Tame expansions of the real and complex fields

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Outline

(1) Expansions of the fields of real and complex numbers

(2) The predimension method

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(1) Expansions of the fields of real and complex numbers

(2) The predimension method

(3) Miller's theorem and question

Expansions of the real ordered field

- $\mathbb{R} = (\mathbb{R}, <, +, 0, -, \cdot, 1)$ has QE and is o-minimal.
- *o-minimal*
  - $(\mathbb{R}, e^x)$ — o-minimal and model complete (Wilkie)
  - $(\mathbb{R}, (x^r)_{r \in K})$, for any $K \subset \mathbb{R}$.
    (where $x^r := e^{r \log x}$ for $x > 0$).
- *non-o-minimal, tame*
  - $(\mathbb{R}, \omega^{\mathbb{Z}})$, for any $\omega > 0$,
  - $(\mathbb{R}, U)$, with $U \subset \mathbb{R}^2$ the set of all roots of unity.
    These have QE up to Boolean combinations of existential formulas (van den Dries, Gunaydin & van den Dries, Belegradek & Zilber)
  - $(\mathbb{R}, S_\omega)$ where $S_\omega = \{(e^t cos(\omega t), e^t sin(\omega t)) : t \in \mathbb{R}\}$.
    This structure is *d-minimal*: every definable subset of $\mathbb{R}$ is a finite union of open intervals and discrete sets. (Miller & Speissegger)
- *wild*
  - $(\mathbb{R}, \mathbb{Z})$ — all projective sets are definable,
    $\sim (\mathbb{R}, \omega^{\mathbb{Z}}, \eta^{\mathbb{Z}})$, for any $\omega, \eta > 0$ such that $\log \omega, \log \eta$ are $\mathbb{Q}$-linearly independent (Hieronymi)

Expansions of the complex field

- $\mathbb{C} = (\mathbb{C}, +, 0, -, \cdot, 1)$ has QE and is strongly minimal.
- *strongly minimal*
  - Very difficult to find an example of a proper expansion of the complex field that is strongly minimal.
    In fact, it was conjectured that this was impossible (part of Zilber's trichotomy conjecture.)
  - Hrushovski constructions gives many examples, but not explicitly.
  - Counterpoint (Peterzil & Starchenko, extending Marker):
    If $\mathcal{C}$ is a proper expansion of the complex field that is definable in an o-minimal expansion of $\mathbb{R}$ (under the the usual identification $\mathbb{C} \sim \mathbb{R}^2$), then the set of real numbers is definable in $\mathcal{C}$.

Expansions of the complex field

- *stable*
  - $(\mathbb{C}, \Gamma)$, with $\Gamma \leq \mathbb{C}^*$ divisible and of finite rank, is ω-stable and has QE up to Boolean combinations of existential formulas (van den Dries, Gunaydin).
  - Let K be a subfield of $\mathbb{C}$. Consider the 2-sorted structure:

$$(\mathbb{C}, +, (r\cdot)_{r \in K}) \xrightarrow{\exp} (\mathbb{C}, +, \cdot),$$

    Assuming Schanuel's Conjecture, for $K \subset \mathbb{R}$ of finite transcendence degree, the structure has QE up to Boolean combiantions of existential formulas and is superstable (Zilber). If $K = \mathbb{Q}(r)$ with $r \in \mathbb{R}$ generic in $\mathbb{R}_{\exp}$, then the result is unconditional (using a theorem of Bays-Kirby-Wilkie).
  - $(\mathbb{C}, G)$ with

$$G = S_\omega \exp(\mathbb{Q}) = \exp((1 + \omega i)\mathbb{R} + \mathbb{Q})$$

    is ω-stable and has QE up to Boolean combinations of existential formulas for ω generic in $\mathbb{R}_{\exp}$. The same holds for all $\omega\mathbb{R} \setminus \{0\}$, assuming Schanuel's Conjecture. (C., Zilber)

Expansions of the complex field

- *unstable, yet tame*
  - Conjecturally, $(\mathbb{C}, \exp)$.
    Conjecture (Zilber): $(\mathbb{C}, \exp)$ is *quasi-minimal*, i.e. every definable subset of $\mathbb{C}$ is countable or has countable complement.
  - $(\mathbb{C}, \mathbb{Z})$ — is quasiminimal.
  - $(\mathbb{C}, \mathbb{R})$ — equivalent to $\mathbb{R}$
- *wild* $(\mathbb{C}, \mathbb{R}, \mathbb{Z})$ — equivalent to $(\mathbb{R}, \mathbb{Z})$.

(1) Expansions of the fields of real and complex numbers

(2) The predimension method

(3) Miller's theorem and question

The predimension method: SC and $\mathbb{C}_{\exp}$

- **Schanuel's conjecture (SC):** If $x_1, \ldots, x_n$ are $\mathbb{Q}$-linearly independent complex numbers, then $\operatorname{tr.d.}_{\mathbb{Q}}(x_1, ..., x_n, \exp(x_1), \ldots, \exp(x_n))$ is at least n.
- Let (F, ex) an exponential field, i.e. a field F together with a homomorphism
$$\mathrm{ex} : (F, +) \to (F^*, \cdot),$$
with F algebraically closed and of characteristic zero.
- For $n \geq 1$ and $x = (x_1, \ldots, x_n) \in F^n$, define
$$\delta_{\mathrm{ex}}(x) := \operatorname{tr.d.}_{\mathbb{Q}}(x, \mathrm{ex}(x)) - \operatorname{lin.d.}_{\mathbb{Q}}(x).$$
- A tuple $c \in F^n$ $(n \geq 0)$ is said to be *self-sufficient* if for every tuple x extending c,
$$\delta_{\mathrm{ex}}(x/c) := \delta_{\mathrm{ex}}(x) - \delta_{\mathrm{ex}}(c) \geq 0.$$
- SC holds **iff** in $(\mathbb{C}, \exp)$, $\delta_{ex}(x) \geq 0$ for all x
iff $\emptyset$ is self-sufficient in $(\mathbb{C}, \exp)$.

The predimension method: Raising to powers

- Fix a subfield K of $\mathbb{C}$ of finite transcendence degree. Let

$$(V, +, (r\cdot)_{r\in K}) \xrightarrow{\mathrm{ex}} (F, +, \cdot)$$

  be a 2-sorted structure where $(V, +, (r\cdot)_{r\in K})$ is a K-vector space, $(F, +, \cdot)$ is an ACF of char 0, and ex is a surjective homomorphism from V to $K^\times$.
- For $n \geq 1$ and $x = (x_1, \ldots, x_n) \in V^n$, define

$$\delta_K(x) := \mathrm{lin.\,d.}_K(x) + \mathrm{tr.\,d.}_{\mathbb{Q}}(\mathrm{ex}(x)) - \mathrm{lin.\,d.}_{\mathbb{Q}}(x).$$

- A tuple $c \in V^n$ is said to be *self-sufficient* if for every tuple x extending c,

$$\delta(x/c) := \delta_K(x) - \delta(c) \geq 0.$$

- SC implies that in $(\mathbb{C}, +, (r\cdot)_{r\in K}) \xrightarrow{\exp} (\mathbb{C}, +, \cdot)$, $\delta_K(x) \geq -\,\mathrm{tr.\,d.}(K)$ for all x.
- Hence, SC implies that there exists a self-sufficient tuple in the structure.

The predimension method: Raising to powers

- For $i = 1, 2$, let

$$\mathcal{V}_i := (V_i, +, (r\cdot)_{r\in K}) \xrightarrow{\mathrm{ex}_i} (F_i, +, \cdot)$$

  be structures as above. Assume the following:
  - there is a partial isomorphism $c_1 \mapsto c_2$ from $\mathcal{V}_1$ to $\mathcal{V}_2$ with c_i a self-sufficient tuple in $\mathcal{V}_i$,
  - each $\mathcal{V}_i$ is "existentially closed (with respect to self-sufficient embeddings)",
  - each $\mathcal{V}_i$ is ω-saturated.
- Then the set of partial isomorpshims

$$\mathcal{F} = \{x_1 \overset{\cong}{\mapsto} x_2 : x_i \text{ self-sufficient in } \mathcal{V}_i\}$$

  is a back-and-forth system from $\mathcal{V}_1$ to $\mathcal{V}_2$.
- Moreover, every finite partial isomorphism that preserves existential formulas extends to a member of $\mathcal{F}$.

The predimension method: Raising to powers

- Suppose c is a self-sufficient tuple in the structure

$$(\mathbb{C}, +, (r\cdot)_{r\in K}) \xrightarrow{\exp} (\mathbb{C}, +, \cdot),$$

  and the structure is "existentially closed".
- Then the complete theory of the structure expanded by constants for c is axiomatized by sentences expressing the following:
  - basic algebraic conditions,
  - "c is self-suficient",
  - "the structure is existentially closed".
- Also, the theory has QE up to Boolean combinations of existential formulas.

The predimension method: Raising to powers

Finding self-sufficient tuples without SC:

Theorem (Bays, Kirby, Wilkie)

If $K = \mathbb{Q}(r)$ with r generic in $\mathbb{R}_{\mathrm{exp}}$, then $\delta_K(x) \geq 0$ for all x.

This means that $\emptyset$ is self-sufficient in the structure

$$(\mathbb{C}, +, (r\cdot)_{r \in K}) \xrightarrow{\exp} (\mathbb{C}, +, \cdot),$$

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Miller's theorem and question

Definition

- A *(real) linear vector* field is an $\mathbb{R}$-linear map $F : \mathbb{R}^n \to \mathbb{R}^n$.
- A *solution* of F is a differentiable map $\gamma : I \to \mathbb{R}^n$ defined on a non-trivial interval $I \subset \mathbb{R}$ such that

$$\gamma'(t) = F(\gamma(t)), \text{ for all } t \in I.$$

- A *trajectory* of F is the image of a solution.

Example

Consider the linear map F given by the matrix

$$\begin{pmatrix} 1 & -\omega \\ \omega & 1 \end{pmatrix}.$$

$\gamma(t) = (e^t \cos \omega t, e^t \sin \omega t)$ is a solution of F on $\mathbb{R}$.
Hence S_ω is a trajectory.

Miller's theorem and question

Theorem (C. Miller)

Let $\mathcal{G}$ be a collection of locally closed trajectories of linear vector fields such that each $\Gamma \in \mathcal{G}$ is the image of a solution on an unbounded interval.

Then $(\mathbb{R}, (\Gamma)_{\Gamma \in \mathcal{G}})$ is, up to interdefinability, one of the following:

- $(\mathbb{R}, (x^r)_{r \in K})$ *for some subfield K of $\mathbb{R}$,*
- $(\mathbb{R}, \exp)$,
- $(\mathbb{R}, S_\omega)$ *for some non-zero $\omega \in \mathbb{R}$,*
- $(\mathbb{R}, \mathbb{Z})$.

Question

Question: What about non-locally closed trajectories?
Basic case to be understood: For irrational $\omega \in \mathbb{R}$, let

$$G_\omega = \{(\cos t, \sin t, \cos \omega t, \sin \omega t) : t \in \mathbb{R}\} \leq (S^1)^2.$$

This is a dense subgroup of $(S^1)^2$. It is a non- locally closed trajectory of a linear vector field.
Note

$$G_\omega = \{(e^{it}, e^{\omega it}) : t \in \mathbb{R}\}$$

i.e. G_ω is the relation of raising to the power ω on S^1.

Theorem: Raising to real powers on S^1.

Theorem (joint with A. Gunaydin and P. Hieronymi)

Let K be a subfield of $\mathbb{R}$ of finite transcendence degee. For $n \geq 1$ and $\omega = (\omega_1, \ldots, \omega_n) \in K^n$, let

$$G_\omega = \{(y_1, \ldots, y_n) \in (S^1)^n : y_1^{\omega_1} \cdots y_n^{\omega_n} = 1\}.$$

Assuming Schanuel's conjecture, the structure $(\mathbb{R}, (G_\omega)_\omega)$ has, after adding constants for appropriate elements, QE up to Boolean combinations of existential formulas.
If $K = \mathbb{Q}(\omega_0)$ with $\omega_0 \in \mathbb{R}$ generic in $\mathbb{R}_{\exp}$, then the result holds unconditionally. (by Bays-Kirby-Wilkie)