GREEN FIELDS WITH TORSION

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Let μ denote the group of roots of unity. Let ν be a divisible subgroup of μ.

Let L be the expansion of the language of rings by a unary predicate G. Let $\mathcal{C}_\nu$ be the class of L-structures $(K, +, \cdot, 0, 1, G)$ satisfying the following conditions:

(i) $(K, +, \cdot, 0, 1)$ is an algebraically closed field of characteristic 0,
(ii) G is a divisible subgroup of $(K^\times, \cdot)$,
(iii) the group of torsion elements of G is isomorphic to ν,
(iv) for all $n \geq 1$ and all $y \in (K^\times)^n$, the value $\delta(y) := 2\,\mathrm{tr.\,d}(y) - \mathrm{mult.\,rk}(y)$ is non-negative.

We shall show below that $\mathcal{C}_0$ is an elementary class. This answers a question left open in [Poi01]; there the same result is proved assuming the *Conjecture on Intersections with Tori (CIT)* (cf. [Poi01, Corollaire 3.5]) and unconditionally only in the case where ν is the trivial group. The idea of replacing the use of the CIT by a combination of the *"Weak CIT"* and *"Manin-Mumford"*, Facts 0.2 and 0.3 below, comes from [Zil04].

In the definition and facts below, K denotes an algebraically closed field of characteristic 0.

Definition 0.1. Let V and W be subvarieties of $(K^\times)^n$ such that $V \cap W$ is non-empty and let S be an irreducible component S of $V \cap W$.

If

$$\dim S > \dim V + \dim W - n,$$

then S is said to be an *atypical component of the intersection of* V *and* W. Otherwise, that is if

$$\dim S = \dim V + \dim W - n,$$

S is said to be a *typical component of the intersection of* V *and* W.

Let T be a coset of an algebraic subgroup of $(K^\times)^n$ with $S \subset T$. If

$$\dim S > \dim V \cap T + \dim W \cap T - \dim T,$$

then S is said to be an *atypical component of the intersection of* V *and* W *with respect to* T. Otherwise, that is if

$$\dim S = \dim V \cap T + \dim W \cap T - \dim T,$$

S is said to be a *typical component of the intersection of* V *and* W *with respect to* T.

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Fact 0.2 ("Weak CIT", [Zil02])**.** *Let $n \geq 1$. For every algebraic subvariety W of $(K^\times)^n$, there exist proper algebraic subgroups $T_1, \ldots, T_s$ of $(K^\times)^n$ with the following property:*
for any coset αT of a proper algebraic subgroup T of $(K^\times)^n$, if S is an atypical component of the intersection of W and αT then there exist $i \in \{1, \ldots, s\}$ and $\alpha' \in (K^\times)^n$ such that $S \subset \alpha' T_i$ and S is a typical component of the intersection of W and αT with respect to $\alpha' T_i$.

Fact 0.3 ("Manin-Mumford")**.** *For every proper algebraic subvariety W of $(K^\times)^n$, there exist proper algebraic subgroups $H_1, \ldots, H_r$ of $(K^\times)^n$ and $\gamma_1, \ldots, \gamma_r \in \mu$ such that*

$$W \cap \mu^n = \bigcup_{j=1}^{r} \gamma_j (H_j \cap \mu^n).$$

Proposition 0.4. *The class $\mathcal{C}_\nu$ is elementary.*

Proof. It is clear that conditions (i) and (ii) can be expressed by a set L-sentences. It is easy to see that condition (iii) can be expressed by a set of sentences requiring that G has non-trivial p-torsion precisely for those primes p for which ν has non-trivial p-torsion.

We shall now see that, modulo (i),(ii),(iii), condition (iv) is equivalent to the following: for each $n \geq 1$, and each algebraic subvariety W of $(K^\times)^n$ defined and irreducible over $\mathbb{Q}$ of dimension $< \frac{n}{2}$,

$$\forall y \left((y \in W \wedge y \in G^n \wedge y \notin W^*) \to \bigvee_{1 \leq i \leq s} \bigvee_{\substack{1 \leq j \leq r_i \\ H_{ij} \text{ proper}}} y^{M^i} \in \mu_{ij} H_{ij} \right)$$

where

- $T_1, \ldots, T_s$ are the proper algebraic subgroups provided by Fact 0.2 for W, and for $i = 1, \ldots, s$, M^i is an $n_i \times n$-matrix with integer entries of rank n_i such that T_i is defined by the system of equations $y^{M^i} = 1$.
- for each $i = 1, \ldots, s$, $\gamma_{i1}, \ldots, \gamma_{ir_i}$ and $H_{i1}, \ldots, H_{ir_i}$ are as provided by Fact 0.3 for the variety W_i, which by definition is the $\mathbb{Q}$-Zariski closure the set W^{M^i}; and for each i, j, μ_{ij} is the set of all roots of unity of the same order as γ_{ij}.
  Note that each set $\mu_{ij} H_{ij}$ is an algebraic subgroup of $(K^\times)^n$ of the same dimension as H_{ij}; in particular, it is defined over $\mathbb{Q}$.
- $W^* = \bigcup_{i=1}^{s} W^{*i}$ and W^{*i} is the $\mathbb{Q}$-Zariski closure of the set
  $$\{b \in W : \dim W \cap bT_i > \dim W - \dim W_i\}.$$
  Note that the above set is the union of the non-generic (i.e. not of minimal dimension) fibres inside W of the map given by $y \mapsto y^{M^i}$. By a standard fact, this set is contained in a proper closed subset of W. Therefore $W^* \subsetneq W$.

Assume (K, G) satisfies the sentences above. To see that (K, G) must then also satisfy (iv), suppose towards a contradiction that $b \in (K^\times)^n$ is such that $\delta(b) < 0$. It is easy to see that we may assume b to be in G^n and multiplicatively independent. Let W be the algebraic locus of b over $\mathbb{Q}$. Then, since $\delta(b) < 0$, we have $\dim W < \frac{n}{2}$. Thus, one of the above sentences corresponds to W. If the disjunction in the sentence is non-empty then we get a multiplicative dependence on b, hence a contradiction. If the disjunction is empty, then the sentence says that the set $(W \setminus W^*) \cap G^n$ is empty, but our b is in this set, thus also a contradiction. This proves that (K, G) satisfies (iv).

Conversely, assume that (K, G) satisfies (iv) and let us see that the above sentences hold in (K, G). Let $n \geq 1$ and let W be an algebraic subvariety of $(K^\times)^n$ defined and irreducible over $\mathbb{Q}$ of dimension $< \frac{n}{2}$. Suppose b is in the set $(W \setminus W^*) \cap G^n$. Since $\operatorname{tr.d}(b) \leq \dim W < n/2$ and by assumption $\delta(b) \geq 0$, the tuple b must be multiplicatively dependent. Thus, let T be a proper algebraic subgroup of $(K^\times)^n$ containing b of dimension $\operatorname{mult.rk}(b)$.

Let S be an irreducible component of $W \cap T$ containing b. Then S is atypical: to see this note that, on the one hand, S is defined over $\mathbb{Q}^{\text{alg}}$ and so $\dim S \geq \operatorname{tr.d}(b) \geq \frac{1}{2} \operatorname{mult.rk}(b) = \frac{1}{2} \dim T$ and, on the other hand, since $\dim W < \frac{n}{2}$, we have $\dim W + \dim T - n < \dim T - \frac{n}{2} \leq \frac{1}{2} \dim T$. Therefore, we can find $i \in \{1, \dots, s\}$ and $\alpha \in (K^\times)^n$ such that $S \subset \alpha T_i$ and S is a typical component of the intersection of W and T with respect to αT_i.

Since S is defined over $\mathbb{Q}^{\text{alg}}$, it has $\mathbb{Q}^{\text{alg}}$ rational points. Hence we may assume $\alpha \in (\mathbb{Q}^{\text{alg}})^n$. Let us now look at the coefficients of the equations defining the coset αT_i, namely $\beta := \alpha^{M^i} \in (\mathbb{Q}^{\text{alg}})^{n_i}$. Also, $\beta = b^{M^i} \in G^{n_i}$. Thus, by (iv), $\beta \in \mu^{n_i}$. We can now conclude that $b^{M^i} \in W_i \cap \mu^{n_i} = \bigcup_{j=1}^{r_i} \gamma_{ij}(H_{ij} \cap \mu^{n_i})$. Therefore we can find $j \in \{1, \dots, r_i\}$ such that $b^{M^i} \in \gamma_{ij} H_{ij} \subset \mu_{ij} H_{ij}$.

It now suffices to show that H_{ij} is a proper subgroup of $(K^\times)^{n_i}$. This follows from the following calculation showing that $\dim W_i < n_i$: First, from the atypicality of S we have

$$\dim S > \dim W + \dim T - n$$

And from the typicality of S with respect to αT_i we have

$$\dim S = \dim W \cap \alpha T_i + \dim T \cap \alpha T_i - \dim \alpha T_i.$$

Combining the last two expressions we get

$$\dim W + \dim T - n < \dim W \cap \alpha T_i + \dim T \cap \alpha T_i - \dim \alpha T_i.$$

Reorganizing terms and noting that $T = bT$ and $\alpha T_i = bT_i$,

$$\begin{aligned} \dim W - \dim W \cap bT_i &< n - \dim bT_i + \dim bT \cap bT_i - \dim bT \\ &= n - \dim T_i + \dim T \cap bT_i - \dim T \\ &\leq n - \dim T_i = n_i. \end{aligned}$$

Since b is not in W^*, we know $\dim W_i = \dim W - \dim W \cap bT_i$. Therefore $\dim W_i < n_i$. □

References

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