Exercises for the lecture *Model Theory of Fields* (WS 15/16) 25.10.2015

Week 2

1. Assume T has the following property: for any models M and N of T and any tuples $a \subset M$ and $b \subset N$, if $\text{qf-tp}_M(a) = \text{qf-tp}_N(b)$ then $\text{tp}_M(a) = \text{tp}_N(b)$.

Show that the map from S_n to S_n^{qf} taking a type p to the set of its quantifier-free formulas is a homeomorphism.

(As usual, S_n is the topological space on the set of complete n-types for T having as closed sets those of the fom $X_\Sigma := \{p \in S_n : \Sigma \subset p\}$ for a set of formulas Σ.

Similarly, S_n^{qf} is the topological space on the set of complete quantifier-free types for T (maximal, consistent sets of quantifier-free n-formulas, wrt. T) having as closed sets those of the form $Y_\Sigma := \{q \in S_n^{qf} : \Sigma \subset q\}$ for a set Σ of quantifier-free formulas.

Notice that these are totally disconnected, Hausdorff, compact spaces.)

Conclude that T has QE.

2. Let X be a topological space. A subset Y of X is said to be *locally closed* if for every y in Y, there is an open neighbourhood U_y of y and a closed set C_y such that $Y \cap U_y = C_y \cap U_y$.

Show that a subset Y of X is locally closed if and only if Y is of the form $U \cap C$ for an open set U and a closed set C.

Show that if $Y \subset X$ has the property that for all $xinX$, there is an open neighbourhood U_x of x and a closed set C_x such that $Y \cap U_x = C_x \cap U_x$, then Y is closed.

3. A topological space is said to be *Noetherian* if it satisfies the descending chain condition (DCC) on closed sets, i.e. there is no infinite descending chain of closed sets (wrt. inclusion). Equivalently (check this), every non-empty set of closed sets has a minimal element.

A closed set C (in an arbitrary topological space) is *irreducible* if it cannot be written as $C_1 \cup C_2$ for any closed sets C_1, C_2 both properly contained in C.

Show that in a Noetherian topological space every closed set can be written as the union of a finite set $\{C_1, ..., C_k\}$ of irreducible closed sets, and that there is a unique such set for which, additionally, there is no inclusion $C_i \subset C_j$ for any $i \neq j$.

4. Prove that a field is existentially closed if and only if it is algebraically closed.