Übungen zur Vorlesung **Modelltheorie** (WS 2012/13) 9.11.2012
Dozenten: PD Dr. Markus Junker, Prof. Dr. Martin Ziegler
Assistent: Dr. Juan Diego Caycedo
Tutor: Christoph Bier B.Sc.

Blatt 3

Aufgabe 1. Two structures $\mathfrak{A}$ and $\mathfrak{B}$ are *partially isomorphic* if there is a non-empty set $\mathcal{I}$ of isomorphisms between substructures of $\mathfrak{A}$ and $\mathfrak{B}$ with the *back-and-forth* property:

1. For every $f \in \mathcal{I}$ and $a \in A$ there is an extension of f in $\mathcal{I}$ with a in its domain.
2. For every $f \in \mathcal{I}$ and $b \in B$ there is an extension of f in $\mathcal{I}$ with b in its image.

Show that partially isomorphic structures are elementarily equivalent.

Hint: Show by induction on the complexity of φ that for all $f \in \mathcal{I}$ and all $\bar{a}$ in the domain of f we have $\mathfrak{A} \models \varphi(\bar{a}) \Leftrightarrow \mathfrak{B} \models \varphi(f(\bar{a}))$.

Aufgabe 2.

1. Show that the usual ordering of $\mathbb{R}$ is definable in the structure $(\mathbb{R}, 0, +, -, 1, \cdot)$.
2. Show that the usual multiplication is not definable in the structure $(\mathbb{R}, 0, +, -)$.
3. Let $L = \{c_i : i \in I\}$ be a language containing only constants and let $\mathfrak{A}$ be an L-structure. Prove that the only subsets of A^1 definable in the structure $\mathfrak{A}$ are of the form $\{c_{i_1}^{\mathfrak{A}}, \ldots, c_{i_m}^{\mathfrak{A}}\}$ or $A \setminus \{c_{i_1}^{\mathfrak{A}}, \ldots, c_{i_m}^{\mathfrak{A}}\}$ for some $m \in \mathbb{N}$ and $i_1, \ldots, i_m \in I$.

Aufgabe 3. Prove the Compactness Theorem (i.e. finitely satisfiable theories are consistent) using ultraproducts.

Hint: Let T be a finitely satisfiable theory. Consider the set I of all finite subsets of T. For every $\Delta \in I$ choose a model $\mathfrak{A}_\Delta$. Find a suitable ultrafilter $\mathcal{F}$ on I such that $\prod_{\Delta \in I} \mathfrak{A}_\Delta / \mathcal{F}$ is a model of T.

Aufgabe 4. Consider a class $\mathcal{C}$ of L-structures. Prove:

1. Let $\mathrm{Th}(\mathcal{C}) = \{\varphi \mid \mathfrak{A} \models \varphi \text{ for all } \mathfrak{A} \in \mathcal{C}\}$ be the *theory of* $\mathcal{C}$. Then $\mathfrak{M}$ is a model of $\mathrm{Th}(\mathcal{C})$ if and only if $\mathfrak{M}$ is elementarily equivalent to an ultraproduct of elements of $\mathcal{C}$.
2. Show that $\mathcal{C}$ is an elementary class if and only if $\mathcal{C}$ is closed under ultraproducts and elementary equivalence.
3. Assume that $\mathcal{C}$ is a class of finite structures containing only finitely many structures of size n for each $n \in \omega$. Then the infinite models of $\mathrm{Th}(\mathcal{C})$ are exactly the models of
$$\mathrm{Th}_{\mathrm{a}}(\mathcal{C}) = \{\varphi \mid \mathfrak{A} \models \varphi \text{ for all but finitely many } \mathfrak{A} \in \mathcal{C}\}.$$

Hint for Part 1: We may assume that $\mathcal{C} = \{\mathfrak{A}_i \mid i \in I\}$ is a set (why?). If $\mathfrak{M}$ is a model of T, choose an ultrafilter $\mathcal{F}$ on I which contains $\mathcal{F}_\varphi = \{i \in I \mid \mathfrak{A}_i \models \varphi\}$ for all $\varphi \in \mathrm{Th}(\mathfrak{M})$.

[0] http://home.mathematik.uni-freiburg.de/caycedo/lehre/ws12_modell/