Übungen zur Vorlesung **Modelltheorie** (WS 2012/13) 14.12.2012
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Blatt 8

Aufgabe 1. Let T be a theory such that for all models $\mathfrak{M}$ and $\mathfrak{N}$ of T and all $n \in \mathbb{N}$, if $\bar{a} \in M^n$ and $\bar{b} \in N^n$ satisfy the same quantifier-free formulas (in $\mathfrak{M}$ and $\mathfrak{N}$ respectively), then they satisfy the same formulas. Show that T has quantifier elimination.

Hint: Given a formula $\varphi(x)$, consider the set

$$\Phi(x) = \{\psi(x) : \psi(x) \text{ is quantifier-free and } T \models \varphi(x) \to \psi(x)\}$$

and show that $T \cup \Phi(x) \models \varphi(x)$.

Aufgabe 2.

1. Let $(K, 0, +, -, 1, \cdot)$ be an algebraically closed field. Show that every subset of K^1 definable with parameters is finite or has finite complement.
2. Let $(R, 0, +, -, 1, \cdot, <)$ be a real closed ordered field. Show that every subset of R^1 definable with parameters is a finite union of singletons and open intervals of the form (a, b) with $a, b \in R \cup \{-\infty, +\infty\}$.

Definition. Let $\mathfrak{M}$ be a structure and A a subset of M. A formula $\varphi(x) \in L(A)$ is called *algebraic* if $\varphi(\mathfrak{M})$ is finite. An element $a \in M$ is algebraic over A if it realises an algebraic $L(A)$-formula. The *(model-theoretic) algebraic closure* of A, $\mathrm{acl}(A)$, is the set of all elements of M algebraic over A. The *definable closure* of A, $\mathrm{dcl}(A)$, is the set of all elements b of M such that the singleton $\{b\}$ is definable in $\mathfrak{M}$ by an $L(A)$-formula.

Aufgabe 3. Let K be a model of ACF, of RCF or of DCF_0 and let A be a subset K. Prove that the model-theoretic algebraic closure of A is

1. (ACF) the algebraic closure of the subfield generated by A,
2. (RCF) the relative algebraic closure of the subfield generated by A,
3. (DCF_0) the algebraic closure of the differential field generated by A.

Aufgabe 4. Let K be a model of ACF_0, ACF_p for $p > 0$, of RCF or of DCF_0 and let A be a subset of K. Prove that the definable closure of A is

1. (ACF_0) the subfield generated by A,
2. (ACF_p) the perfect hull of the subfield generated by A (the *perfect hull* of a subfield k of K is given by $\{a \in K : a^{p^n} \in k \text{ for some } n \in \mathbb{N}\}$),
3. (RCF) the relative algebraic closure of the subfield generated by A,
4. (DCF_0) the differential subfield generated by A.

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[0] `http://home.mathematik.uni-freiburg.de/caycedo/lehre/ws12_modell/`