Exercises for the lecture *Model Theory of Fields* (WS 15/16) 28.11.2015

Week 7

All rings are assumed to be commutative and with 1.

1. Let R be a ring and S be an R-module. The set $\mathrm{Der}(R, S)$ of derivations from R to S inherits an R-module structure from S. If R is a k-algebra via a homomorphism $f : k \to R$, a derivation $d : R \to S$ is said to be a *k-derivation* if $d \circ f = 0$. We write $\mathrm{Der}_k(R, S)$ for the set of k-derivations.

If $S = R$, we write $\mathrm{Der}_k(R)$ for $\mathrm{Der}_k(R, S)$. In this case, derivations can be composed and the following bracket operation is defined:

$$[d, d'] = d \circ d' - d' \circ d.$$

Show that $\mathrm{Der}_k(R)$ is closed under the bracket operation, and that endowed with the bracket it is a Lie algebra.

2. Let S be a ring and $S[\epsilon]$ be the corresponding ring of dual numbers. Prove that for all $f(x) \in S[x]$ and all $a, b \in S$, $f(a + b\epsilon) = f(a) + f'(a)b\epsilon$ in $S[\epsilon]$.

Also, show that the same formula (appropriately interpreted) holds for all $f(x_1, \ldots, x_n) \in S[x_1, \ldots, x_n]$, $a, b \in S^n$.