Übungen zur Vorlesung **Modelltheorie** (WS 2012/13) 25.10.2012
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Blatt 1

Aufgabe 1 (Direct products)**.** Let $\mathfrak{A}_1, \mathfrak{A}_2$ be L-structures. Define an L-structure $\mathfrak{A}_1 \times \mathfrak{A}_2$ with universe $A_1 \times A_2$ such that the natural epimorphisms $\pi_i : \mathfrak{A}_1 \times \mathfrak{A}_2 \longrightarrow \mathfrak{A}_i$ for $i = 1, 2$ satisfy the following universal property: given any L-structure $\mathfrak{D}$ and homomorphisms $\varphi_i : \mathfrak{D} \longrightarrow \mathfrak{A}_i, i = 1, 2$ there is a unique homomorphism $\psi : \mathfrak{D} \longrightarrow \mathfrak{A}_1 \times \mathfrak{A}_2$ such that $\pi_i \circ \psi = \varphi_i, i = 1, 2$, i.e., this is the product in the category of L-structures with homomorphisms.

Aufgabe 2. Let $f \colon \mathfrak{A} \to \mathfrak{A}$ be an embedding. Prove the following: There is an extension $\mathfrak{A} \subseteq \mathfrak{B}$ together with an automorphism g of $\mathfrak{B}$ extending f. We can find B as the union of the chain $A \subseteq g^{-1}(A) \subseteq g^{-2}(A) \subseteq \cdots$. The pair $(\mathfrak{B}, g)$ is uniquely determined by this choice of B.

Aufgabe 3. Let $\mathfrak{A}$ be an L-structure with finite domain A. Show that the number of L-structures on A isomorphic to $\mathfrak{A}$ equals the quotient

$$\text{number of permutations of } A : \text{number of automorphisms of } \mathfrak{A}.$$

Hint: Think of group actions.

Aufgabe 4. Let $\mathfrak{A}, \mathfrak{B}$ be L-structures. Suppose that there exist injective homomorphisms $f : \mathfrak{A} \to \mathfrak{B}$ and $g : \mathfrak{B} \to \mathfrak{A}$. Does it follow that $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic?

Anwesenheitsaufgabe für die zweite Woche (29.10. - 2.11.)

Aufgabe 5 (Ultraproducts and Łos's Theorem)**.** A *filter* on a set I is a non-empty set $\mathcal{F} \subseteq \mathfrak{P}(I)$ which does not contain the empty set and is closed under intersections and supersets, i.e., for $A, B \in \mathcal{F}$, we have $A \cap B \in \mathcal{F}$ and if $A \in \mathcal{F}$ and $A \subseteq C \subseteq I$ we have $C \in \mathcal{F}$. A filter $\mathcal{F}$ is called an *ultrafilter* if for every $A \in \mathfrak{P}$ we have $A \in \mathcal{F}$ or $I \setminus A \in \mathcal{F}$. (By Zorn's Lemma, any filter can be extended to an ultrafilter.)

For a family $(\mathfrak{A}_i \mid i \in I)$ of L-structures and $\mathcal{F}$ an ultrafilter on I we define the *ultraproduct* $\Pi_{i\in I}\mathfrak{A}_i/\mathcal{F}$ as follows. On the Cartesian product $\Pi_{i\in I}\mathfrak{A}_i$, the ultrafilter $\mathcal{F}$ defines an equivalence relation $\sim_{\mathcal{F}}$ by

$$(a_i)_{i\in I} \sim_{\mathcal{F}} (b_i)_{i\in I} \Leftrightarrow \{i \in I \mid a_i = b_i\} \in \mathcal{F}.$$

On the set of equivalence classes $(a_i)_{\mathcal{F}}$ we define an L-structure $\Pi_{i\in I}\mathfrak{A}_i/\mathcal{F}$.

- For constants $c \in L$, put $c^{\Pi_{\mathcal{F}}\mathfrak{A}_i} = (c^{\mathfrak{A}_i})_{\mathcal{F}}$.
- For n-ary function symbols $f \in L$ put
$$f^{\Pi_{\mathcal{F}}\mathfrak{A}_i}((a_i^1)_{\mathcal{F}}, \ldots, (a_i^n)_{\mathcal{F}})) = (f^{\mathfrak{A}_i}(a_i^1, \ldots, a_i^n))_{\mathcal{F}}.$$
- For n-ary relation symbols $R \in L$ put
$$R^{\Pi_{\mathcal{F}}\mathfrak{A}_i}((a_i^1)_{\mathcal{F}}, \ldots, (a_i^n)_{\mathcal{F}})) \Leftrightarrow \{i \in I \mid R^{\mathfrak{A}_i}(a_i^1, \ldots, a_i^n)\} \in \mathcal{F}.$$

1. Show that the ultraproduct $\Pi_{i\in I}\mathfrak{A}_i/\mathcal{F}$ is well-defined.
2. Prove Łos's Theorem: for any L-formula φ we have
$$\Pi_{i\in I}\mathfrak{A}_i/\mathcal{F} \models \varphi((a_i^1)_{\mathcal{F}}, \ldots, (a_i^n)_{\mathcal{F}}) \Leftrightarrow \{i \in I \mid \mathfrak{A}_i \models \varphi(a_i^1, \ldots, a_i^n)\} \in \mathcal{F}.$$
3. For every $i \in I$, the set
$$\mathcal{F}_i = \{J \subseteq I : i \in J\}.$$
is an ultrafilter on I. Ultrafilters of this form are said to be *principal.*

   Show that for every $i_0 \in I$, the ultraproduct $\Pi_{i\in I}\mathfrak{A}_i/\mathcal{F}_{i_0}$ is isomorphic to $\mathfrak{A}_{i_0}$.
4. Show that if I is a finite set, then all ultrafilters on I are principal.
5. Let I be an infinite set. The *Fréchet filter* on I is the set of all subsets J of I such that $I \setminus J$ is finite. Prove that an ultrafilter on I is non-principal if and only if it contains the Fréchet filter.
6. Think about ultraproducts of your favourite structures.

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[0] `http://home.mathematik.uni-freiburg.de/caycedo/lehre/ws12_modell/`