Week 3

1. For a ring R (throughout commutative, with 1), the *spectrum* of R, $\operatorname{Spec} R$, is the topological space on the set of prime ideals of R whose closed sets are those of the form $C_A := \{\mathfrak{p} : A \subset \mathfrak{p}\}$ for an ideal A of $\mathbb{R}$. The topology of $\operatorname{Spec} R$ is called the *Zariski topology*.

Check the following facts:

- The sets C_A are indeed the closed sets of a topology on the set of prime ideals of R.
- $C_A = C_{\operatorname{Rad} A}$ for any ideal A of R.
- $\operatorname{Spec} R$ is (quasi-)compact.
- $\operatorname{Spec} R$ is not in general Hausdorff. For what integral domains R is $\operatorname{Spec} R$ Hausdorff?
- If R is a Noetherian ring, then the space $\operatorname{Spec} R$ is Noetherian.
- If $f : R \to S$ is a ring homomorphism, then $\operatorname{Spec} f : \operatorname{Spec} R \to \operatorname{Spec} S$, defined by $(\operatorname{Spec} f)(\mathfrak{q}) := f^{-1}(\mathfrak{q})$ is a continuous function.
- Spec is a contravariant functor from the category **Rings**, whose objects are rings and whose morphisms are ring homomorphisms, to the category **Top**, with topological spaces as objects and continuous functions as morphisms.

2. A element x of a closed set C (in an arbitrary topological space) is said to be a *generic point* of C if it is not contained in any closed set properly contained in C, i.e. $\overline{\{x\}} = C$.

Check the following:

- If C has a generic point, then C is irreducible.
- Given a field extension $k \subset K$, the *k-Zariski topology* on K^n has as closed sets those of the form

$$\mathcal{V}(A) := \{x \in K^n : f(x) = 0 \text{ for all } f \in A\},$$

for an ideal $A \subset k[X_1, \ldots, X_n]$.

Show that the $\mathbb{Q}$-Zariski closed subset of $\mathbb{R}^2$ given by $\mathcal{V}(X^2+Y^2-1)$ is irreducible and find its generic points. Also for the the $\mathbb{Q}$-Zariski closed subset of $\mathbb{Q}^2$ given by $\mathcal{V}(X^2+Y^2-1)$, show that it is irreducible and find its generic points.

- Every irreducible closed subset of $\operatorname{Spec} R$ has a unique generic point. (You may use the fact that for any ideal A of R, $\operatorname{Rad} A$ is the intersection of all the prime ideals of R containing A.)

3. Let k be a subfield of $K \models \mathrm{ACF}_p$. Consider the map from the space $S_n(k)$ (for the theory ACF_p) to the space $\operatorname{Spec} k[X_1, \ldots, X_n]$ given by

$$p(x) \mapsto (f(x) : f(x) = 0 \in p(x)).$$

Show that this is a continuous bijection but not a homeomorphism.