Übungen zur Vorlesung **Modelltheorie** (WS 2012/13) 23.11.2012
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Blatt 5

Aufgabe 1. Let $\mathfrak{A} = (\mathbb{R}, 0, <, f^{\mathfrak{A}})$, where f is a unary function symbol. Call an element $x \in \mathfrak{A}^* \succ \mathfrak{A}$ *infinitesimal* if $-\frac{1}{n} < x < \frac{1}{n}$ for all positive natural numbers n. Show that if $f^{\mathfrak{A}}(0) = 0$, then $f^{\mathfrak{A}}$ is continuous at 0 if and only if for any elementary extension $\mathfrak{A}^*$ of $\mathfrak{A}$ the map $f^{\mathfrak{A}^*}$ takes infinitesimal elements to infinitesimal elements.

Aufgabe 2.

1. Two functions $f, g : \mathbb{N} \to \mathbb{N}$ are *almost disjoint* if $f(n) \neq g(n)$ for almost all n. Show that there are $2^{\aleph_0}$-many almost disjoint functions from $\mathbb{N}$ to $\mathbb{N}$.

   Hint: For every real r choose a sequence of rational numbers which converges to r.

2. Let $\mathcal{F}$ be the set of all functions $\mathbb{N} \to \mathbb{N}$. Show that $(\mathbb{N}, f)_{f \in \mathcal{F}}$ has no countable proper elementary extension.

   Hint: If e is a new element of an elementary extension and if f and g are almost disjoint, then $f(e)$ and $g(e)$ are different.

3. Let $\mathbb{Q}$ be the ordered field of rational numbers. For every real r introduce two predicates P_r, R_r for $\{q \in \mathbb{Q} \mid q < r\}$ and $\{q \in \mathbb{Q} \mid r \leq q\}$. Show that $(\mathbb{Q}, P_r, Q_r)_{r \in \mathbb{R}}$ has no countable proper elementary extension.

   Hint: Let $\mathfrak{Q}$ be a proper elementary extension. Show first that $\mathfrak{Q}$ contains a positive infinitesimal element e. Then show that for every $r \in \mathbb{R}$ there is an element q_r such that $P_r(q_r)$ and $Q_r(q_r + e)$ are true in $\mathfrak{Q}$.

Aufgabe 3.

1. The theory of K-vector spaces $\mathsf{Mod}(K)$ (see subsection 3.3.3 in the Tent-Ziegler book) is κ-categorical for $\kappa > |K|$.

2. Is ACF_p $\aleph_0$-categorical?

Aufgabe 4. Show that an $\forall\exists$-sentence that holds in all finite fields is also true in all algebraically closed fields.

Hint: Use the facts that, for every p (prime or zero), the theory ACF_p is complete and that, for every prime p, ACF_p has a model which is the union of a chain of finite fields.

[0] http://home.mathematik.uni-freiburg.de/caycedo/lehre/ws12_modell/