\!\!\!|}^{ist}_{A} a_{<i}$ and $\varphi_i(x, a_i)$ divides over $a_{<i}A$. But then by Lemma 3.5.7 bdn$(b/A) \geq |T|^+$, thus T has TP$_2$ by Lemma 3.4.2. So we had to get stuck, and letting $A' = \bigcup a_i$ witnesses the generic local character. $\square$

Remark 3.5.10.
1. The proof of the equivalences shows that in (2) and (3) we may replace $a \mathop{\smile\!\!\!\!\!|}^{ist}_{C} b$ by "tp(a/bC) extends to a global type which is both an heir and a coheir over C".
2. From the proof one immediately gets a similar characterization of strongness. Namely, the following are equivalent:
   (a) T is strong.
   (b) For every M, finite (or even singleton) b and $(a_i)_{i<\omega}$ with $a_i \mathop{\smile\!\!\!\!\!|}^{ist}_{M} a_{<i}$, $b \mathop{\smile\!\!\!\!\!|}^{d}_{M} a_i$ for some $i < \omega$.
   (c) For every $A \subseteq B$ and $p(x) \in S(B)$ there is some *finite* $A' \subseteq B$ such that: for any $\phi(x, b) \in p$, if $b \mathop{\smile\!\!\!\!\!|}^{ist}_{A} A'$, then $\phi(x, b)$ does not divide over AA'.

If we are working over a somewhat saturated model and consider only small sets, then we actually have the generic local character with respect to $\mathop{\smile\!\!\!\!\!|}^{u}$ in the place of $\mathop{\smile\!\!\!\!\!|}^{ist}$.

Lemma 3.5.11. *Let $(\bar{a}_i)_{i<\kappa}$ and C be given, $\bar{a}_i$ starting with a_i. If $\bar{a}_i$ is indiscernible over $\bar{a}_{<i}C$ and $a_i \mathop{\smile\!\!\!\!\!|}^{i}_{C} a_{<i}$, then $(\bar{a}_i)_{i<\kappa}$ is almost mutually indiscernible over C.*

Proposition 3.5.12. *Let T be NTP$_2$. Let M be κ-saturated, $p(x) \in S(M)$ and $A \subset M$ of size $< \kappa$. Then there is $A \subseteq A' \subset M$ of size $< \kappa$ such that for any $\phi(x, a) \in p$, if $a \mathop{\smile\!\!\!\!\!|}^{i}_{A} A'$ then $\phi(x, a)$ does not fork over A'.*

Proof. Assume not, then we can choose inductively on $\alpha < |T|^+$:

1. $\bar{a}_\alpha \subseteq M$ such that $a_{\alpha,0} \mathop{\smile\!\!\!\!\!|}^{i}_{A} A_\alpha$ and $\bar{a}_\alpha$ is A_α-indiscernible, $A_\alpha = A \cup \bigcup_{\beta<\alpha} \bar{a}_\beta$.
2. $\phi_\alpha(x, y_\alpha)$ such that $\phi_\alpha(x, a_{\alpha,0}) \in p$ and $\{\phi_\alpha(x, a_{\alpha,i})\}_{i<\omega}$ is inconsistent.

(1) is possible by saturation of M. But then by Lemma 3.5.11, $(\bar{a}_\alpha)_{\alpha<|T|^+}$ are almost mutually indiscernible. $\square$

3.5.2. Dependent dividing.

Definition 3.5.13. We say that T has *dependent dividing* if given $M \preceq N$ and $p(x) \in S(N)$ dividing over M, then there is a dependent formula $\phi(x, y)$ and $c \in N$ such that $\phi(x, c) \in p$ and $\phi(x, c)$ divides over M.

PROPOSITION 3.5.14. *(1) If* T *has dependent dividing, then it is* NTP$_2$.
(2) If T *has simple dividing, then it is simple.*

PROOF. (1) In fact we will only use that dividing is always witnessed by an instance of an NTP$_2$ formula. Assume that T has TP$_2$ and let $\phi(x,y)$ witness this. Let T_{Sk} be a Skolemization of T, $\phi(x,y)$ still has TP$_2$ in T_{Sk}. Then as in the proof of Theorem 3.5.9, for any κ we can find $(b_i)_{i<\kappa}$, a and M such that $a \models \{\phi(x,b_i)\}_{i<\kappa}$, $\phi(x,b_i)$ divides over M and tp$(b_i/b_{<i}M)$ has a global heir-coheir over M, all in the sense of T_{Sk}. Taking $M_i = \text{Sk}(Mb_i) \models T$, and now working in T, we still have that $a \not\downarrow^{d}_{M} M_i$ and $M_i \downarrow^{\text{ist}}_{M} M_{<i}$ (as tp$(M_i/M_{<i}M)$ still has a global heir-coheir over M). But then for each i we find some $d_i \in M_i$ and NTP$_2$ formulas $\phi_i(x,y_i) \in L$ such that $a \models \{\phi_i(x,d_i)\}$ and $\phi_i(x,d_i)$ divides over M, witnessed by $\bar{d}_i$ starting with d_i. We may assume that $\phi_i = \phi'$, and this contradicts ϕ' being NTP$_2$.
(2) Similar argument shows that if T has simple dividing, then it is simple.

□

Of course, if T is NIP, then it has dependent dividing, and for simple theories it is equivalent to the stable forking conjecture. It is natural to ask if every NTP$_2$ theory T has dependent dividing.

3.5.3. Forking and dividing inside an NTP$_2$ type.

DEFINITION 3.5.15. A partial type $p(x)$ over C is said to be NTP$_2$ if the following does not exist: $(\bar{a}_\alpha)_{\alpha<\omega}$, $\phi(x,y)$ and $k<\omega$ such that $\{\phi(x,a_{\alpha i})\}_{i<\omega}$ is k-inconsistent for every $\alpha<\omega$ and $\left\{\phi(x,a_{\alpha f(\alpha)})\right\}_{\alpha<\omega} \cup p(x)$ is consistent for every $f:\omega\to\omega$. Of course, T is NTP$_2$ if and only if every partial type is NTP$_2$. Also notice that if $p(x)$ is NTP$_2$, then every extension of it is NTP$_2$ and that $q((x_i)_{i<\kappa}) = \bigcup_{i<\kappa} p(x_i)$ is NTP$_2$ (follows from Theorem 3.3.5).

For the later use we will need a generalization of the results from Chapter 1 working inside a partial NTP$_2$ type, and with no assumption on the theory.

LEMMA 3.5.16. *Let* $p(x)$ *be an* NTP$_2$ *type over* M. *Assume that* $p(x)\cup\{\phi(x,a)\}$ *divides over* M, *then there is a global coheir* $q(x)$ *extending* tp(a/M) *such that* $p(x)\cup\{\phi(x,a_i)\}_{i<\omega}$ *is inconsistent for any sequence* $(a_i)_{i<\omega}$ *with* $a_i \models q|_{a_{<i}M}$.

PROOF. The proof of [**CK12**, Lemma 3.12] goes through. □

LEMMA 3.5.17. *Assume that* tp$(a_i/C) = p(x)$ *for all* i *and that* tp$(a_i/a_{<i}C)$ *has a strictly invariant extension to* $p(\mathbb{M})\cup C$. *Then there are mutually* C*-indiscernible* $\left(\bar{b}_i\right)_{i<\kappa}$ *such that* $\bar{b}_i \equiv_{a_i C} \bar{a}_i$.

PROOF. The assumption is sufficient for the proof of Lemma 3.5.4 to work. □

LEMMA 3.5.18. *Let* $p(x)$ *over* M *be* NTP$_2$, $a\in p(\mathbb{M})$, $c\in M$ *and assume that* $p(x)\cup\{\phi(x,ac)\}$ *divides over* M. *Assume that* tp(a/M) *has a strictly invariant extension* $p'(y)\in S(p(\mathbb{M}))$. *Then for any* $(a_i)_{i<\omega}$ *such that* $a_i \models p'|_{a_{<i}M}$, $p(x)\cup\{\phi(x,a_ic)\}_{i<\omega}$ *is inconsistent.*

PROOF. Let $(\bar{a}_0 c)$ with $a_{0,0} = a_0$ be an M-indiscernible sequence witnessing that $p(x) \cup \{\phi(x, a_0 c)\}$ divides over M. Let $\bar{a}_i$ be its image under an M-automorphism sending a_0 to a_i. By Lemma 3.5.4(2) we can find $\left(\bar{b}_i\right)_{i<\omega}$ mutually indiscernible over M and with $\bar{b}_i \equiv_{a_i M} \bar{a}_i$. By the choice of $\bar{b}_i$'s and compactness, there is some $\psi(x) \in p(x)$ such that $\{\psi(x) \wedge \phi(x, b_{i,j} c)\}_{j<\omega}$ is k-inconsistent for all $i < \omega$. It follows that $p(x) \cup \{\phi(x, a_i c)\}_{i<\omega}$ is inconsistent as p is NTP$_2$. $\square$

We need a version of the Broom lemma localized to an NTP$_2$ type.

LEMMA 3.5.19. *Let $p(x)$ be an NTP$_2$ type over M and $p'(x)$ be a partial global type invariant over M. Suppose that $p(x) \cup p'(x) \vdash \bigvee_{i<n} \phi_i(x, c)$ and each $\phi_i(x, c)$ divides over M. Then $p(x) \cup p'(x)$ is inconsistent.*

PROOF. Follows from the proof of [**CK12**, Lemma 3.1]. $\square$

COROLLARY 3.5.20. *Let $p(x)$ be an NTP$_2$ type over M and $a \in p(\mathbb{M})$. Then $\mathrm{tp}(a/M)$ has a strictly invariant extension $p'(x) \in S(p(\mathbb{M}) \cup M)$.*

PROOF. Following the proof of [**CK12**, Proposition 3.7] but using Lemma 3.5.19 in place of the Broom lemma. $\square$

And finally,

PROPOSITION 3.5.21. *Let $p(x)$ be an NTP$_2$ type, $a \in p(\mathbb{M}) \cup M$ and assume that $\{\phi(x, a)\} \cup p(x)$ does not divide. Then there is $p'(x) \in S(p(\mathbb{M}) \cup M)$ which does not divide over M and $\{\phi(x, a)\} \cup p(x) \subset p'(x)$.*

PROOF. By compactness, it is enough to show that if $p(x) \cup \{\phi(x, ac)\} \vdash \bigvee_{i<n} \phi_i(x, a_i c_i)$ with $a, a_i \in p(\mathbb{M})$ and $c, c_i \in M$, then $p(x) \cup \{\phi_i(x, a_i c_i)\}$ does not divide for some $i < n$. As in the proof of [**CK12**, Corollary 3.16], let $\left(a^j a_0^j \ldots a_{n-1}^j\right)_{j<\omega}$ be a strict Morley sequence in $\mathrm{tp}(a a_0 \ldots a_{n-1})$, which exists by Lemma 3.5.20. Notice that $\left(a^j c a_0^j c_0 \ldots a_{n-1}^j c_{n-1}\right)_{j<\omega}$ is still indiscernible over M. Then $p(x) \cup \left\{\phi(x, a^j c)\right\}_{j<\omega}$ is consistent, which implies that $p(x) \cup \left\{\phi_i(x, a_i^j c_i)\right\}_{j<\omega}$ is consistent for some $i < n$. But then by Lemma 3.5.18, $p(x) \cup \{\phi_i(x, a_i c_i)\}$ does not divide over M — as wanted. $\square$

3.6. NIP types

Let T be an arbitrary theory.

DEFINITION 3.6.1. (1) A partial type $p(x)$ over C is called NIP if there is no $\phi(x, y) \in L$, $(a_i)_{i\in\omega}$ with $a_i \models p(x)$ and $(b_s)_{s\subseteq\omega}$ such that $\models \phi(a_i, b_s) \Leftrightarrow i \in s$.

(2) The roles of a's and b's in the definition are interchangeable. It is easy to see that any extension of an NIP type is again NIP, and that the type of several realizations of an NIP type is again NIP.

(3) $p(x)$ is NIP $\Leftrightarrow$ $\mathrm{dprk}(p) < |T|^+ \Leftrightarrow \mathrm{dprk}(p) < \infty$ (see Definition 3.4.6).

LEMMA 3.6.2. *Let $p(x)$ be an NIP type.*

(1) Let $\bar{a} = (a_\alpha)_{\alpha<\kappa}$ be an indiscernible sequence over A with a_α from $p(\mathbb{M})$, and c be arbitrary. If $\kappa = (|a_\alpha| + |c|)^+$, then some non-empty end segment of $\bar{a}$ is indiscernible over Ac.

(2) *Let* $(\bar{a}_\alpha)_{\alpha<\kappa}$ *be mutually indiscernible (over* $\emptyset$*), with* $\bar{a}_\alpha = (a_{\alpha i})_{i<\lambda}$ *from* $p(\mathbb{M})$. *Assume that* $\bar{a} = (a_{0i}a_{1i}...)_{i<\lambda}$ *is indiscernible over* A. *Then* $(\bar{a}_\alpha)_{\alpha<\kappa}$ *is mutually indiscernible over* A.

Standard proofs of the corresponding results for NIP theories go through, see e.g. [**Adl08**].

3.6.1. Dp-rank of a type is always witnessed by an array of its realizations. In [**KS12b**] Kaplan and Simon demonstrate that inside an NTP_2 theory, dp-rank of a type can always be witnessed by mutually indiscernible sequences of realizations of the type. In this section we show that the assumption that the theory is NTP_2 can be omitted, thus proving the following general theorem with no assumption on the theory.

THEOREM 3.6.3. *Let* $p(x)$ *be an* NIP *partial type over* C, *and assume that* $\mathrm{dprk}(p) \geq \kappa$. *Then there is* $C' \supseteq C$, $b \models p(x)$ *and* $(\bar{a}_\alpha)_{\alpha<\kappa}$ *with* $\bar{a}_\alpha = (a_{\alpha i})_{i<\omega}$ *such that:*

- $a_{\alpha i} \models p(x)$ *for all* α, i
- $(\bar{a}_\alpha)_{\alpha<\kappa}$ *are mutually indiscernible over* C'
- *None of* $\bar{a}_\alpha$ *is indiscernible over* bC'.
- $|C'| \leq |C| + \kappa$.

COROLLARY 3.6.4. *It follows that dp-rank of a 1-type is always witnessed by mutually indiscernible sequences of singletons.*

We will use the following result from [**CS10**, Proposition 1.1]:

FACT 3.6.5. *Let* $p(x)$ *be a (partial)* NIP *type,* $A \subseteq p(\mathbb{M})$ *and* $\phi(x, c)$ *given. Then there is* $\theta(x, d)$ *with* $d \in p(\mathbb{M})$ *such that:*

(1) $\theta(A, d) = \phi(A, c)$,
(2) $\theta(x, d) \cup p(x) \to \phi(x, c)$.

We begin by showing that the burden of a dependent type can always be witnessed by mutually indiscernible sequences from the set of its realizations.

LEMMA 3.6.6. *Let* $p(x)$ *be a dependent partial type over* C *of burden* $\geq \kappa$. *Then we can find* $\left(\bar{d}_\alpha\right)_{\alpha<\kappa}$ *witnessing it, mutually indiscernible over* C *and with* $\bar{d}_i \subseteq p(\mathbb{M}) \cup C$.

PROOF. Let λ be large enough compared to $|C|$. Assume that $\mathrm{bdn}(p) \geq \kappa$, then by compactness we can find $\left(\bar{b}_\alpha, \phi_\alpha(x, y_\alpha), k_\alpha\right)_{i<n}$ such that $\bar{b}_\alpha = (b_{\alpha i})_{i<\lambda}$, $\{\phi_\alpha(x, b_{\alpha i})\}_{\alpha<\kappa}$ is k_α-inconsistent and $p(x) \cup \left\{\phi_\alpha(x, b_{\alpha f(\alpha)})\right\}_{i<n}$ is consistent for every $f : \kappa \to \lambda$, let a_f realize it. Set $A = \{a_f\}_{f\in\lambda^\kappa} \subseteq p(\mathbb{M})$.

By Fact 3.6.5, let $\theta_{\alpha i}(x, d_{\alpha i})$ be an honest definition of $\phi_\alpha(x, b_{\alpha i})$ over A (with respect to $p(x)$), with $d_{\alpha i} \in p(\mathbb{M})$. As λ is very large, we may assume that $\theta_{\alpha i} = \theta_\alpha$.

Now, as $\theta_\alpha(x, d_{\alpha i}) \cup p(x) \to \phi_\alpha(x, b_{\alpha i})$, it follows that there is some $\psi_\alpha(x, c) \in p$ such that letting $\chi_\alpha(x, y_1 y_2) = \theta_\alpha(x, y_1) \wedge \psi_\alpha(x, y_2)$, $\{\chi(x, d_{\alpha i} c_\alpha)\}_{i<\omega}$ is k_α-inconsistent.

On the other hand, $\left\{\chi_\alpha(x, d_{\alpha f(\alpha)} c_\alpha)\right\}_{\alpha<\kappa} \cup p(x)$ is consistent, as the corresponding a_f realizes it. Thus this array still witnesses that burden of p is at least κ. $\square$

We will also need the following lemma.

LEMMA 3.6.7. *Let* $p(x)$ *be an* NIP *type over* $M \models T$

(1) Assume that $a \in p(\mathbb{M}) \cup M$ *and* $\phi(x,a)$ *does not divide over* M*, then there is a type* $q(x) \in S(p(\mathbb{M}) \cup M)$ *invariant under* M*-automorphisms and with* $\phi(x,a) \in q$.

(2) Let $p'(x) \supset p(x)$ *be an* M *invariant type such that* $p^{(\omega)}$ *is an heir-coheir over* M*. If* $(a_i)_{i<\omega}$ *is a Morley sequence in* p' *and indiscernible over* bM *with* $b \in p(\mathbb{M})$*, then* $\mathrm{tp}(b/MI)$ *has an* M*-invariant extension in* $S(p(\mathbb{M}) \cup M)$.

PROOF. (1) As NIP type is in particular an NTP_2 type, by Lemma 3.5.21 we find a type $q(x) \in S(p(\mathbb{M}))$ which doesn't divide over M and such that $\phi(x,a) \in q$. It is enough to show that $q(x)$ is Lascar-invariant over M. Assume that we have an M-indiscernible sequence $(a_i)_{i<\omega}$in $p(\mathbb{M})$ such that $\phi(x,a_0) \wedge \neg\phi(x,a_1) \in q$. But then $\{\phi(x,a_{2i}) \wedge \phi(x,a_{2i+1})\}_{i<\omega}$ is inconsistent, so q divides over M — a contradiction. Easy induction shows the same for a_0 and a_1 at Lascar distance n.

(2) By Lemma 3.5.18 and (1). □

Now for the *proof of Theorem 3.6.3.* The point is that first the array witnessing dp-rank of our type $p(x)$ can be dragged inside the set of realizations of p by Lemma 3.6.6. Then, combined with the use of Proposition 3.6.7 instead of the unrelativized version, the proof of Kaplan and Simon [**KS12b**, Section 3.2] goes through working inside $p(\mathbb{M})$.

PROBLEM 3.6.8. Is the analogue of Lemma 3.6.6 true for the burden of an arbitrary type in an NTP_2 theory?

We include some partial observations to justify it.

PROPOSITION 3.6.9. *The answer to the Problem 3.6.6 is positive in the following cases:*

(1) T *satisfies dependent forking (so in particular if* T *is* NIP*).*

(2) T *is simple.*

PROOF. (1): Recall that if $\mathrm{bdn}(p) \geq \kappa$, then we can find $(b_i)_{i<\kappa}$, $a \models p$ and $M \supseteq C$ such that $a \not\downarrow^{d}_{M} b_i$ and $b_i \downarrow^{ist}_{M} b_{<i}$. Notice that $p(x)$ still has the same burden in the sense of a Skolemization T^{Sk}. Choose inductively $M_i \supseteq M \cup b_i$ such that $M_i \downarrow^{ist}_{M} b_{<i}$, let $M_i = Sk(M \cup b_i)$. Let $\phi(x,b_i)$ be witness this dividing with $\phi(x,y)$ an NIP formula, we can make $\bar{b}_i$ mutually indiscernible. Now the proof of Lemma 3.6.6 goes through.

(2): Let $p(x) \in S(A)$, $a \models p(x)$ and let $(b_i)_{i<\kappa}$ independent over A, with $a \not\downarrow_{A} b_i$. Without loss of generality $A = \emptyset$. Consider $\mathrm{tp}(a/b_0)$ and take $I = (a_i)_{i<|T|^+}$ such that $a \frown I$ is a Morley sequence in it. By extension and automorphism we may assume $b_{>0} \downarrow_{ab_0} I$, together with $a \downarrow_{b_0} I$ implies $b_{>0} \downarrow_{b_0} I$, thus $b_{>0} \downarrow I$ (as $b_{>0} \downarrow b_0$).

Assume that I is a Morley sequence over $\emptyset$, then by simplicity $a_i \downarrow b_0$ for some i, contradicting $a_i \equiv_{b_0} a$ and $a \not\downarrow b_0$. Thus by indiscernibility $a \not\downarrow a_{<n}$ for some n, while $\{a_{<n}\} \cup b_{>0}$ is an independent set.

Repeating this argument inductively and using the fact that the burden of a type in a simple theory is the supremum of the weights of its completions (Fact 3.4.10) allows to conclude. □

3.6.2. NIP types inside an NTP_2 theory. We give a characterization of NIP types in NTP_2 theories in terms of the number of non-forking extensions of its completions.

THEOREM 3.6.10. *Let* T *be* NTP_2*, and let* $p(x)$ *be a partial type over* C*. The following are equivalent:*

(1) p *is* NIP.
(2) Every $p' \supseteq p$ *has boundedly many global non-forking extensions.*

PROOF. (1)⇒(2): A usual argument shows that a non-forking extension of an NIP type is in fact Lascar-invariant (see Lemma 3.6.7), thus there are only boundedly many such.

(2)⇒(1): Assume that $p(x)$ is not NIP, that is there are $I = (b_i)_{i \in \omega}$ such that such that for any $s \subseteq \omega$, $p_s(x) = p(x) \cup \{\phi(x, b_i)\}_{i \in s} \cup \{\neg\phi(x, b_i)\}_{i \notin s}$ is consistent. Let $q(y)$ be a global non-algebraic type finitely satisfiable in I. Let $M \supseteq IC$ be some $|IC|^+$-saturated model. It follows that $q^{(\omega)}$ is a global heir-coheir over M by Lemma 3.5.3. Take an arbitrary cardinal κ, and let $J = (c_i)_{i \in \kappa}$ be a Morley sequence in q over M. We claim that for any $s \subseteq \kappa$, $p_s(x)$ does not divide over M. First notice that $p_s(x)$ is consistent for any s, as $\mathrm{tp}(J/M)$ is finitely satisfiable in I. But as for any $k < \omega$, $(c_{ki}c_{ki+1}...c_{k(i+1)-1})_{i<\omega}$ is a Morley sequence in $q^{(k)}$, together with Fact3.5.6 this implies that $p_s(x)|_{c_0...c_{k-1}}$ does not divide over M for any $k < \omega$, thus by indiscernibility of J, $p_s(x)$ does not divide over M, thus has a global non-forking extension by Fact 3.5.6.

As there are only boundedly many types over M, there is some $p' \in S(M)$ extending p, with unboundedly many global non-forking extensions. $\square$

REMARK 3.6.11. (2)⇒(1) is just a localized variant of an argument from Chapter 6.

3.7. Simple types

3.7.1. Simple and co-simple types. Simple types, to the best of our knowledge, were first defined in [**HKP00**, §4] in the form of (2).

DEFINITION 3.7.1. We say that a partial type $p(x) \in S(A)$ is *simple* if it satisfies any of the following equivalent conditions:

(1) There is no $\phi(x, y)$, $(a_\eta)_{\eta \in \omega^{<\omega}}$ and $k < \omega$ such that: $\{\phi(x, a_{\eta i})\}_{i<\omega}$ is k-inconsistent for every $\eta \in \omega^{<\omega}$ and $\{\phi(x, a_{\eta\restriction i})\}_{i<\omega} \cup p(x)$ is consistent for every $\eta \in \omega^\omega$.
(2) Local character: If $B \supseteq A$ and $p(x) \subseteq q(x) \in S(B)$, then $q(x)$ does not divide over AB' for some $B' \subseteq B$, $|B'| \le |T|$.
(3) Kim's lemma: If $\{\phi(x, b)\} \cup p(x)$ divides over $B \supseteq A$ and $(b_i)_{i<\omega}$ is a Morley sequence in $\mathrm{tp}(b/B)$, then $p(x) \cup \{\phi(x, b_i)\}_{i<\omega}$ is inconsistent.
(4) Bounded weight: Let $B \supseteq A$ and $\kappa \ge \beth_{(2^{|B|})^+}$. If $a \models p(x)$ and $(b_i)_{i<\kappa}$ is such that $b_i \mathop{\smile\!\!\!\!\!\!\mid}^f_B b_{<i}$, then $a \mathop{\smile\!\!\!\!\!\!\mid}^d_B b_i$ for some $i < \kappa$.
(5) For any $B \supseteq A$, if $b \mathop{\smile\!\!\!\!\!\!\mid}^f_B a$ and $a \models p(x)$, then $a \mathop{\smile\!\!\!\!\!\!\mid}^d_B b$.

PROOF.

(1)⇒(2): Assume (2) fails, then we choose $\phi_\alpha(x, b_\alpha) \in q(x)$ k_α-dividing over $A \cup B_\alpha$, with $B_\alpha = \{b_\beta\}_{\beta<\alpha} \subseteq B$, $|B_\alpha| \le |\alpha|$ by induction on $\alpha < |T|^+$.

Then w.l.o.g. $\phi_\alpha = \phi$ and $k_\alpha = k$. Now construct a tree in the usual manner, such that $\{\phi(x, a_{\eta i})\}_{i<\omega}$ is inconsistent for any $\eta \in \omega^{<\omega}$ and $\{\phi(x, a_{\eta|i})\}_{i<\omega} \cup p(x)$ is consistent for any $\eta \in \omega^\omega$.

(2)⇒(3): Let $I = (|T|^+)^*$, and $(b_i)_{i\in I}$ be Morley over B in $\mathrm{tp}(b/B)$. Assume that $a \models p(x) \cup \{\phi(x, b_i)\}_{i\in I}$. By (2), $\mathrm{tp}(a/(b_i)_{i\in I}B)$ does not divide over $B(b_i)_{i\in I_0}$ for some $I_0 \subseteq I$, $|I_0| \leq |T|$. Let $i_0 \in I$, $i_0 < I_0$. Then $(b_i)_{i\in I_0} \mathop{\smile\!\!\!\!\!\mid}^f_B b_{i_0}$, and thus $\phi(x, b_{i_0})$ divides over BI_0 - a contradiction.

(3)⇒(4): Assume not, then by Erdös-Rado and finite character find a Morley sequence over B and a formula $\phi(x, y)$ such that $\models \phi(a, b_i)$ and $\phi(x, b_i)$ divides over B, contradiction to (3).

(4)⇒(5): For κ as in (4), let $I = (b_i)_{i<\kappa}$ be a Morley sequence over B, indiscernible over Ba and with $b_0 = b$. By (4), $a \mathop{\smile\!\!\!\!\!\mid}^d_B b_i$ for some $i < \kappa$, and so $a \mathop{\smile\!\!\!\!\!\mid}^d_B b$ by indiscernibility.

(5)⇒(1): Let $(b_\eta)_{\eta\in\omega^{<\omega}}$ witness the tree property of $\phi(x, y)$, such that $\{\phi(x, b_{\eta|i})\}_{i<\omega} \cup p(x)$ is consistent for every $\eta \in \omega^\omega$. Then by Ramsey and compactness we can find $(b_i)_{i\leq\omega}$ indiscernible over a, $\models \phi(a, b_i)$ and $\phi(x, b_i)$ divides over $b_{<i}A$. Taking $B = A \cup \{b_i\}_{i<\omega}$ we see that $a \not\mathop{\smile\!\!\!\!\!\mid}^d_B b_\omega$, while $b_\omega \mathop{\smile\!\!\!\!\!\mid}^f_B a$ (as it is finitely satisfiable in B by indiscernibility) - a contradiction to (5).

□

REMARK 3.7.2. Let $p(x) \in S(A)$ be simple.

(1) Any $q(x) \supseteq p(x)$ is simple.
(2) Let $p(x) \in S(A)$ be simple and $C \subseteq p(\mathbb{M})$. Then $\mathrm{tp}(C/A)$ is simple.

PROOF. (1): Clear, for example by (1) from the definition.

(2): Let $C = (c_i)_{i\leq n}$, and we show that for any $B \supseteq A$, if $b \mathop{\smile\!\!\!\!\!\mid}^f_B C$, then $C \mathop{\smile\!\!\!\!\!\mid}^d_B b$ by induction on the size of C. Notice that $b \mathop{\smile\!\!\!\!\!\mid}^f_{Bc_{<n}} c_n$ and $c_n \models p$, thus $c_n \mathop{\smile\!\!\!\!\!\mid}^d_{Bc_{<n}} b$. By the inductive assumption $c_{<n} \mathop{\smile\!\!\!\!\!\mid}^d_B b$, thus $c_{\leq n} \mathop{\smile\!\!\!\!\!\mid}^d_B b$. □

We give a characterization in terms of local ranks.

PROPOSITION 3.7.3. *The following are equivalent:*

(1) $p(x)$ *is simple in the sense of Definition 3.7.1.*
(2) $D(p, \Delta, k) < \omega$ *for any finite* Δ *and* $k < \omega$.

PROOF. Standard proof goes through. □

LEMMA 3.7.4. *Let* $p(x) \in S(A)$ *be simple,* $a \models p(x)$ *and* $B \supseteq A$ *arbitrary. Then* $a \mathop{\smile\!\!\!\!\!\mid}^f_{B_0} B$ *for some* $|B_0| \leq |T|^+$.

PROOF. Standard proof using ranks goes through. □

It follows that in the Definition 3.7.1 we can replace everywhere "dividing" by "forking".

LEMMA 3.7.5. *Let* $p(x) \in S(A)$ *be simple. If* A *is an extension base, then* $\{\phi(x, c)\} \cup p(x)$ *forks over* A *if and only if it divides over* A.

PROOF. Assume that $\{\phi(x,c)\} \cup p(x)$ does not divide over A, but $\{\phi(x,c)\} \cup p(x) \vdash \bigvee_{i<n} \phi_i(x,c_i)$ and each of $\phi_i(x,c_i)$ divides over A. As A is an extension base, let $(c_i c_{0,i} ... c_{n-1,i})$ be a Morley sequence in $\mathrm{tp}(cc_0...c_{n-1}/A)$. As $p(x) \cup \{\phi(x,c)\}$ does not divide over A, let $a \models p(x) \cup \{\phi(x,c_i)\}$, but then $p(x) \cup \{\phi_i(x,c_{i,j})\}_{j<\omega}$ is consistent for some $i<n$, contradicting Kim's lemma. □

PROBLEM 3.7.6. Let $q(x)$ be a non-forking extension of a complete type $p(x)$, and assume that $q(x)$ is simple. Does it imply that $p(x)$ is simple?

Unlike stability or NIP, it is possible that $\phi(x,y)$ does not have the tree property, while $\phi^*(x',y') = \phi(y',x')$ does. This forces us to define a dual concept.

DEFINITION 3.7.7. A partial type $p(x)$ over A is *co-simple* if it satisfies any of the following equivalent properties:

1. No formula $\phi(x,y) \in L(A)$ has the tree property witnessed by some $(a_\eta)_{\eta\in\omega^{<\omega}}$ with $a_\eta \subseteq p(\mathbb{M})$.
2. Every type $q(x) \in S(BA)$ with $B \subseteq p(\mathbb{M})$ does not divide over AB' for some $B' \subseteq B$, $|B'| \le (|A|+|T|)^+$.
3. Let $(a_i)_{i<\omega} \subseteq p(\mathbb{M})$ be a Morley sequence over BA, $B \subseteq p(\mathbb{M})$ and $\phi(x,y) \in L(A)$. If $\phi(x,a_0)$ divides over BA then $\{\phi(x,a_i)\}_{i<\omega}$ is inconsistent.
4. Let $B \subseteq p(\mathbb{M})$ and $\kappa \ge \beth_{\left(2^{|B|+|A|}\right)^+}$. If $(b_i)_{i<\kappa} \subseteq p(\mathbb{M})$ is such that $b_i \mathop{\smile\!\!\!\!\!\!|}^{f}_{AB} b_{<i}$ and a arbitrary, then $a \mathop{\smile\!\!\!\!\!\!|}^{d}_{AB} b_i$ for some $i<\kappa$.
5. For $B \subseteq p(\mathbb{M})$, if $a \models p$ and $a \mathop{\smile\!\!\!\!\!\!|}^{f}_{AB} b$, then $b \mathop{\smile\!\!\!\!\!\!|}^{d}_{AB} a$.

PROOF. Similar to the proof in Definition 3.7.1. □

REMARK 3.7.8. It follows that if $p(x)$ is a co-simple type over A and $B \subseteq p(\mathbb{M})$, then any $q(x) \in S(AB)$ extending p is co-simple (while adding the parameters from outside of the set of solutions of p may ruin co-simplicity).

It is easy to see that T is simple $\Leftrightarrow$ every type is simple $\Leftrightarrow$ every type is co-simple. What is the relation between simple and co-simple in general?

EXAMPLE 3.7.9. There is a co-simple type over a model which is not simple.

PROOF. Let T be the theory of an infinite triangle-free random graph, this theory eliminates quantifiers. Let $M \models T$, $m \in M$ and consider $p(x) = \{xRm\} \cup \{\neg xRa\}_{a\in M\setminus\{m\}}$ - a non-algebraic type over M. As there can be no triangles, if $a,b \models p(x)$ then $\neg aRb$. It follows that for any $A \subseteq p(\mathbb{M})$ and any B, $B \not\mathop{\smile\!\!\!\!\!\!|}^{d}_{M} A$ $\Leftrightarrow B \cap A \neq \emptyset$. So $p(x)$ is co-simple, for example by checking the bounded weight (Definition 3.7.7(4)).

For each $\alpha < \omega$, take $(b'_{\alpha,i} b''_{\alpha,i})_{i<\omega}$ such that $b'_{\alpha,i} R b''_{\alpha,j}$ for all $i \neq j$, and no other edges between them or to elements of M. Then $\{xRb'_{\alpha,i} \wedge xRb''_{\alpha,i}\}_{i<\omega}$ is 2-inconsistent for every α, while $p(x) \cup \left\{xRb'_{\alpha,\eta(\alpha)} \wedge xRb''_{\alpha,\eta(\alpha)}\right\}_{\alpha<\omega}$ is consistent for every $\eta : \omega \to \omega$. Thus $p(x)$ is not simple by Definition 3.7.1(1). □

However, this T has TP_2.

PROBLEM 3.7.10. Is there a simple, non co-simple type in an arbitrary theory?

3.7.2. Simple types are co-simple in NTP$_2$ theories. In this section we assume that T is NTP$_2$ (although some lemmas remain true without this restriction). In particular, we will write $\mathop{\smile\!\!\!\!\!\!\mid}$ to denote non-forking/non-dividing when working over an extension base as they are the same by Fact 3.5.6(3).

LEMMA 3.7.11. *Weak chain condition: Let A be an extension base, $p(x) \in S(A)$ simple. Assume that $a \models p(x)$, $I = (b_i)_{i<\omega}$ is a Morley sequence over A and $a \mathop{\smile\!\!\!\!\!\!\mid}_A b_0$. Then there is an aA-indiscernible $J \equiv_{Ab_0} I$ satisfying $a \mathop{\smile\!\!\!\!\!\!\mid}_A J$.*

PROOF. Let $a \models \phi(x, b_0)$, then $\{\phi(x, b_0)\} \cup p(x)$ does not divide over A.

CLAIM. $\{\phi(x, b_0) \wedge \phi(x, b_1)\} \cup p(x)$ does not divide over A.

PROOF. As $p(x)$ satisfies Definition 3.7.1(3), $(b_{2i}b_{2i+1})_{i<\omega}$ is a Morley sequence over A and $\{\phi(x, b_i)\}_{i<\omega} \cup p(x)$ is consistent. □

By iterating the claim and compactness, we conclude that $\bigcup_{i<\omega} p(x, b_i)$ does not divide over A, where $p(x, b_0) = \mathrm{tp}(a/b_0)$. As A is an extension base and forking equals dividing, there is $a' \models \bigcup_{i<\omega} p(x, b_i)$ satisfying $a' \mathop{\smile\!\!\!\!\!\!\mid}_A I$. By Ramsey, compactness and the fact that $a'b_i \equiv_A ab_0$ we find a sequence as wanted. □

REMARK 3.7.12. If fact, in Chapter 2 we had demonstrated that in an NTP$_2$ theory this lemma holds over extension bases with I just an indiscernible sequence, not necessarily Morley.

LEMMA 3.7.13. *Let A be an extension base, $p \in S(A)$ simple. For $i < \omega$, Let $\bar{a}_i$ be a Morley sequence in $p(x)$ over A starting with a_i, and assume that $(a_i)_{i<\omega}$ is a Morley sequence in $p(x)$. Then we can find $\bar{b}_i \equiv_{Aa_i} \bar{a}_i$ such that $(\bar{b}_i)_{i<\omega}$ are mutually indiscernible over A.*

PROOF. W.l.o.g. $A = \emptyset$.

First observe that by simplicity of p, $\{a_i\}_{i<\omega}$ is an independent set.

For $i < \omega$, we choose inductively $\bar{b}_i$ such that:

(1) $\bar{b}_i \equiv_{a_i} \bar{a}_i$
(2) $\bar{b}_i$ is indiscernible over $a_{>i}\bar{b}_{<i}$
(3) $a_{>i+1}\bar{b}_{\leq i} \mathop{\smile\!\!\!\!\!\!\mid} a_{i+1}$
(4) $a_{\geq i+1} \mathop{\smile\!\!\!\!\!\!\mid} \bar{b}_{\leq i}$

Base step: As $a_{>0} \mathop{\smile\!\!\!\!\!\!\mid} a_0$ and $\mathrm{tp}(a_{>0})$ is simple by Remark 3.7.2 and Lemma 3.7.11, we find an $a_{>0}$-indiscernible $\bar{b}_0 \equiv_{a_0} \bar{a}_0$ with $a_{>0} \mathop{\smile\!\!\!\!\!\!\mid} \bar{b}_0$.

Induction step: Assume that we have constructed $\bar{b}_0, ..., \bar{b}_{i-1}$. By (3) for $i-1$ it follows that $a_{>i}\bar{b}_{<i} \mathop{\smile\!\!\!\!\!\!\mid} a_i$. Again by Remark 3.7.2 and Lemma 3.7.11 we find an $a_{>i}\bar{b}_{<i}$-indiscernible sequence $\bar{b}_i \equiv_{a_i} \bar{a}_i$ such that $a_{>i}\bar{b}_{<i} \mathop{\smile\!\!\!\!\!\!\mid} \bar{b}_i$.

We check that it satisfies (3): As all tuples are inside $p(\mathbb{M})$, we can use symmetry, transitivity and $\mathop{\smile\!\!\!\!\!\!\mid}^d = \mathop{\smile\!\!\!\!\!\!\mid}^f$ freely. And so, $a_{>i+1}a_{i+1}\bar{b}_{<i} \mathop{\smile\!\!\!\!\!\!\mid} \bar{b}_i \Rightarrow a_{>i+1}\bar{b}_{<i} \mathop{\smile\!\!\!\!\!\!\mid}_{a_{i+1}} \bar{b}_i + a_{>i+1}\bar{b}_{<i} \mathop{\smile\!\!\!\!\!\!\mid} a_{i+1}$(as $a_{>i+1} \mathop{\smile\!\!\!\!\!\!\mid} a_{i+1}$ and $\bar{b}_{<i} \mathop{\smile\!\!\!\!\!\!\mid} a_{\geq i+1}$ by (4) for $i-1$) $\Rightarrow a_{>i+1}\bar{b}_{<i} \mathop{\smile\!\!\!\!\!\!\mid} \bar{b}_i a_{i+1} \Rightarrow a_{>i+1}\bar{b}_{<i} \mathop{\smile\!\!\!\!\!\!\mid}_{\bar{b}_i} a_{i+1} + \bar{b}_i \mathop{\smile\!\!\!\!\!\!\mid} a_{i+1} \Rightarrow a_{>i+1}\bar{b}_{\leq i} \mathop{\smile\!\!\!\!\!\!\mid} a_{i+1}$.

We check that it satisfies (4): As $a_{>i}\bar{b}_{<i} \mathop{\smile\!\!\!\!\!\!\mid} \bar{b}_i \Rightarrow a_{>i} \mathop{\smile\!\!\!\!\!\!\mid}_{\bar{b}_{<i}} \bar{b}_i + a_{>i} \mathop{\smile\!\!\!\!\!\!\mid} \bar{b}_{<i}$ by (4) for $i - 1 \Rightarrow a_{>i} \mathop{\smile\!\!\!\!\!\!\mid} \bar{b}_{\leq i}$.

Having chosen $(\bar{b}_i)_{i<\omega}$ we see that they are almost mutually indiscernible by (1) and (2). Conclude by Lemma 3.2.3. $\square$

LEMMA 3.7.14. *Let* T *be NTP*$_2$*,* A *an extension base and* $p(x) \in S(A)$ *simple. Assume that* $\phi(x, a)$ *divides over* A*, with* $a \models p(x)$*. Then there is a Morley sequence over* A *witnessing it.*

PROOF. As A is an extension base, let $M \supseteq A$ be such that $M \perp^f_A a$. Then $\phi(x, a)$ divides over M. By Fact 3.5.6(1), there is a Morley sequence $(a_i)_{i<\omega}$ over M witnessing it (in particular $(a_i)_{i<\omega} \subseteq p(\mathbb{M})$). We show that it is actually a Morley sequence over A. Indiscernibility is clear, and we check that $a_i \perp_A a_{<i}$ by induction. As $a_i \perp_M a_{<i}$, $a_{<i} \perp_M a_i$ by simplicity of $\operatorname{tp}(a_{<i}/M)$. Noticing that $M \perp_A a_i$, we conclude $a_{<i} \perp_A a_i$, so again by simplicity $a_i \perp_A a_{<i}$. $\square$

PROPOSITION 3.7.15. *Let* T *be NTP*$_2$*,* A *an extension base and* $p(x) \in S(A)$ *simple. Assume that* $a \models p$ *and* $a \perp^f_A b$*. Then* $b \perp^d_A a$*.*

PROOF. Assume that there is $\phi(x, a) \in L(Aa)$ such that $\models \phi(b, a)$ and $\phi(x, a)$ divides over A. Let $(a_i)_{i<\omega}$ be a Morley sequence over A starting with a. Assume that $\{\phi(x, a_i)\}_{i<\omega}$ is consistent. Let $\bar{a}_0$ be a Morley sequence witnessing that $\phi(x, a_0)$ k-divides over A (exists by Lemma 3.7.14), and let $\bar{a}_i$ be its image under an A-automorphism sending a_0 to a_i. By Lemma 3.7.13, we find $\bar{a}_i' \equiv_{a_i A} \bar{a}_i$, such that $(\bar{a}_i')_{i<\omega}$ are mutually indiscernible. But then we have that $\{\phi(x, a_{i,\eta(i)})\}_{i<\omega}$ is consistent for any $\eta \in \omega^\omega$, while $\{\phi(x, a_{i,j})\}_{j<\omega}$ is k-inconsistent for any $i < \omega$ — contradiction to NTP$_2$.

Now let $(a_i)_{i<\omega}$ be a Morley sequence over A starting with a and indiscernible over Ab. Then clearly $b \models \{\phi(x, a_i)\}_{i<\omega}$ for any $\phi(x, a) \in \operatorname{tp}(b/aA)$, so by the previous paragraph $b \perp^d_A a$. $\square$

LEMMA 3.7.16. *Let* $p(x)$ *be a partial type over* A*. Assume that* $p(x)$ *is not co-simple over* A*. Then there is some* $M \supseteq A$*,* $a \models p(x)$ *and* b *such that* $a \perp^u_M b$ *but* $b \not\perp^d_M a$*.*

PROOF. So assume that $p(x)$ is not co-simple over A, then there is an $L(A)$-formula $\phi(x, y)$ and $(a_\eta)_{\eta \in \omega^{<\omega}} \subseteq p(\mathbb{M})$ witnessing the tree property. Let T^{Sk} be a Skolemization of T, then of course $\phi(x, y)$ and a_η still witness the tree property. As in the proof of (5)⇒(1) in Definition 3.7.7, working in the sense of T^{Sk}, we can find an Ab-indiscernible sequence $(a_i)_{i<\omega+1}$ in $p(x)$ such that $\phi(x, a_i)$ divides over $Aa_{<i}$ and $b \models \{\phi(x, a_i)\}_{i<\omega+1}$. Let $I = (a_i)_{i<\omega}$ and $\mathrm{Sk}(AI) = M \models T$. It follows that $a_\omega \perp^u_M b$ (by indiscernibility) and that $b \not\perp^d_M a_\omega$ (as $M \in \operatorname{acl}(Aa_{<\omega})$) — also the sense of T, as wanted. $\square$

THEOREM 3.7.17. *Let* T *be* NTP$_2$*,* A *an arbitrary set and assume that* $p(x)$ *over* A *is simple. Then* $p(x)$ *is co-simple over* A*.*

PROOF. If $p(x)$ over A is not co-simple over A, then by Lemma 3.7.16 we find some $M \supseteq A$, $a \models p$ and b such that $a \perp^u_M b$, but $b \not\perp^d_M a$. As M is an extension base, it follows by Proposition 3.7.15 that $\operatorname{tp}(a/M)$ is not simple, thus $p(x)$ is not simple by Remark 3.7.2(1) — a contradiction. $\square$

COROLLARY 3.7.18. *Let* T *be NTP*$_2$ *and* $p(x) \in S(A)$ *simple.*

(1) If $a \models p(x)$ *then* $a \downarrow_A b \Leftrightarrow b \downarrow_A a$
(2) Right transitivity: If $a \models p(x)$*,* $B \supseteq A$*,* $a \downarrow_A B$ *and* $a \downarrow_B C$ *then* $a \downarrow_A C$*.*

3.7.3. Independence and co-independence theorems.

In [**Kim01**] Kim demonstrates that if T has TP_1, then the independence theorem fails for types over models, assuming the existence of a large cardinal. We give a proof of a localized and a dual versions, showing in particular that the large cardinal assumption is not needed.

DEFINITION 3.7.19. Let $p(x)$ be (partial) type over A.

(1) We say that $p(x)$ *satisfies the independence theorem* if for any $b_1 \downarrow_A^f b_2$ and $c_1 \equiv_A^{\mathrm{Lstp}} c_2 \subseteq p(\mathbb{M})$ such that $c_1 \downarrow_A^f b_1$ and $c_2 \downarrow_A^f b_2$, there is some $c \downarrow_A^f b_1 b_2$ such that $c \equiv_{b_1 A} c_1$ and $c \equiv_{b_2 A} c_2$.
(2) We say that $p(x)$ *satisfies the co-independence theorem* if for any $b_1 \downarrow_A^f b_2$ and $c_1 \equiv_A^{\mathrm{Lstp}} c_2 \models p$ such that $b_1 \downarrow_A^f c_1$ and $b_2 \downarrow_A^f c_2$, there is some $c \models p$ such that $b_1 b_2 \downarrow_A^f c$ and $c \equiv_{A b_1} c_1$, $c \equiv_{A b_2} c_2$.

Of course, both the independence and the co-independence theorems hold in simple theories, but none of them characterizes simplicity.

PROPOSITION 3.7.20. *Let* T *be NTP*$_2$ *and* $p(x)$ *is a partial type over* A*.*

(1) If every $p'(x) \supseteq p$ *with* $p'(x) \in S(M)$*,* $M \supseteq A$ *satisfies the co-independence theorem, then it is simple.*
(2) If $p(x)$ *satisfies the independence theorem, then it is co-simple.*

PROOF. (1) Without loss of generality $A = \emptyset$. Assume that p is not simple, then by Fact 3.4.12 there are some formula $\phi(x, y)$, $(a_\eta)_{\eta \in \omega^{<\omega}}$ such that:

- $\{\phi(x, a_{\eta | i})\}_{i \in \omega} \cup p(x)$ is consistent for every $\eta \in \omega^\omega$.
- $\phi(x, a_\eta) \wedge \phi(x, a_{\eta'})$ is inconsistent for any incomparable $\eta, \eta' \in \omega^{<\omega}$.

By compactness we can find a similar tree of size κ large enough. Let T^{Sk} be some Skolemization of T, and we work in the sense of T^{Sk}.

CLAIM. There is a sequence $(c_i d_i)_{i \in \omega}$ satisfying:

(1) $\{\phi(x, c_i)\}_{i \in \omega} \cup p(x)$ is consistent.
(2) c_i, d_i start an infinite sequence indiscernible over $c_{<i} d_{<i}$.
(3) $\phi(x, d_i) \wedge \phi(x, d_j)$ is inconsistent for any $i \neq j \in \omega$.

PROOF. Why? By induction we let $c_i = a_{s_1 \dots s_{i-1} s_i}$ and $d_i = a_{s_1 \dots s_{i-1} t_i}$ for some $s_i \neq t_i \in \kappa$ such that there is a $c_{<i} d_{<i}$-indiscernible sequence starting with $a_{s_1 \dots s_{i-1} s_i}, a_{s_1 \dots s_{i-1} t_i}$ (exists by Erdos-Rado as κ is large enough), so we get (2). We get (1) and (3) by the assumption on $(a_\eta)_{\eta \in \kappa^{<\kappa}}$. □

By compactness and Ramsey we can find a and $(c_i d_i)_{i \leq \omega+1}$ indiscernible over a, satisfying (1)–(3) and such that $a \models p(x) \cup \{\phi(x, c_i)\}$.

Let $M = \mathrm{Sk}(c_i d_i)_{i<\omega}$, a model of T^{Sk}. Then we have $c_{\omega+1} \downarrow_M^u a$ and $d_\omega \downarrow_M^u c_{\omega+1}$ by indiscernibility. As $c_\omega d_\omega$ start an M-indiscernible sequence, there is $\sigma \in \mathrm{Aut}(\mathbb{M}/M)$ sending c_ω to d_ω. Let $a' = \sigma(a)$, then $a' \equiv_M^{\mathrm{Lstp}} a$, $d_\omega \downarrow_M^u a'$ (as $c_\omega \downarrow_M^u a$ by indiscernibility) and $\phi(a', d_\omega)$. But $\phi(x, c_{\omega+1}) \wedge \phi(x, d_\omega)$ is inconsistent by (3)+(2) — so the co-independence theorem fails for $p' = \mathrm{tp}(a/M)$.

(2) Similar. □

Now we will show that in NTP$_2$ theories simple types satisfy the independence theorem over extension bases. We will need the following fact from Chapter 2.

FACT 3.7.21. *Let* T *be* NTP$_2$ *and* $M \models T$. *Assume that* $c \mathop{\smile\!\!\!\!\!\!|}_M ab$, $b \mathop{\smile\!\!\!\!\!\!|}_M a$, $b' \mathop{\smile\!\!\!\!\!\!|}_M a$, $b \equiv_M b'$. *Then there exists* $c' \mathop{\smile\!\!\!\!\!\!|}_M ab'$ *and* $c'b' \equiv_M cb$, $c'a \equiv_M ca$.

PROPOSITION 3.7.22. *Let* T *be* NTP$_2$ *and* $p(x)$ *a simple type over* $M \models T$. *Then it satisfies the independence theorem: assume that* $e_1 \mathop{\smile\!\!\!\!\!\!|}_M e_2$, $d_i \mathop{\smile\!\!\!\!\!\!|}_M e_i$, $d_1 \equiv_M d_2 \models p(x)$. *Then there is* $d \mathop{\smile\!\!\!\!\!\!|} e_1e_2$ *with* $d \equiv_{e_iA} d_i$.

PROOF. First we find some $e_1' \mathop{\smile\!\!\!\!\!\!|}_M d_2e_2$ and such that $e_1'd_2 \equiv_M e_1d_1$ (Let $\sigma \in \mathrm{Aut}(\mathbb{M}/M)$ be such that $\sigma(d_1) = d_2$, then $\sigma(e_1)d_2 \equiv_M e_1d_1$. As $e_1 \mathop{\smile\!\!\!\!\!\!|}_M d_1$ by simplicity of $\mathrm{tp}(d_1/M)$, $\sigma(e_1) \mathop{\smile\!\!\!\!\!\!|} d_2$. Let e_1' realize a non-forking extension to d_2e_2). Then we also have $d_2 \mathop{\smile\!\!\!\!\!\!|}_M e_1'e_2$ (by transitivity and symmetry using simplicity of $\mathrm{tp}(d_2/M)$).

Applying Fact 3.7.21 with $a = e_2, b = e_1', b' = e_1, c = d_2$ we find some $d \mathop{\smile\!\!\!\!\!\!|}_M e_1e_2$, $de_1 \equiv_M d_2e_1' \equiv_M d_1e_1$ and $de_2 \equiv d_2e_2$ — as wanted. ☐

We conclude with the main theorem of the section.

THEOREM 3.7.23. *Let* T *be* NTP$_2$ *and* $p(x)$ *a partial type over* A. *Then the following are equivalent:*

(1) $p(x)$ *is simple (in the sense of Definition 3.7.1).*
(2) For any $B \supseteq A$, $a \models p$ *and* b, $a \mathop{\smile\!\!\!\!\!\!|}^f_A b$ *if and only if* $b \mathop{\smile\!\!\!\!\!\!|}^f_A a$.
(3) Every extension $p'(x) \supseteq p(x)$ *to a model* $M \supseteq A$ *satisfies the co-independence theorem.*

PROOF. (1) is equivalent to (2) is by Definitions 3.7.1 and Corollary 3.7.18.
(1) implies (3): By Proposition 3.7.22 and Corollary 3.7.18.
(3) implies (1) is by Proposition 3.7.20. ☐

PROBLEM 3.7.24. Is every co-simple type simple in an NTP$_2$ theory?

We point out that at least every co-simple *stably embedded* type (defined over a small set) is simple. Recall that a partial type $p(x)$ defined over A is called stably embedded if for any $\phi(\bar{x}, c)$ there is some $\psi(\bar{x}, y) \in L(A)$ and $d \in p(\mathbb{M})$ such that $p(\mathbb{M})^n \cap \phi(\bar{x}, c) = p(\mathbb{M})^n \cap \psi(\bar{x}, d)$. If $p(x)$ happens to be defined by finitely many formulas, it is easy to see by compactness that $\psi(\bar{x}, y)$ can be chosen to depend just on $\phi(\bar{x}, y)$, and not on c. But for an arbitrary type this is not true.

PROPOSITION 3.7.25. *Let* T *be* NTP$_2$. *Let* $p(x)$ *be a co-simple type over* A *and assume that* p *is stably embedded. Then* $p(x)$ *is simple.*

PROOF. Assume $p(x)$ is not simple, and let $(a_\eta)_{\eta \in \omega^{<\omega}}$, k and $\phi(x, y)$ witness this. We may assume in addition that (a_η) is an indiscernible tree over A (that is, ss-indiscernible in the terminology of [**KKS12**], see Definition 3.7 and the proof of Theorem 6.6 there).

By the stable embeddedness assumption, there is some $\psi(x, z) \in L(A)$ and $b \subseteq p(\mathbb{M})$ such that $\psi(x, b) \cap p(\mathbb{M}) = \phi(x, a_\emptyset) \cap p(\mathbb{M})$. It follows by the indiscernibility over A that for every $\eta \in \omega^{<\omega}$ there is $b_\eta \subseteq p(\mathbb{M})$ satisfying $\psi(x, b_\eta) \cap p(\mathbb{M}) = \phi(x, a_\eta) \cap p(\mathbb{M})$.

As $\{\phi(x, a_{\emptyset i})\}_{i<\omega}$ is k-inconsistent, it follows that $\{\psi(x, b_{\emptyset i})\}_{i<\omega} \cup p(x)$ is k-inconsistent, thus $\{\psi(x, b_{\emptyset i})\}_{i<\omega} \cup \{\chi(x)\}$ is k-inconsistent for some $\chi(x) \in p$ by compactness and indiscernibility. Again by the indiscernibility over A we have that $\{\psi(x, b_{\eta i})\}_{i<\omega} \cup \{\chi(x)\}$ is k-inconsistent for every $\eta \in \omega^{<\omega}$. It is now easy to see that $\psi'(x, z) = \psi(x, z) \wedge \chi(x)$ and $(b_\eta)_{\eta \in \omega^{<\omega}}$ witness that $p(x)$ is not co-simple over A. □

REMARK 3.7.26. If $p(x)$ is actually a definable set, the argument works in an arbitrary theory since instead of extracting a sufficiently indiscernible tree (which seems to require NTP$_2$), we just use the uniformity of stable embeddedness given by compactness.

3.8. Examples

In this section we present some examples of NTP$_2$ theories. But first we state a general lemma which may sometimes simplify checking NTP$_2$ in particular examples.

LEMMA 3.8.1.

(1) *If* $(\bar{a}_\alpha, \phi_{\alpha,0}(x, y_{\alpha,0}) \vee \phi_{\alpha,1}(x, y_{\alpha,1}), k_\alpha)_{\alpha<\kappa}$ *is an* inp*-pattern, then* $(\bar{a}_\alpha, \phi_{\alpha,f(\alpha)}(x, y_{\alpha,f(\alpha)}), k_\alpha)_{\alpha<\kappa}$ *is an* inp*-pattern for some* $f : \kappa \to \{0, 1\}$.

(2) *Let* $(\bar{a}_\alpha, \phi_\alpha(x, y_\alpha), k_\alpha)_{\alpha<\kappa}$ *be an* inp*-pattern and assume that* $\phi_\alpha(x, a_{\alpha 0}) \leftrightarrow \psi_\alpha(x, b_\alpha)$ *for* $\alpha < \kappa$*. Then there is an* inp*-pattern of the form* $\left(\bar{b}_\alpha, \psi_\alpha(x, z_\alpha), k_\alpha\right)_{\alpha<\kappa}$.

3.8.1. Adding a generic predicate. Let T be a first-order theory in the language L. For $S(x) \in L$ we let $L_P = L \cup \{P(x)\}$ and $T^0_{P,S} = T \cup \{\forall x\, (P(x) \to S(x))\}$.

FACT 3.8.2. **[CP98]** *Let* T *be a theory eliminating quantifiers and* $\exists^\infty$*. Then:*

(1) $T^0_{P,S}$ *has a model companion* $T_{P,S}$*, which is axiomatized by* T *together with*

$$\forall \bar{z} \left[\exists \bar{x} \phi(\bar{x}, \bar{z}) \wedge (\bar{x} \cap \mathrm{acl}_L(\bar{z}) = \emptyset) \wedge \textstyle\bigwedge_{i<n} S(x_i) \wedge \bigwedge_{i\neq j<n} x_i \neq x_j\right] \to$$
$$\left[\exists \bar{x} \phi(\bar{x}, \bar{z}) \wedge \textstyle\bigwedge_{i\in I} P(x_i) \wedge \bigwedge_{i\notin I} \neg P(x_i)\right]$$

for every formula $\phi(\bar{x}, \bar{z}) \in L$*,* $\bar{x} = x_0 ... x_{n-1}$ *and every* $I \subseteq n$*. It is possible to write it in first-order due to the elimination of* $\exists^\infty$*.*

(2) $\mathrm{acl}_L(a) = \mathrm{acl}_{L_P}(a)$

(3) $a \equiv^{L_P} b \Leftrightarrow$ *there is an isomorphism between* L_P *structures* $f : \mathrm{acl}(a) \to \mathrm{acl}(b)$ *such that* $f(a) = b$.

(4) *Modulo* $T_{P,S}$*, every formula* $\psi(\bar{x})$ *is equivalent to a disjunction of formulas of the form* $\exists \bar{z} \phi(\bar{x}, \bar{z})$ *where* $\phi(\bar{x}, \bar{z})$ *is a quantifier-free* L_P *formula and for any* $\bar{a}, \bar{b}$*, if* $\models \phi(\bar{a}, \bar{b})$*, then* $\bar{b} \in \mathrm{acl}(\bar{a})$.

THEOREM 3.8.3. *Let* T *be geometric (that is, the algebraic closure satisfies the exchange property, and* T *eliminates* $\exists^\infty$*) and* NTP$_2$*. Then* T_P *is* NTP$_2$*.*

PROOF. Denote $a \mathop{\smile\!\!\!\!\!\vert}^a_c b \Leftrightarrow a \notin \mathrm{acl}(bc) \setminus \mathrm{acl}(c)$. As T is geometric, $\mathop{\smile\!\!\!\!\!\vert}^a$ is a symmetric notion of independence, which we will be using freely from now on.

Let $(\bar{a}_i, \phi(x, y), k)_{i<\omega}$ be an inp-pattern, such that $(\bar{a}_i)_{i<\omega}$ is an indiscernible sequence and $\bar{a}_i$'s are mutually indiscernible in the sense of L_P, and ϕ an L_P-formula.

CLAIM. For any i, $\{a_{ij}\}_{j<\omega}$ is an $\mathop{\smile\!\!\!\!\!\vert}^a$-independent set (over $\emptyset$) and $a_{ij} \notin \mathrm{acl}(\emptyset)$.

PROOF. By indiscernibility and compactness. $\square$

Let $A = \bigcup_{i<\omega} \bar{a}_i$.

CLAIM. There is an infinite A-indiscernible sequence $(b_t)_{t<\omega}$ such that $b_t \models \{\phi(x, a_{i0})\}_{i<\omega}$ for all $t < \omega$.

PROOF. First, there are infinitely many different b_t's realizing $\{\phi(x, a_{i0})\}_{i<\omega}$, as $\{\phi(x, a_{i0})\}_{0<i<\omega} \cup \{\phi(x, a_{0j})\}$ is consistent for any $j < \omega$ and $\{\phi(x, a_{0j})\}_{j<\omega}$ is k-inconsistent. Extract an A-indiscernible sequence from it. $\square$

Let $p_i(x, a_{i0}) = \mathrm{tp}_L(b_0/a_{i0})$.

CLAIM. For some/every $i < \omega$, there is $b \models \bigcup_{j<\omega} p_i(x, a_{ij})$ such that in addition $b \notin \mathrm{acl}(A)$.

PROOF. For any $N < \omega$, let

$$q_i^N(x_0...x_{N-1}, a_{i0}) = \bigcup_{n<N} p_i(x_n, a_{i0}) \cup \{x_{n_1} \neq x_{n_2}\}_{n_1 \neq n_2 < N}$$

As $b_0...b_{N-1} \models \bigcup_{i<\omega} q_i^N(x_0...x_{N-1}, a_{i0})$ and T is NTP$_2$, there must be some $i < \omega$ such that $\bigcup_{j<\omega} q_i^N(x_0...x_{N-1}, a_{ij})$ is consistent for arbitrary large N (and by indiscernibility this holds for every i). Then by compactness we can find $b \models \bigcup_{j<\omega} p_i(x, a_{ij})$ such that in addition $b \notin \mathrm{acl}(A)$. $\square$

Work with this fixed i. Notice that $b_0 a_{i0} \equiv^L b a_{ij}$ for all $j \in \omega$.

CLAIM. The following is easy to check using that $\mathop{\smile\!\!\!\!\!\!|}^a$ satisfies exchange.

(1) $\mathrm{acl}(A) \cap \mathrm{acl}(a_{ij} b) = \mathrm{acl}(a_{ij})$.
(2) $\mathrm{acl}(a_{ij} b) \cap \mathrm{acl}(a_{ik} b) = \mathrm{acl}(b)$ for $j \neq k$.

Now we conclude as in the proof of [**CP98**, Theorem 2.7]. That is, we are given a coloring P on $\bar{a}_i$. Extend it to a P_i-coloring on $\mathrm{acl}(a_{ij} b)$ such that $a_{ij} b$ realizes $\mathrm{tp}_{L_P}(a_{i0} b_0)$, and by the claim all P_i's are consistent. Thus there is some b' such that $b_0 a_{i0} \equiv^{L_P} b' a_{ij}$ for all $j \in \omega$, in particular $b' \models \{\phi_i(x, a_{ij})\}$ — a contradiction. $\square$

EXAMPLE 3.8.4. Adding a (directed) random graph to an o-minimal theory is NTP$_2$.

PROBLEM 3.8.5. Is it true without assuming exchange for the algebraic closure?

3.8.2. Valued fields. In this section we are going to prove the following theorem:

THEOREM 3.8.6. *Let $\bar{K} = (K, \Gamma, k, v : K \to \Gamma, ac : K \to k)$ be a Henselian valued field of characteristic $(0, 0)$ in the Denef-Pas language. Let $\kappa = \kappa_{\mathrm{inp}}^1(k) \times \kappa_{\mathrm{inp}}^1(\Gamma)$. Then $\kappa_{\mathrm{inp}}^1(K) < R(\kappa + 2, \Delta)$ for some finite set of formulas Δ (see Definition 3.2.4). In particular:*

*(1) If k is NTP$_2$, then $\bar{K}$ is NTP$_2$ (as every ordered abelian group is NIP by [**GS84**], thus $\kappa_{\mathrm{inp}}(\Gamma) < \infty$ and NTP$_2$ follows by Lemma 3.4.2).*
(2) If k and Γ are strong (of finite burden), then $\bar{K}$ is strong (resp. of finite burden).

The "in particular" part follows by 3.3.8.

EXAMPLE 3.8.7. (1) Hahn series over pseudo-finite fields are NTP$_2$.

(2) In particular, let $K = \prod_{p \text{ prime}} \mathbb{Q}_p / \mathfrak{U}$ with $\mathfrak{U}$ a non-principal ultra-filter. Then k is pseudo-finite, so has IP by [**Dur80**]. And Γ has SOP of course. It is known that the valuation rings of $\mathbb{Q}_p$ are definable in the pure field language uniformly in p (see e.g. [**Ax65**]), thus the valuation ring is definable in K in the pure field language, so K has both IP and SOP in the pure field language. By Theorem 3.8.6 it is strong of finite burden, even in the larger Denef-Pas language.

COROLLARY 3.8.8. [**Shed**] *If k and Γ are strongly dependent, then K is strongly dependent.*

PROOF. By Delon's theorem [**Del81**], if k is NIP, then K is NIP. Conclude by Theorem 3.8.6 and Fact 3.4.8. □

We start the proof with a couple of easy lemmas about the behavior of $\nu(x)$ and $ac(x)$ on indiscernible sequences which are easy to check.

LEMMA 3.8.9. *Let $(c_i)_{i \in I}$ be indiscernible. Consider function $(i, j) \mapsto \nu(c_j - c_i)$ with $i < j$. It satisfies one of the following:*

(1) It is strictly increasing depending only on i (so the sequence is pseudo-convergent).
(2) It is strictly decreasing depending only on j (so the sequence taken in the reverse direction is pseudo-convergent).
(3) It is constant (we'll call such a sequence "constant").

Contrary to the usual terminology we do not exclude index sets with a maximal element.

LEMMA 3.8.10. *Let $(c_i)_{i \in I}$ be an indiscernible pseudo-convergent sequence. Then for any a there is some $h \in \bar{I} \cup \{+\infty, -\infty\}$ (where $\bar{I}$ is the Dedekind closure of I) such that (taking c_∞ such that $I \frown c_\infty$ is indiscernible):*

For $i < h$: $\nu(c_\infty - c_i) < \nu(a - c_\infty)$, $\nu(a - c_i) = \nu(c_\infty - c_i)$ *and* $ac(a - c_i) = ac(c_\infty - c_i)$.

For $i > h$: $\nu(c_\infty - c_i) > \nu(a - c_\infty)$, $\nu(a - c_i) = \nu(a - c_\infty)$ *and* $ac(a - c_i) = ac(a - c_\infty)$.

Notice that in fact there is a finite set of formulas Δ such that these lemmas are true for Δ-indiscernible sequences. Fix it from now on, and let $\delta = R(\kappa + 2, \Delta)$ for $\kappa = \kappa_k \times \kappa_\Gamma$ with $\kappa_k = \kappa^1_{\text{inp}}(k)$ and $\kappa_\Gamma = \kappa^1_{\text{inp}}(\Gamma)$.

LEMMA 3.8.11. *In K, there is no* inp-*pattern* $\left(\phi_\alpha(x, y_\alpha), \bar{d}_\alpha, k_\alpha\right)_{\alpha < \delta}$ *with mutually indiscernible rows such that x is a singleton and $\phi_\alpha(x, y_\alpha) = \chi_\alpha(\nu(x - y), y^\Gamma_\alpha) \wedge \rho_\alpha(ac(x - y), y^k_\alpha)$, where $\chi_\alpha \in L_\Gamma$ and $\rho_\alpha \in L_k$.*

PROOF. Assume otherwise, and let $d_{\alpha i} = c_{\alpha i} d^\Gamma_{\alpha i} d^k_{\alpha i}$ where $c_{\alpha i} \in K$ corresponds to y, $d^\Gamma_{\alpha i} \in \Gamma$ corresponds to y^Γ_α and $d^k_{\alpha i} \in k$ corresponds to y^k_α. By the choice of δ, there is a Δ-indiscernible sub-sequence of $(c_{\alpha 0})_{\alpha < \delta}$ of length $\kappa + 2$. Take a sub-array consisting of rows starting with these elements – it is still an inp-pattern of depth $\kappa + 2$ – and replace our original array with it. Let $c_{-\infty}$ and c_∞ be such that $c_{-\infty} \frown (c_{\alpha 0})_{\alpha < \kappa} \frown c_\infty$ is Δ-indiscernible and $\left(\bar{d}_\alpha\right)_{\alpha < \kappa}$ is a mutually indiscernible array over $c_{-\infty}c_{\infty}$ (so either find c_{∞} by compactness if κ is infinite, or just let it be $c_{\kappa-1,0}$ and replace our array by $\left(\bar{d}_{\alpha}\right)_{\alpha<\kappa-1}$). Let $a \models \{\phi_{\alpha}(x, d_{\alpha 0})\}_{\alpha<\kappa+1}$.

Case 1. $(c_{\alpha 0})$ is pseudo-convergent. Let $h \in \{-\infty\} \cup \kappa+1 \cup \{\infty\}$ be as given by Lemma 3.8.10.

Case 1.1. Assume $0 < h$. Then $\nu(a - c_{00}) = \nu(c_{\infty} - c_{00})$, $ac(a - c_{00}) = ac(c_{\infty} - c_{00})$. But then actually $c_{\infty} \models \phi(x, d_{00})$, and by indiscernibility of the array over c_{∞}, $c_{\infty} \models \{\phi(x, d_{0i})\}_{i<\omega}$ — a contradiction.

Case 1.2: Thus $\nu(a - c_{\alpha 0}) = \nu(a - c_{\infty})$, $ac(a - c_{\alpha 0}) = ac(a - c_{\infty})$ and $\nu(a - c_{\infty}) < \nu(c_{\infty} - c_{\alpha 0})$ for all $0 < \alpha < \kappa + 1$.

Let $\chi'_{\alpha}(x', e^{\Gamma}_{\alpha i}) := \chi_{\alpha}(x', d^{\Gamma}_{\alpha i}) \wedge x' < \nu(c_{\infty} - c_{\alpha i})$ with $e^{\Gamma}_{\alpha i} = d^{\Gamma}_{\alpha i} \cup \nu(c_{\infty} - c_{\alpha i})$. Finally, for $\alpha < \kappa_{\Gamma}$ let $f^{\Gamma}_{\alpha i} = \bigcup_{\beta<\kappa_k} e_{\kappa_k \times \alpha+\beta, i}$ and $p_{\alpha}(x', f^{\Gamma}_{\alpha i}) = \left\{\chi'_{\beta}(x', e^{\Gamma}_{\kappa_k \times \alpha+\beta, i})\right\}_{\beta<\kappa_k}$. As $\left(f^{\Gamma}_{\alpha i}\right)$ is a mutually indiscernible array in Γ, $\left\{p_{\alpha}(x', f^{\Gamma}_{\alpha 0})\right\}_{\alpha<\kappa_{\Gamma}}$ is realized by $\nu(a - c_{\infty})$ and $\kappa^{1}_{\mathrm{inp}}(\Gamma) = \kappa_{\Gamma}$, there must be some $\alpha < \kappa_{\Gamma}$ and $a_{\Gamma} \in \Gamma$ such that (unwinding) $a_{\Gamma} \models \left\{\chi'_{\beta}(x', e^{\Gamma}_{\kappa_k \times \alpha+\beta, i})\right\}_{\beta<\kappa_k, i<\omega}$.

Analogously letting $\chi'_{\beta}(x', e^{k}_{\beta i}) := \rho_{\kappa_k \times \alpha+\beta}(x', d^{k}_{\kappa_k \times \alpha+\beta, i})$, noticing that $(e^{k}_{\beta i})_{\beta<\kappa_k, i<\omega}$ is an indiscernible array in k and $\kappa_k = \kappa_{\mathrm{inp}}(k)$, there must be some $a_{\rho} \in k$ and $\beta < \kappa_k$ such that $a_{\rho} \models \{\chi'_{\beta}(x', e^{k}_{\beta i})\}_{i<\omega}$.

Finally, take $a' \in K$ with $\nu(a' - c_{\infty}) = a_{\Gamma} \wedge ac(a' - c_{\infty}) = a_{\rho}$ and let $\gamma = \kappa_k \times \alpha + \beta$. As $a_{\Gamma} < \nu(c_{\infty} - c_{\gamma i})$ it follows that $\nu(a' - c_{\gamma i}) = \nu(a' - c_{\infty})$ and $ac(a' - c_{\gamma i}) = ac(a' - c_{\infty})$. But then $a' \models \{\phi_{\gamma}(x, d_{\gamma i})\}_{i<\omega}$ – a contradiction.

Case 2: (c^{α}_{0}) is decreasing - reduces to the first case by reversing the order of rows.

Case 3: (c^{α}_{0}) is constant.

If $\nu(a - c_{\alpha 0}) < \nu(c_{\infty} - c_{\alpha 0})$ $(= \nu(c_{\beta 0} - c_{\alpha 0})$ for $\beta \neq \alpha)$ for some α, then $\nu(a - c_{\alpha 0}) = \nu(a - c_{\beta 0}) = \nu(a - c_{\infty})$ for any β, and $ac(a - c_{\alpha 0}) = ac(a - c_{\infty})$ for all α's and it falls under case 1.2.

Next, there can be at most one α with $\nu(a - c_{\alpha 0}) > \nu(c_{\infty} - c_{\alpha 0})$ (if also $\nu(a - c_{\beta 0}) > \nu(c_{\infty} - c_{\beta 0})$ for some $\beta > \alpha$ then $\nu(c_{\beta 0} - c_{\alpha 0}) = \nu(a - c_{\alpha 0}) > \nu(c_{\infty} - c_{\alpha 0})$, a contradiction). Throw the corresponding row away and we are left with the case $\nu(a - c_{\alpha 0}) = \nu(c_{\infty} - c_{\alpha 0}) = \nu(a - c_{\infty})$ for all $\alpha < \kappa$. It follows by indiscernibility that $\nu(a - c_{\infty}) = \nu(c_{\infty} - c_{\alpha i})$ for all α, i. Notice that it follows that $ac(a - c_{\alpha 0}) \neq ac(c_{\infty} - c_{\alpha 0})$ and $ac(a - c_{\alpha 0}) = ac(a - c_{\infty}) + ac(c_{\infty} - c_{\alpha 0})$.

Let $\rho'_{\alpha}(x', e^{k}_{\alpha i}) := \rho_{\alpha}(x' - ac(c_{\infty} - c_{\alpha i}), d^{k}_{\alpha i}) \wedge x' \neq ac(c_{\infty} - c_{\alpha i})$ with $e^{k}_{\alpha i} = d^{k}_{\alpha i} \cup ac(c_{\infty} - c_{\alpha i})$. Notice that $ac(a - c_{\infty}) \models \left\{\rho'_{\alpha}(x', e^{k}_{\alpha 0})\right\}$ and that $\left(e^{k}_{\alpha i}\right)$ is a mutually indiscernible array in k. Thus there is some $\alpha < \kappa$ and $a_k \models \left\{\rho'_{\alpha}(x', e^{k}_{\alpha i})\right\}_{i<\omega}$.

Take $a' \in K$ such that $\nu(a' - c_{\infty}) = \nu(a - c_{\infty}) \wedge ac(a' - c_{\infty}) = a_k$. By the choice of a_k we have that $\nu(a' - c_{\infty}) = \nu(a - c_{\infty}) = \nu(c_{\infty} - c_{\alpha i})$ and that $ac(a' - c_{\infty}) \neq ac(c_{\infty} - c_{\alpha i})$, thus $\nu(a' - c_{\alpha i}) = \nu(a - c_{\infty})$ and $ac(a' - c_{\alpha i}) = a_k + ac(c_{\infty} - c_{\alpha i})$. It follows that $a' \models \{\phi_{\alpha}(x, d_{\alpha i})\}_{i<\omega}$ — a contradiction. □

LEMMA 3.8.12. *In* K*, there is no* inp*-pattern* $\left(\phi_{\alpha}(x, y_{\alpha}), \bar{d}_{\alpha}, k_{\alpha}\right)_{\alpha<\delta}$ *such that* x *is a singleton and* $\phi_{\alpha}(x, y_{\alpha}) = \chi_{\alpha}(\nu(x - y_1), ..., \nu(x - y_n), y^{\Gamma}_{\alpha}) \wedge \rho_{\alpha}(ac(x - y_1), ..., ac(x - y_n), y^{k}_{\alpha})$*, where* $\chi_{\alpha} \in L_{\Gamma}$ *and* $\rho_{\alpha} \in L_k$.

PROOF. We prove it by induction on n. The base case is given by Lemma 3.8.11. So assume that we have proved it for $n-1$, and let $\left(\phi_\alpha(x, y_\alpha), \bar{d}_\alpha, k_\alpha\right)_{\alpha<\delta}$ be an inp-pattern with $\phi_\alpha(x, y_\alpha) = \chi_\alpha(\nu(x-y_1), ..., \nu(x-y_n), y_\alpha^\Gamma) \wedge \rho_\alpha(ac(x-y_1), ..., ac(x-y_n), y_\alpha^k)$ and $d_{\alpha i} = c_{\alpha i}^1...c_{\alpha i}^n d_{\alpha i}^\Gamma d_{\alpha i}^k$.

So let $a \models \{\phi_\alpha(x, d_{\alpha 0})\}_{\alpha<\delta}$. Fix some $\alpha < \delta$.

Case 1: $\nu(a - c_{\alpha 0}^1) < \nu(c_{\alpha 0}^n - c_{\alpha 0}^1)$.

Then $\nu(a - c_{\alpha 0}^1) = \nu(a - c_{\alpha 0}^n)$ and $ac(a - c_{\alpha 0}^1) = ac(a - c_{\alpha 0}^n)$. We take

$$\begin{aligned}\phi'_\alpha(x, d'_{\alpha i}) \quad &= \quad \left(\chi_\alpha(\nu(x - c_{\alpha i}^1), ..., \nu(x - c_{\alpha i}^1), d_{\alpha i}^\Gamma) \wedge \nu(x - c_{\alpha 0}^1) < \nu(c_{\alpha i}^n - c_{\alpha i}^1)\right)\\ &\quad\quad \wedge \rho_\alpha(ac(x - c_{\alpha i}^1), ..., ac(x - c_{\alpha i}^1), d_{\alpha i}^\rho)\end{aligned}$$

and $d'_{\alpha i} = d_{\alpha i} \cup \nu(c_{\alpha i}^n - c_{\alpha i}^1)$.

Case 2: $\nu(a - c_{\alpha 0}^1) > \nu(c_{\alpha 0}^n - c_{\alpha 0}^1)$.

Then $\nu(a - c_{\alpha 0}^n) = \nu(c_{\alpha 0}^n - c_{\alpha 0}^1)$ and $ac(a - c_{\alpha 0}^n) = ac(c_{\alpha 0}^n - c_{\alpha 0}^1)$. Take

$$\begin{aligned}\phi'_\alpha(x, d_{\alpha i}) \quad &= \quad \left(\chi_\alpha(\nu(x - c_{\alpha i}^1), ..., \nu(c_{\alpha 0}^n - c_{\alpha 0}^1), d_{\alpha i}^\Gamma) \wedge \nu(x - c_{\alpha 0}^1) > \nu(c_{\alpha i}^n - c_{\alpha i}^1)\right)\\ &\quad\quad \wedge \rho_\alpha(ac(x - c_{\alpha i}^1), ..., ac(c_{\alpha 0}^n - c_{\alpha 0}^1), d_{\alpha i}^\rho)\end{aligned}$$

and $d'_{\alpha i} = d_{\alpha i} \cup \nu(c_{\alpha i}^n - c_{\alpha i}^1) \cup ac(c_{\alpha 0}^n - c_{\alpha 0}^1)$.

Case 3: $\nu(a - c_{\alpha 0}^n) < \nu(c_{\alpha 0}^n - c_{\alpha 0}^1)$ and **Case 4**: $\nu(a - c_{\alpha 0}^n) > \nu(c_{\alpha 0}^n - c_{\alpha 0}^1)$ are symmetric to the cases 1 and 2, respectively.

Case 5: $\nu(a - c_{\alpha 0}^1) = \nu(a - c_{\alpha 0}^n) = \nu(c_{\alpha 0}^n - c_{\alpha 0}^1)$.

Then $ac(a - c_{\alpha 0}^n) = ac(a - c_{\alpha 0}^1) - ac(c_{\alpha 0}^n - c_{\alpha 0}^1)$. We take

$$\begin{aligned}\phi'_\alpha(x, d'_{\alpha i}) \quad &= \quad \left(\chi_\alpha(\nu(x - c_{\alpha i}^1), ..., \nu(c_{\alpha 0}^n - c_{\alpha 0}^1), d_{\alpha i}^\Gamma) \wedge \nu(x - c_{\alpha 0}^1) = \nu(c_{\alpha i}^n - c_{\alpha i}^1)\right)\\ &\quad\quad \wedge \left(\rho_\alpha(ac(x - c_{\alpha i}^1), ..., ac(c_{\alpha 0}^n - c_{\alpha 0}^1), d_{\alpha i}^\rho) \wedge ac(x - c_{\alpha 0}^1) \neq ac(c_{\alpha i}^n - c_{\alpha i}^1)\right)\end{aligned}$$

and $d'_{\alpha i} = d_{\alpha i} \cup \nu(c_{\alpha i}^n - c_{\alpha i}^1) \cup ac(c_{\alpha 0}^n - c_{\alpha 0}^1)$.

In any case, we have that $\{\phi'_\alpha(x, d'_{\alpha i})\}_{i<\omega}$ is inconsistent, $\{\phi_\beta(x, d_{\beta,0})\}_{\beta<\alpha} \cup \{\phi'_\alpha(x, d'_{\alpha 0})\} \cup \{\phi_\beta(x, d_{\beta 0})\}_{\alpha<\beta<\delta}$ is consistent, and $\left(\bar{d}_\beta\right)_{\beta<\alpha} \cup \left\{\bar{d}'_\alpha\right\} \cup \left(\bar{d}_\beta\right)_{\alpha<\beta<\delta}$ is a mutually indiscernible array. Doing this for all α by induction we get an inp-pattern of the same depth involving strictly less different $\nu(x - y_i)$'s – contradicting the inductive hypothesis. $\square$

Finally, we are ready to prove Theorem 3.8.6.

PROOF. By the cell decomposition of Pas [**Pas90**], every formula $\phi(x, \bar{c})$ is equivalent to one of the form $\bigvee_{i<n}(\chi_i(x) \wedge \rho_i(x))$ where $\chi_i = \bigwedge \chi_j^i(\nu(x - c_j^i), \bar{d}_j^i)$ with $\chi_j^i(x, \bar{d}_j^i) \in L(\Gamma)$ and $\rho_i = \bigwedge \rho_j^i(ac(x - c_j^i), \bar{e}_j^i)$ with $\rho_j^i(x, \bar{e}_j^i) \in L(k)$. By Lemma 3.8.1, if there is an inp-pattern of depth κ with x ranging over K, then there has to be an inp-pattern of depth κ and of the form as in Lemma 3.8.12, which is impossible. It is sufficient, as Γ and k are stably embedded with no new induced structure and are fully orthogonal. $\square$

PROBLEM 3.8.13.

(1) Can the bound on $\kappa_{\text{inp}}(K)$ given in Theorem 3.8.6 be improved?

(2) Determine the burden of $K = \prod_{p \text{ prime}} \mathbb{Q}_p/\mathfrak{U}$ in the pure field language. In [**DGL11**] it is shown that each of $\mathbb{Q}_p$ is dp-minimal, so combined with Fact 3.4.8 it has burden 1. However K is not inp-minimal, as both v and ac are definable in the pure field language, and the residue field is infinite, so $\{v(x) = v_i\}$, $\{ac(x) = a_i\}$ shows that the burden is at least 2.

CHAPTER 4

Externally definable sets and dependent pairs

This chapter is a joint work with Pierre Simon and is published in the Israel Journal of Mathematics, 2012, DOI: 10.1007/s11856-012-0061-9 [**CS10**].

We prove that externally definable sets in first order NIP theories have *honest definitions*, giving a new proof of Shelah's expansion theorem. Also we discuss a weak notion of stable embeddedness true in this context. Those results are then used to prove a general theorem on dependent pairs, which in particular answers a question of Baldwin and Benedikt on naming an indiscernible sequence.

4.1. Introduction

This paper is organised in two main parts, the first studies externally definable sets in first order NIP theories and the second, using those results, proves dependence of some theories with a predicate, under quite general hypothesis. We believe both parts to be of independent interest. A third section gives some examples of dependent pairs and relates results proved here to ones existing in the literature.

Honest definitions. Let M be a model of a theory T. An *externally definable* subset of M^k is an $X \subseteq M^k$ that is equal to $\phi(M^k, d)$ for some formula ϕ and d in some $N \succ M$. In a stable theory, by definability of types, any externally definable set coincides with some M-definable set. By contrast, in a random graph for example, any subset in dimension 1 is externally definable.

Assume now that T is NIP. A theorem of Shelah ([**Shed**]), generalising a result of Poizat and Baisalov in the o-minimal case ([**BP98**]), states that the projection of an externally definable set is again externally definable. His proof does not give any information on the formula defining the projection. A slightly clarified account is given by Pillay in [**Pil07**].

In section 1, we show how this result follows from a stronger one: existence of honest definitions. An *honest definition* of an externally definable set is a formula $\phi(x, d)$ whose trace on M is X and which implies all M-definable subsets containing X. Then the projection of X can be obtained simply by taking the trace of the projection of $\phi(x, d)$.

Combining this notion with an idea from [**Gui11**], we can adapt honest definitions to make sense over any subset A instead of a model M. We obtain a property of *weak stable-embeddedness* of sets in NIP structures. Namely, consider a pair (M, A), where we have added a unary predicate $\mathbf{P}(x)$ for the set A. Take $c \in M$ and $\phi(x, c)$ a formula. We consider $\phi(A, c)$. If A is stably embedded, then this set is A-definable. Guingona shows that in an NIP theory, this set is externally A-definable, *i.e.*, coincides with $\psi(A, d)$ for some $\psi(x, y) \in L$ and $d \in A'$ where $(M', A') \succ (M, A)$. We strengthen this by showing that one can find such a $\phi(x, d)$ with the additional property that $\psi(x, d)$ never lies, namely $(M', A') \models \psi(x, d) \to \phi(x, c)$. In particular, the projection of $\psi(x, d)$ has the same trace on A as the projection of $\phi(x, c)$. This is the main tool used in Section 2 to prove dependence of pairs.

Dependent pairs. In the second part of the paper we try to understand when dependence of a theory is preserved after naming a new subset by a predicate. We provide a quite general sufficient condition for the dependence of the pair, in terms of the structure induced on the predicate and the restriction of quantification to the named set.

This question was studied for stable theories by a number of people (see [**CZ01**] and [**BB04**] for the most general results). In the last few years there has been a large number of papers proving dependence for some pair-like structures, e.g. [**BDO11**], [**GH11**], [**Box11**], etc. We apologise for adding yet another result to the list. However, our approach differs in an important way from the previous ones, in that we work in a general NIP context and do not make any assumption of minimality of the structure (by asking for example that the algebraic closure controls relations between points). In particular, in the case of pairs of models, we obtain that if M is dependent, $N \succ M$ and (N, M) is bounded (see Section 2 for a definition), then (N, M) is dependent.

Those results seem to apply to most, if not all, of the pairs known to be dependent. It also covers some new cases, in particular answering a question of Baldwin and Benedikt about naming an indiscernible sequence.

The setting. We will not make a blanket assumption that T is NIP, so we work a priori with a general first order theory T in a language L. We use standard notation. We have a monster model $\mathbb{M}$. If A is a set of parameters, $L(A)$ denotes the formulas of L with parameters from A. If $\phi(x)$ is some formula, and A a subset of $\mathbb{M}$, we will write $\phi(A)$ for the set of tuples $a \in A^{|x|}$ such that $\phi(a)$ holds. If A is a set of parameters, by $\phi(x) \to^{A} \psi(x)$, we mean that for every $a \in A$, $\phi(a) \to \psi(a)$ holds. Also $\phi(x) \to^{p(x)} \psi(x)$ stands for $\phi(x) \to^{p(\mathbb{M})} \psi(x)$.

We will often consider pairs of structures. So if our base language is L, we define the language $L_{\mathbf{P}}$ where we add to L a new unary predicate $\mathbf{P}(x)$. If M is an L-structure and $A \subseteq M$, by the pair (M, A) we mean the $L_{\mathbf{P}}$ extension of M obtained by setting $\mathbf{P}(a) \Leftrightarrow a \in A$. Throughout the paper $\mathbf{P}(x)$ will always denote this extra predicate.

As usual $\mathrm{alt}(\phi)$ is the maximal number n such that there exists an indiscernible sequence $(a_i)_{i<n}$ and c satisfying $\phi(a_i, c) \Leftrightarrow i$ is even. Standardly $\phi(x, y)$ is dependent if and only if $\mathrm{alt}(\phi)$ is finite. For more on the basics of dependent theories see e.g. [**Adl08**].

Acknowledgments. We are grateful to Itay Kaplan and the referee for suggesting a number of improvements to the paper, and to Itaï Ben Yaacov and Manuel Bleichner for pointing out some typos and deficiencies.

4.2. Externally definable sets and honest definitions

Recall that a partial type $p(x)$ is said to be *stably embedded* if any definable subset of $p(x)$ is definable with parameters from $p(\mathbb{M})$. It is well known that if $p(x)$ is stable, then $p(x)$ is stably embedded (see e.g. [**OP07**]). We are concerned with an analogous property replacing stable by dependent.

We say that a formula $\phi(x,c)$ is NIP over a (partial) type $p(x)$ if there is no indiscernible sequence $(a_i)_{i<\omega}$ of realisations of p such that $\phi(a_i,c)$ holds if and only if i is even. We say that $\phi(x,y)$ is NIP over $p(x)$ if $\phi(x,c)$ is NIP over $p(x)$ for every c.

The following is the fundamental observation. We assume here that we have two languages $L \subseteq L'$, and we work inside a monster model $\mathbb{M}$ that is an L'-structure. The language L' could be $L_{\mathbf{P}}$ for example.

PROPOSITION 4.2.1. *Let $p(x)$ be a partial L'-type and $\phi(x,c) \in L(\mathbb{M})$ be NIP over $p(x)$. Then for each small $A \subseteq p(\mathbb{M})$ there is $\theta(x) \in L(p(\mathbb{M}))$ such that*

1) $\theta(x) \cap A = \phi(x,c) \cap A$

2) $\theta(x) \rightarrow^{p(x)} \phi(x,c)$

3) $\phi(x,c) \setminus \theta(x)$ *does not contain any A-invariant global L-type consistent with $p(x)$.*

PROOF. Let $q(x) \in S_L(\mathbb{M})$ be A-invariant and consistent with $\{\phi(x,c)\} \cup p(x)$. We try to choose inductively $a_i, b_i \in p(\mathbb{M})$ and $q_i \subseteq q$, for $i < \omega$ such that

- $q_i(x) = q(x)|_{Aa_{<i}b_{<i}}$
- $a_i \models q_i(x) \cup \{\phi(x,c)\} \cup p(x)$ (we can always find one by assumption)
- $b_i \models q_i(x) \cup \{\neg\phi(x,c)\} \cup p(x)$.

Assume we succeed. Consider the sequence $(d_i)_{i<\omega}$ where $d_i = a_i$ if i is even and $d_i = b_i$ otherwise. It is a Morley sequence of q over A, and as such is L-indiscernible. Furthermore, we have $\models \phi(d_i,c)$ if and only if i is even. This contradicts $\phi(x,y)$ being NIP over $p(x)$, so the construction must stop at some finite stage i_0. Then $q_{i_0}(x) \rightarrow^{p(x)} \phi(x,c)$ and by compactness there is $\psi_q(x) \in q_{i_0}$ (so $\psi_q \in L(p(\mathbb{M})))$ such that $\psi_q(x) \rightarrow^{p(x)} \phi(x,c)$. So we see that the set of all such ψ_q's covers the compact space of global L-types invariant over A and consistent with $\{\phi(x,c)\} \cup p(x)$ (so in particular all realised types of elements of A such that $\phi(a,c)$). Let $(\psi_j)_{j<n}$ be a finite subcovering, then taking $\theta(x) = \bigvee_{j<n} \psi_j(x)$ does the job. $\square$

DEFINITION 4.2.2. [Externally definable set] Let M be a model, an externally definable set of M is a subset X of M^k for some k such that there is a formula $\phi(x,y)$ and $d \in \mathbb{M}$ with $\phi(M,d) = X$. Such a $\phi(x,d)$ is called a definition of X.

We can now prove a form of *weak stable embeddedness* for NIP formulas.

COROLLARY 4.2.3. *[Weak stable-embeddedness] Let $\phi(x,y)$ be NIP. Given (M,A) and $c \in M$ there are $(M',A') \succeq (M,A)$ and $\theta(x) \in L(A')$ such that $\phi(A,c) = \theta(A)$ and $\theta(x) \rightarrow^{A'} \phi(x,c)$.*

PROOF. Notice that $\phi(x,y)$ is still NIP in any expansion of the structure. In particular in the $L_{\mathbf{P}}$-structure (M,A). Now apply Proposition 4.2.1 with $L' = L_{\mathbf{P}}$ and $p(x) = \{\mathbf{P}(x)\}$. $\square$

PROBLEM 4.2.4. Do we get uniform weak stable embeddedness ? In other words, is it possible to choose θ depending just on ϕ, or at least just on ϕ and $\mathrm{Th}(M,A)$?

COROLLARY 4.2.5. *Let $f : M \rightarrow M$ be an externally definable function, that is the trace on M of an externally definable relation which happens to be a function on M. Then there is an $\mathbb{M}$-definable partial function $g : \mathbb{M} \rightarrow \mathbb{M}$ with $g|_M = f$.*

Proof. Let $\phi(x, y; c)$ induce f on M, $c \in N \succ M$. By Corollary 4.2.3 we find $(N', M') \succ (N, M)$ and $\theta(x, y) \in L(M')$ satisfying $\theta(M^2) = \phi(M^2, c)$ and $\theta(x, y) \to^{M'} \phi(x, y; c)$. As the extension of pairs is elementary and $M' \models T$, it follows that $\theta(x, y)$ is a graph of a global partial function. $\square$

Definition 4.2.6. [Honest definition] Let $X \subseteq M^k$ be externally definable. Then an honest definition of X is a definition $\phi(x, d)$ of X, $d \in \mathbb{M}$ such that:
$\mathbb{M} \models \phi(x, d) \to \psi(x)$ for every $\psi(x) \in L(M)$ such that $X \subseteq \psi(M)$.

In Section 2, we will need the notion of an honest definition *over* A which is defined at the beginning of that section.

Proposition 4.2.7. *Let* T *be* NIP. *Then every externally definable set* $X \subset M^k$ *has an honest definition.*

Proof. Let $M \prec N$ and $\phi(x) \in L(N)$ be a definition of X, and let $(N', M') \succeq (N, M)$ be $|N|^+$-saturated (in L_P). Let $\theta(x) \in L(M')$ as given by Corollary 4.2.3, so $(N', M') \models (\forall x \in \mathbf{P})\, \theta(x) \to \phi(x)$. If $\psi(x) \in L(M)$ with $X \subseteq \psi(M)$ then $(N', M') \models (\forall x \in \mathbf{P})\, \phi(x) \to \psi(x)$. Combining, we get $(N', M') \models (\forall x \in \mathbf{P})\, \theta(x) \to \psi(x)$. But since $M' \models T$ and $\theta(x), \psi(x) \in L(M')$ we have finally $M' \models \theta(x) \to \psi(x)$. $\square$

We illustrate this notion with an o-minimal example inspired by [**BP98**].

We let M_0 be the real closure of $\mathbb{Q}$ and let $\epsilon > 0$ be an infinitesimal element. Let M be the real closure of $M_0(\epsilon)$. Let π be the usual transcendental number, and finally let N be the real closure of $M(\pi)$.

Lemma 4.2.8. *Let* $0 < b \in N$ *be infinitesimal, then there is* $n \in \mathbb{N}$ *such that* $b < \epsilon^{1/n}$.

Proof. We define a valuation ν on $\mathbb{Q}(\pi, \epsilon)$ by setting $\nu(x) = 0$ for all $x \in \mathbb{Q}(\pi)$ and $\nu(\epsilon) = 1$. We also define a valuation on N with the following standard construction: let $\mathcal{O} \subset N$ be the convex closure of $\mathbb{Q}$ and $\mathfrak{M}$ be the ring of infinitesimals. Then $\mathcal{O}$ is a valuation ring, namely every element of N or its inverse lies in it. It has $\mathfrak{M}$ as unique maximal ideal. There is therefore a valuation ν' on N such that $\nu'(x) \geq 0$ on $\mathcal{O}$ and $\nu'(x) > 0$ on $\mathfrak{M}$. Renaming the value group, we can set $\nu'(\epsilon) = 1$. Then ν' extends the valuation ν. As N is in the algebraic closure of $\mathbb{Q}(\epsilon, \pi)$, by standard results on valuation theory (see for example [**EP05**], Theorem 3.2.4), the value group of ν' is in the divisible hull of the value group of ν.

Let $b \in N$ be a positive infinitesimal. By the previous argument $\nu'(b)$ is rational, so there is $n \in \mathbb{N}$ such that $\nu'(b) > \nu'(\epsilon^{1/n})$. Then $\nu'(b/(\epsilon^{1/n})) > 0$, so $b/(\epsilon^{1/n})$ is infinitesimal and in particular $b < \epsilon^{1/n}$. $\square$

Let $A = \{x \in M : x < \pi\}$. So A is an externally definable initial segment of M. Consider the externally definable set $X = \{(x, y) \in M^2 : x \in A \wedge y \notin A\}$. Let $\phi(x, y; t) = (x < t \wedge y > t)$. Then $\phi(x, y; \pi)$ is a definition of X. However it is not an honest definition because it is not included in the M-definable set $\{(x, y) : y - x > \epsilon\}$. We actually show more.

Claim 1: There is no honest definition of X with parameters in N.

Proof: Assume that $\chi(x, y)$ is such a definition. Consider $c = \inf\{y - x : y - x > 0 \wedge \chi(x, y)\}$. Then $c \in N$. For every $0 < \epsilon \in M$ infinitesimal, we have $c > \epsilon$ by the same argument as above. By the previous lemma, there is $0 < e \in \mathbb{Q}$ such that $c > e$. This is absurd as $\chi(x, y) \supseteq X$.

Let p be the global 1-type such that for $a \in \mathbb{M}$, $p \vdash x > a$ if and only if there is $b \in A \subset M$ such that $a < b$. Thus p is finitely satisfiable in M. Let $a_0 = \pi$ and $a_1 \models p|_N$. Consider the formula $\psi(x, y; a_0, a_1) = (x < a_1 \wedge y > a_0)$.

Claim 2: The formula ψ is an honest definition of X.

Proof: Let $\theta(x, y) \in L(M)$ be a definable set. Assume that $X \subseteq \theta(M^2)$ and for a contradiction that $\mathbb{M} \models (\exists x, y)\psi(x, y; a_0, a_1) \wedge \neg\theta(x, y)$. As p is finitely satisfiable in M, there is $u_0 \in M$ such that $\models (\exists x, y)x < u_0 \wedge y > a_0 \wedge \neg\theta(x, y)$. Consider the M-definable set $\{v : (\exists x, y)x < u_0 \wedge y > v \wedge \neg\theta(x, y)\}$. By o-minimality, this set has a supremum $m \in M \cup \{+\infty\}$. We know $m \geq a_0$, so necessarily there is $v_0 \in M$, $v_0 \notin A$ such that $M \models (\exists x, y)x < u_0 \wedge y > v_0 \wedge \neg\theta(x, y)$. This contradicts the fact that $X \subseteq \theta(M^2)$.

We therefore see that if $\phi(x, y; a)$ is a formula and M a model, then one cannot in general obtain an honest definition of $\phi(M^2; a)$ with the same parameter a. We conjecture that one can find such an honest definition with parameters in a Morley sequence of any coheir of $tp(a/M)$.

As an application, we give another proof of Shelah's expansion theorem from [**She09**].

Proposition 4.2.9. *(T is NIP) Let $X \subseteq M^k$ be an externally definable set and f an M-definable function. Then $f(X)$ is externally definable.*

Proof. Let $\phi(x, c)$ be an honest definition of X. We show that $\theta(y, c) = (\exists x)(\phi(x, c) \wedge f(x) = y)$ is a definition of $f(X)$. First, as $\phi(x, c)$ is a definition of X, we have $f(X) \subseteq \theta(M, c)$. Conversely, consider a tuple $a \in M^k \setminus f(X)$. Let $\psi(x) = (f(x) \neq a)$. Then $X \subseteq \psi(M)$. So by definition of an honest definition, $\mathbb{M} \models \phi(x, c) \rightarrow \psi(x)$. This implies that $\mathbb{M} \models \neg\theta(a, c)$. Thus $\theta(M, c) \subseteq f(X)$.

In fact one can check that $\theta(y, c)$ is an honest definition of $f(X)$. □

Corollary 4.2.10. *[Shelah's expansion theorem] Let $M \models T$, be NIP and let M^{Sh} denote the expansion of M where we add a predicate for all externally definable sets of M^k, for all k. Then M^{Sh} has elimination of quantifiers in this language and is NIP.*

Proof. Elimination of quantifiers follows from the previous proposition, taking f to be a projection. As T is NIP, it is clear that all quantifier free formulas of M^{Sh} are dependent. It follows that M^{Sh} is dependent. □

Note that there is an asymmetry in the notion of an honest definition. Namely if $\theta(x)$ is an honest definition of some $X \subset M$, then $\neg\theta(x)$ is not in general an honest definition of $M \setminus X$. We do not know about existence of *symmetric* honest definitions which would satisfy this. All we can do is have an honest definition contain one (or indeed finitely many) uniformly definable family of sets. This is the content of the next proposition.

Proposition 4.2.11. *(T is NIP) Let $X \subseteq M^k$ be externally definable. Let $\zeta(x, y) \in L$. Define $\Omega = \{y \in M : \zeta(M, y) \subseteq X\}$. Assume that $\bigcup_{y \in \Omega} \zeta(M, y) = X$. Then there is a formula $\theta(x, y)$ and $d \in \mathbb{M}$ such that:*

(1) $\theta(x, d)$ is an honest definition of X,

(2) $\mathbb{M} \models \zeta(x, c) \to \theta(x, d)$ *for every* $c \in \Omega$,
(3) *For any* $c_1, .., c_n \in \Omega$, *there is* $d' \in M$ *such that* $\theta(M, d') \subseteq X$, *and* $\zeta(x, c_i) \to \theta(x, d')$ *holds for all* i.

Proof. Let $M \prec N$ where N is $|M|^+$-saturated. Consider the set $Y \subset M$ defined by

$$y \in Y \Longleftrightarrow (\forall x \in M)(\zeta(x, y) \to x \in X).$$

By Corollary 4.2.10, this is an externally definable subset of M, so there is $\psi(x) \in L(N)$ a definition of it. Let also $\phi(x) \in L(N)$ be a definition of X. Let $(N, M) \prec (N', M')$ be an elementary extension of the pair, sufficiently saturated. Applying Proposition 4.2.1 with $p(y) = \{\mathbf{P}(y)\}$, $A = M$ we obtain a formula $\alpha(y, d) \in L(M')$ such that $\alpha(M, d) = \psi(M)$ and $N' \models \alpha(y, d) \to^{\mathbf{P}(y)} \psi(y)$. Set $\theta(x, d) = (\exists y)(\alpha(y, d) \wedge \zeta(x, y))$. We check that $\theta(x, d)$ satisfies the required properties.

First, let $a \in M'$ such that $N' \models \theta(a, d)$. Then as $M' \prec N'$, there is $y_0 \in M'$ such that $\alpha(y_0, d) \wedge \zeta(a, y_0)$. By construction of $\alpha(y, d)$, this implies that $N' \models \psi(y_0)$. So by definition of $\psi(y)$, $N' \models \phi(a)$, so $N' \models \theta(x, d) \to^{\mathbf{P}(x)} \phi(x)$. Now, assume that $a \in X$. By hypothesis, there is $y_0 \in \Omega$ such that $M \models \zeta(a, y_0)$. Then $\psi(y_0)$ holds, and as $y_0 \in M$, $N' \models \alpha(y_0, d)$. Therefore $N' \models \theta(a, d)$. This proves that $\theta(x, d)$ is an honest definition of X.

Next, if $c \in \Omega$, then $N' \models \alpha(c, d)$, so $N' \models \zeta(x, c) \to \theta(x, d)$.

Finally, let $c_1, ..., c_n \in \Omega$. Then $N' \models (\exists d \in \mathbf{P})(\bigwedge \zeta(x, c_i) \to^{\mathbf{P}(x)} \theta(x, d)) \wedge (\theta(x, d) \to^{\mathbf{P}(x)} \phi(x))$. By elementarity, (N, M) also satisfies that formula. This gives us the required d'. □

Note in particular that the hypothesis on $\zeta(x, y)$ is always satisfied for $\zeta(x, y) = (x = y)$. As an application, we obtain that large externally definable sets contain infinite definable sets.

Corollary 4.2.12. *(T is NIP) Let $X \subseteq M^k$ be externally definable, then if one of the two following conditions is satisfied, X contains an infinite M-definable set.*

(1) X *is infinite and* T *eliminates the quantifier* $\exists^\infty$.
(2) $|X| \geq \beth_\omega$.

Proof. Let $\theta(x, y)$ be the formula given by the previous proposition with $\zeta(x, y) = (x = y)$. If the first assumption holds, then there is n such that for every $d \in M$, if $\theta(M, d)$ has size at least n, it is infinite. Take $c_1, ..., c_n \in X$ and $d' \in M$ given by the third point of 4.2.11. Then $\theta(M, d')$ is an infinite definable set contained in X.

Now assume that $|X| \geq \beth_\omega$. By NIP, there is Δ a finite set of formulas and n such that if $(a_i)_{i<\omega}$ is a Δ-indiscernible sequence and $d \in \mathbb{M}$, there are at most n indices i for which $\neg(\theta(a_i, d) \leftrightarrow \theta(a_{i+1}, d))$. By the Erdös-Rado theorem, there is a sequence $(a_i)_{i<\omega_1}$ in X which is Δ-indiscernible. Define $c_i = a_{\omega.i}$ for $i = 0, .., n$ and let d' be given by the third point of Proposition 4.2.11. Then $\theta(x, d')$ must contain an interval $\langle a_i : \omega \times k \leq i \leq \omega \times k + 1 \rangle$ for some $k \in \{0, .., n-1\}$. In particular it is infinite. □

This property does not hold in general. For example in the random graph, for any κ it is easy to find a model M and $A \subset M$, $|A| \geq \kappa$ such that every M-definable subset of A is finite, while A itself is externally definable.

Also, taking $M = (\mathbb{N} + \mathbb{Z}, <)$ and $X = \mathbb{N}$ shows that $|X|$ has to be bigger than $\aleph_0$ in 4.2.12 in general.

PROBLEM 4.2.13. Is it possible to replace $\beth_\omega$ by $\aleph_1$ in 4.2.12?

4.3. On dependent pairs

Setting. In this section, we assume that T is NIP. We consider a pair (M, A) with $M \models T$. If $\phi(x, a)$ is some formula of $L_P(M)$, then an *honest definition of* $\phi(x, a)$ *over* A is a formula $\theta(x, c) \in L_P$, $c \in P(\mathbb{M})$ such that $\theta(A, c) = \phi(A, a)$ and $\models (\forall x \in P)(\theta(x, c) \to \phi(x, a))$.

(Note that if $M \models T$, $\phi(x, c) \in L(\mathbb{M})$ and $X = \phi(M, c)$, then an honest definition of $\phi(x, c)$ over M in the pair $(\mathbb{M}, M)$ which happens to be an L-formula is an honest definition of X in the sense of Definition 4.2.6.)

We say that an L_P-formula is *bounded* if it is of the form $Q_0 y_0 \in P ... Q_n y_n \in P\, \phi(x, y_0, ..., y_n)$ where $Q_i \in \{\exists, \forall\}$ and $\phi(x, \bar{y})$ is an L-formula, and let L_P^{bdd} be the collection of all bounded formulas. We say that T_P is bounded if every formula is equivalent to a bounded one.

Recall that a formula $\phi(x, y) \in L_P$ is said to be NIP *over* $P(x)$ if there is no L_P-indiscernible (equivalently L-indiscernible if $\phi \in L$) sequence $(a_i)_{i<\omega}$ of points of P and y such that $\phi(a_i, y) \Leftrightarrow i$ is even. If this is the case, then Proposition 4.2.1 applies and in particular there is an honest definition of $\phi(x, a)$ over P for all a.

We say that T (or T_P) is NIP over P if every L (resp. L_P) formula is.

Given a small subset of the monster A and a set of formulas Ω (possibly with parameters) we let $A_{ind(\Omega)}$ be the structure with domain A and a relation added for every set of the form $A^n \cap \phi(\bar{x})$, where $\phi(\bar{x}) \in \Omega$.

Notice that $A_{ind(L_P^{bdd})}$ eliminates quantifiers, while $A_{ind(L)}$ not necessarily does. However $A_{ind(L_P^{bdd})}$ and $A_{ind(L)}$ are bi-interpretable.

LEMMA 4.3.1. *Assume that* $\varphi(xy, c) \in L_P$ *has an honest definition* $\vartheta(xy, d) \in L_P$ *over* A. *Then* $\theta(x, d) = (\exists y \in P)\vartheta(xy, d)$ *is an honest definition of* $\phi(x, c) = (\exists y \in P)\varphi(xy, c)$ *over* A.

PROOF. For $a \in P$, $\theta(a, d) \Rightarrow \vartheta(ab, d)$ for some $b \in P \Rightarrow \varphi(ab, c)$ (as $\vartheta(xy, d)$ is honest and $ab \in P$) $\Rightarrow \phi(a, c)$.

For $a \in A$, $\phi(a, c) \Rightarrow \varphi(ab, c)$ for some $b \in A \Rightarrow \vartheta(ab, d)$ (as $\vartheta(A, d) = \varphi(A, c)) \Rightarrow \theta(a, d)$. □

We will be using λ-big models (see [**Hod93**, 10.1]). We will only use that if N is λ-big, then it is λ-saturated and strongly λ-homogeneous (that is, for every $\bar{a}, \bar{b} \in N^{<\lambda}$ such that $(N, \bar{a}) \equiv (N, \bar{b})$ there is an automorphism of N taking $\bar{a}$ to $\bar{b}$) (see [**Hod93**, 10.1.2 + Exercise 10.1.4]). Every model M has a λ-big elementary extension N.

LEMMA 4.3.2. *1) If* $N \succeq M$, M *is* ω*-big,* N *is* $|M|^+$*-big, and* $a, b \in M^{<\omega}$ *then* $tp_L(a) = tp_L(b) \Leftrightarrow tp_{L_P}(a) = tp_{L_P}(b)$ *in the sense of the pair* (N, M).

2) Let $\phi(x, y) \in L_P$, (M, A) ω*-big,* $(a_i)_{i<\omega} \in M^\omega$ *be* L_P*-indiscernible, and let* $\theta(x, d_0)$ *be an honest definition for* $\phi(x, a_0)$ *over* A *(where* d_0 *is in* P *of the monster model). Then we can find an* L_P*-indiscernible sequence* $(d_i)_{i<\omega} \in P^\omega$ *such that* $\theta(x, d_i)$ *is an honest definition for* $\phi(x, a_i)$ *over* A.

PROOF. 1) We consider here the pair (N, M) as an L_P-structure, where $P(x)$ is a new predicate interpreted in the usual way. Let $\sigma \in Aut_L(M)$ be such that

$\sigma(a) = b$. As N is big, it extends to $\sigma' \in \mathrm{Aut}_L(N)$, with $\sigma'(M) = M$. But then actually $\sigma' \in \mathrm{Aut}_{L_P}(N)$ (since it preserves all L-formulas and $\mathbf{P}$).

2) Let $(N, B) \succeq (M, A)$ be $|M|^+$-big. We consider the pair of pairs $\mathrm{Th}((N, B), (M, A))$ in the language $L_{\mathbf{P},\mathbf{P}'}$, with $\mathbf{P}'(N) = M$. By 1) the sequence $(a_i)_{i<\omega}$ is $L_{\mathbf{P},\mathbf{P}'}$-indiscernible. The fact that $\theta(x, d_0)$ is an honest definition of $\phi(x, a_0)$ over A is expressible by the formula

$$(d_0 \in \mathbf{P}) \wedge ((\forall x \in \mathbf{P}' \cap \mathbf{P})\, \theta(x, d_0) \equiv \phi(x, a_0)) \wedge ((\forall x \in \mathbf{P})\theta(x, d_0) \to \phi(x, a_0)).$$

By $L_{\mathbf{P},\mathbf{P}'}$-indiscernibility, for each i, we can find d_i such that the same formula holds of (a_i, d_i). Then using Ramsey, for any finite $\Delta \subset L_P$, we can find an infinite subsequence $(a_i, d_i)_{i\in I}$, $I \subseteq \omega$ that is Δ-indiscernible. As (a_i) is indiscernible, we can assume $I = \omega$. Then by compactness, we can find the d_i's as required. □

We will need the following technical lemma.

Lemma 4.3.3. *Let* $(M, A) \models T_{\mathbf{P}}$ *be* ω*-big and assume that* $A_{\mathrm{ind}(L_{\mathbf{P}})}$ *is* NIP*.*

Let $(a_i)_{i<\omega} \in M^\omega$ *be* $L_{\mathbf{P}}$*-indiscernible,* $(b_{2i})_{i<\omega} \in A^\omega$ *and* $\Delta((x_i)_{i<n}; (y_i)_{i<n}) \in L_{\mathbf{P}}$ *be such that* $\Delta((x_i)_{i<n}; (a_i)_{i<n})$ *has an honest definition over* A *by an* $L_{\mathbf{P}}$*-formula, and* $\models \Delta(b_{2i_0}, ..., b_{2i_{n-1}}; a_{2i_0}, ..., a_{2i_{n-1}})$ *for any* $i_0, ..., i_{n-1} < \omega$*.*

Then there are $i_0, ..., i_{n-1} \in \omega$ *with* $i_j \equiv j\ (\mathrm{mod}\ 2)$ *and* $(b_{i_j})_{j\equiv 1(\mathrm{mod} 2),<n} \in \mathbf{P}$ *such that* $\models \Delta(b_{i_0}, ..., b_{i_{n-1}}; a_{i_0}, ..., a_{i_{n-1}})$*.*

Proof. To simplify notation assume that n is even. Let

$$\Delta'((x_{2i})_{2i<n}; (y_i)_{i<n}) = (\exists x_1 x_3 ... x_{n-1} \in \mathbf{P})\, \Delta((x_i)_{i<n}; (y_i)_{i<n}).$$

By assumption and Lemma 4.3.1 $\Delta'((x_{2i})_{2i<n}; (a_i)_{i<n})$ has an honest definition over A by some $L_{\mathbf{P}}$-formula, say $\theta((x_{2i})_{2i<n}, d)$ with $d \in \mathbf{P}$. Since $A_{\mathrm{ind}(L_{\mathbf{P}})}$ is NIP, let $N = \mathrm{alt}(\theta)$ inside $\mathbf{P}$.

Choose even $i_0, i_2, ..., i_{n-2} \in \omega$ such that $i_{j+2} - i_j > N$ and consider the sequence $(\bar{a}_i)_{0<i<N}$ with $\bar{a}_i = a_{i_0} a_{i_0+i} a_{i_2} a_{i_2+i} ... a_{i_{n-2}} a_{i_{n-2}+i}$. It is $L_{\mathbf{P}}$-indiscernible (and extends to an infinite $L_{\mathbf{P}}$-indiscernible sequence). By Lemma 4.3.2 we can find an $L_{\mathbf{P}}$-indiscernible sequence $(d_i)_{i<N}$, $d_i \in \mathbf{P}$ such that $\theta((x_{2i})_{2i<n}; d_i)$ is an honest definition for $\Delta'((x_{2i})_{2i<n}; \bar{a}_i)$. By assumption $\theta((b_{i_{2j}})_{2j<n}; d_i)$ holds for all even $i < N$. But then since $N = \mathrm{alt}(\theta)$ inside $\mathbf{P}$, it must hold for some odd $i' < N$. By honesty this implies that $\Delta'((b_{i_{2j}})_{2j<n}; \bar{a}_{i'})$ holds, and decoding we find some $(b_{i_{2j}+i'})_{2j<n} \in \mathbf{P}^{\frac{n}{2}}$ as wanted. □

Now the main results of this section.

Theorem 4.3.4. *Assume* T *is* NIP *and* $T_{\mathbf{P}}$ *is* NIP *over* $\mathbf{P}$*. Then every bounded formula is* NIP*.*

Proof. We prove this by induction on adding an existential bounded quantifier (since NIP formulas are preserved by boolean operations). So assume that $\phi(x, y) = (\exists z \in \mathbf{P})\, \psi(xz, y)$ has IP, where $\psi(xz, y) \in L_{\mathbf{P}}^{bdd}$ is NIP. Then there is an ω-big $(M, A) \models T_{\mathbf{P}}$ and an $L_{\mathbf{P}}$-indiscernible sequence $(a_i)_{i<\omega} \in M^\omega$ and $c \in M$ such that $\phi(a_i, c) \Leftrightarrow i = 0(\mathrm{mod}\ 2)$. Then we can assume that there are $b_{2i} \in A$ such that $(a_{2i} b_{2i})$ is $L_{\mathbf{P}}$-indiscernible and $\models \psi(a_{2i} b_{2i}, c)$.

Notice that from $T_{\mathbf{P}}$ being NIP over $\mathbf{P}$ it follows that $A_{\mathrm{ind}(L_{\mathbf{P}})}$ is NIP and that every $L_{\mathbf{P}}$-formula has an honest definition over A. For $\delta \in L_{\mathbf{P}}$ take $\Delta_\delta((x_i)_{i<n}; (y_i)_{i<n})$ to be an $L_{\mathbf{P}}$-formula saying that $(x_i y_i)_{i<n}$ is δ-indiscernible. Applying Lemma 4.3.3, we obtain $i_0, ..., i_n \in \omega$ with $i_j \equiv j\ (\mathrm{mod}\ 2)$ and $(b_{i_j})_{j\equiv 1(\mathrm{mod}\ 2),<n} \in \mathbf{P}$ such

that $(a_{i_k}b_{i_k})_{k<n}$ is δ-indiscernible. Since $\models \neg(\exists z \in \mathbf{P})\psi(a_{2i+1}z, c)$ for all i, we see that $\psi(a_{i_k}b_{i_k}, c)$ holds if and only if k is even. Taking n and δ large enough, this contradicts dependence of $\psi(xz, y)$. □

COROLLARY 4.3.5. *Assume* T *is* NIP*,* $A_{ind(L)}$ *is* NIP *and* $T_{\mathbf{P}}$ *is bounded. Then* $T_{\mathbf{P}}$ *is* NIP*.*

PROOF. Since $A_{ind(L_P^{bdd})}$ is interpretable in $A_{ind(L)}$ the hypothesis implies that $A_{ind(L_P^{bdd})}$ is NIP. Thus, if $\bar{a} = (a_i)_{i<n}$ is a sequence inside $\mathbf{P}$ then any $\Delta(\bar{x}, \bar{a})$ has an honest definition over A (although we don't yet know that $\Delta(\bar{x}, \bar{y})$ is NIP over $\mathbf{P}$, we do know that $\Delta(\bar{x}, \bar{a})$ is NIP over $\mathbf{P}$, so Proposition 4.2.1 applies). We can then use the same proof as in 4.3.4 to ensure that $T_{\mathbf{P}}$ is NIP over $\mathbf{P}$, and finally apply Theorem 4.3.4 to conclude. □

COROLLARY 4.3.6. *Assume* T *is* NIP*, and let* (M, N) *be a pair of models of* T *(*$N \prec M$*). Assume that* $T_{\mathbf{P}}$ *is bounded, then* $T_{\mathbf{P}}$ *is* NIP*.*

PROOF. $N_{ind(L)}$ is dependent, and so the hypotheses of Corollary 4.3.5 are satisfied. □

Note that the boundedness assumption cannot be dropped, because for example a pair of real closed fields can have IP, and also there is a stable theory such that some pair of its models has IP ([**Poi83**]).

4.4. Applications

In this section we give some applications of the criteria for the dependence of the pair.

4.4.1. Naming an indiscernible sequence. In [**BB00**] Baldwin and Benedikt prove the following.

FACT 4.4.1. *(*T *is* NIP*) Let* $I \subset M$ *be an indiscernible sequence indexed by a dense complete linear order, small in* M *(that is every* $p \in S_{<\omega}(I)$ *is realised in* M*). Then*

1) $Th(M, I)$ *is bounded (*[**BB00**, Theorem 3.3]*),*

2) $(M, I) \equiv (N, J)$ *if and only if* $EM(I) = EM(J)$ *(*[**BB00**, Theorem 8.1]*),*

3) The $L_{\mathbf{P}}$*-induced structure on* $\mathbf{P}$ *is just the equality (if* I *is totally transcendental) or the linear order otherwise (*[**BB00**, Corollary 3.6]*).*

It is not stated in the paper in exactly this form because the bounded formula from [**BB00**, Theorem 3.3] involves the order on the indiscernible sequence. However, it is not a problem. If the sequence $I = (a_i)$ is not totally indiscernible, then the order is L-definable (maybe after naming finitely many constants). Namely, we will have $\phi(a_0, ..., a_k, a_{k+1}, ..., a_n) \wedge \neg\phi(a_0, ..., a_{k+1}, a_k, ..., a_n)$ for some $k < n$ and $\phi \in L$ (as the permutation group is generated by transpositions). But then the order on I is given by $y_1 < y_2 \leftrightarrow \phi(a'_0...a'_{k-1}, y_1, y_2, a'_{k+2}, ..., a'_n)$, for any $a'_0...a_{k-1}Ia'_{k+2}...a'_n$ indiscernible (and we can find such $a'_0...a_{k-1}a'_{k+2}...a'_n$ in M by the smallness assumption). If I is an indiscernible set, then the stable counterpart of their theorem [**BB00**, 3.3] applies giving a bounded formula using just the equality (as the proof in [**BB00**, Section 4] only uses that for an NIP formula $\phi(x, y)$ and an arbitrary c, $\{a_i : \phi(a_i, c)\}$ is either finite or cofinite, with size bounded by $alt(\phi)$).

The following answers Conjecture 9.1 from that paper.

PROPOSITION 4.4.2. *Let* (M, I) *be a pair as described above, obtained by naming a small, dense, complete indiscernible sequence. Then* T_P *is* NIP.

PROOF. By 1) and 3) above, all the assumptions of Corollary 4.3.5 are satisfied. $\square$

It also follows that every unstable dependent theory has a dependent expansion with a definable linear order.

Recall the following definition (one of the many equivalent) from [**Shed**].

DEFINITION 4.4.3. [**Shed**, Observations 2.1 and 2.10] T is strongly (resp. strongly$^+$) dependent if for any infinite indiscernible sequence $(\bar{a}_i)_{i\in I}$ with $\bar{a}_i \in \mathbb{M}^\omega$, I a complete linear order, and finite tuple c there is a finite $u \subset I$ such that for any two $i_1 < i_2 \in u, (i_1, i_2) \cap u = \emptyset$ the sequence $(\bar{a}_i)_{i\in(i_1,i_2)}$ is indiscernible over c (resp. $c \cup (\bar{a}_i)_{i\in(-\infty,i_1]\cup[i_2,\infty)}$).

T is dp-minimal (resp. dp^+-minimal) when for a singleton c there is such a u of size 1.

For a general NIP theory, the property described in the definition holds, but with $u \subset I$ of size $|T|$, instead of finite. We can take u to be the set of *critical points* of I defined by: $i \in I$ is critical for a formula $\phi(x; y_1, ..., y_n, c) \in L$ if there are $j_1, ..., j_n \neq i$ such that $\phi(a_i; a_{j_1}, ..., a_{j_n}, c)$ holds, but in every open interval of I containing i, we can find some i' such that $\neg\phi(a_{i'}; a_{j_1}, ..., a_{j_n}, c)$ holds. One can show (see [**Adl08**, Section 3]) that given such a formula $\phi(x; y_1, .., y_n, c)$, the set of critical points for ϕ is finite. Also T is strongly$^+$ dependent if and only if for every finite set c of parameters, the total number of critical points for formulas in $L(c)$ is finite.

Unsurprisingly dp-minimality is not preserved in general after naming an indiscernible sequence. By [**Goo10**, Lemma 3.3] in an ordered dp-minimal group, there is no infinite definable nowhere-dense subset, but of course every small indiscernible sequence is like this.

There are strongly dependent theories which are not strongly$^+$ dependent, for example p-adics ([**Shed**]). In such a theory, strong dependence is not preserved by naming an indiscernible sequence.

PROPOSITION 4.4.4. *Let* T *be not strongly* $^+$ *dependent, witnessed by a dense complete indiscernible sequence* $(\bar{a}_i)_{i\in I}$ *of finite tuples. Let* $\mathbf{P}$ *name that sequence in a big saturated model. Then* T_P *is not strongly dependent.*

PROOF. So let $(\bar{a}_i)_{i\in I}, c$ witness failure of strong$^+$ dependence. By dependence of T, let $u \subset I$ be chosen as above. Notice that for every $\phi(x; y_1, ..., y_n, c)$, the finite set of its critical points in I is L_P-definable over c (and possibly finitely many parameters, using order on I in the non-totally indiscernible case, and just the equality otherwise). As in our situation u is infinite, we get infinitely many different finite subsets of $(\bar{a}_i)_{i\in I}$ definable over c, in T_P. As $(\bar{a}_i)_{i\in I}$ is still indiscernible in T_P by Fact 4.4.1, 3), this contradicts strong dependence. $\square$

PROBLEM 4.4.5. Is strong$^+$ dependence preserved by naming an indiscernible sequence ?

4.4.2. Dense pairs and related structures. Van den Dries proves in [**vdD98**] that in a dense pair of o-minimal structures, formulas are bounded. This is generalised in [**Ber**] to lovely pairs of geometric theories of þ-rank 1. From Theorem 4.3.6, we conclude that such pairs are dependent.

This was already proved by Berenstein, Dolich and Onshuus in [**BDO11**] and generalised by Boxall in [**Box11**]. Our result generalises [**BDO11**, Theorem 2.7], since the hypothesis there (acl is a pregeometry and A is "innocuous") imply boundedness of $T_{\mathbf{P}}$. To see this take any two tuples a and b and assume that they have the same bounded types. Let $a' \in \mathbf{P}$ be such that aa' is a $\mathbf{P}$-independent tuple. Then by hypothesis, we can find b' such that $\mathrm{tp}_{L_{\mathbf{P}}^{bdd}}(bb') = \mathrm{tp}_{L_{\mathbf{P}}^{bdd}}(aa')$. Now the fact that aa' is $\mathbf{P}$-independent can be expressed by bounded formulas. In particular bb' is also $\mathbf{P}$-independent. So by innocuous, $\mathrm{tp}_{L_{\mathbf{P}}}(aa') = \mathrm{tp}_{L_{\mathbf{P}}}(bb')$ and we are done.

It is not clear to us if Boxall's hypothesis imply that formulas are bounded. (However, note that in the same paper Boxall applies his theorem to the structure of $\mathbb{R}$ with a named subgroup studied by Belegradek and Zilber, where we know that formulas are bounded.)

The paper [**BDO11**] gives other examples of theories of pairs for which formulas are bounded, including dense pairs of p-adic fields and weakly o-minimal theories, recast in the more general setting of *geometric topological structures*.

Similar theorems are proved by Günaydin and Hieronymi in [**GH11**]. Their Theorem 1.3 assumes that formulas are bounded along with other hypothesis, so is included in Theorem 4.3.6. They apply it to show that pairs of the form $(\mathbb{R}, \Gamma)$ are dependent, where $\Gamma \subset \mathbb{R}^{>0}$ is a dense subgroup with the *Mann property*. We refer the reader to [**GH11**] for more details.

In this same paper the authors also consider the case of tame pairs of o-minimal structures. This notion is defined and studied in [**vdDL95**]. Let T be an o-minimal theory. A pair (N, M) of models of T is *tame* if $M \prec N$ and for every $a \in N$ which is in the convex hull of M, there is $\mathrm{st}(a) \in M$ such that $|a - \mathrm{st}(a)| < b$ for every $b \in M^{>0}$. It is proved in [**vdDL95**] that formulas are bounded is such a pair, so again it follows from Theorem 4.3.6 that $T_{\mathbf{P}}$ is dependent. Note that Günaydin and Hieronymi prove this using their Theorem 1.4 involving quantifier elimination in a language with a new function symbol. This theorem does not seem to factorise trivially through 4.3.5. They also prove in that same paper that the pair $(\mathbb{R}, 2^{\mathbb{Z}})$ is dependent.

Let C be an elliptic curve over the reals, defined by $y^2 = x^3 + ax + b$ with $a, b \in \mathbb{Q}$, and let $\mathbf{P} \subseteq \mathbb{Q}^2$ name the set of its rational points. This theory is studied in [**GnH11**], where it is proved in particular that

FACT 4.4.6. *1)* $\mathrm{Th}(\mathbb{R}, C(\mathbb{Q}))$ *is bounded (follows from* [**GnH11**, Theorem 1.1]*)*
2) $A_{\mathrm{ind}(L_{\mathbf{P}})}$ *is* NIP *(follows from* [**GnH11**, Proposition 3.10]*)*

Applying Corollary 4.3.5 we conclude that the pair is dependent.

CHAPTER 5

Externally definable sets and dependent pairs II

This chapter is a joint work with Pierre Simon and is submitted to the Transactions of the American Mathematical Society as "Externally definable sets and dependent pairs II" [**CS12**].

We continue investigating the structure of externally definable sets in NIP theories and preservation of NIP after expanding by new predicates. Most importantly: types over finite sets are uniformly definable; over a model, a family of non-forking instances of a formula (with parameters ranging over a type-definable set) can be covered with finitely many invariant types; we give some criteria for the boundedness of an expansion by a new predicate in a distal theory; naming an arbitrary small indiscernible sequence preserves NIP, while naming a large one doesn't; there are models of NIP theories over which all 1-types are definable, but not all n-types.

5.1. Introduction

A characteristic property of stable theories is the definability of types. Equivalently, every externally definable set is internally definable. In unstable theories this is no longer true. However, as was observed early on by Shelah (e.g. [**She09**]), the class of externally definable sets in NIP theories satisfies some nice properties resembling those in the stable case (e.g. it is closed under projection). In this chapter we continue the investigation of externally definable sets in NIP theories started in Chapter 4.

As it was established there, every externally definable set $X = \phi(x, b) \cap A$ has an *honest definition*, which can be seen as the existence of a uniform family of internally definable subsets approximating X. Formally, there is $\theta(x, z)$ such that for any *finite* $A_0 \subseteq X$ there is some $c \in A$ satisfying $A_0 \subseteq \theta(A, c) \subseteq A$. The first section of this paper is devoted to establishing the existence of *uniform* honest definitions. By uniform we mean that $\theta(x, z)$ can be chosen depending just on $\phi(x, y)$ and not on A or b. We achieve this assuming that the whole theory is NIP, combining careful use of compactness with a strong combinatorial result of Alon-Kleitman [**AK92**] and Matousek [**Mat04**]: the (p, k)-theorem. As a consequence we conclude that in an NIP theory types over finite sets are uniformly definable (UDTFS). This confirms a conjecture of Laskowski.

In the next section we consider an implication of the (p, k)-theorem for forking in NIP theories. Combined with the results on forking and dividing in NIP theories from Chapter 1, we deduce the following: working over a model M, let $\{\phi(x, a) : a \models q(y)\}$ be a family of non-forking instances of $\phi(x, y)$, where the parameter a ranges over the set of solutions of a partial type q. Then there are finitely many global M-invariant types such that each $\phi(x, a)$ from the family belongs to one of them.

In Section 3 we return to the question of naming subsets with a new predicate. In Chapter 4 we gave a general condition for the expansion to be NIP: it is enough that the theory of the pair is *bounded*, i.e. eliminates quantifiers down to the predicate, and the induced structure on the predicate is NIP. Here, we try to complement the picture by providing a general sufficient condition for the boundedness of the pair. In the stable case the situation is quite neatly resolved using the notion of nfcp. However nfcp implies stability, so one has to come up with some generalization of it that is useful in unstable NIP theories. Towards this purpose we introduce *dnfcp*, i.e. no finite cover property for definable sets of parameters, and its relative version with respect to a set. We also introduce dnfcp' – a weakening of dnfcp with separated variables. Using it, we succeed in the distal, stably embedded, case: if one names a subset of M which is small, uniformly stably embedded and the induced structure satisfies dnfcp', then the pair is bounded.

In section 4 we look at the special case of naming an indiscernible sequence. On the one hand, we complement the result in Chapter 4 by showing that naming a small indiscernible sequence of *arbitrary* order type is bounded and preserves NIP. On the other hand, naming a large indiscernible sequence does not.

In the last section we consider models over which all types are definable. While in general even o-minimal theories may not have such models, many interesting NIP theories do (RCF, $ACVF$, $\mathrm{Th}(\mathbb{Q}_p)$, Presburger arithmetic...). In practice, it is often much easier to check definability of 1 types, as opposed to n-types, so it is natural to ask whether one implies the other. Unfortunately, this is not true – we give an NIP counter-example. Can anything be said on the positive side? Pillay [**Pil11**] had established: let M be NIP, $A \subseteq M$ be definable with rosy induced structure. Then if it is 1-stably embedded, it is stably embedded. We observe that Pillay's results holds when the definable set A is replaced with a model, assuming that it is *uniformly* 1-stably embedded. This provides a generalization of the classical theorem of Marker and Steinhorn about definability of types over models in o-minimal theories. We also remark that in NIP theories, there are arbitrary large models with "few" types over them (i.e. such that $|S(M)| \leq |M|^{|T|}$).

5.2. Preliminaries

5.2.1. VC dimension, co-dimension and density. Let $\mathcal{F}$ be a family of subsets of some set X. Given $A \subseteq X$, we say that it is *shattered* by $\mathcal{F}$ if for every $A' \subseteq A$ there is some $S \in \mathcal{F}$ such that $A \cap S = A'$.

A family $\mathcal{F}$ is said to have finite VC-*dimension* if there is some $n \in \omega$ such that no subset of X of size n can be shattered by $\mathcal{F}$. In this case we let $\mathrm{VC}(\mathcal{F})$ be the largest integer n such that some subset of X of size n is shattered by it.

The VC co-dimension of $\mathcal{F}$ is the largest integer n for which there are $S_1, ..., S_n \in \mathcal{F}$ such that for any $u \subseteq n$ there is $b_u \in X$ satisfying $b_u \in S_i \Leftrightarrow i \in u$. It is well known that $\mathrm{coVC}(\mathcal{F}) < 2^{\mathrm{VC}(\mathcal{F})+1}$.

5.2.2. NIP and alternation. We are working in a monster model $\mathbb{M}$ of a complete first-order theory T.

Recall that a formula $\phi(x, y)$ is NIP if there are no $(a_t)_{t\in\omega}$ and $(b_s)_{s\subseteq\omega}$ such that $\phi(a_t, b_s) \Leftrightarrow t \in s$. Equivalently, for any indiscernible sequence $(a_t)_{t\in I}$ and b, there can be only finitely many $t_0 < ... < t_n \in I$ such that $\phi(a_{t_i}, b) \Leftrightarrow i$ is even.

The following is a very important refinement of this statement, see e.g. [**Adl08**, Theorem 14].

Let $(a_t)_{t\in I}$ be an indiscernible sequence and let E be a convex equivalence relation on I. If $\bar{t} = (t_i)_{i<\kappa}$ and $\bar{s} = (s_i)_{i<\kappa}$ are tuples of elements from I, we will write $\bar{t} \sim_E \bar{s}$ if $\bar{t}$ and $\bar{s}$ have the same quantifier-free order type and $t_i E s_i$ for all $i < \kappa$.

FACT 5.2.1. *Let $(a_t)_{t\in I}$ be an indiscernible sequence and let b be any finite tuple. Let $\phi(x_0, ..., x_n; y)$ be* NIP*. Then there is a convex equivalence relation E on* I *with finitely many classes such that for any $(s_i)_{i\leq n} \sim_E (t_i)_{i\leq n}$ from* I *we have* $\phi(a_{s_0}, ..., a_{s_n}; b) \leftrightarrow \phi(a_{t_0}, ..., a_{t_n}; b)$.

REMARK 5.2.2. In particular, if I is a complete linear order and $\phi(x_0, ..., x_n; y)$ is NIP, then all ϕ-types over I are definable, possibly after adding finitely many elements extending I on both sides. Why? If I is totally indiscernible, then all ϕ-types over it are in fact definable using just equality. If it is not, then there is some formula giving the order on the sequence, and by Fact 5.2.1, ϕ-types over I are definable using this order (see Chapter 4, Section 3.1).

In a natural way we define the VC dimension of a formula in a model M as $\mathrm{VC}(\phi(x, y)) = \mathrm{VC}\{\phi(M, a) : a \in M^n\}$. Notice that this value does not depend on the model, so we'll talk about VC dimension of ϕ in T. Similarly we define VC co-dimension.

It was observed early on by Laskowski that $\phi(x, y)$ is NIP if and only if it has finite VC dimension, if and only if it has finite VC co-dimension [**Las92**]. We also recall an early result of Shelah about counting types over finite sets.

FACT 5.2.3. *[Shelah/Sauer] The following are equivalent:*

(1) $\phi(x, y)$ *is* NIP.

(2) There are $k, d \in \omega$ *such that for all finite* A, $|S_\phi(A)| \leq d \cdot |A|^k$.

Then one defines the VC *density* of ϕ to be the infimum of all reals r such that for some d, $|S_\phi(A)| \leq d \cdot |A|^r$ for all finite A.

5.2.3. Invariant types. Let $p(x)$ be a global type over a monster model $\mathbb{M}$, invariant over some small submodel M. Then one naturally defines $p^{(\omega)}(x) \in S_\omega(\mathbb{M})$, the type of a Morley sequence in it (see [**HP11**, Section 2] for details).

FACT 5.2.4. *Let T be* NIP*. Assume that $p(x), q(x)$ are global types invariant over a small model M. If $p^{(\omega)}|_M = q^{(\omega)}|_M$, then $p = q$.*

We will use the following lemma, see [**Sim11a**, Lemma 2.18] for a proof.

LEMMA 5.2.5. *Assume that T is NIP. Let a be given and $q(x) \in S(A')$ be invariant over $C \subset A'$. Then there is D of size $\leq |C| + |x| + |a| + |T|$ such that $C \subseteq D \subseteq A'$ and for any $b, b' \in A'$ realizing $q(x)|D$, $tp(ab/D) = tp(ab'/D)$.*

5.2.4. (p,k)-theorem. We will need the following theorem from [**Mat04**].

FACT 5.2.6. *[(p, k)-theorem] Let $\mathcal{F}$ be a family of subsets of some set X. Assume that the VC co-dimension of $\mathcal{F}$ is bounded by k. Then for every $p \geq k$, there is an integer N such that: for every finite subfamily $\mathcal{G} \subset \mathcal{F}$, if $\mathcal{G}$ has the (p, k)-property meaning that among any p subsets of $\mathcal{G}$ some k intersect, then there is an N-point set intersecting all members of $\mathcal{G}$.*

REMARK 5.2.7. Although the theorem is stated this way in [**Mat04**], N depends only on p and k and not on the family $\mathcal{F}$. To see this, assume that for every N, we had a family $\mathcal{F}_N$ on some set X_N of VC co-dimension bounded by k and for which the (p, k) theorem fails for this N. Then consider X to be the disjoint union of the sets X_N and $\mathcal{F}$ the union of the families $\mathcal{F}_N$. Then clearly $\mathcal{F}$ has VC co-dimension bounded by k and the theorem fails for it. Also, it follows from the proof.

5.2.5. Expansions and stable embeddedness. Let A be a subset of $M \models T$ and let $L_{\mathbf{P}} = L \cup \{\mathbf{P}(x)\}$, where $\mathbf{P}(x)$ is a new unary predicate. We define the structure (M, A) as the expansion of M to an $L_{\mathbf{P}}$-structure where $\mathbf{P}(M) = A$. Recall that $\mathrm{Th}(M, A)$ is $\mathbf{P}$-*bounded* if every $L_{\mathbf{P}}$ formula is equivalent to one of the form

$$Q_1 y_1 \in \mathbf{P} \dots Q_n y_n \in \mathbf{P}\, \phi(x, \bar{y}),$$

where $Q_i \in \{\exists, \forall\}$ and ϕ is an L-formula. We may just say bounded when it creates no confusion.

Given $A \subseteq M \models T$ and a set of formulas F, possibly with parameters, we let $A_{\mathrm{ind}(F)}$ be the structure in the language $L(T) \cup \left\{D_{\phi(x)}(x) : \phi(x) \in F\right\}$ with $D_\phi(x)$ interpreted as the set $\phi(A)$. When $F = L$, we may omit it. Given $A \subseteq M$ and a tuple $b \in M$, let $A_{[b]}$ be shorthand for $A_{\mathrm{ind}(F)}$ with $F = \{\phi(x, b) : \phi \in L\}$.

A set $A \subset M$ is called *small* if for every finite $b \in M$, every finitary type over Ab is realized in M. Finally, a set $A \subset M$ is *stably embedded* if for every $\phi(x, y)$ and $c \in M$ there is $\psi(x, z)$ and $b \in A$ such that $\phi(A, c) = \psi(A, b)$. We say that it is *uniformly stably embedded* if ψ can be chosen depending just on ϕ, and not on c. A definable set is stably embedded if and only if it is uniformly stably embedded, by compactness.

5.3. Uniform honest definitions

5.3.1. Uniform honest definitions.

We recall the following result about existence of honest definitions for externally definable sets in NIP theories established in Chapter 4.

FACT 5.3.1. *[Honest definition] Let T be* NIP *and let M be a model of T and $A \subseteq M$ any subset. Let $\phi(x, a)$ have parameters in M. Then there is an elementary extension (M', A') of the pair (M, A) and a formula $\theta(x, b) \in L(A')$ such that $\phi(A, a) = \theta(A, b)$ and $\theta(A', b) \subseteq \phi(A', a)$.*

It can be reformulated as existence of a uniform family of internally definable subsets approximating our externally definable set.

COROLLARY 5.3.2. *Let M, A and $\phi(x, a)$ be as above. Then there is $\theta(x, t)$ such that for any finite subset $A_0 \subseteq \phi(A, a)$, there is $b \in A$ such that $A_0 \subseteq \theta(A, b) \subseteq \phi(A, a)$.*

PROOF. Immediately follows from Fact 5.3.1 because the extension $(M, A) \prec (M', A')$ is elementary and the condition on b can be stated as a single formula in the theory of the pair. Note that conversely this implies Fact 5.3.1 by compactness. □

It is natural to ask whether θ can be chosen in a uniform way depending just on ϕ, and not on A and a (Question 1.4 from Chapter 4). The aim of this section is to answer this question positively.

First, compactness gives a weak uniformity statement.

PROPOSITION 5.3.3. *Fix a formula* $\phi(x, y)$. *For every formula* $\theta(x, t)$ *(in the same variable* x*, but* t *may vary), fix an integer* n_θ. *Then there are finitely many formulas* $\theta_1(x, t_1), ..., \theta_k(x, t_k)$ *such that the following holds:*

For every $M \models T$ *and* $A \subset M$*, for every* $a \in M$ *there is* $i \leq k$ *such that for every subset* $A_0 \subseteq \phi(A, a)$ *of size at most* n_{θ_i}*, there is* $b \in A$ *satisfying* $A_0 \subseteq \theta_i(A, b) \subseteq \phi(A, a)$.

PROOF. Consider the theory T' in the language $L' = L \cup \{P(x), c\}$ saying that if $(M, A) \models T'$ (where $A = P(M)$), then $M \models T$ and for every $\theta \in L$, there is a subset A_0 of $\phi(A, c)$ of size at most n_θ for which there does not exist a $b \in A$ satisfying $A_0 \subseteq \theta(A, b) \subseteq \phi(A, a)$. By Corollary 5.3.2, T' is inconsistent. By compactness, we find a finite set of formulas as required. □

Combining this with the (p, k)-theorem we get the full result.

THEOREM 5.3.4. *Let* T *be* NIP *and* $\phi(x, y)$ *given. Then there is a formula* $\chi(x, t)$ *such that for every set* A *of size* ≥ 2*, tuple* a *and finite subset* $A_0 \subseteq A$*, there is* $b \in A$ *satisfying:*

(1) $\phi(A_0, a) = \chi(A_0, b)$,
(2) $\chi(A, b) \subseteq \phi(A, a)$.

PROOF. By the usual coding tricks, using $|A| \geq 2$, it is enough to find a finite set of formulas $\{\chi_i\}_{i<n}$ such that for every finite set, one of them works.

For every formula $\theta(x, t)$, let n_θ be its VC dimension. Proposition 5.3.3 gives us a finite set $\{\theta_1, ..., \theta_k\}$ of formulas. Using the previous remark, we may assume $k = 1$ and write $\theta(x, t) = \theta_1(x, t)$. Let N be given by Fact 5.2.6 taking $p = k = n_\theta$ (using Remark 5.2.7).

Let $A_0 \subseteq A \subseteq M \models T$ and $a \in M$ be given, A_0 is finite. Set $B \subseteq A^{|t|}$ be the set of tuples $b \in A^{|t|}$ such that $\theta(A, b) \subseteq \phi(A, a)$. Consider the family $\mathcal{F} = \{\theta(d, B) : d \in \phi(A_0, a)\}$ of subsets of B. This is a finite family, and by hypothesis the intersection of any k members of it is non-empty. Therefore the (p, k)-theorem applies and gives us N tuples $b_1, ..., b_N \in B$ such that $\{b_1, ..., b_N\}$ intersects any set in $\mathcal{F}$. Unwinding, we see that $\phi(A_0, a) = \bigvee_{i \leq N} \theta(A_0, b_i)$ and $\bigvee_{i \leq N} \theta(A, b_i) \subseteq \phi(A, a)$. So taking $\chi(x, t_1...t_N) = \bigvee_{i \leq N} \theta(x, t_i)$ works. □

5.3.2. UDTFS.

Recall the following classical fact characterizing stability of a formula.

FACT 5.3.5. *The following are equivalent:*

(1) $\phi(x, y)$ *is stable.*
(2) There is $\theta(x, z)$ *such that for any* A *and* a*, there is* $b \in A$ *satisfying* $\phi(A, a) = \theta(A, b)$.
(3) There are $m, n \in \omega$ *such that* $|S_\phi(A)| \leq m \cdot |A|^n$ *for any set* A.

DEFINITION 5.3.6. We say that $\phi(x, y)$ has UDTFS (Uniform Definability of Types over Finite Sets) if there is $\theta(x, z)$ such that for every finite A and a there is $b \in A$ such that $\phi(A, a) = \theta(A, b)$. We say that T satisfies UDTFS if every formula does.

REMARK 5.3.7. If $\phi(x, y)$ has UDTFS, then it is NIP (by Fact 5.2.3).

Comparing Fact 5.3.5 and Fact 5.2.3 naturally leads to the following conjecture of Laskowski: assume that $\phi(x, y)$ is NIP, then it satisfies UDTFS. It was proved for weakly o-minimal theories in [**JL10**] and for dp-minimal theories in [**Gui10**]. An immediate corollary of Theorem 5.3.4 is that if the whole T is NIP, then every formula satisfies UDTFS.

THEOREM 5.3.8. *Let* T *be* NIP. *Then it satisfies* UDTFS.

PROOF. Follows from Theorem 5.3.4 taking $A_0 = A$. □

REMARK 5.3.9. This does not fully answer the original question as our argument is using more than just the dependence of $\phi(x, y)$ to conclude UDTFS for $\phi(x, y)$. Looking more closely at the proof of Fact 5.3.1, we can say exactly how much NIP is needed. Depending on the VC dimension of ϕ, there is a finite set Δ_ϕ of formulas for which we have to require NIP consisting of formulas of the form $\psi(x_1, ..., x_k) = \exists y \bigwedge_i \phi(x_i, y)^{\epsilon(i)}$, where k is at most $\mathrm{VC}(\phi) + 1$.

UDTFS implies that in the statement of the (p, k)-theorem for sets inside an NIP theory consistent pieces are uniformly definable.

COROLLARY 5.3.10. *Let* T *be* NIP. *For any* $\phi(x, y)$ *there is* $\psi(y, z)$ *and* $k \leq N < \omega$ *such that: for every finite* A*, if* $\{\phi(x, a) : a \in A\}$ *is* k*-consistent, then there are* $c_0, ..., c_{N-1} \in A$ *such that* $A = \bigcup_{i<N} \psi(A, c_i)$ *and* $\{\phi(x, a) : a \in \psi(A, c_i)\}$ *is consistent for every* $i < N$.

5.3.3. Strong honest definitions and distal theories.

DEFINITION 5.3.11. A theory T is called *distal* if it satisfies the following property: Let $I + b + J$ be an indiscernible sequence, with I and J infinite. For arbitrary A, if $I + J$ is indiscernible over A, then $I + b + J$ is indiscernible over A.

The class of distal theories was introduced in [**Sim11a**], in order to capture the class of dependent theories which do not contain any "stable part". Examples of distal theories include ordered dp-minimal theories and $\mathbb{Q}_p$.

We will say that $p(x), q(y) \in S(A)$ are *orthogonal* if $p(x) \cup q(y)$ determines a complete type over A.

PROPOSITION 5.3.12. *[Strong honest definition] Let* T *be distal,* $A \subset M$ *and* $a \in M$ *arbitrary. Let* $(M', A') \succ (M, A)$ *be* $|M|^+$*-saturated. Then for any* $\phi(x, y)$ *there are* $\theta(x, z)$ *and* $b \in A'$ *such that* $\models \theta(a, b)$ *and* $\theta(x, b) \vdash tp_\phi(a/A)$.

PROOF. Let $(M', A') \succ (M, A)$ be $\kappa = |M|^+$-saturated, we show that $p = \mathrm{tp}_L(a/A')$ is orthogonal to any L-type $q \in S(A')$ finitely satisfiable in a subset of size $< \kappa$. So take such a q, finitely satisfiable in $C \subset A'$. By Lemma 5.2.5, there is some D of size $< \kappa$, $C \subseteq D \subset A'$, such that for any two realizations $I, I' \subset A'$ of $q^{(\omega)}|D$, we have $\mathrm{tp}_L(aI/C) = \mathrm{tp}_L(aI'/C)$. Take some $I \models q^{(\omega)}|D$ in A' (exists by saturation of (M', A') and finite satisfiability) and $J \models q^{(\omega)}|\mathbb{M}$.

CLAIM. $I + J$ is indiscernible over aC.

PROOF. As $q^{(\omega)}|\mathbb{M}$ is finitely satisfiable in C, by compactness and saturation of (M', A') there is $J' \models q^{(\omega)}|aDI$ in A'.

If $I + J$ is not aC-indiscernible, then $I' + J'$ is not aC-indiscernible for some finite $I' \subset I$. As both $I' + J'$ and J' realize $q^{(\omega)}|D$ in A', it follows that J' is not indiscernible over aC – a contradiction. □

Now, if $b \in \mathbb{M}$ is any realization of q, then $I + b + J$ is C-indiscernible. By the claim and distality, $I+b$ is aC-indiscernible. It follows that $\mathrm{tp}(b/Ca)$ is determined by $\mathrm{tp}(a/A')$. As we can always take a bigger C, $\mathrm{tp}(b/A'a)$ is determined, so p is orthogonal to q as required.

Consider the set $S^{fs}(A', A)$ of L-types over A' finitely satisfiable in A. It is a closed subset of $S_L(A')$. By compactness, there is $\theta(x, b) \in p(x)$ such that for any $a' \models \theta(x, b)$ and any $c \models q(y) \in S^{fs}(A', A)$, $\models \phi(a, c) \leftrightarrow \phi(a', c)$. This applies, in particular, to every $c \in A$ and thus $\theta(x, b) \vdash tp_\phi(a/A)$. $\square$

REMARK 5.3.13. In fact, the argument is only using that every indiscernible sequence in A' is distal.

THEOREM 5.3.14. *The following are equivalent:*

(1) T *is distal.*
(2) For any $\phi(x, y)$ *there is* $\theta(x, z)$ *such that: for any* A, a *and a finite* $C \subseteq A$, *there is* $b \in A$ *such that* $\models \theta(a, b)$ *and* $\theta(x, b) \vdash tp_\phi(a/C)$

PROOF. (1)⇒(2): It follows from Proposition 5.3.12 that we have: For any finite $C \subset A$, there is $b \in A$ such that $\models \theta(a, b)$ and $\theta(x, b) \vdash \mathrm{tp}_\phi(a/C)$. Similarly to the proof of Theorem 5.3.4, we can choose θ depending just on ϕ.

(2)⇒(1): Let $I + d + J$ be an indiscernible sequence, with I and J infinite. Assume that $I+J$ is indiscernible over A, and we show that $I+d+J$ is indiscernible over A.

Let a be a finite tuple from A and $\phi(x, y_0...y_n...y_{2n}) \in L$, and let $\theta(x, z)$ be as given for ϕ by (2). Without loss of generality $\models \phi(a, b_0...b_n...b_{2n})$ holds for all $b_0 < ... < b_{2n} \in I + J$. Let $I_0 \subset I$ be finite. Then for some $b \subset I_0$, $\models \theta(a, b)$ and $\theta(a, b) \vdash \mathrm{tp}_\phi(a/I_0)$. If we take I_0 to be large enough compared to $|z|$, then there will be some $b_0 < ... < b_n < ... < b_{2n}$ such that $\{b_i\}_{i \leq 2n} \cap b = \emptyset$. As we have $\models \forall x\, \theta(x, b) \to \phi(x, b_0...b_n...b_{2n})$, by indiscernibility of $I + d + J$ for any $\{b'_i\}_{i \leq 2n, i \neq n}$ in $I + J$ there is a corresponding b' in $I + J$ such that $\models \forall x\, \theta(x, b') \to \phi(x, b'_0...d...b'_{2n})$. As $\models \theta(a, b')$ holds by indiscernibility of $I + J$ over a, it follows that $\models \phi(a, b_0...d...b_{2n})$ holds – as wanted. $\square$

REMARK 5.3.15. It follows from this theorem that types over finite sets in distal theories admit uniform definitions of a special "coherent" form as considered in [**ADH⁺11**, Section 7.1].

5.4. (p,k)-theorem and forking

We recall some properties of dividing and forking in NIP theories.

FACT 5.4.1. *Let* T *be* NIP.

(1) If $M \models T$, *then* $\phi(x, a)$ *divides over* M ⇔ *it forks over* M ⇔ *the set* $\{\phi(x, a') : a \equiv_M a' \in \mathbb{M}\}$ *is inconsistent.*
(2) For any $\phi(x, y)$, *the set* $\{a : \phi(x, a)$ *forks over* $M\}$ *is type-definable over* M.
(3) If $(a_i)_{i<\omega}$ *is indiscernible over* M *and* $\phi(x, a_0)$ *does not fork over* M, *then* $\{\phi(x, a_i)\}_{i<\omega}$ *does not fork over* M.
(4) $\phi(x, a)$ *does not fork over* M ⇔ *there is a global* M*-invariant type* p *with* $\phi(x, a) \in p$.

Proof. (1) and (2) are by Chapter 1, Theorem 1.1 and Chapter 1, Remark 3.33, (4) is from [**Adl08**]. Finally, (3) is well-known and follows from (4). Indeed, if $\phi(x, a_0)$ does not fork over M then it is contained in some global type $p(x)$ invariant over M. But then by invariance $\{\phi(x, a_i)\}_{i<\omega} \subseteq p(x)$, thus does not fork over M. $\square$

Definition 5.4.2. Let M be a small model. We say that $(\phi(x, y), q(y))$ (where $\phi \in L(M)$ and q is a partial type over M) is a *non-forking family over* M if for every $a \models q(y)$, the formula $\phi(x, a)$ does not fork over M.

Notice that by Fact 5.4.1(2), if $(\phi(x, y), q(y))$ is a non-forking family, then there is some formula $\psi(y) \in q$ such that $(\phi(x, y), \psi(y))$ is a non-forking family.

Proposition 5.4.3. *Let* $(\phi(x; y), q(y))$ *be a non-forking family over* M*, then there are finitely many global* M*-invariant types* $p_1, ..., p_{n-1}$ *such that for every* $a \models q(y)$*, there is* $i < n$ *with* $p_i \vdash \phi(x; a)$*.*

Proof. Let $M \prec N$ be such that N is $|M|^+$-saturated.

Consider the set $X = \{x \in \mathbb{M} : \text{tp}(x/N) \text{ is } M\text{-invariant}\}$, it is type-definable over N by $\{\phi(x, a) \leftrightarrow \phi(x, b) : a, b \in N,\ a \equiv_M b,\ \phi \in L\}$. Let $\mathcal{F} \stackrel{\text{def}}{=} \{Y \subseteq X : Y = X \cap \phi(x, a),\ a \in q(N)\}$, and notice that the dual VC-dimension of $\mathcal{F}$ is finite, say k (as $\phi(x, y)$ is NIP).

Assume that for any $p < \omega$, $\mathcal{F}$ does not satisfy the (p, k)-property. As by Fact 5.4.1(2) the set $\{(a_0...a_{k-1}) : \phi(x, a_0...a_{k-1}) \text{ forks over } M\}$ is type-definable, by Ramsey, compactness and Fact 5.4.1(4) we can find an M-indiscernible sequence $(a_i)_{i<\omega} \subseteq q(N)$ such that $\bigwedge_{i<k} \phi(x, a_i)$ forks over M, contradicting Fact 5.4.1(3) and the assumption on q.

Thus $\mathcal{F}$ satisfies the (p, k)-property for some p. Let n be as given by Fact 5.2.6 and define

$$Q(x_0, ..., x_{n-1}) \stackrel{\text{def}}{=} \{x_i \in X\}_{i<n} \cup \left\{ \bigvee_{i<n} \phi(x_i, a) : a \in q(N) \right\}.$$

As every finite part of Q is consistent by Fact 5.2.6, there are $b_0...b_{n-1}$ realizing it, take $p_i \stackrel{\text{def}}{=} \text{tp}(b_i/N)$. $\square$

Remark 5.4.4. If $q(x)$ is a complete type then this holds with $n = 1$, just by taking some M-invariant $p_0(x)$ containing $\phi(x, a)$.

However, we cannot hope to replace invariant ϕ-types by definable ϕ-types in the proposition.

Example 5.4.5. Let T be the theory of a complete discrete binary tree with a valuation map. Let M_0 be the prime model, and take c an element of valuation larger than $\Gamma(M_0)$. Let d be the smallest element in M_0. Let $\phi(x; y, z)$ say "if $z = d$, then $\text{val}(x) > \text{val}(y)$, if $z \neq d$, then $x > y$" (where $>$ is the order in the tree). Let $\psi(y, z) =$ "$z = d$". Then (ϕ, ψ) is a non-forking family over M, however there is no definable ϕ-type consistent with $\phi(x; c, d)$.

Remark 5.4.6. In [**CS11**] it is proved that if T is a VC-minimal theory with unpacking and $M \models T$, then $\phi(x, a)$ does not fork over M if and only if there is a global M-definable type $p(x)$ such that $\phi(x, a) \in p$. The previous example shows that the same result cannot hold in a general NIP theory.

PROBLEM 5.4.7. Assume $\phi(x, a)$ does not fork over M. Is there a formula $\psi(y) \in \mathrm{tp}(a/M)$ such that $\{\phi(x, a) : \models \psi(a)\}$ is consistent (and thus does not fork over M)?

5.5. Sufficient conditions for boundedness of T_P

In Chapter 4 we have demonstrated the following result.

FACT 5.5.1. (1) *Let* (M, A) *be bounded. If* M *is* NIP *and* A_{ind} *is* NIP, *then* (M, A) *is* NIP.
(2) *Let* (M, A) *be bounded and* $A \prec M$. *If* M *is* NIP *then* (M, A) *is* NIP.

However, a general sufficient condition for the boundedness of an expansion by a predicate for NIP theories is missing. In the stable case, a satisfactory answer is given in [**CZ01**]. Recall:

DEFINITION 5.5.2. (1) T satisfies nfcp (no finite cover property) if for any $\phi(x, y)$ there is $k < \omega$ such that for any A, if $\{\phi(x, a)\}_{a\in A}$ is k-consistent, then it is consistent.
(2) We say that $M \models T$ satisfies nfcp over $A \subset M$ if for any $\phi(x, y, z)$ there is $k < \omega$ such that for any $A' \subseteq A$ and $b \in M$, if $\{\phi(x, a, b)\}_{a\in A'}$ is k-consistent, then it is consistent.

And then one has:

FACT 5.5.3. *Let* T *be stable.*
(1) [**CZ01**, Proposition 2.1] *Assume that* $A \subset M \models T$ *is small and* M *has nfcp over* A. *Then* (M, A) *is bounded.*
(2) [**CZ01**, Proposition 4.6] *In fact, "nfcp over* A*" can be relaxed to "*A_{ind} *is nfcp".*

In this section we present results towards a possible generalization for unstable NIP theories.

5.5.1. Dnfcp (nfcp for definable sets of parameters).

DEFINITION 5.5.4. We say that M satisfies dnfcp over $A \subseteq M$ if for any $\phi(x, y, z)$ there is $k \in \omega$ such that: for any $b \in M$, if $\{\phi(x, a, b) : a \in A\}$ is k-consistent, then it is consistent.

We remark that dnfcp over A is an elementary property of the pair (M, A).

LEMMA 5.5.5. (1) *nfcp over* $A \Rightarrow$ dnfcp *over* A.
(2) *If* T *is stable and* $M \models T$, *then nfcp* $\Leftrightarrow$ *nfcp over* $M \Leftrightarrow$ dnfcp *over* M.

PROOF. (1) Clear.
(2) Assume that T is stable. Then nfcp and nfcp over M are easily seen to be equivalent. Assume that T has fcp, then by Shelah's nfcp theorem [**She90**, Theorem 4.4] there is a formula $E(x, y, z)$ such that $E(x, y, c)$ is an equivalence relation for every c and for each $k \in \omega$ there is c_k such that $E(x, y, c_k)$ has more than k, but finitely many equivalence classes. Taking $\phi(x, y, z) = \neg E(x, y, z)$ and M big enough we see that $\{\phi(x, a, c_k) : a \in M\}$ is k-consistent, but inconsistent. $\square$

LEMMA 5.5.6. *If every formula of the form* $\phi(x, y, z)$ *with* $|x| = 1$ *is* dnfcp *over* A, *then* T *is* dnfcp *over* A.

PROOF. Assume we have proved that all formulas with $|x| \leq m$ are dnfcp, and we prove it for $|x| = m+1$. So assume that for every $n < \omega$ we have some $c_n \in M$ such that $\{\phi(x_0...x_m, a, c_n)\}_{a \in A}$ is n-consistent, but not consistent. Let $\psi(x_1...x_m, y_i...y_l, z) = \exists x_0 \bigwedge_{i \leq l} \phi(x_0...x_m, y_i, z)$, of course still $\{\psi(\bar{x}, \bar{a}, c_n)\}_{\bar{a} \in A}$ is $\lfloor n/l \rfloor$-consistent, so consistent for n large enough by the inductive assumption. Let $b_1...b_m$ realize it. Then consider $\Gamma = \{\theta(x_0, a, c_n b_1...b_m)\}_{a \in A}$ where $\theta(x_0, a, c_n b_1...b_m) = \phi(x_0 b_1...b_m, a, c_n)$. It is l-consistent. Again by the inductive assumption, if l was chosen large enough, there is some b_0 realizing Γ, but then $b_0...b_m \models \{\phi(x_0...x_m, a, c_n)\}_{a \in A}$ - a contradiction. □

EXAMPLE 5.5.7. DLO has dnfcp over models.

The following criterion for boundedness follows from the proof of [**CZ01**].

THEOREM 5.5.8. *Let $A \subset M$ be small and uniformly stably embedded. Assume that M has* dnfcp *over A. Then (M, A) is bounded.*

The problem with dnfcp is that it does not seem possible to conclude dnfcp over A from properties of the induced structure on A. To remedy this, we introduce a weaker variant with separated variables.

DEFINITION 5.5.9. We say that M satisfies dnfcp$'$ over $A \subseteq M$ if for any $\phi(x, y)$ and $\psi(y, z)$, there is $k < \omega$ such that for any $b \in M$, if $\{\phi(x, a) : a \in \psi(A, b)\}$ is k-consistent, then it is consistent. We say that T has dnfcp$'$ if for any $M \prec N$, N has dnfcp$'$ over M.

REMARK 5.5.10. Let (M, A) be a pair, and assume that A is small and A_{ind} is saturated. Then if formulas are bounded, M has dnfcp$'$ over A.

PROOF. By assumption $\exists y \forall a \in \mathbf{P}, \psi(a; z) \to \phi(a; y)$ is equivalent to a bounded formula $\theta(z)$, for any ϕ and ψ. If dnfcp$'$ does not hold, then there is a consistent bounded type satisfying $\neg\theta(z)$ and for all n, $\forall a_1, ..., a_n \in \mathbf{P}\, \exists y, \bigwedge \psi(a_i; z) \to \phi(a_i; y)$. As A_{ind} is saturated, it is resplendent, and we can find a type over A which satisfies this bounded type. By smallness of A in M, this type is realized by some $c \in M$. Then again by smallness, there is $b \in M$ such that $\psi(a; c) \to \phi(a; b)$ for all $a \in A$. This contradicts the hypothesis on θ. □

We can now prove some preservation result.

LEMMA 5.5.11. *Let T be NIP, $A \subseteq M \models T$ and assume that $\mathrm{Th}\left(A_{\text{ind}(L_P)}\right)$ has* dnfcp$'$. *Then M has* dnfcp$'$ *over A.*

PROOF. Let $\phi(x, y)$ and $\psi(y, b)$ be given. Let $\theta_\phi(y, s)$ be a uniform honest definition for ϕ and $\theta_\psi(y, t)$ a uniform honest definition for ψ (by Theorem 5.3.4). Let $(M', A') \succ (M, A)$ be a sufficiently saturated elementary extension, then naturally $A'_{ind(L_P)} \succ A_{ind(L_P)}$. There is $c_\psi \in A'$ such that $\psi(A, b) = \theta_\psi(A, c_\psi)$.

Let $\chi(s)$ be the formula $\exists d \forall y \in P \theta_\phi(y, s) \to \phi(d, y)$ and let $k \in \omega$ be as given for $\theta_\phi(y, s) \wedge \chi(s)$, $\theta_\psi(y, t)$ by dnfcp$'$ of $A_{ind(L_P)}$ for it. Assume that $\{\phi(x, a) : a \in \psi(A, b)\}$ is k-consistent, then $\{\theta_\phi(a, s) \wedge \chi(s) : a \in \theta_\psi(A, c_\psi)\}$ is k-consistent (let $d \models \{\phi(x, a_i)\}_{i<k}$, and choose $c_\phi \in A$ such that $\{a_i\}_{i<k} \subseteq \theta_\phi(A, c_\phi) \subseteq \phi(d, A)$). As $A_{ind(L_P)}$ is dnfcp$'$, we conclude that it is consistent. In particular, for any $n \in \omega$ and $a_0, ..., a_n \in \theta_\psi(A, c_\psi) = \psi(A, b)$, there is $c_\phi \in A$ such that $\bigwedge_{i<n} \theta_\phi(a_i, c_\phi) \wedge \chi(c_\phi)$, thus unwinding there is some $d \models \{\phi(x, a_i)\}_{i<n}$. □

5.5.2. Boundedness of the pair for distal theories. We now aim at giving an analog of Theorem 5.5.3 for distal theories and stably embedded predicates.

First, we improve Lemma 5.5.11.

LEMMA 5.5.12. *Let* T *be distal,* $A \subseteq M \models T$ *and assume that* $\mathrm{Th}\left(A_{\mathrm{ind}(L)}\right)$ *has* dnfcp$'$. *Then* M *has* dnfcp$'$ *over* A.

PROOF. Follow the proof of Lemma 5.5.11, except that we define $\chi(s)$ as $\exists x \forall y \theta_\phi(y,s) \to \phi(d,y)$, which we can by strong honest definitions (Lemma 5.3.14). □

Let A_0 be a small subset of M_0, and take a $|T|^+$-saturated $(M,A) \succ (M_0,A_0)$.

LEMMA 5.5.13. *Assume that* T *is distal and* M *has* dnfcp$'$ *over* A. *Let* $a \in M$*,*$\zeta(x,y) \in L$ *and* $q(y) \in S(A)$ *be an* a*-definable type. Then the following are equivalent:*

(1) There is $b \models q$ *in* $\mathbb{M}$ *such that* $\models \zeta(a,b)$.
(2) There is $b \models q$ *in* M *such that* $\models \zeta(a,b)$.

PROOF. By L_P-saturation of (M,A) and definability of $q(y)$ over a, it is enough to find such a b realizing the $\phi(y,z)$-part of $q(y)$. Assume that it is definable by $d_\phi(z,a)$. Let $\theta(y,t)$ be given by Proposition 5.3.12 for ϕ, and let $d_\theta(t,a)$ define the θ-part of q. By dnfcp$'$, the fact that $d_\phi(z,a), d_\theta(t,a)$ define a consistent ϕ,θ-type q_a over P is expressible by a bounded formula $\psi_1(a)$ saying:

$$\forall z_1 ... z_n \in P \, \forall t_1 ... t_n \in P \; \exists y \left(\bigwedge_{i \le n} \phi(y,z) \leftrightarrow d_\phi(z,a) \wedge \bigwedge_{i \le n} \theta(y,t) \leftrightarrow d_\theta(t,a) \right),$$

where n is given by dnfcp$'$ for $\phi'(y,z_1z_2t_1t_2) = \phi(y,z_1) \wedge \neg\phi(y,z_2) \wedge \theta(y,t_1) \wedge \neg\theta(y,t_2)$ and $\psi'(z_1z_2t_1t_2,\alpha) = d_\phi(z_1,\alpha) \wedge \neg d_\phi(z_2,\alpha) \wedge d_\theta(t_1,\alpha) \wedge \neg d_\theta(t_2,\alpha)$.

Observe that for any $d \in d_\theta(A,a)$, $M \models \exists b\, \theta(b,d) \wedge \zeta(a,b)$ (as $q(y) \wedge \zeta(a,y)$ is consistent). It can be expressed by a bounded formula $\psi_2(a)$.

Let $a_0 \in M_0$ be such that $(M_0,A_0) \models \psi_1(a_0) \wedge \psi_2(a_0)$. Assume that there is a finite $C \subseteq A_0$ such that $q_{a_0}(y)|_C \wedge \zeta(a_0,y)$ is inconsistent. Let $d \in d_\theta(A_0,a_0)$ be as given by Theorem 5.3.14. Then find some $b \in M_0$ such that $\models \theta(b,d) \wedge \zeta(a_0,b)$ (by $\psi_2(a_0)$). By the hypothesis on θ, we have $b \models q_{a_0}|C$ – a contradiction.

So $q_{a_0}(y) \wedge \zeta(a_0,y)$ is consistent, and it follows by smallness of A_0 in M_0 that $(M_0,A_0) \models \forall x\, \psi_1(x) \wedge \psi_2(x) \to \exists b \models q_x(y) \wedge \zeta(x,y)$. It follows that (M,A) satisfies the same sentence, and unwinding we conclude. □

THEOREM 5.5.14. *Let* T *be distal,* $A \subseteq M$ *is small and (uniformly) stably embedded, and* A_{ind} *has* dnfcp$'$. *Then* T_P *is bounded.*

PROOF. By Lemma 5.5.12, M has dnfcp$'$ over A. Take (M,A) a $|T|^+$-saturated elementary extension of the pair. Let $a,a' \in M$ be such that $A_{[a]} \equiv A_{[a']}$. We have to show that $\mathrm{tp}_{L_P}(a) = \mathrm{tp}_{L_P}(a')$. We do a back-and-forth. Take $b \in M$.

Case 1: $b \in A$. As $A_{[a]} \equiv A_{[a']}$, by L_P-saturation we can find $b' \in P$ such that $A_{[ab]} \equiv A_{[a'b']}$.

Case 2: $b \in M \setminus A$. By stable embeddedness and Case 1, we may assume that $\mathrm{tp}(ab/A)$ is a-definable. It is enough to find $b' \in M \setminus A$ such that $\mathrm{tp}(b',a') = \mathrm{tp}(b,a)$ and $\mathrm{tp}(ab'/A)$ is defined over a' the same way $\mathrm{tp}(ab/A)$ is over a. Now the previous lemma (and saturation) applies and gives such a b'. □

5.6. Naming indiscernible sequences, again

We recall briefly the story of the question. In [**BB00**] Baldwin and Benedikt had established the following.

FACT 5.6.1. *Let* T *be NIP. Let* $I \subset M$ *be a small indiscernible sequence indexed by a dense complete linear order. Then* $Th(M, I)$ *is bounded and the* L_P*-induced structure on* I *is just the linear order.*

We have demonstrated (Chapter 4, Proposition 3.2) that in this case (M, I) is still NIP. In this section we are going to complement the picture by resolving some of the remaining questions: naming a small indiscernible sequence of *arbitrary* order type preserves NIP, while naming a large indiscernible sequence may create IP.

5.6.1. Naming an arbitrary small indiscernible sequence.

LEMMA 5.6.2. *Let* I *be small in* M *and* $N \succ M$ *such that* I *is small in* N*. Then* (M, I) *and* (N, I) *are elementary equivalent.*

PROOF. We do a back and forth starting with the identity mapping from I to I, and inductively choosing $A = \{a_i\}_{i<\omega} \subset M$ and $B = \{b_i\}_{i<\omega} \subset N$ such that $tp_L(AI) = tp_L(BI)$. Assume we have chosen $\{a_m b_m : m < n\}$ and we pick $a_n \in M$. Consider $p(x, AI) = tp_L(a_n/AI)$. By the inductive assumption, $p(x, BI)$ is consistent. Let $b_n \in N$ realize it (possible by smallness). In the end, in particular, $AI \equiv^{qf-L_P} BI$. $\square$

Assume that D is an L-definable set which is uniformly stably embedded in the sense of T (and T eliminates quantifiers in a relational language L), let $\mathbf{P}$ name a subset of D. Now let (N, P) be a saturated model of the pair.

A formula is D-bounded if it is equivalent to one of the form $\psi(\bar{x}) = Q_1 z_1 \in D ... Q_n z_n \in D \bigvee_{i<m} \phi_i(\bar{x}, \bar{z}) \wedge \chi_i(\bar{x}, \bar{z})$, where $\phi_i(\bar{x}, \bar{z})$ is a qf-L-formula and $\chi_i(\bar{x}, \bar{z})$ is a qf-$\mathbf{P}$-formula (follows from the relationality of L).

LEMMA 5.6.3. *Let* $a, a' \in N$ *have the same* D*-bounded type, then* $a \equiv^{L_P} a'$*.*

PROOF. We do a back-and -forth. Assume that $a \equiv^{L^{D-bdd}} a'$, and let $b \in N$ be arbitrary.

Case 1. $b \in D$: Consider $p(x, a) = tp_{L^{D-bdd}}(ba)$. For any finite $p_0(x, a) \subseteq p(x, a)$ we have $\models \exists x \in D\, p_0(x, a)$, which is a D-bounded formula, thus $\models \exists x \in D\, p_0(x, a')$, and by saturation of N there is $b' \in D$ satisfying $ab \equiv^{L^{D-bdd}} a'b'$.

Case 2. $b \notin D$: Possibly adding some points in D using (1), we may assume that $tp_L(ab/D)$ is L-definable over $c = a \cap D$. Take some $b' \in N$ such that $ab \equiv^L a'b'$, then $tp_L(a'b'/D)$ is L-definable over $c' = a' \cap D$ using the same formulas. We want to check that $ab \equiv^{L^{D-bdd}} a'b'$. Let $\psi(\bar{x})$ be a D-bounded formula, say $\psi(\bar{x}) = Q_1 z_1 \in D ... Q_n z_n \in D \bigvee_{i<m} \phi_i(\bar{x}, \bar{z}) \wedge \chi_i(\bar{x}, \bar{z})$. Then we have:
$\models Q_1 x_1 \in D ... Q_n x_n \in D \bigvee_{i<m} \phi_i(ab, \bar{x}) \wedge \chi_i(ab, \bar{x}) \Leftrightarrow \models \bar{Q}\bar{x} \in D \bigvee_{i<m} d_{\phi_i}(c, \bar{x}) \wedge \chi_i'(\bar{x})$ (as we know the truth values of $\mathbf{P}(x)$ on ab) $\Leftrightarrow \models \bar{Q}\bar{x} \in D \bigvee_{i<m} d_{\phi_i}(c', \bar{x}) \wedge \chi_i'(\bar{x})$ (as $c \equiv^{L_P^{D-bdd}} c'$) $\Leftrightarrow \models Q_1 x_1 ... Q_n x_n \bigvee_{i<m} \phi_i(a'b', \bar{x}) \wedge \chi_i(a'b', \bar{x})$ (as the truth values of $\mathbf{P}(x)$ on $a'b'$ are the same by the choice of b' and assumption on a'). $\square$

LEMMA 5.6.4. *Assume that* $Th(D_{\mathrm{ind}}, P)$ *is bounded. Then* $Th(M, P)$ *is bounded.*

PROOF. Let (N, P) be saturated. Assume that $P_{[a]} \equiv P_{[a']}$ and let b be given.

If $b \in D$, then we find a $b' \in D$ such that $P_{[ab]} \equiv P_{[a'b']}$ by the assumption that (D, P) is bounded and saturation.

If $b \notin D$, then we take the same b' as in (2) of the previous lemma and conclude that $bb' \equiv^{L_P^{D-bdd}} aa'$ in the same way (using that $c \equiv^{L_P^{P-bdd}} c' \Rightarrow c \equiv^{L_P^{D-bdd}} c'$), which is sufficient (clearly, if two tuples have the same D-bounded $L_{\mathbf{P}}$-type, then they have the same $\mathbf{P}$-bounded $L_{\mathbf{P}}$-type). ☐

LEMMA 5.6.5. *In the situation as above, if* T *is* NIP *and* (D, P) *with the induced quantifier-free structure is* NIP*, then* $T_{\mathbf{P}}$ *is* NIP*.*

PROOF. As $D_{\mathrm{ind}(L_P^{qf})}$ is NIP, it follows that $D_{\mathrm{ind}(L_P^{D-bdd})}$ is NIP. Conclude as in Corollary 2.5 in Chapter 4 (and even easier as D is actually stably embedded). ☐

THEOREM 5.6.6. *Let* (M, I) *be small and* M *be* NIP*. Then* (M, I) *is* NIP*.*

PROOF. Let (M, I) be small. By Lemma 5.6.2 we may assume that M is $\left(2^{|I|}\right)^+$-saturated. Let $I \subseteq J \subset M$, where J is a dense complete indiscernible sequence such that (M, J) is still small. Name J by D, and let T' be a Morleyzation of T_D. Then by Fact 5.6.1, T' is NIP and D is stably embedded. Thus formulas in $T'_{\mathbf{P}}$ are D-bounded by Lemma 5.6.3. It is easy to check directly that (J_{ind}, I) is bounded, thus $T'_{\mathbf{P}}$ is $\mathbf{P}$-bounded by Lemma 5.6.4. Conclude by Fact 5.5.1 (as the structure induced on I is still NIP). ☐

5.6.2. Large indiscernible sequence producing IP**.** Take $L = \{<, E\}$ and T saying that $<$ is DLO and E is an equivalence relation with infinitely many classes, all of which are dense co-dense with respect to $<$. It is easy to check by back-and-forth that this theory eliminates quantifiers and that it is NIP. Let M/E denote the imaginary sort of E-equivalence classes.

Let D be an equivalence class, pick some $x_0 \in M$ outside of it and take $\mathbf{P}$ to name $D \cap (-\infty, x_0)$. Consider the formula

$$\phi(x) = \exists y \forall s < y \exists t \in \mathbf{P}, yEx \wedge s < t < y \wedge (\neg \exists u > y, u \in \mathbf{P}).$$

Then $\phi(x)$ picks out exactly the points equivalent to x_0. Easily, that formula is not equivalent to a D-bounded one (simply because all imaginary elements of equivalence classes different from D have exactly the same type over D).

Now consider the following formula:

$$S(x_1, x_2) = \exists y_1, y_2, y_1 E x_1 \wedge y_2 E x_2 \wedge L_0(y_1) \wedge R_0(y_2) \wedge (\forall y_1 < z < y_2, \neg \mathbf{P}(z))$$

where $L_0(y) = \exists t \in \mathbf{P} \forall s \in \mathbf{P}, t < y \wedge (s > t \to y < s)$ and same for $R_0(y)$, but reversing the inequalities.

CLAIM 5.6.7. (1) Let D be an equivalence class. Then any increasing sequence contained in D is indiscernible.

(2) Let G be an arbitrary countable graph. Then we can choose $P \subseteq D$ such that $(M/E, S) \cong G$.

PROOF. (1) is immediate by the quantifier elimination.

(2) By induction, for every edge $e_1 e_2 \in (M/E)^2$ that we want to put, chose a pair of representatives $a_1, a_2 \in \mathbb{Q}$ such that the interval (a_1, a_2) is disjoint from all the previously chosen intervals. Let P name the set of points in D outside of the union of these intervals. ☐

In particular we can choose $\mathbf{P}$ so that $T_{\mathbf{P}}$ interprets the random graph.

REMARK 5.6.8. We also observe that naming two small indiscernible sequences at once can create IP. This time we name sequences which satisfy $\neg xEy$ for any two points x and y in them. So pick any small I_0. Let $A = A[I_0] = \{t \in M/E : \exists x \in I_0, xEt\}$. Then A gets an order $<_0$ form I_0 induced by $<$. Fix $<_1$ any other order on A. Then we can find another sequence I_1 such that $A[I_1] = A$ and the order induced on A by I_1 is $<_1$. With two linear orders, we can code pseudo-finite arithmetic as in [**SS11**]. In particular we have IP.

5.7. Models with definable types

Classically,

FACT 5.7.1. T *is stable* $\Leftrightarrow$ *for every* $M \models T$, $|S(M)| \leq |M|^{|T|}$ $\Leftrightarrow$ *for every* $M \models T$, *all types over it are definable* $\Leftrightarrow$ *there is a saturated* $M \models T$ *with all types over it definable.*

We start by observing that if T is NIP, then it has models of arbitrary size with few types over them.

PROPOSITION 5.7.2. *Let* T *be* NIP. *For any* $\kappa \geq |T|$ *there is a model* M *with* $|M| = \kappa$ *such that* $|S(A)| \leq |A|^{|T|}$ *for every* $A \subseteq M$.

PROOF. If T is stable then every model of size κ works. Otherwise assume T is unstable and let $I = (a_\alpha)_{\alpha<\kappa}$ be linearly ordered by $< (x, y) \in L$. Let T^{Sk} be a Skolemization of T, and let $M = \mathrm{Sk}(I)$, $|M| \leq \kappa + |T|$.

We show that $S^L(M) \leq \kappa^{|T|}$. Consider

$$\widetilde{L} := \{\phi(x, f(\bar{y})) : \phi \in L \text{ and fis an } L^{\mathrm{Sk}}\text{-definable function}\}.$$

Notice that every $\psi(x, y) \in \widetilde{L}$ is NIP. But then (by Remark 5.2.2) for every $\psi \in \widetilde{L}$, every ψ-type over I is $<$-definable, in particular $|S^{\widetilde{L}}(I)| \leq |I|^{|T|}$.

Given $p, q \in S^L(M)$ choose some $p', q' \in S^{\widetilde{L}}(M)$ with $p \subseteq p', q \subseteq q'$. It is easy to see that $p'|_I = q'|_I \Rightarrow p = q$ (for any $a \in M$ and $\phi \in L$ we have $\phi(x, a) \in p \Longleftrightarrow \phi(x, f(\bar{b})) \in p'|_I$ where $\bar{b} \subseteq I$ and $f(\bar{b}) = a$), thus $|S^L(M)| \leq |S^{\widetilde{L}}(I)| \leq \kappa^{|T|}$. $\square$

REMARK 5.7.3. Slightly elaborating on the argument, we may construct such an M which is in addition gross (M is called *gross* if every infinite subset definable with parameters from M is of cardinality $|M|$, see [**LP04**]).

In general one cannot find a model such that all types over it are definable (for example, take RCF and add a new constant for an infinitesimal). However, some interesting NIP theories have models with all types over them definable.

EXAMPLE 5.7.4.
1. $\mathbb{R}$ as a model of RCF (and this is the only model of RCF with all types definable).
2. In ACVF there are arbitrary large models with all types definable (maximally complete fields with $\mathbb{R}$ as a value group).
3. $(\mathbb{Z}, +, <)$ is a model of Presburger arithmetic with all types definable (but there are no larger models).
4. $(\mathbb{Q}_p, +, \times, 0, 1)$ (by [**Del89**]).

When looking at a particular example, it is usually much easier to check that 1-types are definable, rather than n-types, and one can ask if this is actually the same thing.

DEFINITION 5.7.5. Let A be a set. We say that it is (n,m)-*stably embedded* if every subset of A^n which can be defined as $\phi(A,a)$ with $|a| \leq m$, can actually be defined as $\psi(A,b)$ with $b \in A$. We say that it is *uniformly* (n,m)-stably embedded if ψ can be chosen depending just of ϕ (and not on a). A compactness argument shows that for a definable set A, it is (n,m)-stably embedded if and only if it is uniformly (n,m)-stably embedded. Obviously, (∞,n)-stable embeddedness is equivalent to definability of n-types over A.

Of course, (n,m)-stable embeddedness implies (n',m')-stable embeddedness for $n' \leq n, m' \leq m$.

PROPOSITION 5.7.6. *Let* T *be* NIP *and assume that* M *is* (∞,n)-*stably embedded. Then it is* (n,∞)-*stably embedded.*

PROOF. By definability, every type $p \in S_n(M)$ has a unique heir.

Claim 1: If $p \in S_n(M)$ has a unique heir, then it has a unique coheir.

Let $p'(x)$ be the unique global heir of p. Let $p_1(x)$ be a global coheir of p, and $(a_i)_{i<\omega}$ a Morley sequence in it over M. Given $\bar{m} \in M$ and noticing that $tp(a_0/a_1...a_nM)$ is an heir over M (so is contained in a global heir as $M \models T$) we have that $\models \phi(a_0,...,a_n,\bar{m})$ if and only if $\phi(x,a_1...a_n\bar{m}) \in p'(x)$. Thus by Fact 5.2.4, p has a unique global coheir.

Claim 2: Every $p \in S_n(A)$ has a unique coheir $\Leftrightarrow A$ is (n,∞)-stably embedded.

$\Rightarrow$: Let $\phi(x,c) \in L(\mathbb{M})$ and consider $p(x) \in S_n(A)$ finitely satisfiable in $\phi(x,c) \cap A$. If it was finitely satisfiable in $\neg\phi(x,c) \cap A$ as well, then p would have two coheirs, thus there is some $\psi_p(x) \in p(x)$ with $\psi_p(x) \to^A \phi(x,c)$. By compactness we have $\bigvee \psi_{p_i}(x) \leftrightarrow^A \phi(x,c)$ for finitely many p_i's.

$\Leftarrow$: Let p_1, p_2 be two global coheirs of $p \in S_n(A)$, and assume that $\phi(x,a) \in p_1, \neg\psi(x,a) \in p_2$. Let $\psi(x) \in L(A)$ be such that $\psi(A^n) = \phi(A^n,a)$. It follows that $\psi(x) \in p$. But this implies that p_2 cannot be a coheir as $\psi(x) \wedge \neg\phi(x,a)$ is not realized in A. $\square$

And so it is natural to ask whether $(\infty,1)$-stable embeddedness of M implies (∞,n)-stable embeddedness. The answer is yes in stable theories, for the obvious reason, and yes in o-minimal theories, where by a theorem of Marker and Steinhorn [**MS94**], $(1,1) \to (\infty,\infty)$ for models. However, we show in the next section that this is not true in NIP theories in general. The question remains open for C-minimal theories.

5.7.1. Example of $(\infty,1) \not\to (\infty,m)$.

5.7.1.1. *General construction.* Start with a theory T in a language L containing an equivalence relation $E(x,y)$. Assume T has a model M_0 composed of ω-many E-equivalence classes, each one finite of increasing sizes. So that any model M of T contains M_0 as a sub-model and all the E-classes disjoint from M_0 are infinite.

We consider the language L' defined as follows:

- For each relation $R(x_1,...,x_n)$ in L, L' contains a relation $R'(x_1,y_1,...,x_n,y_n)$.
- Also L' contains an equivalence relation $\tilde{E}(u,v)$, a binary relation $S(u,v)$ and a quaternary relation $U(u_1,v_1,u_2,v_2)$. The relation S will code a graph and U will be an equivalence relation on S-edges.

We build an L' structure N_0 as follows:

N_0 has ω-many $\tilde{E}$-equivalence classes, corresponding to the E-equivalence classes of M_0. Let $\mathfrak{e}$ be an E-class, and let n be its size. Then the corresponding $\tilde{E}$ class $\tilde{\mathfrak{e}}$ in N_0 is a finite regular graph, with S as the edge relation, of degree n (every vertex has degree n) and with no cycles of length $\leq n$ (such graphs exist, see e.g. [**Bol78**, III.1, Theorem 1.4']). The predicate U is interpreted as an equivalence relation between edges so that every vertex is adjacent to exactly one edge from each equivalence class. We fix a bijection π between U-equivalence classes and elements of the E-class $\mathfrak{e}$. This being done, for each relation $R(x_1, ..., x_n)$ we say that $R'(x_1, y_1, ..., x_n, y_n)$ holds in N_0 if $\bigwedge_{i \leq n} S(x_i, y_i)$ and if $R(\pi(x_1, y_1), ..., \pi(x_n, y_n))$ holds in M_0.

Note that any model of $T' = Th(N_0)$ contains N_0 as submodel and its $\tilde{E}$-classes not in N_0 are infinite and composed of disjoint unions of trees with infinite branching. So the graph structure does not interact in any way with the structure coming from the R' relations.

Given a model of T' we can recover a model of M_0 by looking at U-equivalence classes and we obtain in this way every model of T. So there are at least as many 2-types over N_0 as there are 1-types over M_0. However, the non-realized 1-types over N_0 correspond to imaginary types of non-realized E-classes over M_0. See below.

Assume that L contains a constant for every element of M_0. Let $N \models T'$ and denote by M the model of T which we get from N. We build a language $L'' \supset L'$:

- We add a constant for every element of N_0.
- For every $n \in \omega$, we add a relation $d_n(u, v)$ which holds if and only if u and v are at distance n (in the sense of the shortest path in graph $S(u, v)$).
- For every $\emptyset$-definable set $\phi(x_1, ..., x_n, y_1, ..., y_m)$ of M_0 which is E-congruent with respect to the variables x_i (*i.e.*, for $a_i E a_i'$ and b_i's, we have $\phi(a_1, ..., a_n, b_1, ..., b_m) \leftrightarrow \phi(a_1', ..., a_n', b_1, ..., b_m))$ we add a predicate $W_\phi(x_1, ..., x_n, y_1, z_1, ..., y_m, z_m)$ which we interpret as:
  $N \models W_\phi(a_1, ..., a_n, b_1, c_1, ..., b_m, c_m)$ if and only if $\bigwedge_{i \leq m} S(b_i, c_i)$ and for some $e_1, ..., e_n \in M$ with e_i in the E-class corresponding to the $\tilde{E}$-class of a_i, we have $M \models \phi(e_1, ..., e_n, \pi(b_1, c_1), ..., \pi(b_m, c_m))$.

Claim 5.7.7. If T eliminates quantifiers in L, then T' eliminates quantifiers in L''.

Proof. By easy back-and-forth. □

Corollary 5.7.8. *If T is NIP, then T' is NIP.*

Corollary 5.7.9. *Assume that all (imaginary) types of a new E class in M_0 are definable, then all 1-types over N_0 are definable.*

5.7.1.2. *An example of M_0 with* NIP. Let $L_0 = \{\leq, E\}$. We build an L_0-structure M_0 as follows:

- The reduct to $\leq$ is a binary tree with a root (every element has exactly two immediate successors, there is a unique element with no predecessor). The tree is of height ω, so every element is at finite distance from the root.

- Two elements are E-equivalent if they are at the same distance from the root.

This theory eliminates quantifiers in the language L obtained from L_0 by adding a constant for every element of M_0, a binary function symbol $\wedge$ interpreted as $x \wedge y$ is the maximal element z such that $z \leq x$ and $z \leq y$ and for each n a predicate $d_n(x, y)$ saying that the difference between the heights of x and y is n. Note that those predicates are E-congruent.

Clearly, M_0 is NIP, there is a unique imaginary type of a new E-class over M_0 and this type is definable. However, not all types over M_0 are definable.

So we obtain the required counter-example.

REMARK 5.7.10. Together with Proposition 5.7.6 it follows that also $(1, \infty) \not\to (n, \infty)$ in a general NIP theory. Another example due to Hrushovski witnessing this is presented in Pillay [**Pil11**] – a proper dense elementary pair of ACVF's $F_1 \prec F_2$ with the same residue field and value group. Then F_1 is $(1, \infty)$-stably embedded in F_2, but if $a \in F_2 \setminus F_1$, then the function taking $x \in F_1$ to $v(x - a)$ is not F_1-definable.

5.7.2. Some positive results. In [**Pil11**] Pillay had established the following.

FACT 5.7.11. *Let A be a definable subset of M. Assume that A_{ind} is rosy, M is NIP over A and A is $(1, \infty)$-stably embedded. Then A is stably embedded.*

In fact, one can replace the definable set A with a model, at the price of requiring that $(1, \infty)$-stable embeddedness is uniform. We explain briefly how to modify Pillay's argument.

THEOREM 5.7.12. *Let $A \preceq M$. Assume that A_{ind} is rosy, M is NIP over A and A is **uniformly** $(1, \infty)$-stably embedded. Then A is uniformly stably embedded.*

PROOF. Assume that $A \preceq M$ is a counterexample to the theorem. We consider (M, A) as a pair with $\mathbf{P}$ naming A. As A is a model, it follows that A_{ind} eliminates quantifiers, thus *every set definable in A_{ind} is given by the trace of an L-formula.* As there are two languages L and $L_{\mathbf{P}}$ around, we make a terminology clarification: induced structure is always meant to be with respect to L formulas, and (n, m)-stable embeddedness always means that sets externally definable by L-formulas are internally definable by L-formulas.

CLAIM. We may assume that (M, A) is saturated (as a pair in the $L_{\mathbf{P}}$ language).

PROOF. Just let $(N, B) \succ (M, A)$ be a saturated extension. Of course, A is uniformly (n, ∞)-stably embedded in M if and only if B is uniformly (n, ∞)-stably embedded in N. Notice that $B_{\text{ind}} \succ A_{\text{ind}}$, thus B_{ind} is rosy. Finally, N is still NIP over B with respect to L-formulas. $\square$

CLAIM. Let $f : A \to Z$ be an $L(M)$-definable function (namely the trace on A of an $L(M)$-definable relation which happens to define a function on A), where Z is some sort in $A^{\text{eq}}_{\text{ind}}$. Then there is an $L(A)$-definable relation $R(x, y)$ and $k < \omega$ such that $(M, A) \models \forall x \in \mathbf{P}\ \left(R(x, f(x)) \wedge \exists^{\leq k} y \in \mathbf{P}\ R(x, y)\right)$.

PROOF. Let the graph of f be defined by $f(x, y, e) \in L(M)$. Let κ be large enough. Working entirely in A_{ind}, assume that we could choose $(a_i b_i)_{i<\kappa}$ in A such that $b_i = f(a_i)$ and $b_i \notin \operatorname{acl}_L\left((a_j b_j)_{j<i}\, a_i\right)$ for all i. Following Pillay's

proof of [**Pil11**, Lemma 3.2] and using saturation of A_{ind}, we may assume that $(a_i b_i)$ is L-indiscernible and then find (b_i') in A such that $b_i' = b_i$ if and only if i is even, and $\mathrm{tp}_L\left((a_i b_i)_{i<\kappa}\right) = \mathrm{tp}_L\left((a_i b_i')_{i<\kappa}\right)$, so still L-indiscernible. But then $(M, A) \models f(a_i, b_i', e)$ if and only if i is even – a contradiction to M being NIP over A with respect to L-formulas.

So, by compactness we find some $R(x, y) \in L(A)$ and $k < \omega$ such that $(M, A) \models \forall x \in \mathbf{P}\ R(x, f(x)) \wedge \exists^{\leq k} y \in \mathbf{P}\ R(x, y)$. $\square$

Claim. In the previous claim, we can take $k = 1$.

Proof. Pillay's proof of [**Pil11**, Lemma 3.3] goes through again, with acl, dcl and forking all considered inside of the L-induced structure on A (which is saturated and eliminates quantifiers). $\square$

Finally, we conclude by induction on the dimension of the externally definable sets. So let $X = A^{n+1} \cap \phi(x_0, ..., x_n, x_{n+1}, c)$ be given, and assume inductively that A is uniformly (n, ∞)-stably embedded (the base case given by the assumption). For any $a \in A$, let $X_a = A^n \cap \phi(x_0, ..., x_n, a, c)$. By the inductive assumption, there is some $\psi(x_0, ..., x_n, z)$ such that for any $a \in A$, $X_a = A^n \cap \psi(x_0, ..., x_n, b)$ for some $b \in A$. By Shelah's expansion theorem, the function $f : A \to Z$ sending a to $[b]_\psi$ (the canonical parameter of $\psi(x_0, ..., x_n, b)$) is externally definable. Thus, by the previous claim, it is actually definable with parameters from A. It follows that X is defined by $\psi(x_0, ..., x_n, f(x_{n+1}))$. $\square$

As an application, we obtain a new proof of a theorem of Marker and Steinhorn [**MS94**].

Corollary 5.7.13. *Let T be o-minimal and $M \models T$. Assume that the order on M is complete. Then all types over M are uniformly definable.*

CHAPTER 6

On non-forking spectra

This chapter is a joint work with Itay Kaplan and Saharon Shelah as is submitted to the Transactions of the American Mathematical Society as "On non-forking spectra" [**CKS12**].

Non-forking is one of the most important notions in modern model theory capturing the idea of a generic extension of a type (which is a far-reaching generalization of the concept of a generic point of a variety).

To a countable first-order theory we associate its *non-forking spectrum* — a function of two cardinals κ and λ giving the supremum of the possible number of types over a model of size λ that do not fork over a sub-model of size κ. This is a natural generalization of the stability function of a theory.

We make progress towards classifying the non-forking spectra. On the one hand, we show that the possible values a non-forking spectrum may take are quite limited. On the other hand, we develop a general technique for constructing theories with a prescribed non-forking spectrum, thus giving a number of examples. In particular, we answer negatively a question of Adler whether NIP is equivalent to bounded non-forking.

In addition, we answer a question of Keisler regarding the number of cuts a linear order may have. Namely, we show that it is possible that $\operatorname{ded}\kappa < (\operatorname{ded}\kappa)^{\omega}$.

6.1. Introduction

The notion of a non-forking extension of a type (see Definition 6.2.3) was introduced by Shelah for the purposes of his classification program to capture the idea of a "generic" extension of a type to a larger set of parameters which essentially doesn't add new constraints to the set of its solutions. In the context of stable theories non-forking gives rise to an independence relation enjoying a lot of natural properties (which in the special case of vector spaces amounts to linear independence and in the case of algebraically closed fields to algebraic independence) and is used extensively in the analysis of models. In a subsequent work of Shelah [**She80**], Kim and Pillay [**Kim98, KP97**] the basic properties of forking were generalized to a larger class of simple theories. Recent work of the first and second authors shows that many properties of forking still hold in a larger class of theories without the tree property of the second kind (Chapter 1).

Here we consider the following basic question: how many non-forking extensions can there be? More precisely, given a complete first-order theory T, we associate to it its non-forking spectrum, a function $f_T(\kappa,\lambda)$ from cardinals $\kappa \leq \lambda$ to cardinals defined as:

$$f_T(\kappa,\lambda) = \sup\left\{ S^{\mathrm{nf}}(N,M) \mid M \preceq N \models T,\ |M| \leq \kappa,\ |N| \leq \lambda \right\},$$

where $S^{\mathrm{nf}}(A,B) = \{p \in S_1(A) \mid p \text{ does not fork over } B\}$ (counting 1-types rather than n-types is essential, as the value may depend on the arity, see Section 6.5.8).

This is a generalization of the classical question "how many types can a theory have?". Recall that the stability function of a theory is defined as $f_T(\kappa) = \sup\{S(M) \mid M \models T, |M| = \kappa\}$. It is easy to see that $f_T(\kappa,\kappa) = f_T(\kappa)$. This function has been studied extensively by Keisler [**Kei76**] and the third author [**She71**], where the following fundamental result was proved:

Fact 6.1.1. *For any complete countable first-order theory* T*,* f_T *is one of the following:* κ*,* $\kappa + 2^{\aleph_0}$*,* $\kappa^{\aleph_0}$*,* $\operatorname{ded}(\kappa)$*,* $\operatorname{ded}(\kappa)^{\aleph_0}$*,* 2^{κ}*.*

Where $\operatorname{ded}(\kappa)$ is the supremum of the number of cuts that a linear order of size κ may have (see Definition 6.6.1). While this result is unconditional, in some models of ZFC, some of these functions may coincide. Namely, if GCH holds, $\operatorname{ded}(\kappa) = \operatorname{ded}(\kappa)^{\aleph_0} = 2^{\kappa}$. By a result of Mitchell [**Mit73**], it was known that for any cardinal κ with $\operatorname{cof}\kappa > \aleph_0$ consistently $\operatorname{ded}(\kappa) < 2^{\kappa}$. In 1976, Keisler [**Kei76**, Problem 2] asked whether $\operatorname{ded}(\kappa) < \operatorname{ded}(\kappa)^{\aleph_0}$ is consistent with ZFC. We give a positive answer in Section 6.6.

The aim of this paper is to classify the possibilities of $f_T(\kappa,\lambda)$. The philosophy of "dividing lines" of the third author suggests that the possible non-forking spectra are quite far from being arbitrary, and that there should be finitely many possible functions, distinguished by the lack (or presence) of certain combinatorial configurations. We work towards justifying this philosophy and arrive at the following picture.

Theorem 6.1.2. *Let* T *be countable complete first-order theory. Then for* $\lambda \gg \kappa$*,* $f_T(\kappa,\lambda)$ *can be one of the following, in increasing order (meaning that we have an example for each item in the list except for (11), and "???" means that we don't know if there is anything between the previous and the next item, while the lack of "???" means that there is nothing in between):*

(1) κ
(2) $\kappa + 2^{\aleph_0}$
(3) $\kappa^{\aleph_0}$
(4) $\operatorname{ded}\kappa$
(5) ???
(6) $(\operatorname{ded}\kappa)^{\aleph_0}$
(7) $2^{2^{\kappa}}$
(8) λ
(9) $\lambda^{\aleph_0}$
(10) ???
(11) $\lambda^{<\beth_{\aleph_1}(\kappa)}$
(12) $\operatorname{ded}\lambda$
(13) ???
(14) $(\operatorname{ded}\lambda)^{\aleph_0}$
(15) ???
(16) 2^{λ}

In particular, note that the existence of an example of $f_T(\kappa,\lambda) = 2^{2^{\kappa}}$ answers negatively a question of Adler [**Adl08**, Section 6] whether NIP is equivalent to bounded non-forking.

The restriction $\lambda \gg \kappa$ is in order to make the statement clearer. It can be taken to be $\lambda \geq \beth_{\aleph_1}(\kappa)$. In fact we can say more about smaller λ in some cases. In the class of NTP$_2$ theories (see Section 6.4), we have a much nicer picture, meaning that there is a gap between (6) and (16).

In the first part of the paper, we prove that the non-forking spectra cannot take values which are not listed in the Main Theorem. The proofs here combine techniques from generalized stability theory (including results on stable and NIP theories, splitting and tree combinatorics) with a two cardinal theorem for $L_{\omega_1,\omega}$.

The second part of the paper is devoted to examples.

We introduce a general construction which we call *circularization*. Roughly speaking, the idea is the following: modulo some technical assumptions, we start with an arbitrary theory T_0 in a finite relational language and an (essentially) arbitrary prescribed set of formulas F. We expand T by putting a circular order on the set of solutions of each formula in F, iterate the construction and take the limit. The point is that in the limit all the formulas in F are forced to fork, and we have gained some control on the set of non-forking types. This construction turns out to be quite flexible: by choosing the appropriate initial data, we can find a wide range of examples of non-forking spectra previously unknown.

6.2. Preliminaries

Our notation is standard: κ, λ, μ are cardinals; $\alpha, \beta, \ldots$ are ordinals; $M, N, \ldots$ are models; $\mathbb{M}$ is always a monster model of the theory in question; $B^{[\kappa]}$ is the set of subsets of B of size $\leq \kappa$; T is a complete countable first-order theory; for a sequence $\bar{a} = \langle a_i \,|\, i < \alpha \rangle$, EM $(\bar{a}/A)$ denotes its Ehrenfeucht-Mostowski type over A.

6.2.1. Basic properties of forking and dividing.

We recall the definition of forking and dividing (e.g. see Chapter 1, Section 2 for more details).

Definition 6.2.1. (Dividing) Let A be be a set, and a a tuple. We say that the formula $\varphi(x, a)$ *divide*s over A if and only if there is a number $k < \omega$ and tuples $\{a_i \,| i < \omega\}$ such that

(1) $\operatorname{tp}(a_i/A) = \operatorname{tp}(a/A)$.
(2) The set $\{\varphi(x, a_i) \,|\, i < \omega\}$ is k-inconsistent (i.e. every subset of size k is not consistent).

In this case, we say that a formula k-divides.

Remark 6.2.2. From Ramsey and compactness it follows that $\varphi(x, a)$ divides over A if and only if there is an indiscernible sequence over A, $\langle a_i \,| i < \omega \rangle$ such that $a_0 = a$ and $\{\varphi(x, a_i) \,|\, i < \omega\}$ is inconsistent.

Definition 6.2.3. (Forking) Let A be be a set, and a a tuple.

(1) Say that the formula $\varphi(x, a)$ *forks* over A if there are formulas $\psi_i(x, a_i)$ for $i < n$ such that $\varphi(x, a) \vdash \bigvee_{i<n} \psi_i(x, a_i)$ and $\psi_i(x, a_i)$ divides over A for every $i < n$.
(2) Say that a type p forks over A if there is a finite conjunction of formulas from p which forks over A.

It follows immediately from the definition that if a partial type $p(x)$ does not fork over A then there is a global type $p'(x) \in S(\mathbb{M})$ extending $p(x)$ that does not fork over A.

Lemma 6.2.4. *Let* $(A, \leq)$ *be a* κ^+*-directed order and let* $f : A \to \kappa$. *Then there is a cofinal subset* $A_0 \subseteq A$ *such that* f *is constant on* A_0.

Proof. Assume not, then for every $\alpha < \kappa$ there is some $a_\alpha \in A$ such that $f(a) \neq \alpha$ for any $a \geq a_\alpha$. By κ^+-directedness there is some $a \geq a_\alpha$ for all $\alpha < \kappa$. But then whatever $f(a)$ is, we get a contradiction. $\square$

LEMMA 6.2.5. *Assume that* $p(x) \in S(A)$ *does not fork over* B. *Then there is some* $B_0 \subseteq B$ *such that* $|B_0| \leq |A| + |T|$ *and* $p(x)$ *does not fork over* B_0.

PROOF. Let $\kappa = |A|+|T|$, and assume the converse. Then $p(x)$ forks over every $C \subseteq B$ with $|C| \leq \kappa$. That is, for every $C \in B^{[\kappa]}$ there are $p_C \subseteq p$ with $|p_C| < \omega$, $\psi_0^C(x, y_0), \ldots, \psi_{m_C-1}^C(x, y_{m_C}) \in L$ and $k_C < \omega$ such that for some $d_0^C, ..., d_{m_C-1}^C$, $p_C(x) \vdash \bigvee_{i<m_C} \psi_i^C(x, d_i^C)$ and each of $\psi_i^C(x, d_i^C)$ is k_C-dividing over C. As $B^{[\kappa]}$ is κ^+-directed under inclusion and $|p(x)| \leq \kappa$, it follows by Lemma 6.2.4 that for some finite $p_0 \subseteq p$, $\{\psi_i \mid i < m\}$ and k this holds for every $C \in B^{[\kappa]}$. But then by compactness $p_0(x)$ forks over B — a contradiction. □

6.2.2. The non-forking spectra.

DEFINITION 6.2.6.
(1) For a countable first-order T and infinite cardinals $\kappa \leq \lambda$, let
$$f_T(\kappa, \lambda) = \sup \left\{ S^{\mathrm{nf}}(N, M) \mid M \preceq N \models T, |M| \leq \kappa, |N| \leq \lambda \right\},$$
where $S^{\mathrm{nf}}(A, B) = \{p \in S_1(A) \mid p \text{ does not fork over } B\}$. We call this function the *non-forking spectrum* of T.
(2) For $n > 1$, we may also define $f_T^n(\kappa, \lambda)$ and S_n^{nf} similarly where we replace 1-types with n-types.

All the proofs in Section 6.3 remain valid for f_T replaced by f_T^n.

REMARK 6.2.7. A special case $f_T(\kappa, \kappa)$ is the well-known stability function $f_T(\kappa)$ because $S^{\mathrm{nf}}(N, N) = S(N)$ (Because every type over a model M does not fork over M).

Some easy observations:

LEMMA 6.2.8. *For all* $\kappa \leq \lambda$,
(1) $f_T(\kappa) \leq f_T(\kappa, \lambda)$
(2) $\kappa \leq f_T(\kappa, \lambda) \leq 2^\lambda$
(3) If $f_T(\kappa, \lambda) \geq \mu$ *and* $\kappa \leq \kappa'$ *then* $f_T(\kappa', \lambda) \geq \mu$.
(4) $f_T^n(\kappa, \lambda) \leq f_T^{n+1}(\kappa, \lambda)$

For set theoretic preliminaries, see Section 6.6.

6.3. Gaps

In the following series of subsections, we exclude all the possibilities for f_T which are not in our list (except when "???" is indicated).

6.3.1. On (1) – (4).

DEFINITION 6.3.1. Recall that a theory T is called stable if $f_T(\kappa) \leq \kappa^{\aleph_0}$ for all κ (see [**She90**, Theorem II.2.13] for equivalent definitions).

REMARK 6.3.2. If T is stable then every type over a model M has a unique non-forking extension to any model containing M, so $f_T(\kappa) = f_T(\kappa, \lambda)$ for all $\lambda \geq \kappa \geq \aleph_0$.

If T is unstable, then $f_T(\kappa) \geq \mathrm{ded}(\kappa)$ for all κ (see [**She90**, Theorem II.2.49]), so $f_T(\kappa, \lambda) \geq \mathrm{ded}(\kappa)$ for all $\lambda \geq \kappa$.

PROPOSITION 6.3.3. *The following holds:*

(1) *If* $f_T(\kappa, \lambda) > \kappa$ *for some* $\lambda \geq \kappa$ *then* $f_T(\kappa, \lambda) \geq \kappa + 2^{\aleph_0}$ *for all* $\lambda \geq \kappa$.
(2) *If* $f_T(\kappa, \lambda) > \kappa + 2^{\aleph_0}$ *for some* $\lambda \geq \kappa$ *then* $f_T(\kappa, \lambda) \geq \kappa^{\aleph_0}$ *for all* $\lambda \geq \kappa$.
(3) *If* $f_T(\kappa, \lambda) > \kappa^{\aleph_0}$ *for some* $\lambda \geq \kappa$ *then* $f_T(\kappa, \lambda) \geq \operatorname{ded}(\kappa)$ *for all* $\lambda \geq \kappa$.

PROOF. (3): Suppose $f_T(\kappa, \lambda) > \kappa^{\aleph_0}$ for some $\lambda \geq \kappa$. Then T is unstable, then by Remark 6.3.2 and so $f_T(\kappa, \lambda) \geq \operatorname{ded}(\kappa)$ for all $\lambda \geq \kappa$.

(1): Suppose $f_T(\kappa, \lambda) > \kappa$ for some $\lambda \geq \kappa$. Without loss of generality T is stable. So $f_T(\kappa) = f_T(\kappa, \lambda) > \kappa$. By Fact 6.1.1, $f_T(\kappa) \geq \kappa + 2^{\aleph_0}$ for all κ, and we are done.

(2): Similar to (1). $\square$

6.3.2. The gap between (6) and (7).

DEFINITION 6.3.4. (1) A formula $\varphi(x, y)$ has the independence property (IP) if there are $\{a_i \mid i < \omega\}$ and $\{b_s \mid s \subseteq \omega\}$ in $\mathbb{M}$ such that $\varphi(a_i, b_s)$ holds if and only if $i \in s$ for all $i < \omega$ and $s \subseteq \omega$.
(2) A theory T is NIP (dependent) if no formula $\varphi(x, y)$ has IP.

See [**Adl08**] for more about NIP.

FACT 6.3.5. *If* T *is NIP and* $M \models T$ *then the* $|S(M)| \leq (\operatorname{ded}|M|)^{\aleph_0}$ [**She71**] *and if* $M \prec N$ *and* $p \in S(M)$ *then* p *has at most* $(\operatorname{ded}|M|)^{\aleph_0}$ *non-forking extensions (e.g. follows from the proof of* [**Adl08**, Theorem 42]*, noticing that* $|S_\omega(M)| \leq (\operatorname{ded}|M|)^{\aleph_0}$*). It follows that* $\left|S^{nf}(N, M)\right| \leq (\operatorname{ded}|M|)^{\aleph_0}$.

A generalization of a result due to Poizat [**Poi81**].

PROPOSITION 6.3.6. *Assume that* $f_T(\kappa, \lambda) > (\operatorname{ded}\kappa)^{\aleph_0}$ *for some* $\lambda \geq \kappa$. *Then* $f_T(\kappa, \lambda) \geq 2^{\min\{\lambda, 2^\kappa\}}$ *for all* $\lambda \geq \kappa$.

PROOF. By Fact 6.3.5, some formula $\varphi(x, y)$ in T has IP.

Recall that a set $S \subseteq \mathcal{P}(\kappa)$ is called independent if every finite intersection of elements of S or their complements is non-empty. By a theorem of Hausdorff there is such a family of size 2^κ. Fix some κ and $\mu \leq 2^\kappa$, and let S be a family of independent subset of κ, such that $|S| = \mu$.

Let $A = \{a_i \mid i < \kappa\}$ be such that $b_s \models \left\{\varphi(x, a_i)^{\text{if } i \in s} \mid i < \kappa\right\}$ for every $s \subseteq \kappa$. Let M be a model of size κ containing A and N of size μ containing $M \cup \{b_s \mid s \in S\}$. Now for every $D \subseteq S$, there is an ultrafilter on κ containing D, and let $p_D \in S(N)$ be

$$\{\psi(x, c) \mid c \in N,\ \psi \in L,\ \{a \in M \mid \psi(a, c)\} \in D\},$$

so it is finitely satisfiable in A. Notice that if $D_1 \neq D_2$ then $p_{D_1} \neq p_{D_2}$, as $\varphi(x, b_s) \in p_{D_1} \wedge \neg\varphi(x, b_s) \in p_{D_2}$ for any $s \in D_1 \setminus D_2$. Thus $S^{\text{nf}}(N, M) \geq 2^\mu$.

If $\lambda \leq 2^\kappa$, then let $\mu = \lambda$ and we have that $f_T(\lambda, \kappa) \geq 2^\lambda$.

If $\lambda > 2^\kappa$, then let $\mu = 2^\kappa$, so $f_T(\kappa, \lambda) \geq 2^{2^\kappa}$ and we are done. $\square$

Note that in the Main Theorem we assumed that $\lambda \geq 2^{2^\kappa}$, so in this case we have $f_T(\kappa, \lambda) \geq 2^{2^\kappa}$.

6.3.3. The gap between (7) and (8).
We recall the basic properties of splitting.

DEFINITION 6.3.7. Suppose $A \subseteq B$ are sets. A type $p(x) \in S(B)$ splits over A if there is some formula $\varphi(x, y)$ such and $b, c \in B$ such that $\mathrm{tp}(b/A) = \mathrm{tp}(c/A)$ and $\varphi(x, b) \wedge \neg\varphi(x, c) \in p$.

FACT 6.3.8. *(See e.g.* **[Adl08**, Sections 5, 6]*) Let $M \prec N$ be models*

(1) *The number of types in $S(N)$ that do not split over M is bounded by $2^{2^{|M|}}$.*
(2) *If N is $|M|^+$-saturated and $p \in S(N)$ splits over M then there is an M-indiscernible sequence $\langle a_i \mid i < \omega \rangle$ in N such that $\varphi(x, a_0) \wedge \neg\varphi(x, a_1) \in p$ for some φ.*
(3) *If T is NIP, and $p \in S^{nf}(N, M)$, then p does not split over M.*

DEFINITION 6.3.9. A *non-forking pattern* of depth θ over a set A consists of an array $\{\bar{a}_\alpha \mid \alpha < \theta\}$ where $\bar{a}_\alpha = \langle a_{\alpha,i} \mid i < \omega \rangle$ and formulas $\{\varphi_\alpha(x, y) \mid \alpha < \theta\}$ such that

- $\bar{a}_{\alpha_0}$ is indiscernible over $\{\bar{a}_\alpha \mid \alpha < \alpha_0\} \cup A$.
- $\{\varphi_\alpha(x, a_{\alpha,0}) \wedge \neg\varphi_\alpha(x, a_{\alpha,1}) \mid \alpha < \theta\}$ does not fork over A.

DEFINITION 6.3.10. For an infinite cardinal κ, let $g_T(\kappa)$ be the smallest cardinal θ such that there is no non-forking pattern of depth θ over some model of size κ.

REMARK 6.3.11. It is clear that $g_T(\kappa') \geq g_T(\kappa)$ whenever $\kappa' \geq \kappa$. In addition, from Lemma 6.2.5 it follows that if $g_T(\kappa) > \theta$ then $g_T(\theta + \aleph_0) > \theta$.

LEMMA 6.3.12. *If $g_T(\kappa) > \theta$ then there is M of size κ such that for any λ we can find a non-forking pattern $\{\bar{a}_\alpha, \varphi_\alpha \mid \alpha < \theta\}$ such that in addition:*

- $\bar{a}_\alpha = \langle a_{\alpha,i} \mid i < \lambda \rangle$
- $\{\varphi_\alpha(x, a_{\alpha,0}) \mid \alpha < \theta\} \cup \{\neg\varphi_\alpha(x, a_{\alpha,i}) \mid \alpha < \theta, 0 < i < \lambda\}$ *does not fork over* M.

PROOF. By assumption we have some non-forking pattern $\{\bar{a}_\alpha, \varphi_\alpha \mid \alpha < \theta\}$ over some M of size κ. By compactness, we may assume that $\bar{a}_\alpha$ is of length λ for all $\alpha < \theta$. Let $p(x) \in S(\mathbb{M})$ be a non-forking extension of $\{\varphi_\alpha(x, a_{\alpha,0}) \wedge \neg\varphi_\alpha(x, a_{\alpha,1}) \mid \alpha < \theta\}$. By omitting some elements from each sequence $\bar{a}_\alpha$ and maybe changing φ_α to $\neg\varphi_\alpha$ we may assume

$$\{\varphi_\alpha(x, a_{\alpha,0}) \mid \alpha < \theta\} \cup \{\neg\varphi_\alpha(x, a_{\alpha,i}) \mid \alpha < \theta, 0 < i < \lambda\} \subseteq p.$$

∎

PROPOSITION 6.3.13. *The following are equivalent:*

(1) *For some κ, $g_T(\kappa) > 1$.*
(2) *For every $\lambda \geq \kappa \geq \aleph_0$, $f_T(\kappa, \lambda) = 2^\lambda$ if $\lambda \leq 2^\kappa$ and $f_T(\kappa, \lambda) \geq \lambda$ otherwise.*
(3) *For some $\lambda \geq \kappa$, $f_T(\kappa, \lambda) > 2^{2^\kappa}$.*

PROOF. (1) implies (2): By remark 6.3.11, we may assume that $\kappa = \aleph_0$. By Lemma 6.3.12 there is some countable M such that for any λ there is some $\bar{b} = \langle b_i \mid i < \lambda \rangle$ such that $\{\varphi(x, b_0)\} \cup \{\neg\varphi(x, b_i) \mid i < \lambda\}$ does not fork over M. So, for every $i < \lambda$, $p_i(x) = \left\{ \varphi(x, b_j)^{\text{if } j=i} \mid i \leq j < \lambda \right\}$ does not fork over M.

Taking some model $N \supseteq \bar{b}$ of size λ we can expand each p_i to some $q_i \in S^{\mathrm{nf}}(N, M)$. Notice that for any $i < j < \lambda$, $q_i \neq q_j$ as $\neg\varphi(x, a_j) \in p_i$, but

$\varphi(x, a_j) \in p_j$. So we conclude that $S^{\mathrm{nf}}(N, M) \geq \lambda$. By Lemma 6.2.8, we get that $f_T(\kappa, \lambda) \geq \lambda$ for every $\lambda \geq \kappa$.

Note that by Fact 6.3.5, we know that T is not NIP, so if $\lambda \leq 2^{\kappa}$, then by Proposition 6.3.6 $f_T(\kappa, \lambda) = 2^{\lambda}$.

(2) implies (3) is clear.

(3) implies (1): Let $M \prec N$ witness that $f_T(\kappa, \lambda) > 2^{2^{\kappa}}$. By Fact 6.3.8(1), there is some $p \in S^{\mathrm{nf}}(N, M)$ that splits over M.

Let $N' \succ N$ be $|M|^{+}$-saturated and $p' \in S^{\mathrm{nf}}(N', M)$, a non-forking extension of p. By Fact 6.3.8(2) we find an indiscernible sequence $\bar{a} = \langle a_i \mid i < \omega \rangle$ in N' and a formula $\varphi(x, a_0) \wedge \neg\varphi(x, a_1) \in p$ — and we get (1). $\square$

6.3.4. The gap between (8) and (9).

LEMMA 6.3.14. *For any cardinals λ and θ, if θ is regular or $\lambda \geq 2^{<\theta}$ then $\left(\lambda^{<\theta}\right)^{<\theta} = \lambda^{<\theta}$.*

PROOF. By [**She86**, Observation 2.11 (4)], if $\lambda \geq 2^{<\theta}$, then $\lambda^{<\theta} = \lambda^{\nu}$ for some $\nu < \theta$. So $\left(\lambda^{<\theta}\right)^{<\theta} = (\lambda^{\nu})^{<\theta} = \lambda^{<\theta}$. If θ is regular, then, letting $\lambda' = \lambda^{<\theta}$, since $\lambda' \geq 2^{<\theta}$, $(\lambda')^{<\theta} = (\lambda')^{\nu}$ for some $\nu < \theta$ so

$$(\lambda')^{<\theta} = (\lambda')^{\nu} = \left(\lambda^{<\theta}\right)^{\nu} = \left(\sum_{\mu<\theta} \lambda^{\mu}\right)^{\nu} = \sum_{\mu<\theta} (\lambda^{\mu\cdot\nu}) = \lambda^{<\theta} = \lambda'.$$

$\square$

LEMMA 6.3.15. *Suppose $f_T(\kappa, \lambda) > \lambda^{<\theta}$, and $\lambda \geq \sum_{\mu<\theta} 2^{2^{\kappa+\mu}}$ then $g_T(\kappa) > \theta$.*

PROOF. Let $\lambda' = \lambda^{<\theta}$. By Lemma 6.3.14, $(\lambda')^{<\theta} = \lambda'$. So, we have $f_T(\kappa, \lambda') \geq f_T(\kappa, \lambda) > \lambda^{<\theta} = (\lambda')^{<\theta}$, so we may replace λ with λ' and assume $\lambda^{<\theta} = \lambda$.

Let (N, M) be a witness to $f_T(\kappa, \lambda) > \lambda$. For every $u \subseteq N$ of size $< \theta$, let $M_u \subseteq \mathbb{M}$ be a $(\kappa + |u|)^{+}$-saturated model of size $\leq 2^{|u|+\kappa}$ containing $M \cup u$. Let $N_0 = \bigcup_{u \in N^{[<\theta]}} M_u$. So $N_0 \supseteq N$, and $|N_0| \leq \lambda \cdot 2^{<\theta+\kappa} = \lambda$. Repeating the construction with respect to (N_0, M), construct N_1, and more generally N_i for $i \leq \theta$, taking union in limit steps. So $|N_\theta| \leq \lambda \cdot \theta = \lambda$ and for every subset $u \subseteq N_\theta$ such that $|u| < \theta$, there is some model $M \cup u \subseteq M_u \subseteq N_\theta$ which is $(\kappa + |u|)^{+}$-saturated.

Fix $p(x) \in S^{\mathrm{nf}}(N_\theta, M)$. We try to choose by induction on $\alpha < \theta$, formulas $\varphi^p_\alpha(x, y)$ and sequences $\bar{a}^p_\alpha = \langle a^p_{\alpha,i} \mid i < \omega \rangle$ in $N_{\alpha+1}$ such that $\bar{a}^p_\alpha$ is indiscernible over $\left\{ \bar{a}^p_\beta \mid \beta < \alpha \right\} \cup M$ and $\varphi^p_\alpha(x, a^p_{\alpha,0}) \wedge \neg\varphi^p_\alpha(x, a^p_{\alpha,1}) \in p$. If we succeed, then we found a non-forking pattern of depth θ over M as desired. Otherwise, we are stuck in some $\alpha_p < \theta$. Let $A_p = \bigcup \left\{ \bar{a}^p_\beta \mid \beta < \alpha_p \right\}$.

Let $F \subseteq S^{\mathrm{nf}}(N_\theta, M)$ be a set of size $> \lambda$ such that for $p \neq q \in F$, $p|_N \neq q|_N$. As the size of the set $\{A_p \mid p \in F\}$ is bounded by $\lambda^{<\theta} = \lambda$ there is some A and α such that, letting $S = \{p \in F \mid A_p = A \wedge \alpha_p = \alpha\}$, $|S| > \lambda$. Let $M_0 \subseteq N_\alpha$ be some model containing $A \cup M$ of size $\kappa + |A|$. Suppose $p \in S$ and $p|_{N_\alpha}$ splits over M_0. Then there is some $(\kappa + |A|)^{+}$-saturated model $N' \subseteq N_{\alpha+1}$ containing M_0 such that $p|_{N'}$ splits over M_0. By Fact 6.3.8(2), we can find an M_0-indiscernible sequence $\langle a^p_{\alpha,i} \mid i < \omega \rangle$ in $N' \subseteq N_{\alpha+1}$ such that $\varphi(x, a^p_{\alpha,0}) \wedge \neg\varphi(x, a^p_{\alpha,1}) \in p$ —

contradicting the choice of α. So, for every $p \in S$, $p|_{N_\alpha}$ does not split over M_0. But then by the choice of F and Fact 6.3.8(1), $|S| \le 2^{2^{\kappa+|A|}}$ — contradiction. $\square$

LEMMA 6.3.16. *If* $g_T(\kappa) > \theta$ *then* $f_T(\kappa,\lambda) \ge \lambda^{\langle\theta\rangle_{\rm tr}}$ *for all* $\lambda \ge \kappa + \sum_{\mu<\theta} \mu$ *(see Definition 6.6.3).*

PROOF. Fix $\lambda \ge \kappa + \theta$. By Lemma 6.3.12, there is some non-forking pattern $\{\bar{a}_\alpha, \varphi_\alpha \mid \alpha < \theta\}$ over a model M of size κ such that $\bar{a}_\alpha = \langle a_{\alpha,i} \mid i < \lambda\rangle$ and $p(x) = \{\varphi_\alpha(x, a_{\alpha,0}) \mid \alpha < \theta\} \cup \{\neg\varphi_\alpha(x, a_{\alpha,i}) \mid \alpha < \theta, 0 < i < \lambda\}$ does not fork over M. By induction on $\beta \le \theta$ we define elementary mappings F_η, $\eta \in \lambda^\beta$, with $\mathrm{dom}(F_\eta) = A_\beta = M \cup \{\bar{a}_\alpha \mid \alpha < \beta\}$:

- $F_\emptyset$ is the identity on M.
- If β is a limit ordinal, then let $F_\eta = \bigcup_{\alpha<\beta} F_{\eta\restriction\alpha}$.
- If $\beta = \alpha + 1$, let $F_{\eta 0}$ be an arbitrary extension of F_η to $A_{\alpha+1}$. For $i < \lambda$, $F_{\eta i}$ be an arbitrary elementary mapping extending F_η such that $F_{\eta i}(a_{\alpha,j}) = F_{\eta 0}(a_{\alpha,i+j})$. This could be done by indiscerniblity.

Let $p_\eta = F_\eta(p)$. So,

- $p_\eta(x)$ does not fork over M — as F_η is an elementary map fixing M.
- If $\eta \ne \nu \in \lambda^\theta$, then $p_\eta \ne p_\nu$. To see it, let $\alpha = \min\{\beta < \theta \mid \eta \restriction \beta \ne \nu \restriction \beta\}$ and suppose $\alpha = \beta + 1$, $\rho = \eta \restriction \beta = \nu \restriction \beta$. Assume $\eta(\beta) = i < j = \nu(\beta)$ and $0 < k < \lambda$ is such that $i + k = j$. Then $\varphi(x, a_{\alpha,0}) \in p \Rightarrow \varphi(x, F_\nu(a_{\alpha,0})) \in p_\nu$. Similarly, $\neg\varphi(x, a_{\alpha,k}) \in p \Rightarrow \neg\varphi(x, F_\eta(a_{\alpha,k})) \in p_\eta$. But,

$$F_\nu(a_{\alpha,0}) = F_{\rho j}(a_{\alpha,0}) = F_{\rho 0}(a_{\alpha,j}) = F_{\rho 0}(a_{i+k}) = F_{\rho i}(a_{\alpha,k}) = F_\eta(a_{\alpha,k}),$$

so $p_\eta \ne p_\nu$.

Let $T \subseteq \lambda^{<\theta}$ be a tree of size $\le \lambda$ such that if $x \in T$ and $y < x$ then $y \in T$. Let $B = \bigcup\{F_\eta(\bar{a}_\alpha) \mid \alpha < \lg(\eta) \wedge \eta \in T\} \cup M$, so $|B| \le \lambda + \kappa + \sum_{\alpha<\theta} |\alpha| = \lambda$. Let N be some model containing B of size λ. Thus, $\left|S^{\rm nf}(N, M)\right|$ is at least the number of branches in T of length θ. By the definition of $\lambda^{\langle\theta\rangle_{\rm tr}}$ we are done. $\square$

PROPOSITION 6.3.17. *If* $f_T(\kappa,\lambda) > \lambda$ *for some* $\lambda \ge 2^{2^\kappa}$*, then* $f_T(\kappa,\lambda) \ge \lambda^{\aleph_0}$ *for all* $\lambda \ge \kappa$.

PROOF. By Lemma 6.3.15, taking $\theta = \aleph_0$, $g_T(\kappa) > \aleph_0$ and then by Remark 6.3.11, $g_T(\aleph_0) > \aleph_0$. By Lemma 6.3.16, $f_T(\aleph_0,\lambda) > \lambda^{\langle\aleph_0\rangle}$ for all λ but $\lambda^{\langle\aleph_0\rangle} = \lambda^{\aleph_0}$ (see Remark 6.6.4). By Remark 6.2.8, $f_T(\kappa,\lambda) \ge f_T(\aleph_0,\lambda) \ge \lambda^{\aleph_0}$ so we are done. $\square$

6.3.5. On (10).

PROPOSITION 6.3.18. *If* $f_T(\kappa,\lambda) > \lambda^\mu$ *for some* $\lambda \ge 2^{2^{\kappa+\mu}}$*, then* $f_T(\kappa,\lambda) \ge \lambda^{\langle\mu^+\rangle_{\rm tr}}$ *for all* $\lambda \ge \kappa \ge \mu^+$.

PROOF. By Lemma 6.3.15, $g_T(\kappa) > \mu^+$. By Lemma 6.2.5, $g_T(\mu^+) > \mu^+$. By Lemma 6.3.16, $f_T(\mu^+,\lambda) \ge \lambda^{\langle\mu^+\rangle_{\rm tr}}$ for all $\lambda \ge \aleph_{n+1}$, and so by Lemma 6.2.8 , $f_T(\kappa,\lambda) \ge \lambda^{\langle\mu^+\rangle_{\rm tr}}$ for any $\lambda \ge \kappa \ge \mu^+$. $\square$

COROLLARY 6.3.19. *If* $f_T(\kappa,\lambda) > \lambda^{\aleph_n}$ *for some* $\lambda \ge 2^{2^{\kappa+\aleph_n}}$*, then* $f_T(\kappa,\lambda) \ge \lambda^{\langle\aleph_{n+1}\rangle_{\rm tr}}$ *for all* $\lambda \ge \kappa \ge \aleph_{n+1}$.

This corollary says that morally there are gaps between λ and $\lambda^{\aleph_0}$, $\lambda^{\aleph_0}$ and $\lambda^{\aleph_1}$ etc.

6.3.6. On the gap between (11) and (12).

The following fact follows from the proof of Morley's two cardinal theorem. For details, see [**Kei71**, Theorem 23].

FACT 6.3.20. *Suppose $\psi \in L_{\omega_1,\omega}$, $<$ is a binary relation, P and Q are predicates in L and ψ implies that "$<$ is a linear order on Q". If for every countable ordinal ε there is a structure B such that*

- $B \models \psi$
- *There is an embedding of the order $\beth_\varepsilon\left(\left|P^B\right|\right)$ into $\left(Q^B, <^B\right)$.*

Then for every cardinal λ there is some structure B such that

- $B \models \psi$
- $\left|P^B\right| = \aleph_0$
- *there is an embedding of $(\lambda, <)$ into $\left(Q^B, <^B\right)$.*

LEMMA 6.3.21. *Let $M \prec N$ and $a \in N$. Then the following are equivalent:*

1. *$\varphi(x, a)$ forks over M.*
2. *The following holds in N:*

$$\bigvee_{\{\psi_0,\dots,\psi_{m-1}\}\subseteq L} \bigvee_{k_i<\omega, i<m} \bigwedge_{\Delta\subseteq L\, \mathit{finite}} \bigwedge_{n<\omega} \forall c_0,\dots,c_{n-1} \in M \exists \bar{y}_0,\dots,\exists \bar{y}_{m-1}$$
$$\left(\varphi(x,a) \vdash \bigvee_{i<n} \psi(x, y_{i,0}) \wedge \bigwedge_{i<m, j<n}\left(y_{i,j} \equiv^{\Delta}_{\bar{c}} y_{i,0}\right) \wedge \bigwedge_{i<m,\, s\in n^{[k_i]}} \forall x\left(\neg \bigwedge_{j\in s} \varphi(x, y_{i,j})\right)\right)$$

where $\bar{y}_i = \langle y_{i,j} \,|\, j < n\rangle$ for $i < m$ and $\bar{c} = \langle c_i \,|\, i < n\rangle$.

PROOF. By compactness. □

LEMMA 6.3.22. *If $g_T(\kappa) > \mu > \aleph_0$, then there is a non-forking pattern $\{\varphi_\alpha, \bar{a}_\alpha \,|\, \alpha < \mu\}$ such that $\varphi_\alpha = \varphi$ for some formula φ.*

PROOF. By pigeon-hole. □

PROPOSITION 6.3.23. *If for all $\varepsilon < \aleph_1$, there is some κ such that $g_T(\kappa) > \beth_\varepsilon(\kappa)$ then $g_T(\aleph_0) = \infty$.*

PROOF. By Lemma 6.3.22, for every $\varepsilon < \aleph_1$ there is some formula φ_ε and a non-forking pattern $\{\varphi_\varepsilon, \bar{a}^\varepsilon_\alpha \,|\, \alpha < \beth_\varepsilon(\kappa)\}$ over a model M_ε of size κ. We may assume that $\varphi_\varepsilon = \varphi$ for all $\varepsilon < \aleph_1$.

Let ψ be the following $L_{\omega_1,\omega}$ sentence in the language

$$\{P(x), S(x), Q(\alpha), <(\alpha,\beta), R(x,\alpha), <_R(x,y,\alpha)\} \cup L(T)$$

saying:

1. $S \models T$
2. P is an L-elementary substructure of S.
3. $S \cap Q = \emptyset$
4. The universe is $S \cup Q$.
5. Q is infinite and$<$ is a linear order on Q.
6. For each $\alpha \in Q$, $R(-,\alpha)$ is infinite and contained in S and $<_R(-,-,\alpha)$ is discrete linear order with a first element on $R(-,\alpha)$.
7. For each $\alpha \in Q$, $R(-,\alpha)$ is an L-indiscernible sequence over $P \cup \bigcup_{j<i} R(-,\alpha)$ ordered by $<_R(-,-,\alpha)$.

(8) The set $\{\varphi(x, y_{\alpha,0}) \wedge \neg\varphi(x, y_{\alpha,1}) \mid \alpha \in Q\}$ does not fork over P (in the sense of L), where $y_{\alpha,0}$ and $y_{\alpha,1}$ are the first elements in the sequence $R(-, \alpha)$.

Note that (6) can be expressed in $L_{\omega_1,\omega}$ by Lemma 6.3.21.

As the assumptions of Fact 6.3.20 are satisfied, for each λ we find a model B of ψ such that:

- $|P^B| = \aleph_0$
- There is an embedding h of $(\lambda, <)$ into $(Q^B, <^B)$.

For all $\alpha < \lambda$ let $\bar{a}_\alpha$ be an infinite sub-sequence of $R(B, h(\alpha))$ and let $M = P(B)$. By (1) – (8), it follows that $\{\varphi, \bar{a}_\alpha \mid \alpha < \lambda\}$ is a non-forking pattern of depth λ over M — as wanted. $\square$

Corollary 6.3.24. *(1) If for all $\varepsilon < \aleph_1$, there is some κ such that $g_T(\kappa) > \beth_\varepsilon(\kappa)$ then $f_T(\lambda, \kappa) \geq \operatorname{ded}(\lambda)$ for all $\lambda \geq \kappa$.*

(2) If for every $\varepsilon < \aleph_1$ there is some $\lambda \geq \beth_\varepsilon(\kappa)$ such that $f_T(\lambda, \kappa) > \lambda^{<\beth_\varepsilon(\kappa)}$ then $f_T(\lambda, \kappa) \geq \operatorname{ded}(\lambda)$ for all $\lambda \geq \kappa$.

(3) If $f_T(\lambda, \kappa) > \lambda^{<\beth_{\aleph_1}(\kappa)}$ for some $\lambda \geq \beth_{\aleph_1}(\kappa)$, then $f_T(\lambda, \kappa) \geq \operatorname{ded}(\lambda)$ for all $\lambda \geq \kappa$.

Proof. (1) By Lemma 6.3.23, we know that $g_T(\aleph_0) = \infty$. For any $\lambda \geq \kappa$, by Lemma 6.3.16 we have that $f_T(\kappa, \lambda) \geq \lambda^{\langle\theta\rangle_{tr}}$ for all $\theta \leq \lambda$. As $\operatorname{ded}(\lambda) = \sup\{\lambda^{\langle\theta\rangle_{tr}} \mid \theta \leq \lambda, \text{ is regular}\}$ by Proposition 6.6.5 (6) we get $f_T(\kappa, \lambda) \geq \operatorname{ded}(\lambda)$.

(2) It is enough to show that for every $\varepsilon < \aleph_1$, there is some κ such that $g_T(\kappa) > \beth_\varepsilon(\kappa)$. Let $\varepsilon < \aleph_1$ be a limit ordinal and $\theta = \beth_\varepsilon(\kappa)$. Then

$$\sum_{\mu<\theta} 2^{2^{\kappa+\mu}} = \sum_{\alpha<\varepsilon} 2^{2^{\beth_\alpha(\kappa)}} = \sum_{\alpha<\varepsilon} \beth_{\alpha+2}(\kappa) = \beth_\varepsilon(\kappa).$$

By Lemma 6.3.15, $g_T(\kappa) > \beth_\varepsilon(\kappa)$. So we can apply (1) to conclude.

(3) follows from (2). $\square$

6.3.7. Further observations.

Proposition 6.3.25. *If $f_T(\kappa, \lambda) > \lambda^{\aleph_0}$ for some $\lambda \geq$ and $\lambda \geq \sum_{\mu<\theta} 2^{2^{\kappa+\mu}}$ then $g_T(\kappa) > \theta$.*

6.4. Inside NTP$_2$

NTP$_2$ is a large class of first-order theories containing both NIP and simple theories introduced by Shelah. For a general treatment, see Chapter 3. In this section we show that for theories in this class, the non-forking spectra is well behaved, i.e. it cannot take values between (6) and (16).

Fact 6.4.1. *(see e.g. [**HP11**]) Let $p(x)$ be a global type non-splitting over a set A. For any set $B \supseteq A$, and an ordinal α, let the sequence $\bar{c} = \langle c_i \mid i < \alpha\rangle$ be such that $c_i \models p|_{Bc_{<i}}$. Then $\bar{c}$ is indiscernible over B and its type over B does not depend on the choice of $\bar{c}$. Call this type $p^{(\alpha)}|_B$, and let $p^{(\alpha)} = \bigcup_{B\supseteq A} p^{(\alpha)}|_B$. Then $p^{(\alpha)}$ also does not split over A.*

Definition 6.4.2. (strict invariance) Let $p(x)$ be a global type. We say that p is strictly invariant over a set A if p does not split over A, and if $B \supseteq A$ and $c \models p|_B$ then $\operatorname{tp}(B/cA)$ does not fork over A.

Lemma 6.4.3. *Let p be a global type finitely satisfiable in A. Then there is some model $M \supseteq A$ with $|M| \leq |A| + \aleph_0$ such that $p^{(\omega)}$ is strictly invariant over M.*

Proof. Let M_0 be some model containing A of size $|A| + \aleph_0$. Construct by induction an increasing sequence of models M_i for $i < \omega$, such that $|M_i| = |M_0|$ and for every formula $\varphi(x, y)$ over M if $\varphi(x, c) \in p^{(\omega)}$ for some c, then there is some $c' \in M_{i+1}$ such that $\varphi(x, c') \in p^{(\omega)}$. Let $M = \bigcup_{i<\omega} M_i$. $\square$

In lieu of giving a definition of NTP$_2$, we only state the properties which we will be using from Chapter 1.

Fact 6.4.4. *Let T be NTP$_2$ and $M \models T$, then:*

(1) $\varphi(x, c)$ divides over M if and only if $\varphi(x, c)$ forks over M.
(2) Let $p(x)$ is a global type strictly invariant over M and $\langle c_i \mid i < \omega \rangle \models p^{(\omega)}|_M$. Then for any formula $\varphi(x, c_0)$ dividing over M, $\{\varphi(x, c_i) \mid i < \omega\}$ is inconsistent.

Improving on Chapter 1, Theorem 4.3 we establish the following:

Theorem 6.4.5. *Let T be NTP$_2$. Then the following are equivalent:*

(1) $f_T(\kappa, \lambda) > (\mathrm{ded}\,\kappa)^{\aleph_0}$ for some $\lambda \geq \kappa$.
(2) T has IP.
(3) $f_T(\kappa, \lambda) = 2^\lambda$ for every $\lambda \geq \kappa$.

Proof. (1) implies (2) follows from Fact 6.3.5 and (3) implies (1) is clear.

(2) implies (3): Fix $\lambda \geq \kappa$. Let $\varphi(x, y)$ have IP, and $\bar{a} = \langle a_i \mid i < \omega \rangle$ be an indiscernible sequence such that $\forall U \subseteq \omega \exists b_U \, \varphi(a_i, b_U) \Leftrightarrow i \in U$. Let $p(x)$ be a global non-algebraic type finitely satisfiable in $\bar{a}$. By Lemma 6.4.3, there a model $M \supseteq \bar{a}$ be such that $|M| \leq \aleph_0$ and $p^{(\omega)}$ is strictly invariant over M.

Let $\bar{b} = \langle b_i \mid i < \lambda \rangle$ realize $p^{(\lambda)}|_M$. We show that $p_\eta(x) = \left\{\varphi(x, b_i)^{\text{if } \eta(i)=1} \mid i < \lambda \right\}$ does not divide over M for any $\eta \in 2^\lambda$. First note that $p_\eta(x)$ is consistent for any η, as $\mathrm{tp}\left(\bar{b}/M\right)$ is finitely satisfiable in $\bar{a}$. But as for any $k < \omega$, $\left\langle \left(b_{k \cdot i}, b_{k \cdot i+1}, \ldots, b_{k \cdot (i+1)-1}\right) \mid i < \omega \right\rangle$ realizes $\left(p^{(k)}\right)^{(\omega)}$, Fact 6.4.4(2) implies that $p_\eta(x)|_{b_0 \ldots b_{k-1}}$ does not divide over M for any $k < \omega$. Thus by indiscernibility of $\bar{b}$, $p_\eta(x)$ does not divide over M.

Take $N \supseteq \bar{b} \cup M$ of size λ. By Fact 6.4.4(1) every p_η extends to some $p'_\eta \in S^{\mathrm{nf}}(N, M)$, thus $f_T(\kappa, \lambda) = 2^\lambda$. $\square$

6.5. Examples

6.5.1. Examples of (1) – (6).

Proposition 6.5.1. *(1) If T is the theory of equality, then $f_T(\kappa, \lambda) = \kappa$ for all $\lambda \geq \kappa$.*

(2) Let T be the model companion of the theory of countably many unary relations then $f_T(\kappa, \lambda) = \kappa + 2^{\aleph_0}$ for all $\lambda \geq \kappa$.
(3) Let T be the model companion of the theory of countably many equivalence relations then $f_T(\kappa, \lambda) = \kappa^{\aleph_0}$ for all $\lambda \geq \kappa$.
(4) Let $T = \mathrm{DLO}$. Then $f_T(\kappa, \lambda) = \mathrm{ded}(\kappa)$ for all $\lambda \geq \kappa$.
(5) Let T be the model companion of infinitely many linear orders. Then $f_T(\kappa, \lambda) = \mathrm{ded}(\kappa)^{\aleph_0}$.

PROOF. (1) – (3): it is well known that these examples have the corresponding $f_T(\kappa)$'s, and that they are stable. It follows from Remark 6.3.2 that they have the corresponding $f_T(\kappa, \lambda)$.

(4): It is easy to check that every type has finitely many non-splitting global extensions, but DLO is NIP so by Fact 6.3.8 every non-forking extension is non-splitting. Since $f_T(\kappa) = \mathrm{ded}(\kappa)$ for this theory, we are done.

(5): This theory is NIP so $f_T(\kappa, \lambda) \leq \mathrm{ded}(\kappa)^{\aleph_0}$ by Fact 6.3.5, and clearly $f_T(\kappa) = (\mathrm{ded}\,\kappa)^{\aleph_0}$. □

6.5.2. Circularization.

We shall first describe a general construction for examples of non-forking spectra functions.

For this section, a "formula" means an $\emptyset$-definable formula unless otherwise specified. Most formulas we work with are partitioned formulas, $\varphi(\bar{x}; \bar{y})$, where the variables are broken into two distinct sets. We write φ instead of $\varphi(\bar{x}; \bar{y})$ when the partition is clear from the context. We let $\varphi^1 = \varphi$ and $\varphi^0 = \neg\varphi$. We assume that our languages relational in this section (so a subset is a substructure).

6.5.2.1. *Circularization: Base step.*

The dense circular order was used as an example of a theory where forking is not the same as dividing (see e.g. [**Kim96**, Example 2.11]). The reason is that with circular ordering around, it is hard not to fork.

DEFINITION 6.5.2. A circular order on a finite set is a ternary relation obtained by placing the points on a circle and taking all triples in clockwise order. For an infinite set, a circular order is a ternary relation such that the restriction to any finite set is a circular order. Equivalently, a circular order is a ternary relation C such that for every x, $C(x, -, -)$ is a linear order on $\{y \mid y \neq x\}$ and $C(x, y, z) \to C(y, z, x)$ for all x, y, z. Denote the theory of circular orders by T_C.

The following definitions are well-known.

DEFINITION 6.5.3. Let K be a class of L-structures (where L is relational).

(1) We say that K has the strong amalgamation property (*SAP*) if for every $A, B, C \in K$ and embeddings $i_1 : A \to B$ and $i_2 : A \to C$ there exist both a structure $D \in K$ and embeddings $j_1 : B \to D$, $j_2 : C \to D$ such that
   (a) $j_1 \circ i_1 = j_2 \circ i_2$ and
   (b) $j_1(B) \cap j_2(C) = (j_1 \circ i_1)(A) = (j_2 \circ i_2)(A)$.
(2) We say that K has the disjoint embedding property (*DEP*) if for any 2 structures $A, B \in K$, there exists a structure $C \in K$ and embeddings $j_1 : B \to C$, $j_2 : A \to C$ such that $j_1(A) \cap j_2(B) = \emptyset$.
(3) We say that a first-order theory T has these properties if its class of (finite) models has them.

Note that

REMARK 6.5.4. T_C is universal and it has DEP and SAP.

FACT 6.5.5. *Let T be a universal theory with DEP and SAP in a finite relational language L, then:*

(1) [**Hod93**, Theorem 7.4.1] *It has a model completion T_0 which is ω-categorical and eliminates quantifiers.*
(2) [**Hod93**, Theorem 7.1.8] *If $A \subseteq M \models T_0$ then* $\mathrm{acl}(A) = A$.

Corollary 6.5.6. *Suppose that* $\varphi(\bar{x};\bar{y})$ *is a formula in* L, $\bar{a} \in M \models T_0$. *If* $M \models \exists \bar{z}\varphi(\bar{z};\bar{a}) \wedge \bar{z} \nsubseteq \bar{a}$ *then* $\left\{\bar{t} \in M \,\middle|\, \varphi\left(\bar{t};\bar{a}\right)\right\}$ *is infinite.*

Definition 6.5.7. For any formula $\varphi(\bar{x};\bar{y})$ in L where $\bar{x}$ is not empty , let $C[\varphi(\bar{x};\bar{y})]$ be a new $\lg(\bar{y}) + 3 \cdot \lg(\bar{x})$-place relation symbol. Denote $L[\varphi(\bar{x};\bar{y})] = L \cup \{C[\varphi(\bar{x};\bar{y})]\}$.

Definition 6.5.8. Suppose $\varphi(\bar{x};\bar{y})$ is a quantifier free formula in L with $\bar{x}$ not empty. Let $T[\varphi(\bar{x};\bar{y})]$ be the theory in $L[\varphi(\bar{x};\bar{y})]$ containing T and the following axioms:

- For all $\bar{t}$ in the length of $\bar{y}$, the set:
$$S[\varphi(\bar{x};\bar{y})]\left(\bar{t}\right) := \left\{\bar{s} \,\middle|\, \bar{s} \cap \bar{t} = \emptyset \wedge \lg(\bar{s}) = \lg(\bar{x}) \wedge \varphi\left(\bar{s};\bar{t}\right)\right\}$$
is circularly ordered by the relation:
$$C[\varphi(\bar{x};\bar{y})]\left(\bar{t}\right) := \left\{(\bar{s}_1,\bar{s}_2,\bar{s}_3) \,\middle|\, C[\varphi(\bar{x},\bar{y})]\left(\bar{t},\bar{s}_1,\bar{s}_2,\bar{s}_3\right)\right\}$$
(i.e. $C[\varphi(\bar{x};\bar{y})]$ with index $\bar{t}$ orders this set in a circular order). Call $\bar{t}$ the index variables, and $\bar{s}$ the main variables.
- If $C[\varphi(\bar{x};\bar{y})]\left(\bar{t}\right)(\bar{s}_1,\bar{s}_2,\bar{s}_3)$ then $\bar{s}_1,\bar{s}_2,\bar{s}_3 \in S[\varphi(\bar{x};\bar{y})]\left(\bar{t}\right)$.

Claim 6.5.9. If φ is as in the definition, then

(1) $T[\varphi]$ is universal.
(2) $T[\varphi]$ has DEP.
(3) $T[\varphi]$ has SAP.

Proof. As T_C is universal, (1) is clear (note that this uses the fact that φ is quantifier free).

(3): Let M'_0, M'_1 and M'_2 be models of $T[\varphi]$ such that $M'_0 = M'_1 \cap M'_2$. Let $M_i = M'_i \restriction L$ for $i < 3$. By assumption, there is a model $M_3 \models T$ such that $M_1 \cup M_2 \subseteq M_3$. We define M'_3 as an expansion of M_3. Let $\bar{t} \in M_3$ be a tuple of length $\lg(\bar{y})$. Split into cases:

Case 1. $\bar{t} \in M'_0$. In this case, $\left(S^{M'_i}[\varphi]\left(\bar{t}\right), C^{M'_i}[\varphi]\left(\bar{t}\right)\right)$ are circular orders for $i < 3$ and $S^{M'_1}[\varphi]\left(\bar{t}\right) \cap S^{M'_2}[\varphi]\left(\bar{t}\right) = S^{M'_0}[\varphi]\left(\bar{t}\right)$ so we can amalgamate them as circular orders and extend it arbitrarily to $S^{M_3}[\varphi]\left(\bar{t}\right)$, and that will be $C^{M'_3}[\varphi]\left(\bar{t}\right)$.

Note that in the special case where $S^{M_0}[\varphi]\left(\bar{t}\right) = \emptyset$, there are no restrictions on the place of $S^{M_i}[\varphi]\left(\bar{t}\right)$ for $i < 3$ in this order.

Case 2. $\bar{t} \in M_1 \backslash M_2$. Then $\left(S^{M'_1}[\varphi]\left(\bar{t}\right), C^{M'_1}[\varphi]\left(\bar{t}\right)\right)$ is a circular order. Extend it so that its domain would be $S^{M_3}[\varphi]\left(\bar{t}\right)$ arbitrarily.

Case 3. $\bar{t} \in M_2 \backslash M_1$ — the same.

Case 4. $\bar{t} \notin M_1$ and $\bar{t} \notin M_2$. Then $C^{M'_3}[\varphi]\left(\bar{t}\right)$ is any circular order on $S^{M_3}[\varphi]\left(\bar{t}\right)$.

(2): Similar to (3), but easier. □

Remark 6.5.10. It is follows from the proof of amalgamation, that if $M \models T$ contains models $M_0 \subseteq M_i \subseteq M$ for $i < n$ such that $M_0 = M_i \cap M_j$ for $i < j < n$ and for each M_i, there is an expansion M'_i to a model of $T[\varphi]$ such that $M'_0 \subseteq M'_i$ then there is an expansion M' of M to a model of $T[\varphi]$ such that $M'_i \subseteq M'$.

Claim 6.5.11.

(1) If $M \models T$, then we can expand it to a model M' of $T[\varphi]$.
(2) Moreover: if $B \subseteq M$ and there is already an expansion B' of B to a model of $T[\varphi]$, then we can expand M in such a way that $B' \subseteq M'$.
(3) Moreover: suppose that
- $A \subseteq M$
- $\langle \bar{c}_i \,|\, i < n \rangle$ is a finite sequence of finite tuples from M, such that $\bar{c}_i \cap \bar{c}_j \subseteq A$, $\mathrm{tp}_{\mathrm{qf}}(\bar{c}_i/A) = \mathrm{tp}_{\mathrm{qf}}(\bar{c}_j/A)$ for all $i < j < n$.
- M'_0 is an expansion of $A\bar{c}_0$ to a model of $T[\varphi]$.

Then we can find an expansion M' such that the quantifier free types are still equal in the sense of $L[\varphi]$ and $M'_0 \subseteq M'$.

Proof. (2): For any $\bar{t}$ in the length of $\bar{y}$, if $\bar{t} \in B$ then we choose a circular order $C^{M'}[\varphi](\bar{t})$ that extends $C^{B'}[\varphi](\bar{t})$ on $S^M[\varphi](\bar{t})$. If not, then define it arbitrarily.

(3): Let $M_i = A\bar{c}_i$. As $\bar{c}_0 \equiv^{\mathrm{qf}}_A \bar{c}_i$ for $i < n$, there are isomorphisms $f_i : M_0 \to M_i$ of L that fix A and take $\bar{c}_0$ to $\bar{c}_i$. So f_i induces expansions M'_i of M_i, isomorphic (via f_i) to M'_0. As the intersection of any two models M_i is exactly A, by Remark 6.5.10, there is an expansion M' of M to a model of $T[\varphi]$ that contains M'_i. In this expansion the quantifier free types will remain the same because f_i are $L[\varphi]$-isomorphisms. $\square$

Corollary 6.5.12. *Suppose that* $M' \models T[\varphi]$, $M' \restriction L \subseteq N \models T$. *Then there is an expansion of* N *to a model* N' *of* $T[\varphi]$ *such that* $M' \subseteq N'$. *In particular, if* $M' \models T[\varphi]$ *is existentially closed, then* $M' \restriction L$ *is an existentially closed model of* T. *Denote by* $T_0[\varphi]$ *the model completion of* $T[\varphi]$. *We will call it the* φ*-circularization of* T_0. *It follows that* $T_0[\varphi] \restriction L = T_0$ *(for more see* [**Hod93**, Theorem 8.2.4]*).*

We turn to dividing:

Claim 6.5.13. Assume that $M \models T_0[\varphi]$, $A \subseteq M$, $\bar{a} \in M$, $S^M[\varphi](\bar{a}) \cap A^{\lg(\bar{x})} = \emptyset$, and $\bar{c} \neq \bar{d} \in S^M[\varphi](\bar{a})$. Then the formula $\psi\left(\bar{z};\bar{a},\bar{c},\bar{d}\right) = C[\varphi]\left(\bar{a},\bar{c},\bar{z},\bar{d}\right)$ 2-divides over $A\bar{a}$.

Proof. Let $M_0 = A\bar{a}$, $M_1 = M_0\bar{c}\bar{d}$ and $M_2 = M_0\bar{c}'\bar{d}'$ where $M_1 \cap M_2 = M_0$ and there is an isomorphism $f : M_1 \to M_2$ that fixes M_0 and takes $\bar{c}\bar{d}$ to $\bar{c}'\bar{d}'$.

By SAP, there is a model $M_3 \models T[\varphi]$ that contains $M_1 \cup M_2$. We wish to choose it carefully: in the proof of Claim 6.5.9, we saw that there are no constraints on the amalgamation of $C^{M_1}[\varphi](\bar{a})$ and $C^{M_2}[\varphi](\bar{a})$ (because $S^{M_0}[\varphi](\bar{a}) = \emptyset$, see the definition of $S[\varphi]$). In particular we can put $\bar{c}'$ and $\bar{d}'$ so that in the circular order we have $\bar{c} \to \bar{d} \to \bar{c}' \to \bar{d}' \to \bar{c}$, and in this case there is no $\bar{z}$ such that $C[\varphi](\bar{a})\left(\bar{c},\bar{z},\bar{d}\right)$ and $C[\varphi](\bar{a})\left(\bar{c}',\bar{z},\bar{d}'\right)$.

Applying the same technique n times, there is a model of $T[\varphi]$ with a sequence $\left\langle \bar{c}_i, \bar{d}_i \,|\, i < n \right\rangle$ that contains M_1 and satisfies $\mathrm{tp}_{\mathrm{qf}}\left(\bar{c}_i\bar{d}_i/A\bar{a}\right) = \mathrm{tp}_{\mathrm{qf}}\left(\bar{c}\bar{d}/A\bar{a}\right)$, so that in the circular order $C[\varphi](\bar{a})$ the tuples will be ordered as follows: $\bar{c} \to \bar{d} \to \bar{c}_1 \to \bar{d}_1 \to \ldots \to \bar{c}_n \to \bar{d}_n \to \bar{c}$. Hence, there is a model of $T_0[\varphi]$ and an infinite such sequence, and this sequence witnesses the 2-dividing of $\psi\left(\bar{z};a,\bar{c},\bar{d}\right)$.

Note that the tuples $\bar{c}_i\bar{d}_i$ were chosen so that the intersection of each pair $\bar{c}_i\bar{d}_i$, $\bar{c}_j\bar{d}_j$ is contained in A. $\square$

The last sentence justifies the following auxiliary definition which will make life a bit easier:

DEFINITION 6.5.14. Say that a formula $\varphi(\bar{x}, \bar{a})$ k-*divides disjointly* over A if there is an indiscernible sequence $\langle \bar{a}_i \mid i < \omega \rangle$ that witnesses k-dividing and moreover $\bar{a}_i \cap \bar{a}_j \subseteq A$.

REMARK 6.5.15. Note that if $\varphi(\bar{x}, \bar{a})$ divides over A, then it divide disjointly over some $B \supseteq A$ (if I is an indiscernible sequence witnessing dividing, then $B = A \cup \bigcap I$).

We shall also need some kind of a converse to the last claim. More precisely, we need to say when a formula does not divide.

CLAIM 6.5.16. Suppose

(1) $A \subseteq M \models T_0[\varphi]$
(2) $p(\bar{x}) = p_1(\bar{x}) \cup p_2(\bar{x})$ is a complete quantifier-free type over M.
(3) $p_1(\bar{x})$ is a complete L type over M and $p_2(\bar{x})$ is a complete $\{C[\varphi]\}$ type over M.
(4) $p_1(\bar{x})$ does not divide over A (as an L-type so also as an $L[\varphi]$-type).
(5) For all $\bar{t} \in M^{\lg(\bar{y})}$, $p_2(\bar{x}) \restriction \{C[\varphi](\bar{t}, -, -, -)\}$ does not divide over $A\bar{t}$ (this means all formulas in $p_2(\bar{x})$ of the form $C[\varphi](\bar{t}, \bar{z}_1, \bar{z}_2, \bar{z}_3)$ where $\bar{x}$ substitutes the $\bar{z}$'s in some places and in the others there are parameters from M).

Then $p(\bar{x})$ does not divide over A.

In particular, if both $p_1(\bar{x})$, $p_2(\bar{x})$ do not divide over A, then $p(\bar{x})$ does not divide over A.

PROOF. Denote $\bar{x} = (x_0, \dots, x_{m-1})$, $p(\bar{x}, M) = p(\bar{x})$. We may assume that $p \restriction x_i$ is non-algebraic for all $i < m$ (otherwise, by Fact 6.5.5, $(x_i = c) \in p$ for some $c \in M$, so $c \in A$ as $x = c$ divides over A, and we can replace x_i by c). Suppose $\langle M_i \mid i < \omega \rangle$ is an $L[\varphi]$-indiscernible sequence over A in some model $N \supseteq M$ such that $M_0 = M$. We will show that $\bigcup\{p(\bar{x}, M_i) \mid i < \omega\}$ is consistent.

Let $\bar{c} \models \bigcup\{p_1(\bar{x}, M_i)\}$ (exists by (4)), and $B = \bigcup\{M_i \mid i < \omega\}$ and let $B' = B\bar{c} \restriction L$ (i.e. forget $C[\varphi]$). Also let $\bar{d} \models p(\bar{x})$ be in some other model $N' = M\bar{d}$ of $T[\varphi]$.

For $\bar{t} \in (B\bar{c})^{\lg(\bar{y})}$ we define a circular order on $S[\varphi](\bar{t})$ to make B' into a model U of $T[\varphi]$ extending B such that $\bar{c} \models \bigcup\{p(\bar{x}, M_i)\}$.

Case 1. $\bar{t} \nsubseteq M_i\bar{c}$ for any $i < \omega$. In this case, there is no information on $C[\varphi](\bar{t})$ in $\bigcup\{p_2(\bar{x}, M_i)\}$, so let $C[\varphi]^U(\bar{t})$ be any circular order on $S[\varphi](\bar{t})$ that extends the circular order $C[\varphi]^B(\bar{t})$ (in case $\bar{t} \subseteq B$).

Case 2. $\bar{t} \subseteq M_i\bar{c}$ for some $i < \omega$, but $\bar{t} \nsubseteq M_j\bar{c}$ for some other $j \neq i$. By indiscernibility, it follows that $\bar{t} \nsubseteq M_j\bar{c}$ for all $j \neq i$. Let $\sigma : M_i\bar{c} \to M\bar{d}$ be an L-isomorphism. There are two sub-cases:

Case i. $\bar{t} \cap \bar{c} \neq \emptyset$. Define $C[\varphi]^U(\bar{t})$ as any extension of $\sigma^{-1}\left(C[\varphi]^{N'}(\sigma(\bar{t}))\right)$ to $S^U[\varphi](\bar{t})$.

Case ii. $\bar{t} \cap \bar{c} = \emptyset$. Then $C[\varphi]^B(\bar{t})$ is already some circular order on $S^B[\varphi](\bar{t})$. On the other hand, $\sigma^{-1}\left(C[\varphi]^{N'}(\sigma(\bar{t}))\right)$ defines some circular order on $S^{M_i\bar{c}}[\varphi](\bar{t})$. The intersection is

$S^{M_i}[\varphi](\bar{t})$ on which they agree, so we can amalgamate the two circular orders.

Case 3. $\bar{t} \subseteq \bigcap M_i$. In this case, by (5), $p_2(\bar{x}) \restriction \{C[\varphi](\bar{t}, -, -, -)\}$ does not divide over $A\bar{t}$, so let $\bar{c}' \models \bigcup \{p_2(\bar{x}, M_i) \restriction C[\varphi](\bar{t}, -, -, -) \,|\, i < \omega\}$. Let U' be the $L[\varphi]$ structure $B\bar{c}'$. Let $f : B\bar{c} \to B\bar{c}'$ fix B and take $\bar{c}$ to $\bar{c}'$. Now, $C^{U'}[\varphi](f(\bar{t}))$ induces a circular order on

$$S = f^{-1}\left(S^{U'}[\varphi](f(\bar{t}))\right) \cap S^{B'}[\varphi](\bar{t}).$$

Extend it to some circular order on $S^U[\varphi](\bar{t})$ and let it be $C^U[\varphi](\bar{t})$.

Case 4. $\bar{t} \subseteq \bigcap M_i\bar{c}$, and $\bar{t} \cap \bar{c} \neq \emptyset$. Let $\sigma_i : M_i\bar{c} \to M\bar{d}$ be the L-isomorphism fixing $\bigcap M_i$ and taking $\bar{c}$ to $\bar{d}$. σ_i induces a circular order on $S^{M_i\bar{c}}[\varphi](\bar{t})$, and the intersection of any two $S^{M_i\bar{c}}[\varphi](\bar{t})$ and $S^{M_j\bar{c}}[\varphi](\bar{t})$ is $S^{\bigcap M_i\bar{c}}[\varphi](\bar{t})$ on which these circular orders agree. By amalgamation, we have a circular order on the union $\bigcup_i S^{M_i\bar{c}}[\varphi](\bar{t})$ that we can expand to a circular order on $S^U[\varphi](\bar{t})$.

□

Claim 6.5.17. Let $A \subseteq M \models T_0[\varphi]$ be $|A|^+$-saturated and $M' = M \restriction L$. Suppose that $\psi(\bar{z}, \bar{a})$, a quantifier free L-formula, k-divides disjointly over A in M'. Then the same is true in M.

Proof. Suppose that $I = \langle \bar{a}_i \,|\, i < \omega \rangle \subseteq M$ witnesses k-dividing disjointly of $\psi(\bar{z}, \bar{a})$ over A in the sense of L. Assume that $\bar{a}_0 = \bar{a}$.

By Claim 6.5.11 (3) and compactness, we can expand and extend M' to $M'' \models T_0[\varphi]$ that will keep the equality of types of the tuples in the sequence. In addition, the interpretation of the new relation $C[\varphi]$ on $A\bar{a}$ remains as it was in M. In particular, in M'', $\psi(\bar{z}, \bar{a})$ still k-divides over A. We may amalgamate a copy of M'' with M over $A\bar{a}$ to get a bigger model in which $\psi(\bar{z}, \bar{a})$ still k-divides disjointly and by saturation this is still true in M. □

6.5.2.2. *Circularization: Iterations.*
Suppose we have a sequence of theories $\mathcal{T} = \langle T_i^{\forall} \,|\, i \leq \omega \rangle$ and formulas $\langle \varphi_i(\bar{x}_i; \bar{y}_i) \,|\, i < \omega \rangle$ in the finite relational languages $\langle L_i \,|\, i \leq \omega \rangle$ where:

- $T_0^{\forall}$ is a universal theory with SAP and DEP in L_0.
- $T_i^{\forall}$ is a theory in L_i for $i \leq \omega$.
- $\varphi_i(\bar{x}_i; \bar{y}_i)$ is a quantifier free formula in L_i.
- $L_i = L_i[\varphi_i(\bar{x}_i; \bar{y}_i)]$ and $T_{i+1}^{\forall} = T_i^{\forall}[\varphi_i(\bar{x}_i; \bar{y}_i)]$.
- $L_\omega = \bigcup \{L_i \,|\, i < \omega\}$ and $T_\omega^{\forall} = \bigcup \{T_i^{\forall} \,|\, i < \omega\}$.

Proposition 6.5.18. *In the situation above,* $T_i^{\forall}$ *has a model completion* T_i, $T_i \subseteq T_{i+1}$ *and* $T_i \subseteq T_\omega$ *which is the model completion of* $T_\omega^{\forall}$ *for all* $i < \omega$.

Proof. Follows from Claim 6.5.9 and Claim 6.5.12. □

From now on, we work in $T := T_\omega$. Call T_ω the $\bar{\varphi}$-circularization of T_0 where $\bar{\varphi} = \langle \varphi_i \,|\, i < \omega \rangle$. Let $M \models T$ and $A \subseteq M$.

Claim 6.5.19. Suppose $\varphi(\bar{x}; \bar{y}) = \varphi_i(\bar{x}_i; \bar{y}_i)$ for some $i < \omega$. Then for all $\bar{a} \in M^{\lg(\bar{y})}$, $\varphi(\bar{z}, \bar{a}) \wedge (\bar{z} \cap (\bar{a} \cap A) = \emptyset)$ forks over A if and only if it is not satisfied in A.

Proof. Denote $\bar{a}' = \bar{a} \cap A$, and $\alpha(\bar{z}, \bar{a}) = \varphi(\bar{z}, \bar{a}) \wedge (\bar{z} \cap \bar{a}' = \emptyset)$. Obviously if α is satisfied in A it does not fork over A.

Suppose α is not satisfied in A. Consider the formula $\psi(\bar{z}, \bar{a}) = \varphi(\bar{z}, \bar{a}) \wedge (\bar{z} \cap \bar{a} = \emptyset)$. First we prove that ψ forks. It defines $S[\varphi]^M(\bar{a})$, and by assumption $S[\varphi]^M(\bar{a}) \cap A = \emptyset$. Note that for all $\bar{c} \neq \bar{d} \in S^M[\varphi](\bar{a})$, since $C^M[\varphi](\bar{a})$ orders this set in a circular order,

$$S[\varphi](\bar{a})(\bar{z}) \vdash C[\varphi](\bar{a})\left(\bar{c}, \bar{z}, \bar{d}\right) \vee C[\varphi](\bar{a})\left(\bar{d}, \bar{z}, \bar{c}\right) \vee \bar{z} = \bar{c} \vee \bar{z} = \bar{d}.$$

If $S[\varphi]^M(\bar{a}) = \emptyset$ we are done. If not, (by Corollary 6.5.6) this set is infinite and there are such $\bar{c}, \bar{d}$.

By Claim 6.5.13 and Claim 6.5.17, it follows that $C[\varphi](\bar{a})\left(\bar{c}, \bar{z}, \bar{d}\right)$, $C[\varphi](\bar{a})\left(\bar{d}, \bar{z}, \bar{c}\right)$ divides over $A\bar{a}$. By Corollary 6.5.6, both $\bar{z} = \bar{c}$ and $\bar{z} = \bar{d}$ divides over $A\bar{a}$. This means that $S[\varphi](\bar{a})(\bar{z}) = \psi(\bar{z}, \bar{a})$ forks over A.

Now, $\alpha(\bar{z}, \bar{a}) \vdash \psi(\bar{z}, \bar{a}) \vee \bigvee_{i,j}(z_i = a_j)$ (where z_i, a_j run over all the variables and parameters from $\bar{a} \backslash A$ in φ). But the formula $z_i = a_j$ divides over A when $a_j \notin A$ (By Corollary 6.5.6), so we are done. $\square$

On the other hand, we have:

Claim 6.5.20. Suppose that $p(\bar{x})$ is a (quantifier free) type over M such that:

- $p_0(\bar{x}) = p \restriction L_0$ does not divide over A.
- $p_i(\bar{x}) = p \restriction L_{i+1} \backslash L_i$ does not divide over A.

Then p does not divide over A.

Proof. By induction on $i < \omega$ we show that $p_i' = p \restriction L_i$ does not divide over A. For $i = 0$ it is given. For $i + 1$ use Claim 6.5.16. $\square$

The following definition is a bit vague

Proposition 6.5.21. *Let $\mathcal{F}$ be a function defined on the class of all countable relational first-order languages such that $\mathcal{F}(L)$ is a set of quantifier free partitioned formulas in L. Let T_0 be a universal theory in the language L_0 satisfying SAP and DEP. We define:*

- *For $n < \omega$, let $L_{n+1} = \bigcup\{L_n[\varphi(\bar{x}; \bar{y})] \mid \varphi(\bar{x}; \bar{y}) \in \mathcal{F}(L_n)\}$ and $L_\omega = \bigcup\{L_n \mid n < \omega\}$.*
- *For $n < \omega$, let $T_n^\forall$ be a universal theory in L_n defined by induction on $n \leq \omega$:*
  - $T_0^\forall = T_0$
  - $T_{n+1}^\forall = \bigcup\left\{T_n^\forall[\varphi(\bar{x}; \bar{y})] \mid \varphi \in \mathcal{F}(L_n)\right\}$
  - $T_\omega^\forall = \bigcup\left\{T_n^\forall \mid n < \omega\right\}$

Then $T_\omega^\forall$ has a model completion which we denote by $\circlearrowleft_{T_0, L_0, \mathcal{F}}$. Moreover, it is a $\bar{\varphi}$-circularization for some choice of $\bar{\varphi}$.

Proof. By carefully choosing an enumeration of the formulas in L_ω, we can reconstruct $T_\omega^\forall, L_\omega$ in such a way that in each step we deal with one formula and it has a model completion by Proposition 6.5.18. $\square$

6.5.3. Example of (7).

Definition 6.5.22. Let $L_0 = \{=\}$ and T_0 be empty. Let $\mathcal{F}(L)$ be the set of all quantifier free partitioned formulas from L. Let $T = \circlearrowright_{T_0, L_0, \mathcal{F}}$.

Remark 6.5.23. T has IP: Let $\varphi(x, y) = (x \neq y)$. Then $C[\varphi](y; x_1, x_2, x_3)$ has IP.

Corollary 6.5.24. *For any set A, a type $p(\bar{x}) \in S(\mathbb{M})$ does not fork over A if and only if p is finitely satisfiable in A. In particular, by Fact 6.3.8, $f_T(\kappa, \lambda) \leq 2^{2^{\kappa}}$.*

Proof. Suppose $p(\bar{x})$ is a global type that is not finitely satisfiable in A. By quantifier elimination, there is a quantifier free formula $\varphi(\bar{x}; \bar{y})$ and $\bar{a} \in \mathbb{M}$ such that $\varphi(\bar{x}, \bar{a}) \in p$ and this formula is not satisfiable in A. If $\bar{a} \cap A \neq \emptyset$, and $x_i = a \in p$ for some $a \in \bar{a} \cap A$, replace x_i by a in φ, and change the partition of the variables so that we get $\varphi(\bar{z}, \bar{a}) \wedge \bar{z} \cap (\bar{a} \cap A) = \emptyset \in p$. By Claim 6.5.19, this formula forks over A and we are done. □

Proposition 6.5.25. *We have $f_T(\kappa, \lambda) = 2^{\min\{2^{\kappa}, \lambda\}}$.*

Proof. By the proof of Proposition 6.3.6 and Remark 6.5.23. □

6.5.4. Example of (8).

In this section we are going to construct an example of a theory T with $f_T(\kappa, \lambda) = \lambda$. The idea is to start with the random graph and circularize it in order to ensure that any non-forking type $p \in S^{\mathrm{nf}}(N, M)$ can be R-connected to at most one point of N.

Definition 6.5.26. Suppose L is a relational language which includes a binary relation symbol R. For a quantifier free L-formula $\psi(\bar{x}; \bar{y})$ and atomic formulas $\theta_0(\bar{x}; \bar{y}_0)$, $\theta_1(\bar{x}, \bar{y}_1)$, where $\lg(\bar{x}) > 0$, and both $\bar{x}$ and $\bar{y}_i$ occur in them, define the formula:

$$\begin{aligned} \varphi_{\psi}^{\theta_0, \theta_1}(\bar{x}; \bar{y}') &= \\ \varphi_{\psi}^{\theta_0, \theta_1}(\bar{x}; \bar{y}, \bar{y}_0, \bar{y}_1, z_0, z_1, z_2) &= \theta_0(\bar{x}, \bar{y}_0) \wedge \theta_1(\bar{x}, \bar{y}_1) \wedge \\ & \psi(\bar{x}, \bar{y}) \wedge \\ & \bigwedge_{i<j<3} R(z_i, z_j) \wedge \bigwedge_{i<3, y \in \bar{y}\bar{y}_0\bar{y}_1} R(z_i, y) \\ & \bar{y}_0 \neq \bar{y}_1. \end{aligned}$$

So z_0, z_1, z_2 form a triangle and are connected to all other parameters. The reason for this will be made clearer in the proof of Claim 6.5.28.

Definition 6.5.27. For a countable first-order relational language L containing a binary relation symbol R, Let $\mathcal{F}(L)$ be the set of all formulas of the form $\varphi_{\psi}^{\theta_0, \theta_1}$ from L as above. Let $L_0 = \{R\}$ where R is a binary relation symbol. Let T_0 say that R is a graph (symmetric and non-reflexive). Let $T = \circlearrowright_{T_0, L_0, \mathcal{F}}$.

Claim 6.5.28. Let $b \in M$. Let $p_b(z)$ be a non-algebraic type over M in one variable saying that $R(z, a)$ just when $a = b$. Then p_b isolates a complete type over M.

Proof. We will show:

(1) $p_b \restriction L_0$ is complete.

(2) If $L \supseteq L_0$ is some subset of L_ω and for all atomic formulas $\theta(z) \in L \backslash L_0$ over M, $p_b(z) \models \neg\theta(z)$, then for all $\varphi \in L$ used in the circularization (as in Definition 6.5.26) and atomic formulas $\theta(z, \bar{y}) \in L[\varphi] \backslash L$ and $\bar{c} \in M^{\lg(\bar{y})}$, $p_b(z) \models \neg\theta(z, \bar{c})$.

From (1) and (2) it follows by induction that p_b is complete.

(1) is immediate.

(2): Suppose $\theta(z, \bar{y})$ is an atomic formula in $L[\varphi] \backslash L$. Then it is of the form $C[\varphi](\ldots)$ where $\varphi = \varphi_\psi^{\theta_0, \theta_1}(\bar{x}; \bar{y}')$ for some $\psi(\bar{x}; \bar{y})$ and $\theta_i(\bar{x}; \bar{y}_i)$ from L. Suppose z appears in $\theta(z, \bar{y})$ among the index variables. Then by the choice of φ, it follows that $\theta(z, \bar{c})$ implies that z is R-connected to at least two different elements from M, and this contradicts the choice of p_b (this is why we added the extra parameters forming an R-triangle in Definition 6.5.26). So assume that z appears only in the main variables.

Case 1. One of θ_0, θ_1 is not from L_0, say θ_0. Since $C[\varphi](\bar{y}', \bar{x}_1, \bar{x}_2, \bar{x}_3) \models \bigwedge \varphi(\bar{x}_i, \bar{y}')$, and $p_b(z) \models \neg\theta_0(\ldots z \ldots)$ by induction (this notation means: substituting some variables of θ_0 with z, and putting parameters from M elsewhere), $p_b(z) \models \neg\theta(z, \bar{c})$.

Case 2. Both $\theta_0, \theta_1 \in L_0$. Suppose $\bar{c} \in M^{\lg(\bar{y}')}$ and show that $p_b(z) \models \neg C[\varphi](\bar{c}; \ldots z \ldots)$. There are two possibilities for θ_i: $R(z, y)$ and $z = y$. If $C[\varphi](\bar{c}; \ldots z \ldots)$ holds, then we would get that either $R(z, c_0) \wedge R(z, c_1)$ for some $c_0 \neq c_1 \in M$, or some equation $x = s'$ for $s' \in M$ is in p_b (here we use the fact that both x and $\bar{y}_i$ occur in θ_0, θ_1) — contradiction.

□

Claim 6.5.29. $f_T(\kappa, \lambda) \geq \lambda$.

Proof. Let $M \prec N \models T$, $|M| = \kappa, |N| = \lambda$. For each $b \in M$, let p_b be the type defined in the previous claim. Then p_b extends naturally to a global type q_b (i.e. the type over $\mathbb{M}$ that is R-connected only to b). This type does not divide over M (in fact it does not divide over $\emptyset$). This is by Claim 6.5.20 and the proof of Claim 6.5.28 (all atomic formulas in L_n have exactly the same truth value for $n > 0$). □

Claim 6.5.30. $f_T^n(\kappa, \lambda) = \lambda$ for all n and all $\lambda \geq 2^{2^\kappa}$.

Proof. Suppose $f_T^n(\kappa, \lambda) > \lambda$. Let $M \prec N \models T$ where $|M| = \kappa, |N| = \lambda$ and $\left|S_n^{\mathrm{nf}}(N, M)\right| > \lambda$.

Let $\{p_i(\bar{x}) \mid i < \lambda^+\} \subseteq S_n^{\mathrm{nf}}(N, M)$ be pairwise distinct. By possibly replacing $\bar{x}$ with a sub-tuple and throwing away some i's, we may assume that for all $i < \lambda^+$, $p_i \models \bar{x} \cap M = \emptyset$. Since $\lambda \geq 2^{2^\kappa}$, we may assume that for all $i < \lambda^+$, p_i is not finitely satisfiable in M.

Then, an easy computation shows that there must be some some $i < \lambda^+$ such that p_i contains two positive occurrences of atomic formulas $\theta_0(\bar{x}, \bar{a}_0)$ and $\theta_1(\bar{x}, \bar{a}_1)$ for some $\bar{a}_0 \neq \bar{a}_1 \in N$. Let $p = p_i$. There is some quantifier free formula $\psi(\bar{x}, \bar{c}) \in p$ such that ψ is not realized in M. Let $\bar{a}$ be the tuple of parameters $\langle \bar{c}, \bar{a}_0, \bar{a}_1 \rangle$ and let $d_0, d_1, d_2 \in N$ be an R-triangle such that $R(d_i, a)$ for all $a \in \bar{a}$. Finally, let $\bar{a}' = \bar{a}d \cap M$ and $\varphi_\psi^{\theta_0, \theta_1}(\bar{x}; \bar{c}, \bar{a}_0, \bar{a}_1, d) \wedge \bar{x} \cap \bar{a}' = \emptyset \in p$ forks over M by Claim 6.5.19. □

6.5.5. Example of (9).

In this subsection we prove the following Proposition:

PROPOSITION 6.5.31. *For any theory* T*, there is a theory* T_* *such that* $f_{T_*}(\kappa,\lambda) = f_T(\kappa,\lambda)^{\aleph_0}$ *for all* $\lambda \geq \kappa$.

Let T be a theory in the language L and assume that T eliminates quantifiers. For each $n < \omega$, let L_n be a copy of L such that $L_n \cap L_m = \emptyset$ for $n < m$, and $L_n = \{R_n \mid R \in L\}$. Let $\langle M_n \mid n < \omega \rangle$ be a sequence of models of T. We define a structure M in the language $\{P_n(x), Q(x), f_n : Q \to P_n \mid n < \omega\} \cup \bigcup L_n$:

(1) $M = \bigsqcup_{n<\omega} M_n \sqcup \left(\prod_{n<\omega} M_n\right)$ ($\sqcup$ means disjoint union).
(2) $P_n^M = M_n$, $Q^M = \prod_{n<\omega} M_n$
(3) If $R(\bar{x}) \in L(T)$ then for every $n < \omega$, $R_n^M \subseteq \left(P_n^M\right)^{\lg(\bar{x})}$ and P_n^M is the structure M_n.
(4) $f_n^M : Q^M \to P_n^M$, $f_n^M(\eta) = \eta(n)$ — the projection onto the n-th coordinate.

Let $T_* = \mathrm{Th}(M)$.

REMARK 6.5.32. The following properties are easy to check by back-and-forth:

(1) Doing the same construction with respect to any sequence of models $\langle M_n \mid n < \omega \rangle$ of T gives the same T_*.
(2) Moreover, if we have $M_n \preceq N_n$ for all n and do the construction, then $M \preceq N$.
(3) T_* eliminates quantifiers.

Now let $M \preceq N \models T$ with $|M| = \kappa$, $|N| = \lambda$.

LEMMA 6.5.33. *Given* $p(x) \in S_1(N)$ *such that* $Q(x) \in p$ *, for each* $n < \omega$ *we let* $p_n(y) = \{\varphi(y) \mid \varphi \in L_n,\ \varphi(f_n(x)) \in p\}$.

(1) $p(x)$ *is equivalent to* $\bigcup_{n<\omega} p_n(f_n(x))$.
(2) For each $n < \omega$*, let* $q_n(y)$ *be a complete* L_n*-type over* P_n^N*. Then the type* $\left(\bigcup_{n<\omega} q_n(f_n(x))\right) \cup \{Q(x)\}$ *is consistent and complete.*
(3) P_n *is stably embedded and the induced structure on* P_n *is just the* L_n*-structure. Moreover, for any* $n < \omega$ *and* L_**-formula* $\varphi(\bar{x}, \bar{y}_1, \bar{y}_2, \bar{z})$ *there is some* L_n*-formula* $\psi(\bar{x}, \bar{y}_1, \bar{z}')$ *such that for any e* $\bar{c}_1 \in P_n$, $\bar{c}_2 \in \bigcup_{m\neq n} P_m$ *and* $\bar{d} \in Q$*, the set* $\left\{\bar{a} \in P_n \,\middle|\, \models \varphi\left(\bar{a}, \bar{c}_1, \bar{c}_2, \bar{d}\right)\right\} = \bigcup \left\{\bar{a} \in P_n \,\middle|\, \models \psi\left(\bar{a}, \bar{c}_1, f_n\left(\bar{d}\right)\right)\right\}$.
(4) $p(x)$ *forks over* M *if and only if for some* $n < \omega$*,* $p_n(y) \restriction L_n$ *forks over* P_n^M *(in the sense of* T*).*

PROOF. (1), (2) and (3) follows by quantifier elimination and (4) follows from (1)–(3). ☐

PROOF. (of Proposition 6.5.31). We may assume that T eliminates quantifiers (by taking its Morleyzation). Consider T_*as above, and let us compute $f_{T_*}(\kappa,\lambda)$. Let $M \preceq N \models T_*$.

Let $S_n = \left\{p \in S^{\mathrm{nf}}(N, M) \mid P_n(x) \in p\right\}$.

From Lemma 6.5.33, it follows that $|S_n| = \left|S^{\mathrm{nf}, L_n}\left(P_n^N, P_n^M\right)\right|$.

Let $S_Q = \left\{p \in S^{\mathrm{nf}}(N, M) \mid Q(x) \in p\right\}$.

From Lemma 6.5.33, it follows that $|S_Q| = \prod_{n<\omega} \left|S^{\mathrm{nf}, L_n}\left(P_n^N, P_n^M\right)\right|$.

Let $S_{\neg} = \left\{p \in S^{\mathrm{nf}}(N, M) \mid \neg Q(x), \forall n < \omega (\neg P_n(x))\right\}$.

Since there is no structure on elements outside of all the P_n and Q, $|S_{\neg}| \leq |M|$.

Note that $S^{\mathrm{nf}}(N, M) = \bigcup_{n<\omega} S_n \cup S_Q \cup S_{\neg}$. From this and Remark 6.5.32(2), it follows that $f_{T^*}(\kappa, \lambda) = f_T(\kappa, \lambda)^{\aleph_0}$. $\square$

REMARK 6.5.34. This analysis easily generalizes to show that $f^n_{T_*}(\kappa, \lambda) = f^n_T(\kappa, \lambda)^{\aleph_0}$.

6.5.6. Examples of (12) and (14).

Here we construct an example of a theory T with $f_T(\kappa, \lambda) = \operatorname{ded}\lambda$. The idea is that we start with an ordered random graph, and we circularize in order to ensure that for any $p \in S^{\mathrm{nf}}(N, M)$ there is some cut of N such that $R(x, a)$ is in p if any only if a is in the cut.

(1) Here the language L contains an order relation $<$ which induces the natural lexicographic order on tuples, so abusing notation, we may write $\bar{y} < \bar{z}$.
(2) In this section, we say that two atomic formulas $\theta_1(\bar{x}; \bar{y}_1)$ and $\theta_2(\bar{x}; \bar{y}_2)$ are different when the relation symbol in different (rather than just the variables are different).
(3) Also, when we say atomic formula in the definition below, we mean that it does <u>not</u> use the order relation $<$.

DEFINITION 6.5.35. Suppose L is a relational language which includes a binary relation symbol R, a unary predicate P and an order relation $<$.

(1) For a quantifier free L-formula $\psi(\bar{x}; \bar{y})$ and two <u>different</u> atomic formulas $\theta_0(\bar{x}; \bar{y}_0)$, $\theta_1(\bar{x}, \bar{y}_1)$, where $\lg(\bar{x}) > 0$, and both $\bar{x}$ and $\bar{y}_i$ occur in them, define the formula, define the formula
$$\begin{aligned} \varphi^{\theta_0,\theta_1}_{\psi}(\bar{x}; \bar{y}') &= \\ \varphi^{\theta_0,\theta_1}_{\psi}(\bar{x}; \bar{y}, \bar{y}_0, \bar{y}_1, z_0, z_1) &= \theta_0(\bar{x}, \bar{y}_0) \wedge \theta_1(\bar{x}, \bar{y}_1) \wedge \\ &\quad \psi(\bar{x}, \bar{y}) \wedge \\ &\quad z_0 < z_1 \wedge P(z_0) \wedge P(z_1) \wedge \\ &\quad \bigwedge_{y \in \bar{y}\bar{y}_0\bar{y}_1, i<2} (y \neq z_i) \wedge R(y, z_1) \wedge \neg R(y, z_0). \end{aligned}$$
(2) For an L-formula $\psi(\bar{x}; \bar{y})$ and an atomic formula $\theta(\bar{x}; \bar{y}_0)$ (in which $\bar{y}_0$ appears) , define the formula
$$\begin{aligned} \varphi^{\theta}_{\psi}(\bar{x}; \bar{y}') &= \\ \varphi^{\theta}_{\psi}(\bar{x}; \bar{y}, \bar{y}_0, \bar{y}_1, z_0, z_1) &= \neg\theta(\bar{x}, \bar{y}_0) \wedge \theta(\bar{x}, \bar{y}_1) \wedge \\ &\quad \psi(\bar{x}, \bar{y}) \wedge \\ &\quad z_0 < z_1 \wedge P(z_0) \wedge P(z_1) \wedge \\ &\quad \bigwedge_{y \in \bar{y}\bar{y}_0\bar{y}_1, i<2} (y \neq z_i) \wedge R(y, z_1) \wedge \neg R(y, z_0) \\ &\quad \bar{y}_0 < \bar{y}_1. \end{aligned}$$

DEFINITION 6.5.36. For a countable first-order relational language L containing a binary relation symbol R, Let $\mathcal{F}(L)$ be the set of all formulas from L of the form

$\varphi_{\psi}^{\theta_0,\theta_1}$ or φ_{ψ}^{θ} as above. Let $L_0 = \{R, <\}$ where R and $<$ are binary relation symbols. Let T_0 say that R is a graph and that $<$ is a linear order. Let $T = \circlearrowleft_{T_0,L_0,\mathcal{F}}$.

Suppose $M \models T$.

Claim 6.5.37. Let I be initial segments in M. Let $p_I(x)$ be a non-algebraic type over M saying that $x > M$, $\neg P(x)$ and $R(x, a)$ just when $a \in I$. Then p_I isolates a complete type over M.

Proof. In fact, $p_I \restriction L_0$ is complete, and for all atomic formulas $\theta(x) \notin L_0$ over M, $p_I \models \neg\theta(x)$. The proof is very similar to the proof of Claim 6.5.28. $\square$

Claim 6.5.38. $f_T(\kappa, \lambda) \geq \text{ded}(\lambda)$.

Proof. Let $M \prec N \models T$, $|M| = \kappa, |N| = \lambda$. For each cut I in N, let p_I be the type defined in the previous claim. Then p_I extends naturally to a global type q_I (i.e. the type over $\mathbb{M}$ defined by $p_{I'}$ where $I' = \{c \in \mathbb{M} \mid \exists a \in I (c < a)\}$). This type does not divide over M (in fact it does not divide over $\emptyset$) by Claim 6.5.20 and by the proof of the previous claim (all atomic formulas have exactly the same truth value in L_n for $n > 0$). $\square$

Claim 6.5.39. $f_T^n(\kappa, \lambda) = \text{ded}(\lambda)$ for all n and all $\lambda \geq 2^{2^\kappa}$.

Proof. Suppose $f_T^n(\kappa, \lambda) > \text{ded}(\lambda)$. Let $M \prec N \models T$ where $|M| = \kappa, |N| = \lambda$.

Let $\left\{p_i(\bar{x}) \,\middle|\, i < \text{ded}(\lambda)^+\right\} \subseteq S^{\text{nf}}(N, M)$ is a set of pairwise distinct types. As in the proof of Claim 6.5.30, we may assume that $p_i \models \bar{x} \cap M = \emptyset$ for all i, and that p_i is not finitely satisfiable in N. Also we may assume that $p_i \restriction \{<\}$ is constant.

Then, by the choice of $\varphi_{\psi}^{\theta_0,\theta_1}$, for every $i < \text{ded}(\lambda)^+$ there is at most one atomic formula of the form $\theta(\bar{x}; \bar{y})$ such that there is some positive instance $\theta(\bar{x}, \bar{a}) \in p_i$ (if not, suppose $\theta_0(\bar{x}, \bar{a}_0) \wedge \theta_1(\bar{x}, \bar{a}_1) \in p$. There is some quantifier free formula $\psi(\bar{x}, \bar{c}) \in p_i$ such that ψ is not realized in M. Let $\bar{a}$ be the tuple of parameters $\langle \bar{c}, \bar{a}_0, \bar{a}_1 \rangle$ and let $d_0, d_1, d_2 \in N$ be an R-triangle such that $R(d, b)$ for all $b \in \bar{a}$. Finally, let $\bar{a}' = \bar{a}d \cap M$ and $\varphi_{\psi}^{\theta_0,\theta_1}(\bar{x}; \bar{c}, \bar{a}_0, \bar{a}_1, d) \wedge \bar{x} \cap \bar{a}' = \emptyset \in p$ forks over M by Claim 6.5.19).

Similarly, by the choice of φ_{ψ}^{θ}, this formula induces a cut $I = \{\bar{a} \mid \theta(\bar{x}, \bar{a}) \in p_i\}$.

This formula and the cut it induces determine the type. But this is a contradiction to the definition of ded. $\square$

Corollary 6.5.40. *There is a theory* T_* *such that* $f_{T_*}(\lambda, \kappa) = \text{ded}(\lambda)^{\aleph_0}$.

Proof. By Proposition 6.5.31. $\square$

6.5.7. Example of (16).

As a pleasant surprise to the reader who managed to get this far, the example is just the theory of the random graph (it is NTP$_2$ and has IP, see Proposition 6.4.5).

6.5.8. Example of $f_T^1(\kappa, \lambda) \leq 2^{2^\kappa}$ but $f_T^2(\kappa, \lambda) = 2^\lambda$.

Again we use circularizations, but instead of considering all formulas, we consider only formulas with one variable.

Definition 6.5.41. Let $L_0 = \{=\}$ and T_0 be empty. Let $\mathcal{F}(L)$ be the set of all quantifier free partitioned formulas from L of the form $\varphi(x; \bar{y})$ where x is a singleton. Let $T = \circlearrowleft_{T_0,L_0,\mathcal{F}}$.

Let $A \subseteq M \models T$. By Claim 6.5.19 and as in the proof of Proposition 6.5.25,

COROLLARY 6.5.42. *If* $p(x) \in S_1(M)$ *then* p *does not fork over* A *if and only if it is finitely satisfiable in* A. *So* $f_T^1(\kappa, \lambda) \leq 2^{2^\kappa}$ *for all*

On the other hand, if we consider types in two variables, then there is no reason for them to fork.

CLAIM 6.5.43. $f_T^2(\kappa, \lambda) \geq 2^\lambda$.

PROOF. Suppose $|M| = \lambda$, so $M = \{a_i \mid i < \lambda\}$, and $A \subseteq M$ of size κ. Let $q(z) \in S_1(M)$ be any 1-type which is finitely satisfiable in A but not algebraic over A. For $S \subseteq \lambda$, let $p_S(x, y)$ be a partial type over M such that

(1) $p_S \upharpoonright x = q(x)$, $p_S \upharpoonright y = q(y)$.
(2) $R(x, y, a_i) \in p_S$ if and only if $i \in S$.

First, p_S is indeed a type. The proof is by induction, i.e. one proves that $p_S \upharpoonright L_0$ is a type (which is clear), and that if L is some subset of L_ω such that $p_S \upharpoonright L$ is a type and $\varphi(x; \bar{y})$ is some partitioned L-formula with $\lg(x) = 1$, then also $p_S \upharpoonright L[\varphi]$ is a type, and this follows from Claim 6.5.11.

Let $N \supseteq M$ be an $|A|^+$-saturated model and $q' \supseteq q$ be a global type which is finitely satisfiable in A. Fix $c \models q'|_N$ and $d \models q'|_{Nc}$.

We want to construct a completion $r_S(x, y) \in S_2(N)$ containing p_S which does not divide over A. We start by $r_S \upharpoonright x = q'|_N(x)$, $r_S \upharpoonright y = q'_{N\bar{}}(y)$ and $r_S \upharpoonright L_0$ is any completion of $p_S \upharpoonright L_0$. For each atomic formulas $\theta\left(x, y, \bar{t}\right)$ over N of the form $C[\varphi]\left(\bar{t}, -, -, -\right)$ (so $\bar{t} \in N$) such that $\varphi(x, t) \in q'(x)$ define $\theta(x, y) \in r_S$ if and only if $\theta(c, d)$ holds. This is a type (by induction again, by Claim 6.5.11 (3), but follow the proof a bit more carefully, and choose the amalgamation of the circular orders corresponding to $\bar{t}$ according to the choice of c, d). Let r_S by any completion.

Finally, r_S does not divide over A by Claim 6.5.16 (by induction and by the choice of c, d). $\square$

6.6. On $\operatorname{ded}\kappa < (\operatorname{ded}\kappa)^{\aleph_0}$

6.6.1. On $\operatorname{ded}(\lambda)$.

DEFINITION 6.6.1. Let $\operatorname{ded}(\lambda)$ be the supremum of the set

$$\{|I| \mid I \text{ is a linear order with a dense subset of size } \leq \lambda\}.$$

FACT 6.6.2. *It is well known that* $\lambda < \operatorname{ded}\lambda \leq (\operatorname{ded}\lambda)^{\aleph_0} \leq 2^\lambda$. *If* $\operatorname{ded}\lambda = 2^\lambda$, *then* $\operatorname{ded}\lambda = (\operatorname{ded}\lambda)^{\aleph_0} = 2^\lambda$. *This is true for* $\lambda = \aleph_0$, *or more generally for any* λ *such that* $\lambda = \lambda^{<\lambda}$. *So in particular this holds for any* λ *under GCH.*

In addition, if $\operatorname{ded}\lambda$ *is not attained (i.e. it is a supremum rather than a maximum), then* $\operatorname{cof}(\operatorname{ded}\lambda) > \lambda$. *See also Corollary 6.6.13.*

DEFINITION 6.6.3. (1) Given a linear order I and two regular cardinals θ, μ, we say that S is a (θ, μ)-cut when it has cofinality θ from the left and cofinality μ from the right.

(2) By a tree we mean a partial order $(T, <)$ such that for every $a \in T$, $T_{<a} = \{x \in T \mid x < a\}$ is well ordered.
(3) For two cardinals λ and μ, let $\lambda^{\langle\mu\rangle_{\mathrm{tr}}}$ be

$$\sup\{\kappa \mid \text{there is some tree } T \text{ with } \lambda \text{ many nodes and } \kappa \text{ branches of length } \mu\}.$$

REMARK 6.6.4. Note that $\lambda^{\langle\mu\rangle_{\mathrm{tr}}} \leq \lambda^{\mu}$ and if $\lambda = \lambda^{<\mu}$ then $\lambda^{\langle\mu\rangle_{\mathrm{tr}}} = \lambda^{\mu}$ (consider the tree $\lambda^{<\mu}$ ordered lexicographically).

PROPOSITION 6.6.5. *The following cardinalities are the same:*

(1) $\mathrm{ded}(\lambda)$
(2) $\sup\{\kappa \mid$ *there is a linear order* I *of size* λ *with* κ *many cuts*$\}$
(3) $\sup\{\kappa \mid$ *there is a regular* μ *and a linear orded* I *of size* $\leq \lambda$ *with* κ *many* (μ,μ)*-cuts*$\}$
(4) $\sup\{\kappa \mid$ *there is a regular* μ *and a tree* T *with* κ *branches of length* μ *and* $|T| \leq \lambda\}$
(5) $\sup\{\kappa \mid$ *there is a regular* μ *and a binary tree* T *with* κ *branches of length* μ *and* $|T| \leq \lambda\}$
(6) $\sup\left\{\lambda^{\langle\mu\rangle_{\mathrm{tr}}} \mid \mu \leq \lambda \text{ is regular}\right\}$

PROOF. (1)=(2), (4)=(6): obvious.

(2)=(3): By [**KSTT05**, Theorem 3.9], given a linear order I and two regular cardinals $\theta \neq \mu$ the number of (θ,μ)-cuts in I is at most $|I|$. Given I and a regular cardinal μ, let $D_{\mu}(I)$ be the set of (μ,μ)-cuts, and let $D(I)$ be the set of all cuts. Suppose $|I| = \lambda$, then $|D(I)| = \sup\{|D_{\mu}(I)| \mid \mu = \mathrm{cof}(\mu) \leq \lambda\}$ holds whenever $|D(I)| > \lambda$. By Fact 6.6.2, $\mathrm{ded}(\lambda) = \sup\{D_{\mu}(I) \mid \mu = \mathrm{cof}(\mu) \leq \lambda, |I| \leq \lambda\}$.

(2)=(4): Follows from [**Bau76**, Theorem 2.1(a)].

(4)=(5): Obviously (4)$\geq$(5). Suppose T is a tree as in (4). We may assume $T \subseteq \lambda^{<\mu}$ as a complete sub-tree (i.e. if $\eta \in \lambda^{<\mu}$ and ν is an initial segment of η, then $\nu \in T$). Let $(\mu \times \lambda \cup \{(\mu,0)\}, <)$ be the lexicographic order ($(\beta,j) < (\alpha,i) \Leftrightarrow [\beta < \alpha \vee (\beta = \alpha \wedge j < i)]$) and let $f : \lambda^{\leq\mu} \to 2^{\leq(\mu\times\lambda)}$ be such that for $\alpha \leq \mu$ and $\eta \in \lambda^{\alpha}$, $f(\eta) \in 2^{\alpha\times\lambda}$, and $f(\eta)(\beta,i) = 1$ if and only if $\eta(\beta) = i$. (So by $2^{\leq(\mu\times\lambda)}$ we mean all functions of the form $\eta : \{(\beta,j) < (\alpha,i)\} \to 2$ for some $(\alpha,i) \in \mu\times\lambda \cup \{(\mu,0)\}$). It is easy to see that f is a tree embedding and $f(T)$ is a sub-tree of $2^{<(\mu\times\lambda)}$. So $f(T)$ is a binary tree with λ many nodes, and for each branch $\varepsilon : \mu \to \lambda$ of T (i.e. such that $\varepsilon \restriction \alpha \in T$ for all $\alpha < \mu$), $\{f(\varepsilon \restriction \alpha) \mid \alpha < \mu\}$ is a branch of $f(T)$ of height μ. $\square$

REMARK 6.6.6. Any tree of size $\leq \lambda$ of height $< \theta$ is isomorphic to a sub-tree of $\lambda^{<\theta}$ such that if $x \in T$ and $y \leq x$ then $y \in T$.

6.6.2. Consistency of $\mathrm{ded}\,\kappa < (\mathrm{ded}\,\kappa)^{\aleph_0}$.

In [**Kei76**], the following fact is mentioned (without proof), attributed to Kunen:

REMARK 6.6.7. [Kunen] If $\kappa^{\aleph_0} = \kappa$ then $(\mathrm{ded}\,\kappa)^{\aleph_0} = \mathrm{ded}\,\kappa$.

PROOF. Suppose I is a linear order, and $J \subseteq I$ is dense, $|J| = \kappa$. Let $\mathcal{U}$ be a non-principal ultrafilter on ω. Then the linear order $I^{\omega}/\mathcal{U}$ has $J^{\omega}/\mathcal{U}$ as a dense subset. Now[1], $|J^{\omega}/\mathcal{U}| = \kappa^{\aleph_0} = \kappa$ and $|I^{\omega}/\mathcal{U}| = |I|^{\aleph_0}$. The remark follows from Fact 6.6.2. $\square$

Answering a question of Keisler [**Kei76**, Problem 2], we show:

THEOREM 6.6.8. *It is consistent with ZFC that* $\mathrm{ded}\,\kappa < (\mathrm{ded}\,\kappa)^{\aleph_0}$.

Our proof uses Easton forcing, so let us recall:

[1] If A is infinite then $A^{\omega}/\mathcal{U}$ has size $|A|^{\aleph_0}$: let $g_n : A^n \to A$ be bijections. Then take $f \in \lambda^{\omega}$ to $\bar{f} = \langle g_n(f(0),\ldots,f(n-1)) \mid n < \omega\rangle$, so that if $f \neq g$ then $\bar{f} \neq \bar{g}$ from some point onwards, and in particular, modulo $\mathcal{U}$.

THEOREM 6.6.9. *[Easton] Let* M *be a transitive model of ZFC and assume that the Generalized Continuum Hypothesis holds in* M. *Let* F *be a function (in* M*) whose arguments are regular cardinals and whose values are cardinals, such that for all regular* κ *and* λ*:*

(1) $F(\kappa) > \kappa$
(2) $F(\kappa) \leq F(\lambda)$ *whenever* $\kappa \leq \lambda$.
(3) $\operatorname{cof}(F(\kappa)) > \kappa$

Then there is a generic extension $M[G]$ *of* M *such that* M *and* $M[G]$ *have the same cardinals and cofinalities, and for every regular* κ*,* $M[G] \models 2^{\kappa} = F(\kappa)$.

See [**Jec03**, Theorem 15.18].

Easton forcing is a class forcing but we can just work with a set forcing, i.e. when F is a set. The following is the main claim:

CLAIM 6.6.10. Suppose M is a transitive model of ZFC that satisfies GCH, and furthermore:

- κ is a regular cardinal.
- $\langle \theta_i \mid i < \kappa \rangle$, $\langle \mu_i \mid i < \kappa \rangle$ are strictly increasing sequences of cardinals, $\theta = \sup_{i<\kappa} \theta_i$, $\mu = \sup_{i<\kappa} \mu_i$.
- $\kappa < \theta_0$, $\theta_i < \mu_0$ for all $i < \kappa$.
- θ_i are regular for all $i < \kappa$.

Then, letting P be Easton forcing with $F : \{\theta_i \mid i < \kappa\} \to \mathbf{card}$, $F(\theta_i) = \mu_i$ and G a generic for P, in $M[G]$, $\operatorname{ded}\theta = \mu$ and the supremum is attained.

REMARK 6.6.11. Note that in $M[G]$, we also get by Easton's Theorem 6.6.9 that $2^{\theta_i} = \mu_i$; $\operatorname{cof}(\theta) = \operatorname{cof}(\mu) = \kappa < \theta$ and $\mu^{\kappa} > \mu$.

PROOF. First let us show that $\operatorname{ded}\theta \geq \mu$. Recall,

- $\operatorname{Add}(\kappa, \lambda)$ is the forcing notion that adjoins λ subsets to κ, i.e. it is the set of partial functions $p : \kappa \times \lambda \to 2$ such that $|\operatorname{dom}(p)| < \kappa$.
- The Easton forcing notion P is the set of all elements in $\prod_{i<\kappa} \operatorname{Add}(\theta_i, \mu_i)$ such that the for every regular cardinal $\gamma \leq \kappa$, and for each $p \in P$, the support $s(p)$ satisfies $|s(p) \cap \gamma| < \gamma$.

If G is a generic of P, then the projection of G to i, G_i, is generic in $\operatorname{Add}(\theta_i, \mu_i)$.

For $i < \kappa$, consider the tree $T_i = \left(2^{<\theta_i}\right)^M$. Since M satisfies GCH, $M[G] \models |T_i| = \theta_i$. But for all $\beta < \mu_i$, we can define a branch $\eta_\beta : \theta_i \to 2$ of T_i by $\eta_\beta(\alpha) = p(\alpha, \beta)$ for some $p \in G_i$ such that $(\alpha, \beta) \in \operatorname{dom}(p)$. If $\alpha < \theta_i$, then $\eta_\beta \restriction \alpha \in M$ (consider the dense set $D = \{p \in \operatorname{Add}(\theta_i, \mu_i) \mid \alpha \times \{\beta\} \subseteq \operatorname{dom}(p)\}$), and if $\beta_1 \neq \beta_2$ then $\eta_{\beta_1} \neq \eta_{\beta_2}$. Together, by Proposition 6.6.5 we have $\operatorname{ded}\theta_i = \mu_i = 2^{\theta_i}$ in $M[G]$. Since $\operatorname{ded}\theta \geq \operatorname{ded}\theta_i$ for all $i < \kappa$, we are done.

Now let us show that $\operatorname{ded}(\theta) \leq \mu$. Let I be some linear order such that $|I| = \theta$. For any choice of cofinalities (κ_1, κ_2), we look at the set of all (κ_1, κ_2)-cuts of I, C_{κ_1,κ_2}. Obviously for it to be nonempty, $\kappa_1, \kappa_2 \leq \theta$, so let us assume that $\kappa_1, \kappa_2 \leq \theta_i$ for some i. We map each such cut to a pair of cofinal sequences (from the left and from the right). Hence we obtain $|C_{\kappa_1,\kappa_2}| \leq \theta^{\kappa_1+\kappa_2} \leq \theta^{\theta_i}$. Since $\theta \leq \mu_0$, $\theta^{\theta_i} \leq \mu_0^{\theta_i} \leq 2^{\theta_0+\theta_i} = \mu_i < \mu$. The number of regular cardinals below θ is $\leq \theta$, so we are done. □

COROLLARY 6.6.12. *Suppose GCH holds in* M. *Choose* $\kappa = \aleph_0$*,* $\theta_i = \aleph_{i+1}$ *and* $\mu_i = \aleph_{\omega+i}$. *Then in the generic extension,* $\aleph_{\omega+\omega} = \operatorname{ded}\aleph_\omega < (\operatorname{ded}\aleph_\omega)^{\aleph_0}$.

In fact, since the Singular Cardinal Hypothesis holds under Easton forcing (see [**Jec03**, Exercise 15.12]*),* $(\operatorname{ded} \aleph_\omega)^{\aleph_0} = \aleph_{\omega+\omega+1}$.

COROLLARY 6.6.13. *It is consistent with ZFC that* $\operatorname{cof}(\operatorname{ded} \lambda) < \lambda$.

PROBLEM 6.6.14. Is it consistent with ZFC that $\operatorname{ded} \kappa < (\operatorname{ded} \kappa)^{\aleph_0} < 2^\kappa$?

We remark that our construction is not sufficient for that: in the context of Claim 6.6.10, $(\operatorname{ded} \theta)^\kappa \leq 2^\theta$, but $2^\theta = \prod_{i<\kappa} 2^{\theta_i} \leq \prod_{i<\kappa} \mu_i \leq \mu^\kappa = (\operatorname{ded} \theta)^\kappa$.

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