Hinweise zu Blatt 14

Aufgabe 1.

1. Define an $L(M)$-formula $\varphi(x)$ to be *large* if $|\varphi(\mathcal{M})| \geq \mu$ and proceed as in the original proof (Theorem 5.4.1 in Tent-Ziegler). 2. Assume towards a contradiction that for some uncountable $\lambda < \kappa$ there is a model $\mathcal{M}$ of T of cardinality λ that is not saturated. Let $\nu < \lambda$ be the cardinality of a parameter set $A \subseteq M$ such that some type p in $S(A)$ is omitted in $\mathcal{M}$. Let μ to be ν^+. Hence μ is an uncountable regular cardinal and $\mu \leq \lambda$. Proceeding as in the proof of Corollary 5.4.2 in Tent-Ziegler and using part 1, one obtains an elementary extension of M of size κ that omits all types over sets of size $< \mu$ omitted in M, in particular p.

Aufgabe 2.

For the first part combine Vaught's two cardinal theorem (Theorem 5.5.2 in Tent-Ziegler) with part 1 of Aufgabe 1. The second part follows easily from the first, using that for a κ-categorical theory every model of size κ is saturated.

Aufgabe 3.

In fact, every model of size $\aleph_1$ has the required property. Suppose there is a model that does not. Then, using the downward Löwenheim-Skolem Theorem, one can obtain a Vaughtian pair, hence a contradiction to Aufgabe 2.

Aufgabe 4.

Let $\mathcal{N}$ be a countable model of RG. Let a be any element of N and consider the formula $\varphi(x) := xRa$ with parameter a. Let b be an element in N with $\neg bRa$. It is easy to see that the substucture $\mathcal{M}$ of $\mathcal{N}$ with domain $M := N \setminus \{b\}$ is also a model of RG. By quantifier elimination, $\mathcal{M}$ is therefore an elementary substructure of $\mathcal{N}$. Finally note $\varphi(\mathcal{M}) = \varphi(\mathcal{N})$ is infinite.

Aufgabe 5.

1. Note that for every parameter set A there are injections

$$S_T(A\bar{a}) \to S_{T(q)}(A) \to S_T(A).$$

2. The implication from right to left is easy to see. For the implication from left to right, use a construction as in the proof of Theorem 5.5.2 in Tent-Ziegler to find a Vaughtian pair $\mathcal{M} \prec \mathcal{N}$ such that q is realised in $\mathcal{N}$. 3. Use the fact that a theory is κ-categorical if and only if all its models of cardinality κ are saturated.

Hinweise zu Blatt 15

Aufgabe 1.

If $p \in \mathrm{S}(A)$ is not algebraic, then all n-types

$$q(x_1, \ldots, x_n) = \{x_i, i = 1, \ldots, n, \text{ satisfies } p \text{ and the } x_i \text{ are pairwise distinct}\}$$

are consistent and hence realised in $\mathfrak{M}$.

Aufgabe 3.

Prove that the theory eliminates quantifiers.