Expansions of 1-dimensional algebraic groups by a predicate for a subgroup

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Abstract

This thesis considers theories of expansions of the natural algebraic structure on the multiplicative group and on an elliptic curve by a predicate for a subgroup that are constructed by Hrushovski's predimension method. In the case of the multiplicative group, these are the theories of fields with green points constructed by Poizat. The convention of calling the elements of the distinguished subgroup green points is maintained throughout this work, also in the elliptic curve case, and we speak of theories of green points.

In the first part of the thesis, we give a detailed account of the construction of the theories of green points. The work of Poizat is extended to the case of elliptic curves and an open question is answered in order to complete the construction in the cases where the distinguished subgroup is allowed to have torsion. Proofs of the main model-theoretic properties of the theories, ω-stability and near model-completeness, are included, as well as rank calculations.

In the second part, following ideas of Zilber, we find natural models of the constructed theories on the complex points of the corresponding algebraic group. In the case of elliptic curves, this is done under the assumption that the curve has no complex multiplication and is defined over the reals. In general, we also need to assume a consequence of the Schanuel Conjecture, in the multiplicative group case, and an analogous statement in the elliptic curve case. For the multiplicative group, the assumption is known to hold in generic cases by a theorem of Bays, Kirby and Wilkie; our result is therefore unconditional in these cases.

Motivated by Zilber's work on connections between model theory and non-commutative geometry, we prove similar results for variations of the above theories in which the distinguished subgroup is elementarily equivalent to the additive group of the integers, which we call theories of emerald points.

Contents

Chapter 1

Introduction

This thesis aims to contribute to the model-theoretic study of analytic structures. In particular, the present work fits in Boris Zilber's programme of finding mathematically natural models for model-theoretically well-behaved theories ([47]).

We consider theories of expansions of the natural algebraic structure on the multiplicative group and on an elliptic curve by a predicate for a subgroup that are constructed by Hrushovski's predimension method. In the case of the multiplicative group, these are the theories of fields with green points constructed by Bruno Poizat in [33]. Following the convention introduced by Poizat, we call the elements of the distinguished subgroup *green points* and the other elements of the structure *white points*, also in the elliptic curve case. The constructed theories are called *theories of green points*. In models of the theories the dimension of the distinguished subgroup is half of that of the ambient algebraic group. Of course, dimension here is not algebro-geometric dimension but rather a model-theoretic rank and the subgroup is indeed far from being an algebraic subgroup.

These theories are very well-behaved from the viewpoint of model theory (they are ω-stable), but a priori do not have mathematically natural models. In what follows, we explicitly find models for these theories on the complex points of the corresponding algebraic group, under certain assumptions. This is done following the strategy by Zilber in [46], where he considered the same question in the case of the multiplicative group and whose results are here improved. In the case of an elliptic curve, the green subgroup in the complex models is a dense 1-parameter subgroup. In the multiplicative group case, the green subgroup is a union of real logarithmic spirals in the complex plane. In order to prove that these structures are models of the constructed theories, in general, we need to assume a consequence of the Schanuel Conjecture, in the multiplicative group case, and of an analogous conjecture, in the case of elliptic curves. For the multiplicative group, the assumption is known to hold in generic cases by a theorem of Bays, Kirby and Wilkie ([5]); our result is therefore unconditional in these cases.

1.1 Pure model theory: constructing theories

For already more than twenty years, Hrushovski's predimension method has been the most widely used tool to produce theories exhibiting exotic properties. The method was first used by Ehud Hrushovski in the late 1980s to produce counterexamples to Zilber's Trichotomy conjecture.

The Trichotomy conjecture proposed a classification of strongly minimal theories by the properties of their associated pregeometries.

Strongly minimal theories are first-order theories such that all their models are infinite and have the property that every definable subset of the universe is finite or has finite complement. Such theories are at the tamest end of the hierarchy developed in stability theory. Moreover, they are the building blocks of all theories of finite Morley rank and also have an important role in the study of uncountably categorical theories. Basic examples of strongly minimal theories are the theory of infinite sets, the theory of infinite vector spaces over a any given field, and the theory of algebraically closed fields of any given characteristic. In every strongly minimal theory the model-theoretic algebraic closure operator is a pregeometry and thus induces a dimension function on subsets of the universe. In the case of infinite sets, the algebraic closure of every subset of the universe is the subset itself and the associated dimension is cardinality. In vector spaces, the algebraic closure is given by the linear span and the associated dimension function by linear dimension. In algebraically closed fields, the model-theoretic algebraic closure coincides with the field-theoretic algebraic closure and the dimension function is therefore given by the transcendence degree. These three cases exemplify different geometric behaviours: in the first case, the pregeometry is *trivial*, meaning that for every set A, one has $\mathrm{cl}(A) = \bigcup_{a \in A} \mathrm{cl}(\{a\})$; in the second case, the pregeometry is non trivial and *locally modular*, a notion that formalises the idea of being "linear"; in the third case, the pregeometry is not locally modular. The Trichotomy conjecture states that the above are essentially all examples of pregeometries of strongly minimal theories. Formally, the conjecture can be seen as having two parts: on the one hand, it claims that every non-locally modular strongly minimal theory interprets the theory of an algebraically closed field, and, on the other hand, that in every strongly minimal theory of expansions of algebraically closed fields, every definable set is already definable in the language of rings.

Hrushovski showed that both parts of the conjecture are, in fact, false. He constructed a strongly minimal theory with a non-locally modular pregeometry which does not interpret any infinite group, thus refuting the first part of the conjecture ([14]). To disprove the second part of the conjecture, he also constructed strongly minimal theories of expansions of algebraically closed fields having definable sets that are not definable in the ring language; to do this he in fact proved that the union of any two strongly minimal theories in disjoint languages can be completed to a strongly minimal theory, called their *fusion*, provided the two theories have the *definable multiplicity property*, which, he also showed, holds in theories of algebraically closed fields ([13]).

Despite being false in its original form, the Trichotomy conjecture has been greatly influential in the development of geometric model theory. First of all, the trichotomy was proved to hold in the context of Zariski geometries, by Hrushovski and Zilber ([17]), and this result was applied in Hrushovski's proof of the function field version of the Mordell-Lang conjecture ([17]). But also, the conjecture has inspired developments in several other contexts, where similar classifications have been proved (e.g. [26]).

Hrushovski's method for constructing strongly minimal theories, or more generally theories of finite rank, can be thought of as having two separate stages: a first construction, that generally produces theories of infinite rank, which is usually called the *free amalgamation construction*; and a second stage, called the *collapse*, in which the first construction is refined to obtain a theory of finite rank.

In this thesis we only consider theories obtained from the the first part of Hrushovski's method. Among the many theories constructed in this fashion, our main interest is in Poizat's theories of fields with green points, from [33].

Poizat constructed ω-stable theories of expansions of algebraically closed fields by a unary predicate such that the Morley rank of the domain is twice the Morley rank of the predicate. In the initial construction ([31]) he called the elements of the predicate *black points* and the other elements of the domain *white points*. In this case the Morley rank of the predicate is ω and the Morley rank of the domain is $\omega \cdot 2$. This refuted a conjecture of Berline and Lascar, which stated that all superstable theories of algebraically closed fields had U-rank ω^α, for some ordinal α. He also carried out the collapse, obtaining a theory of Morley rank two, where the predicate has rank one. Later Poizat also constructed similar theories in which the interpretation of the predicate is a subgroup of the additive group of the field, case in which he called the elements of the subgroup *red points*, and theories in which the interpretation of the predicate is a multiplicative subgroup, case in which he called the elements of the subgroup *green points* ([33]). In the case of red points, his theory has Morley rank $\omega \cdot 2$ in the case of fields of positive characteristic and rank $\omega^2 \cdot 2$ in the characteristic zero case. In the green case, the field always has characteristic zero and the theory has Morley rank $\omega \cdot 2$.

Before giving some details about the construction, let us briefly discuss the two main reasons for interest in the theories of fields with green points.

The first reason is that the construction has direct relevance for the question of existence of so-called *bad fields*. A bad field is an expansion of an algebraically closed field by a predicate for a multiplicative subgroup whose Morley rank is finite and greater than one. The question of whether such structures exist arose in work on the Algebraicity Conjecture due to Cherlin and Zilber, which states that every simple group of finite Morley rank is an algebraic group over an algebraically closed field and which is open since the late 1970s. The non-existence of bad fields would have simplified some of the work on the conjecture (see, for example, the introduction of [2]). Eventually, however, it was suspected that bad fields of characteristic zero could be obtained by Hrushovski's method. Poizat's construction of fields with green points was effectively the first step in this direction. It produced an analogue of a bad field of infinite rank. Moreover, carrying out the corresponding collapse would complete the construction of a bad field. This was subsequently done by Baudisch, Hils, Martin-Pizarro and Wagner in [2]. Let us also mention that, by a result of Wagner ([41]), the existence of bad fields of positive characteristic p would imply that there are only finitely many p-Mersenne primes. Bad fields of positive characteristic are therefore not expected to exist, but proving their nonexistence is, for the same reason, difficult.

The second reason is that the construction of the theories of fields with green points involves strong algebro-geometric results in an essential way. Besides being interesting in its own right, this technical difficulty also exposes strong connections with Zilber's construction of fields with pseudo-exponentiation, where similar results have to be applied. In both cases questions of intersections of algebraic subvarieties of algebraic tori with algebraic subgroups naturally appear. In this regard, Zilber's Conjecture on intersections with tori (CIT), from [44], answers all the questions involved, but its general validity remains an open question. The conjecture was indeed motivated by Zilber's work on the model theory complex exponentiation. A partial result, usually called Weak CIT, follows from a theorem of Ax in differential algebra by a model-theoretic argument, as showed by Poizat in [33] and by Zilber in [44]. Remarkably, for the construction of theories of green points, Weak CIT suffices. This was shown by

Poizat in [33] in the case where the theory requires the green subgroup to be torsion-free and he stated the question of whether this was also true for arbitrary torsion in the green subgroup. We answer this question positively in Chapter 4. Another strong algebraic result that is needed in the construction is the Thumbtack Lemma, which was first proved by Zilber in [52] (see also [6]) and which is now available in a very general form by a theorem of Bays, Gavrilovich and Hils [4]. The Thumbtack Lemma is used in an essential way to prove that the theories of green points are ω-stable. This point is not explicit in Poizat's original paper and has only become clear in more recent works on the topic, e.g. [11].

The first part of this thesis, which consists of chapters 2 to 4, is dedicated to the construction of the theories of green points, extending the work of Poizat to include the case where the multiplicative group is replaced by an elliptic curve. Chapter 2 contains a general presentation of the first part of Hrushovski's construction method. Chapter 3 collects the main algebro-geometric facts needed for the construction. In Chapter 4, the construction of the theories is carried out and their main model-theoretic properties are proved.

Schematically, the construction of the theories of green points is as follows:

Fix $\mathbb{A}$ defined over a field k_0 of characteristic zero; throughout $\mathbb{A}$ is assumed to be the multiplicative group or an elliptic curve. Note that for every algebraically closed field K extending k_0, there is a natural structure on the set $A = \mathbb{A}(K)$ of K-points of $\mathbb{A}$ in the language having a predicate for each Zariski closed subset of a cartesian power A^n of A. Also, every such A is a divisible $\mathrm{End}(\mathbb{A})$-module, where $\mathrm{End}(\mathbb{A})$ denotes the ring of algebraic endomorphisms of $\mathbb{A}$.

Consider the class $\mathcal{C}$ of structures of the form (A, G), where $A = \mathbb{A}(K)$ for some algebraically closed field K extending k_0 (carrying the structure described above) and G is a divisible $\mathrm{End}(\mathbb{A})$-submodule of A. We introduce the following *predimension function* δ on structures in $\mathcal{C}$: given $(A, G) \in \mathcal{C}$ and a finite subset Y of A,

$$\delta(Y) = 2\,\mathrm{tr.\,d.}_{k_0}(Y) - \mathrm{lin.\,d.}(\mathrm{span}(Y) \cap G),$$

where $\mathrm{tr.\,d.}_{k_0}(Y)$ denotes the transcendence degree of the field extension $k_0(Y)/k_0$, $\mathrm{span}(Y)$ is the divisible hull of the $\mathrm{End}(\mathbb{A})$-submodule of A generated by Y, and lin. d. denotes $\mathrm{End}(\mathbb{A})$-linear dimension. For any finite-($\mathrm{End}(\mathbb{A})$-)dimensional subset Y of A, the value $\delta(Y)$ is defined to be $\delta(Y_0)$ for any finite Y_0 such that $\mathrm{span}(Y_0) = \mathrm{span}(Y)$.

With the above, the set up for the construction of the theories of green points is already determined. The construction then follows the general method, which is presented in detail in Chapter 2 and which we now briefly summarize. Associated to the predimension function there is the notion of self-sufficient sets. A finite-dimensional subset Y of A is said to be *self-sufficient* in a subset Z of A containing Y, if for every finite dimensional subset Y' of Z containing Y, $\delta(Y') \geq \delta(Y)$. Given any finite-dimensional substructure $\mathcal{X}_0$ of a structure $\mathcal{A}_0$ in $\mathcal{C}$ such that its domain X_0 is self-sufficient in $\mathcal{A}_0$, consider the class $\mathcal{C}_0$ of structures $\mathcal{B}$ in $\mathcal{C}$ such that there is a self-sufficient embedding of $\mathcal{X}_0$ into $\mathcal{B}$ (that is an embedding with self-sufficient image). The structures $\mathcal{A}$ in $\mathcal{C}_0$ having the property that all self-sufficient embeddings between finite-dimensional substructures of structures in $\mathcal{C}_0$ can be realised, up to isomorphism, by substructures of $\mathcal{A}$ are said to be *rich*. Provided the class of substructures of structures in $\mathcal{C}_0$ satisfies some natural properties with respect to self-sufficient embeddings, most notably the amalgamation property, rich structures can be found in $\mathcal{C}_0$ by the construction of Fraïsse limits. The universality property of rich structures implies that all rich structures in $\mathcal{C}_0$ are back-and-forth equivalent, and hence elementarily equivalent. Moreover, it also follows that the complete first-order theory common to all rich structures in $\mathcal{C}_0$ has a form of quantifier elimination; this is stated in Proposition 2.3.10, and further elaborated in Proposition 2.3.11, in the general context. Section 4.1 contains the proofs of the necessary provisions for the above scheme to go through in the case of green points.

Regarding the constructed theory, besides the aspects of completeness and quantifier elimination, there is also the question of finding an explicit axiomatization. There is a general scheme for doing this: first finding axioms for the class $\mathcal{C}_0$ and then further axioms such that the ω-saturated models of both sets of axioms are precisely the rich structures in $\mathcal{C}_0$. It follows that the two sets of axioms together form an axiomatization of the theory.

However, the task of finding these two sets of axioms depends strongly on the particular instance of the construction at hand and is not included in the general presentation in Chapter 2. This aspect is only treated for the theories of green points, in Section 4.2.

Section 4.3 contains the proofs of the main model-theoretic properties of the theories. They are ω-stable by Theorem 4.3.3 and near model-complete by Theorem 4.3.6. Theorem 4.3.12 shows that the Morley rank of the domain is $\omega \cdot 2$ and that of the distinguished subgroup is ω.

1.2 Analytic structures: finding models

With the disproof of the Trichotomy conjecture, the question of finding natural models for the newly found theories naturally arises. This can be thought of as an initial step towards a revised classification of strongly minimal theories in the spirit of the conjecture.

In this direction, the observation that the Schanuel Conjecture from transcendental number theory can be seen as the natural inequality for a predimension function on the complex exponential field $\mathbb{C}_{\exp} = (\mathbb{C}, +, \cdot, \exp)$ has proved to be very fertile. Indeed, it was the starting point for the programme of relating the new examples to classical analytic structures laid out in [47]. The series of works culminating with Zilber's construction of fields with pseudo-exponentiation ([44], [52], [48], [51]) can be seen as the most central instance of the programme. A full realisation of the programme in this case would amount to proving the conjecture that the unique field with pseudo-exponentiation of cardinality continuum is indeed the complex exponential field. This would include proving the Schanuel conjecture, and is therefore thought to be out of reach. Also, even assuming the Schanuel conjecture, only very partial results towards a proof of the conjecture are known.

In [46], Zilber finds natural models for Poizat's theory of fields with black points, unconditionally, and for the theory of fields with green points, assuming the Schanuel Conjecture. These are some of few instances where the programme has been fully realised, modulo the hard transcendence questions.

In the second part of this thesis, Chapters 5 to 7, we improve and extend the work of Zilber in [46] to find natural models for the theories of green points in the cases of the multiplicative group and of an elliptic curve without complex multiplication defined over the reals. For this, we need to assume instances of the Schanuel Conjecture for raising to powers, a consequence of the Schanuel Conjecture, in the multiplicative group case, and analogous statements in the elliptic curve case.

Chapter 6 deals with the multiplicative group case. The main result of this chapter is Theorem 6.1.1. The following is a shorter statement for the theorem:

Theorem. *Let $\epsilon = 1 + \beta i$, with β a non-zero real number, and let Q be a non-trivial divisible subgroup of $(\mathbb{R}, +)$ of finite rank. Let*

$$G = \exp(\epsilon\mathbb{R} + Q).$$

Assume the Schanuel Conjecture for raising to powers in $K = \mathbb{Q}(\beta i)$.

Then the structure $(\mathbb{C}^*, G)$ *can be expanded by constants to a model of a theory of green points. In particular,* $(\mathbb{C}^*, G)$ *is* ω*-stable.*

In the cases where β is generic in the o-minimal structure $\mathbb{R}_{\exp}$, the Schanuel Conjecture for raising to powers in $K = \mathbb{Q}(\beta i)$ is known to hold by a theorem of Bays, Kirby and Wilkie ([5]). In these cases, the above result is therefore unconditional.

The elliptic curve case is treated in Chapter 7. The main theorem of Chapter 7 is Theorem 7.1.2, which we now state, also in a shortened version.

Theorem. *Let* $\mathbb{E}$ *be an elliptic curve without complex multiplication and let* $E = \mathbb{E}(\mathbb{C})$*. Assume the corresponding lattice* Λ *has the form* $\mathbb{Z} + \tau\mathbb{Z}$ *and* $\Lambda = \Lambda^c$*.*

Let $\epsilon = 1 + \beta i$*, with* β *a non-zero real number, be such that* $\epsilon\mathbb{R} \cap \Lambda = \{0\}$*. Put* $G = \exp_{\mathbb{E}}(\epsilon\mathbb{R})$*.*

Assume the Weak Elliptic Schanuel Conjecture for raising to powers in $K := \mathbb{Q}(\beta i)$ *(wESC*$_K$*) holds for* $\mathbb{E}$*.*

Then the structure (E, G) *can be expanded by constants to a model of a theory of green points. In particular,* (E, G) *is* ω*-stable.*

Intuitively, that the distinguished subset has real dimension one and the ambient complex field has real dimension two corresponds to the fact that in the theory the Morley rank of the predicate is half of that of the domain. In all cases, the definition of the subgroup has been chosen in such a way that, the first set of axioms in the theories of green points, those defining the class $\mathcal{C}_0$, can be easily shown to hold in the structure, with the help of existing Schanuel-style conjectures. The main part of the proof of the theorems is to show that the second set of axioms, the one that relate to richness and to what we later call the *existential closedness property*, also hold. To do this we use the same strategy as in [46], with corrections and improvements. The density of the subgroup plays an important role; in a certain sense, this density is in fact necessary for the structure not to define the reals, which would imply instability (this follows from the proof of a theorem of Marker, [22, Theorem 3.1], which has been extended by Peterzil and Starchenko, [27, Theorem 1.3]).

In [49] Zilber suggests considering variations of the theories of green points, in the multiplicative group case, in which the distinguished subgroup is elementarily equivalent to the additive group of the integers. These theories are called *theories of emerald points*. We do so in Section 6.5 and prove analogous results to those in the green case in Theorem 6.5.13 and Theorem 6.5.13. The new theories are superstable, non-ω-stable. As in the green case, the U-rank of the domain is $\omega \cdot 2$ and that of the distinguished subgroup is ω.

Let us finally note that the above results are also interesting because they give new examples of stable expansions of the complex field. Most known examples of such structures are covered by the theorems on expansions by *small sets* in [9] and the green subgroups are not small.

1.3 Outlook

Let us now briefly comment on some possible continuations of the work in this thesis.

One of the motivations for this work was that it may contribute to the work on connections between model theory and non-commutative geometry. More precisely, it is hoped that the structures considered in this thesis may be useful in finding a tame model-theoretic setting for non-commutative tori, which are basic examples of non-commutative spaces. Non-commutative tori can be defined by a construction of *non-commutative quotients* of tori by dense 1-parameter subgroups.

In the case of complex elliptic curves, we have found a model-theoretically tame, namely ω-stable, structure on a torus, which on top of a natural algebraic structure also has a dense 1-parameter subgroup as a definable set. We thus have a structure where the basic ingredients for the construction of a non-commutative torus are definable. However, standard model-theoretic tools do not seem to be able to account for the construction of non-commutative quotients. Finding appropriate tools for this purpose seems to be an interesting problem in the bigger programme of understanding non-commutative phenomena in model-theory, which can also be seen, for example, in the examples of non-classical Zariski geometries of Hrushovski and Zilber ([17]).

Regarding this project, let us simply note that the notion of generalised imaginaries, introduced by Hrushovski in [15], seems to be a good candidate for a suitable framework.

Another interesting project is to explore the possibility that the methods that have been used in this thesis to obtain results about expansions of the complex field can be applied in the study of similar expansions of the real field. The starting point for the project is the question about tameness of the expansions of the real field by the groups of green points in the complex numbers, as defined in chapters 6 and 7.

We suspect that the arguments in Chapter 6 can be strengthened to prove a real version of the existential closedness result. This suggests the conjecture that, under the same Schanuel-type assumption, the theories of these expansions of the real field are near model complete.

This project is particularly interesting because it connects with the programme of studying expansions of o-minimal structures by trajectories of definable vector fields. There is a classification of the expansions of the real field by locally closed trajectories of linear vector fields: such structures are either o-minimal, they are essentially the expansion of the real field by a logarithmic spiral in $\mathbb{R}^2$ (and thus are not o-minimal, but at least d-minimal), or they define the integers ([25]). The situation for non-locally closed trajectories is much less clear. The simplest case yet to be understood is the following: consider the expansion of the real field by a subgroup G of the torus $S^1 \times S^1$ in $\mathbb{R}^4$ of the following form:

$$G = \{\exp(it), \exp(irt) : t \in \mathbb{R}\},$$

where r is an irrational real number. Conjecturally, our methods could be used to show that if r is generic in $\mathbb{R}_{\exp}$, then the theory of this structure is near model complete and, consequently, does not define the integers. This would show that in the case of non-locally closed trajectories new kinds of structures appear, with respect to the above classification of expansions by locally closed trajectories (see Section 3 of [25]).

Chapter 2

Predimension constructions

This chapter contains a general account of some aspects of Hrushovski's predimension construction method. Since we shall only consider theories of infinite rank, we do not treat the part of the method that is usually called *collapse*, but only what is commonly referred to as the *free amalgamation construction*. Our choice of terminology is meant to stress the importance of the predimension function, and the related notion of self-sufficient embeddings, in obtaining a complete theory with a certain quantifier elimination, over that of the intermediate stage of constructing a generic structure by means of amalgamation.

The framework laid out in this chapter will be specialised to several instances later in the thesis. Our aim is to isolate some hypotheses and results inherent to the general method, rather than to the different particular cases considered in the thesis. The scope of our general approach is, however, limited, both in terms of generality and of strength of results, to the needs of the present work. In this chapter we only deal with some algebraic aspects of the construction, leading to a quantifier elimination result (Proposition 2.3.10) as the main statement; the key aspect of finding axioms for the constructed complete theory is only treated in the main instance of application of the method, in Chapter 4.

Everything in this chapter is well-known. It is hoped, however, that the exposition will offer clarification of some subtle details. The proofs of the most straightforward facts are omitted.

2.1 Pregeometries and dimension functions

The notion of dimension is fundamental in all what follows. In particular, the dimension theory in a pregeometry will be part of our basic language throughout this work. We start by introducing the basic definitions thereof and fixing some conventions.

This is in part intended to make the introduction of the notion of a predimension function more natural. The proofs of standard facts are generally omitted; some of them can be found, for example, in Section 8.1 of [23].

Let A be a set.

Definition 2.1.1. A *closure operator* on A is a map $\mathrm{cl} : \mathcal{P}(A) \to \mathcal{P}(A)$ satisfying the following conditions:

PG1 (Expansivity) For all $X \subset A$, $X \subset \mathrm{cl}(X)$.

PG2 (Monotonicity) For all $X, Y \subset A$, if $X \subset Y$ then $\mathrm{cl}(X) \subset \mathrm{cl}(Y)$.

PG3 (Idempotency) For all $X \subset A$, $\mathrm{cl}(\mathrm{cl}(X)) = \mathrm{cl}(X)$.

A closure operator on A is said to be a *pregeometry* if it satisfies the further conditions:

PG4 (Finite character) For all $X \subset A$, $\mathrm{cl}(X) = \bigcup_{X' \subset_{\mathrm{fin}} X} \mathrm{cl}(X')$.

PG5 (Exchange principle) For all $X \subset A$ and $x, y \in A$, if $y \in \mathrm{cl}(Xx) \setminus \mathrm{cl}(X)$ then $x \in \mathrm{cl}(Xy)$.

Remark 2.1.2. Let cl be a pregeometry on A.

A set $X \subset A$ is said to be cl-*closed* if $\mathrm{cl}(X) = X$. Equivalently, X is cl-closed if there exists a set Y such that $\mathrm{cl}(Y) = X$.

A set $X \subset A$ is said to be cl-*independent* if for every $x \in X$, $x \notin \mathrm{cl}(X \setminus \{x\})$.

Let $X, Y \subset A$ be such that $Y \subset X$. We say that Y is a cl-*generating subset* of X if $X \subset \mathrm{cl}(Y)$; equivalently, if $\mathrm{cl}(X) = \mathrm{cl}(Y)$.

We say that X is a cl-*basis* of Y if X is a maximal cl-independent subset of Y, or, equivalently, X is a minimal cl-generating subset of Y.

For every subset X of A there exists a cl-basis and all cl-basis of X have the same cardinality. This cardinality is the cl-*dimension* of X.

We therefore have an associated function $\mathrm{d}_{\mathrm{cl}} : \mathcal{P}(A) \to \mathrm{Card}$ that assigns to each subset X of A its cl-dimension. Due to the finite character of pregeometries, the function d_{cl} is determined by its values at finite subsets of A and the fact that it is the dimension function of a pregeometry. With this in mind, we will also refer to the restriction of d_{cl} to $\mathcal{P}_{\mathrm{fin}}(A)$ by d_{cl}.

We shall now make explicit the correspondence between pregeometries and dimension functions. We start with a definition of the notion of *dimension function* suitable for our purposes.

Definition 2.1.3. A *dimension function* on A is a map $\mathrm{d} : \mathcal{P}_{\text{fin}}(A) \to \mathbb{N}$ such that the following conditions hold.

D1 $\mathrm{d}(\emptyset) = 0$.

D2 For all $X, Y \subset_{\text{fin}} A$, if $X \subset Y$ then $\mathrm{d}(X) \leq \mathrm{d}(Y)$.

D3 For all $X, Y, Z \subset_{\text{fin}} A$, if $\mathrm{d}(XY) = \mathrm{d}(Y)$ then $\mathrm{d}(XYZ) = \mathrm{d}(YZ)$. [1]

D4 For all $x \in A$, $\mathrm{d}(\{x\}) \leq 1$.

Lemma 2.1.4. *If* cl *is a pregeometry on* A*, then* d_{cl} *is a dimension function on* A*.*

In proving the converse of 2.1.4, we will make use of the following definitions of and remarks on localisation of pregeometries and dimension functions.

Definition 2.1.5. Given a closure operator cl on A and a subset Y of A, we define a map $\mathrm{cl}_Y : \mathcal{P}(A) \to \mathcal{P}(A)$ by

$$\mathrm{cl}_Y(X) = \mathrm{cl}(X \cup Y).$$

We call cl_Y the *localisation of* cl *at* Y. We also write $\mathrm{cl}(X/Y)$ for $\mathrm{cl}_Y(X)$.

Lemma 2.1.6. *Let* Y *be a subset of* A*. If* cl *is a closure operator on* A*, then so is* cl_Y*.*

Moreover, if cl *is a pregeometry on* A*, then* cl_Y *is also a pregeometry on* A*.*

Definition 2.1.7. Let d be a dimension function on A. For $Y \subset_{\text{fin}} A$, we define the *localisation of* d *at* Y as the function $\mathrm{d}_Y : \mathcal{P}_{\text{fin}}(A) \to \mathbb{N}$ given by

$$\mathrm{d}_Y(X) := \mathrm{d}(XY) - \mathrm{d}(Y),$$

for any $X \subset_{\text{fin}} A$. We also write $\mathrm{d}(X/Y)$ for $\mathrm{d}_Y(X)$, and refer to this value as the *dimension of* X *over* Y.

For an arbitrary subset Y of A, we define the localisation of the dimension function d at Y as follows: for $X \subset_{\text{fin}} A$,

$$\mathrm{d}_Y(X) := \min_{Y' \subset_{\text{fin}} Y} \mathrm{d}(X/Y').$$

Lemma 2.1.8. *For every dimension function* d *on* A *and every* $Y \subset A$*,* d_Y *is a dimension function on* A*.*

[1] Equivalently, for all $Y, Z \subset_{\text{fin}} A$ and all $x \in X$, if $\mathrm{d}(xY) = \mathrm{d}(Y)$ then $\mathrm{d}(xYZ) = \mathrm{d}(YZ)$.

Lemma 2.1.9. *For any dimension function* d *on* A *we have the following* addition formula*: for all* $X, Y, Z \subset_{fin} A$ *with* $X \supset Y \supset Z$*,*

$$d(X/Z) = d(X/Y) + d(Y/Z).$$

Proof. Using just Definition 2.1.7 we have:

$$d(X/Z) = d(X) - d(Z) = d(X) - d(Y) + d(Y) - d(Z) = d(X/Y) + d(Y/Z).$$

□

Lemma 2.1.10. *Suppose* d *is a dimension function on* A*. Then there is a corresponding pregeometry* $\mathrm{cl_d}$ *on* A *defined by:*

$$x \in \mathrm{cl_d}(X) \iff \exists X' \subset_{fin} X,\ d(x/X') = 0.$$

Note that, by D3, if X *is finite then*

$$x \in \mathrm{cl_d}(X) \iff d(x/X) = 0.$$

Proof. We shall check that the conditions in the definition of pregeometry hold for $\mathrm{cl_d}$:

1. (Expansivity) $X \subset \mathrm{cl_d}(X)$:
   Clear.

2. (Monotonicity) $X \subset Y \implies \mathrm{cl_d}(X) \subset \mathrm{cl_d}(Y)$:
   Clear.

3. (Idempotency) $\mathrm{cl_d}(\mathrm{cl_d}(X)) = \mathrm{cl_d}(X)$:
   The inclusion from right to left follows immediately from 1 and 2.

   To prove the other inclusion, suppose $x \in \mathrm{cl_d}(\mathrm{cl_d}(X))$ and let us show that $x \in \mathrm{cl}(X)$. It is easy to see that we may assume X is finite. Then let X' be a finite subset of $\mathrm{cl_d}(X)$ such that $d(x/X') = 0$. Let $X' = \{x_1, \ldots, x_n\}$. Since X' is contained in $\mathrm{cl_d}(X)$, for every $1 \leq i \leq n$, $d(x_i/X) = 0$. Then, using the addition formula for d,

   $$d(x/X) = d(x/X') + d(X'/X) = d(X'/X) = \sum_{1 \leq i \leq n} d(x_i/Xx_1 \cdots x_{i-1}).$$

   But, for every i, we know that $d(x_i/X) = 0$. Hence, by D3, $d(x_i/Xx_1 \cdots x_{i-1}) = 0$. Thus, $d(x/X) = 0$, i.e. $x \in \mathrm{cl_d}(X)$.

4. (Finite character) $\mathrm{cl_d}(X) = \bigcup_{X' \subset_{\mathrm{fin}} X} \mathrm{cl_d}(X')$:
   Clear.

5. (Exchange principle) $y \in \mathrm{cl_d}(Xx) \setminus \mathrm{cl_d}(X) \implies x \in \mathrm{cl_d}(Xy)$:
   Suppose towards a contradiction that there exist $X \subset A$ and $x, y \in A$ such that $y \in \mathrm{cl_d}(Xx) \setminus \mathrm{cl_d}(X)$ but $x \notin \mathrm{cl_d}(Xy)$. By the definition of $\mathrm{cl_d}$, this means $\mathrm{d}(xyX) = \mathrm{d}(Xx)$, $\mathrm{d}(yX) \neq \mathrm{d}(X)$, and $\mathrm{d}(xyX) \neq \mathrm{d}(yX)$. Therefore also $\mathrm{d}(xX) \neq \mathrm{d}(X)$; because otherwise $\mathrm{d}(xyX) = \mathrm{d}(xX) = \mathrm{d}(X)$, and then, by D2, $\mathrm{d}(yX) = \mathrm{d}(X)$, which would yield a contradiction. Hence, by D4, $\mathrm{d}(y/X) = \mathrm{d}(x/X) = \mathrm{d}(x/Xy) = 1$. But then, on the one hand,

   $$\mathrm{d}(xy/X) = \mathrm{d}(y/Xx) + \mathrm{d}(x/X) = 0 + 1 = 1.$$

   and, on the other hand,

   $$\mathrm{d}(xy/X) = \mathrm{d}(x/Xy) + \mathrm{d}(y/X) = 1 + 1 = 2.$$

   A contradiction.

□

Remark 2.1.11. In the proof of the above lemma we have only used the fact that the function d has property D4 in order to show that $\mathrm{cl_d}$ satisfies the Exchange principle. Indeed, we see that whenever d has properties D1-D3, $\mathrm{cl_d}$ is a closure operator with finite character on A and its restriction to the set $A_1 = \{x \in A : \mathrm{d}(x) \leq 1\}$ is a pregeometry on A_1.

Finally, let us note that the correspondence between dimension functions and pregeometries is bijective; indeed, $\mathrm{cl}_{\mathrm{d}_{\mathrm{cl}}} = \mathrm{cl}$ and $\mathrm{d}_{\mathrm{cl_d}} = \mathrm{d}$. Also, we have the following commutativity property between localisation and passing between dimension functions and pregeometries: $\mathrm{cl}_{\mathrm{d}_Y} = (\mathrm{cl_d})_Y$ and $\mathrm{d}_{\mathrm{cl}_Y} = (\mathrm{d_{cl}})_Y$.

2.2 Predimension functions

2.2.1 Predimension functions

Let A be a set and cl_0 be a pregeometry on A.

Definition 2.2.1. A *predimension function δ on A* is a function $\delta : \mathcal{P}_{\mathrm{fin}}(A) \to \mathbb{Z}$.

The expression *predimension function* refers to functions that have a special role in our constructions and arguments, it does not, however, imply any properties of the function in question beyond having domain $\mathcal{P}_{\text{fin}}(A)$ and taking values in $\mathbb{Z}$.

Definition 2.2.2. Let δ be a predimension function on A.

Let $y \subset A$. Let δ_y, the *localisation of* δ *at* y, be the function defined by

$$\delta_y(x) = \delta(xy) - \delta(y),$$

for all $x \subset A$. We also write $\delta(x/y)$ for the value $\delta_y(x)$, which we call the *predimension of* x *over* y.

Definition 2.2.3. Let δ be a predimension function on A.

The function δ is said to be *well-defined with respect to* cl_0 if δ factors through the quotient $\mathcal{P}_{\text{fin}}(A)/\sim_0$, where, by definition, $x \sim_0 y$ if $\mathrm{cl}_0(x) = \mathrm{cl}_0(y)$.

If δ is a predimension function, well-defined with respect to cl_0, then for any finite dimensional cl_0-closed set X the expression $\delta(X)$ will denote the value $\delta(x)$ for a finite x such that $\mathrm{cl}_0(x) = X$.

Definition 2.2.4. Let δ be a predimension function on A, well-defined with respect to cl_0.

For a cl_0-finite dimensional set Y, we define the *localisation of* δ *at* Y, δ_Y, to be the function δ_y for any finite y with $\mathrm{cl}_0(y) = \mathrm{cl}_0(Y)$.

We also write $\delta(x/Y)$ for the value $\delta_Y(x)$, and call it the *predimension of* x *over* Y.

Lemma 2.2.5. *Let* δ *be a predimension function, well-defined with respect to* cl_0, *and let* Y *be a* cl_0*-finite dimensional set. Then the predimension function* δ_Y *on* A *is well-defined with respect to* $(\mathrm{cl}_0)_Y$.

Definition 2.2.6. A predimension function δ on A is said to be *submodular with respect to* cl_0 if it is well-defined with respect to cl_0 and it satisfies the following *submodularity inequality* with respect to the lattice of finitely generated cl_0-closed subsets of A: for all finite dimensional cl_0-closed sets X, Y,

$$\delta(X \vee Y/Y) \leq \delta(X/X \wedge Y),$$

where $X \wedge Y = X \cap Y$ and $X \vee Y = \mathrm{cl}_0(X \cup Y)$. Notice that $X \wedge Y$ and $X \vee Y$ are the natural lattice operations on the collections of finitely generated cl_0-closed subsets of A.

Remark 2.2.7. Let us note that for a predimension function δ, well-defined with respect to cl_0, the property of submodularity with respect to cl_0 can also be expressed as follows: for all $x, y \subset A$,

$$\delta(xy/Y) \leq \delta(x/X \cap Y),$$

where $X = \mathrm{cl}_0(x)$ and $Y = \mathrm{cl}_0(Y)$.

Remark 2.2.8. We also note that every dimension function d is submodular with respect to the corresponding pregeometry cl, and also with respect to any weaker pregeometry cl' (i.e. any pregeometry cl' such that $\mathrm{cl}'(X) \subset \mathrm{cl}(X)$ for all X).

Definition 2.2.9. A pregeometry cl is said to be *modular* if the corresponding dimension function d has the property that for all cl-closed sets X, Y,

$$\mathrm{d}(X \vee Y/Y) = \mathrm{d}(X/X \wedge Y).$$

Remark 2.2.10. A pregeometry cl is modular if and only if the lattice of finite dimensional cl-closed sets, ordered by inclusion, is a modular lattice. Recall that a lattice $(L, \vee, \wedge)$ is said to be *modular* if for all $x, y, z \in L$, if $x \leq z$ implies $x \vee (y \wedge z) = (x \vee y) \wedge z$.

It is easy to see that the pregeometry given by the linear span in a module is a modular pregeometry.

Lemma 2.2.11. *Let δ be a predimension function, submodular with respect to* cl_0, *and let Y be a* cl_0*-finite dimensional set. Then the predimension function δ_Y on A is submodular with respect to* $(\mathrm{cl}_0)_Y$.

For the rest of this section assume δ is a predimension function on A, submodular with respect to cl_0.

Lemma 2.2.12. *The following* addition formula *holds: for all finite dimensional* cl_0*-closed sets X, Y, Z with $X \supset Y \supset Z$,*

$$\delta(X/Z) = \delta(X/Y) + \delta(Y/Z).$$

Proof. Indeed, $\delta(X/Z) = \delta(X) - \delta(Z) = \delta(X) - \delta(Y) + \delta(Y) - \delta(Z) = \delta(X/Y) + \delta(Y/Z)$. □

Remark 2.2.13. It is easy to see that the above lemma can be rephrased as follows: for all finite dimensional cl_0-closed sets X, Y, Z,

$$\delta(X \vee Y/Z) = \delta(Y/Z) + \delta(X/Y \vee Z).$$

We shall use the expression *addition formula* to refer to either of the two equivalent formulations.

Lemma 2.2.14. *Suppose* cl_0 *is a modular pregeometry on* A *and* δ *is a submodular predimension function with respect to* cl_0*. Then the following inequality holds: for all finite dimensional* cl_0*-closed sets* X*,* Y*,* Z *with* $Z \subset Y$*,*

$$\delta(X/Z) \geq \delta(X \wedge Y/Z) + \delta(X/Y)$$

Proof.

$$\begin{aligned}
\delta(X/Z) &= \delta(X \vee (X \wedge Y)/Z) (\text{as } X = X \vee (X \wedge Y)) \\
&= \delta(X \wedge Y/Z) + \delta(X/Z \vee (X \wedge Y)) \text{ (by the addition formula)} \\
&= \delta(X \wedge Y/Z) + \delta(X/(Z \vee X) \wedge Y)) \text{ (by the modularity of } \mathrm{cl}_0) \\
&= \delta(X \wedge Y/Z) + \delta(X \vee Z/(Z \vee X) \wedge Y)) \\
&\quad \text{(since } Z \subset (Z \vee X) \wedge Y \text{ and } \delta_{(Z \vee X) \wedge Y} \text{ is is well-defined w.r.t. } (\mathrm{cl}_0)_{(Z \vee X) \wedge Y}) \\
&\geq \delta(X \wedge Y/Z) + \delta((X \vee Z) \vee Y/Y) \text{ (by submodularity of } \delta_Y \text{ w.r.t. } (\mathrm{cl}_0)_Y) \\
&= \delta(X \wedge Y/Z) + \delta(X/Y) \text{ (since } Z \subset Y \text{ and } \delta_Z \text{ is well defined w.r.t. by } (\mathrm{cl}_0)_Z)
\end{aligned}$$

□

2.2.2 Self-sufficient sets

Assume δ is a submodular pregeometry on A with respect to cl_0 and the pregeometry cl_0 is modular.

Definition 2.2.15. Let X be a cl_0-closed subset of A.

A finite dimensional cl_0-closed subset Y of X is *self-sufficient* in X, written $Y \leq X$, if for all $x \subset_{\mathrm{fin}} X$, $\delta(x/Y) \geq 0$.

An arbitrary cl_0-closed subset Y of X is self-sufficient in X, also written $Y \leq X$, if it is the union of a directed system of finite dimensional self-sufficient cl_0-closed subsets of X with respect to inclusions. Equivalently, Y is self-sufficient in X if the collection of all its finite dimensional cl_0-closed subsets that are self-sufficient in X is directed with respect to inclusions and has Y as union.

For arbitrary subsets Y of X, we say that Y is self-sufficient in X if $\mathrm{cl}_0(Y)$ is self-sufficient in X in the above sense.

Lemma 2.2.16. *1. (Transitivity) Let X, Y, Z be cl_0-closed subsets of A. If $Z \leq Y$ and $Y \leq X$ then $Z \leq X$.*

2. (Unions of self-sufficient chains) Let $(X_i)_{i\in I}$, be an increasing $\leq$-chain of subsets of A, i.e. for all $i, j \in I$, if $i \leq j$ then $X_i \leq X_j$. Put $X = \bigcup_{i\in I} X_i$. Then for all $i \in I$, $X_i \leq X$.

Proof. (Transitivity.) Suppose $Z \leq Y$ and $Y \leq X$. Assume further that Y, Z are finite dimensional. Let X' be a finite dimensional cl_0-closed subset of X. By Lemma 2.2.14, we have

$$\delta(X'/Z) \geq \delta(X' \wedge Y/Z) + \delta(X'/Y)$$

The first summand on the right hand side is nonnegative as $Z \leq Y$. The second summand is nonnegative as $Y \leq X$. Hence $\delta(X'/Z) \geq 0$. Thus, $Z \leq X$.

The result follows for arbitrary Y, Z. Indeed, if $Z \leq Y$ and $Y \leq X$, then Z is the union of the directed system of its finite dimensional cl_0-closed subsets that are self-sufficient in Y and Y is the union of the directed system of its finite dimensional cl_0-closed subsets that are self-sufficient in X. By finite dimensionality and directedness, every element Z_i of the first system is contained in some element Y_j of the second system; also $Z_i \leq Y_j$, because $Z_i \leq Y$. Since also $Y_j \leq X$, by the transitivity property proved above we get $Z_i \leq X$. Thus, Z is the union of a directed system of finite dimensional cl_0-closed self-sufficient subsets of X, and hence Z is self-sufficient in X.

(Unions of self-sufficient chains.) If X_i is finite dimensional, then simply note that every $x \subset_{\text{fin}} X$ is contained in some X_j with $i \leq j$ and, since $X_i \leq X_j$, $\delta(x/X_i) \geq 0$; thus, $X_i \leq X$.

If X_i is infinite dimensional, then, for each $j \geq i$, that X_i is self-sufficient in X_j means that the finite dimensional cl_0-closed sets of X_i that are self-sufficient in X_j form a directed system with union X_i. But then all finite dimensional cl_0-closed sets of X_i that are self-sufficient in some X_j with $j \geq i$ also form a directed system with union X_i. Using finite dimensionality and directedness as above, one sees that every finite dimensional cl_0-closed subset of X_i that is self-sufficient in some X_j with $j \geq i$

is self-sufficient in X. Thus, X_i is the union of a directed system of finite dimensional cl_0-closed subsets that are self-sufficient in X. Hence $X_i \leq X$. □

Lemma 2.2.17. *If Y_1 and Y_2 are finite dimensional self-sufficient cl_0-closed subsets of A, then so is $Y_1 \cap Y_2$.*

Proof. Let $Y = Y_1 \cap Y_2$. Let X be a finite dimensional cl_0-closed subset of A containing Y. Applying Lemma 2.2.14 and the submodularity of δ directly, we have:

$$\begin{aligned}\delta(X/Y) &\geq \delta(X \cap Y_1/Y) + \delta(X/Y_1)\\ &= \delta(X \cap Y_1/(X \cap Y_1) \cap Y_2) + \delta(X/Y_1)\\ &\geq \delta((X \cap Y_1) \vee Y_2/Y_2) + \delta(X/Y_1).\end{aligned}$$

Since Y_1 and Y_2 are self-sufficient, the two summands in the last expression are non-negative. Therefore $\delta(X/Y) \geq 0$. Thus, Y is self-sufficient. □

2.2.3 The self-sufficient closure

Definition 2.2.18. Let $X \subset A$. The *self-sufficient closure of* X, denoted $\text{sscl}(X)$, is the smallest cl_0-closed self-sufficient subset of A containing X.

Lemma 2.2.19. *Let δ be a submodular predimension function on A with respect to the modular pregeometry cl_0. Assume the values of δ are bounded from below in $\mathbb{Z}$. Then for every set X, the self-sufficient closure of X exists.*

Proof. Assume X has finite cl_0-dimension. Since the values of δ are bounded from below in $\mathbb{Z}$, among all the finite dimensional cl_0-closed sets Y containing X we can find one such that $\delta(Y)$ is minimal, and such Y is then clearly self-sufficient. Thus, the collection $\mathcal{S}_X$ of all self-sufficient finite dimensional cl_0-closed sets containing X is non-empty. By Lemma 2.2.17, $\mathcal{S}_X$ is closed under finite intersections. Also, since all the elements of $\mathcal{S}_X$ are finite dimensional cl_0-closed sets, any intersection of elements of $\mathcal{S}_X$ is the intersection of finitely many of them. Thus, the intersection of all elements of $\mathcal{S}_X$ is in $\mathcal{S}_X$, and it is indeed the self-sufficient closure of X.

For arbitrary X, we have

$$\text{sscl}(X) = \bigcup\{\text{sscl}(X') : X' \subset_{\text{fin}} X\}.$$

To see this note that the set on the right hand side is self-sufficient in A because it is the union of a directed system of finite-dimensional cl_0-closed self-sufficient subsets of A; also, it is contained in any self-sufficient subset of A that contains X, for it is clear from the definitions that such a set must contain each of the $\text{sscl}(X')$ for $X' \subset_{\text{fin}} X$. □

2.2.4 Proper predimension functions and dimension

Definition 2.2.20. A predimension function δ on a set A is said to be *proper* with respect to a pregeometry cl_0 if it is submodular with respect to cl_0; for every finite x, $\delta(x) \geq 0$; and $\delta(\emptyset) = 0$.

Remark 2.2.21. Given a submodular predimension function δ on A, with respect to a pregeometry cl_0, and a self-sufficient subset Z of A, the localisation δ_Z of δ at Z is a proper predimension function with respect to the pregeometry $(\mathrm{cl}_0)_Z$. In particular, if δ is bounded from below, then one can obtain a proper predimension function by localising at the self-sufficient closure of the empty set.

Let us also note that if δ is a proper predimension function on A, then the set $\mathrm{cl}_0(\emptyset)$ is self-sufficient in A.

We now show how to define a dimension function d from the proper predimension function δ.

Definition 2.2.22. The *dimension function* d *associated to* δ is defined, for all finite $x \subset A$, by the formula

$$\mathrm{d}(x) = \min\{\delta(x') : x \subset x' \subset_{\text{fin}} A\}.$$

Remark 2.2.23. We note that d satisfies conditions D1-D3 of the definition of dimension function on A (Definition 2.1.3). Hence d is a dimension function on the set $A_1 = \{x \in A : \mathrm{d}(x) \leq 1\}$ in the sense of 2.1.3.

Thus, associated to d we have a closure operator with finite character cl_{d} on A which restricts to a pregeometry on A_1.

For $X \subset_{\text{fin}} A$ and $x_0 \in A$,

$$\begin{aligned}
& x_0 \in \mathrm{cl}_{\mathrm{d}}(X) \\
\iff & \mathrm{d}(x_0/X) = 0 \\
\iff & \mathrm{d}(x_0X) = \mathrm{d}(X) \\
\iff & \text{There exists } x \supset x_0 \text{ such that } \delta(x/\operatorname{sscl}(X)) = 0 \\
\iff & x_0 \in \operatorname{sscl}(X) \text{ or} \\
& \text{there exists } x \supset x_0, \ \mathrm{cl}_0\text{-independent over } \operatorname{sscl}(X), \text{ such that } \delta(x/\operatorname{sscl}(X)) = 0.
\end{aligned}$$

Remark 2.2.24. Note that if x is self-sufficient, then $\delta(x) = \mathrm{d}(x)$. Also, for all x, $\mathrm{d}(x) = \delta(\operatorname{sscl}(x))$.

2.3 Rich structures

We now leave the context of a fixed set A and consider classes of first-order structures.

Let $\mathcal{C}$ be a class of first-order L-structures, for some first-order language L.

Definition 2.3.1. Suppose we have family of pregeometries $\{\mathrm{cl}_0^{\mathcal{A}} : \mathcal{A} \in \mathcal{C}\}$ satisfying the following conditions:

- For all $\mathcal{A} \in \mathcal{C}$, $\mathrm{cl}_0^{\mathcal{A}}$ is a pregeometry on A.
- Partial isomorphisms preserve the pregeometries, that is: for all $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{C}$, if f is a partial isomorphism from $\mathcal{A}_1$ to $\mathcal{A}_2$ with domain X_1 and image X_2, then for every (finite) set $Y \subset X_1$ and every element $y \in X_1$, $y \in \mathrm{cl}_0^{\mathcal{A}_1}(Y)$ if and only if $f(y) \in \mathrm{cl}_0^{\mathcal{A}_2}(f(Y))$.

We then say that *the class $\mathcal{C}$ has a pregeometry* cl_0, and write cl_0 for each $\mathrm{cl}_0^{\mathcal{A}}$, $\mathcal{A} \in \mathcal{C}$.

The second condition in the above definition asks for compatibility among the different pregeometries $\mathrm{cl}_0^{\mathcal{A}}$. Indeed, it guarantees that for any (finite) subset X of a structure $\mathcal{A}$ in $\mathcal{C}$, the value $\mathrm{cl}_0(X) := \mathrm{cl}_0^{\mathcal{A}}(X)$ does not depend on the choice of $\mathcal{A}$. The same kind of compatibility condition appears in the following definition of a predimension function for the class $\mathcal{C}$.

Definition 2.3.2. Suppose the class $\mathcal{C}$ has a pregeometry cl_0. We say that *the class $\mathcal{C}$ has a submodular (resp. proper) predimension function δ with respect to* cl_0 if there exists a family of functions $\{\delta^{\mathcal{A}} : \mathcal{A} \in \mathcal{C}\}$, satisfying the following:

- For every $\mathcal{A} \in \mathcal{C}$, $\delta^{\mathcal{A}}$ is a submodular (resp. proper) predimension function on A with respect to the pregeometry $\mathrm{cl}_0^{\mathcal{A}}$.
- For every $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{C}$, $X_1 \subset_{\text{fin}} A_1$ and $X_2 \subset_{\text{fin}} A_2$, if there exists a partial isomorphism from $\mathcal{A}_1$ to $\mathcal{A}_2$ mapping X_1 onto X_2 then $\delta^{\mathcal{A}_1}(X_1) = \delta^{\mathcal{A}_2}(X_2)$.

If δ is a predimension function for $\mathcal{C}$, then for every $x \subset \mathcal{A} \in \mathcal{C}$ we write $\delta(x)$ for the value $\delta^{\mathcal{A}}(x)$.

Henceforth, we assume $\mathcal{C}$ to be a class with a modular pregeometry cl_0 and a proper predimension function δ with respect to cl_0.

Let $\mathrm{Sub}\,\mathcal{C}$ be the class of substructures of structures in $\mathcal{C}$ whose domain is a cl_0-closed set. Let $\mathrm{Fin}\,\mathcal{C}$ be the class of structures in $\mathrm{Sub}\,\mathcal{C}$ whose domain has finite cl_0-dimension.

We consider cl_0 and δ as a modular pregeometry and proper predimension function for the class $\mathrm{Sub}\,\mathcal{C}$, extending their definitions by taking restrictions.

Definition 2.3.3. Let $\mathcal{X}, \mathcal{Y}$ be structures in $\operatorname{Sub}\mathcal{C}$.

If $\mathcal{X}$ is a substructure of $\mathcal{Y}$ and X is a self-sufficient subset of $\mathcal{Y}$, then we say that $\mathcal{Y}$ is a *strong extension* of $\mathcal{X}$ and write $\mathcal{X} \leq \mathcal{Y}$.

An embedding $f : \mathcal{X} \to \mathcal{Y}$ is said to be a *self-sufficient embedding*, or a *strong embedding*, if $f(X)$ is a self-sufficient subset of Y. If $Z \subset X$, we say that f is an embedding *over* Z if $f(z) = z$ for all $z \in Z$.

Definition 2.3.4. A structure $\mathcal{A} \in \mathcal{C}$ is said to be *rich (with respect to* δ *and* cl_0*)* if for every $\mathcal{X}, \mathcal{Y} \in \operatorname{Fin}\mathcal{C}$ with $\mathcal{X} \leq \mathcal{A}$ and $\mathcal{X} \leq \mathcal{Y}$, there exists a self-sufficient embedding of $\mathcal{Y}$ into $\mathcal{A}$ over $\mathcal{X}$.

Definition 2.3.5. For $\mathcal{A}_1$ and $\mathcal{A}_2$ in $\mathcal{C}$, let $\mathcal{F}(\mathcal{A}_1, \mathcal{A}_2)$ to be the following set of partial isomorphisms

$$\mathcal{F}(\mathcal{A}_1, \mathcal{A}_2) = \{f : X_1 \xrightarrow{\cong} X_2 : \ X_i \leq A_i,\, X_i \text{ fin. dim. } \mathrm{cl}_0\text{-closed}, i = 1, 2\}.$$

Remark 2.3.6. Remember that a collection $\mathcal{F}$ of partial isomorphisms from a structure $\mathcal{A}_1$ to a structure $\mathcal{A}_2$ is said to be a *back-and-forth system* if $\mathcal{F}$ is non-empty and has the following properties:

- *(Forth)* For all $f \in \mathcal{F}$, for all $a_1 \in A_1$, there exists $g \in \mathcal{F}$ extending f such that a_1 is in the domain of g.
- *(Back)* For all $f \in \mathcal{F}$, for all $a_2 \in A_1$, there exists $g \in \mathcal{F}$ extending f such that a_2 is in the image of g.

If there exists a back-and-forth system of partial isomorphisms from $\mathcal{A}_1$ to $\mathcal{A}_2$, then we say that the structures $\mathcal{A}_1$ and $\mathcal{A}_2$ are back-and-forth equivalent.

It is a theorem of Karp that two structures are back-and-forth equivalent if and only if they satisfy the same $L_{\infty\omega}$-sentences (recall that $L_{\infty\omega}$ is the extension of first-order logic where conjunctions of arbitrary sets of formulas in a given finite set of variables are allowed as formulas). From this, it follows that every element of a back-and-forth system is a partial elementary map and that back-and-forth equivalent structures are elementarily equivalent.

Remark 2.3.7. Note that for all $\mathcal{A}_1, \mathcal{A}_2$ in $\mathcal{C}$, $\mathcal{F}(A_1, A_2)$ is non-empty. Indeed, there is a partial isomorphism with domain $\mathrm{cl}_0^{\mathcal{A}_1}(\emptyset)$ and image $\mathrm{cl}_0^{\mathcal{A}_2}(\emptyset)$, due to the assumption that cl_0 is a pregeometry for the class $\mathcal{C}$, and these sets are self-sufficient in $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively, because δ is a proper predimension function.

Note that for any $\mathcal{A} \in \mathcal{C}$, $\mathcal{A}$ is rich if and only if for every $\mathcal{X}, \mathcal{Y} \in \operatorname{Fin}\mathcal{C}$ and every self-sufficient embedding f of $\mathcal{X}$ into $\mathcal{Y}$, there exits a self-sufficient embedding of $\mathcal{Y}$ into $\mathcal{A}$ extending f^{-1}. It is then easy to see that if $\mathcal{A}_1$ and $\mathcal{A}_2$ are rich structures in $\mathcal{C}$, then $\mathcal{F}(\mathcal{A}_1, \mathcal{A}_2)$ is a back-and-forth system. It follows that all rich structures in $\mathcal{C}$ are $L_{\infty\omega}$-equivalent and, in particular, elementarily equivalent.

We further note the following:

Lemma 2.3.8. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be rich structures in $\mathcal{C}$. Let $f : X_1 \overset{\cong}{\to} X_2$ be a partial isomorphism from $\mathcal{A}_1$ to $\mathcal{A}_2$ such that X_i is a self-sufficient* cl_0*-closed subset of A_i, for $i = 1, 2$, respectively. Then f is an elementary map.*

Proof. If X_1 and X_2 are finite dimensional, then f is in the back-and-forth system $\mathcal{F}(\mathcal{A}_1, \mathcal{A}_2)$ and is hence an elementary map.

In general, X_1 is the union of a directed system of finite dimensional sets (X_1^i), self-sufficient in A. Note that X_2 is the union of the directed system of sets $f(X_1^i)$. Since each X_1^i is self-sufficient in $\mathcal{A}_1$, X_1^i is, in particular, self-sufficient in X_1. Since f is a partial isomorphism, each $f(X_1^i)$ is therefore self-sufficient in X_2. Therefore, by transitivity of self-sufficiency, each $f(X_1^i)$ is self-sufficient in $\mathcal{A}_2$. Thus, every restriction of f to an X_1^i is a partial isomorphism between a self-sufficient subset of $\mathcal{A}_1$ and a self-sufficient subset of $\mathcal{A}_2$ and hence is an elementary map. Since f is the union of the directed system of its restrictions to the X_1^i, it follows that f is an elementary map. □

2.3.1 Existence of rich structures

The following lemma gives sufficient conditions for the existence of rich structures in the class $\mathcal{C}$.

Lemma 2.3.9. *Assume the following:*

- (Sub $\mathcal{C}$ has the amalgamation property for self-sufficient embeddings) *For all $\mathcal{Y}_0, \mathcal{Y}_1, Y_2 \in \operatorname{Sub}\mathcal{C}$ with $\mathcal{Y}_0 \leq \mathcal{Y}_1$ and $\mathcal{Y}_0 \leq \mathcal{Y}_2$, there exist $\mathcal{Y} \in \operatorname{Sub}\mathcal{C}$ and self-sufficient embeddings $j_1 : \mathcal{Y}_1 \to \mathcal{Y}$ and $j_2 : \mathcal{Y}_2 \to \mathcal{Y}$ with $j_1|_{\mathcal{Y}_0} = j_2|_{\mathcal{Y}_0}$.*

- ($\mathcal{C}$ is closed under unions of self-sufficient increasing chains) *If $(\mathcal{A}_i)_{i\in I}$ is a self-sufficient increasing sequence of structures in $\mathcal{C}$, then the structure $\mathcal{A} = \bigcup_{i\in I} \mathcal{A}_i$ is in $\mathcal{C}$.*

- (Extension property) *for all $\mathcal{Y} \in \operatorname{Sub}\mathcal{C}$ there exists $\mathcal{A} \in \mathcal{C}$ with $\mathcal{Y} \leq \mathcal{A}$.*

Then for every $\mathcal{Y} \in \operatorname{Sub} \mathcal{C}$ there exists a rich structure $\mathcal{A} \in \mathcal{C}$ with $\mathcal{Y} \leq \mathcal{A}$.

Proof. It is sufficient to show the following:

Claim: for all $\mathcal{Y} \in \operatorname{Sub} \mathcal{C}$, there exists $\mathcal{A} \in \mathcal{C}$ with $\mathcal{Y} \leq \mathcal{A}$ such that for all $\mathcal{X}, \mathcal{X}' \in \operatorname{Fin} \mathcal{C}$ with $\mathcal{X} \leq \mathcal{Y}$ and $\mathcal{X} \leq \mathcal{X}'$, there is a self-sufficient embedding of $\mathcal{X}'$ into $\mathcal{A}$ over $\mathcal{X}$.

Indeed, the above claim implies the lemma: given any $\mathcal{Y} \in \operatorname{Sub} \mathcal{C}$, we can construct a self-sufficient increasing chain of structures $(\mathcal{A}_i)_{i<\omega}$ in $\mathcal{C}$ starting with any $\mathcal{A}_0$ in $\mathcal{C}$ with $\mathcal{Y} \leq \mathcal{A}_0$ (such exist by the extension property), and inductively taking $\mathcal{A}_{i+1}$ to be as provided by the claim for $\mathcal{A}_i$. Since $\mathcal{C}$ is closed under unions of self-sufficient increasing chains, the structure $\mathcal{A} = \bigcup_i \mathcal{A}_i$ is in $\mathcal{C}$. It is easy to see that $\mathcal{A}$ is rich and $\mathcal{Y} \leq \mathcal{A}$.

Proof of the claim: Let $\mathcal{Y} \in \operatorname{Sub} \mathcal{C}$. Let $((\mathcal{X}_i, \mathcal{X}_i'))_{i<\lambda}$ be an enumeration of all pairs (X, X') with $\mathcal{X}, \mathcal{X}' \in \operatorname{Fin} \mathcal{C}$, $\mathcal{X} \leq \mathcal{Y}$ and $\mathcal{X} \leq \mathcal{X}'$. We define a self-sufficient increasing chain $(\mathcal{A}_i)_{i<\lambda}$ of structures in $\mathcal{C}$ as follows: Let $\mathcal{A}_0$ be any structure in $\mathcal{C}$ with $\mathcal{Y} \leq \mathcal{A}_0$ (extension property). For each limit ordinal $0 < \sigma < \lambda$, let $\mathcal{A}_\sigma := \bigcup_{i<\sigma} \mathcal{A}_i$. For each $i < \lambda$, let $\mathcal{Z} \in \operatorname{Sub} \mathcal{C}$ be as provided by the amalgamation property of $\operatorname{Sub} \mathcal{C}$ for the extensions $\mathcal{X}_i \leq \mathcal{A}_i$, $\mathcal{X}_i \leq \mathcal{X}_i'$; after an identification, we may assume $\mathcal{A}_i \leq \mathcal{Z}$; let $\mathcal{A}_{i+1}$ be a structure in $\mathcal{C}$ with $\mathcal{Z} \leq \mathcal{A}_{i+1}$ (extension property). The structure $\mathcal{A} := \bigcup_{i<\lambda} \mathcal{A}_i$ is then as required. □

2.3.2 Quantifier elimination

Let T be a first-order L-theory whose class of models $\mathcal{C}$ has a modular pregeometry cl_0 and a proper predimension function δ with respect to cl_0. Furthermore, assume that every ω-saturated model of T is rich.

Let L^* be an expansion of the language L by predicates for some L-definable relations and let T^* denote the canonical extension of T to an L^*-theory such that every model $\mathcal{A}$ of T expands to a unique model $\mathcal{A}^*$ of T^* according to the definitions of the new predicates.

Proposition 2.3.10. *1. The theory T is complete.*

2. Suppose that for any rich structure $\mathcal{A}$ in $\mathcal{C}$, every finite L^*-partial isomorphism from $\mathcal{A}^*$ to $\mathcal{A}^*$ extends to a partial isomorphism in $\mathcal{F}(\mathcal{A}, \mathcal{A})$. Then T^* has quantifier elimination in the language L^*.

Proof. 1. The completeness of T follows immediately from Remark 2.3.7.

2. Suppose L^* is an expansion of the language by predicates for L-definable relations such that for any rich structure $\mathcal{A}$ in $\mathcal{C}$, every finite L^*-partial isomorphism from $\mathcal{A}^*$ to $\mathcal{A}^*$ extends to a partial isomorphism in $\mathcal{F}(\mathcal{A}, \mathcal{A})$. Thus, for every rich $\mathcal{A}$ in $\mathcal{C}$, every L^*-partial isomorphism from $\mathcal{A}$ to $\mathcal{A}$ is an L-elementary map. Since L^* is an expansion by definable predicates, every L-elementary map is also an L^*-elementary map. Thus, every L^*-partial isomorphism is L^*-elementary, which means that the the quantifier-free L^*-type of any tuple in a model of T determines its L^*-type. Quantifier elimination for T^*, in the language L^*, follows by a standard compactness argument.

□

We now consider the particular case where L^* is the expansion of the language L by a predicate for each existentially definable set in L, which is often of interest, and isolate a sufficient condition for quantifier elimination of the theory T^* in the expanded language L^*.

Recall that an L-theory T is said to be *near model complete* if, for the above language L^*, T^* has quantifier elimination.

Proposition 2.3.11. *Assume the following property holds: For all $a \subset \mathcal{A} \models T$, there exists an existential L-formula $\tau_a^\delta(x)$ such that*

- $\mathcal{A} \models \tau_a^\delta(a)$, *and*
- *for all $a' \subset \mathcal{A}' \models T$, if $\mathcal{A}' \models \tau_a^\delta(a')$ then $\delta(a') \leq \delta(a)$.*

Then:

1. *For all $a \subset \mathcal{A} \models T$, there exists an existential L-formula $\tau_a^{\mathrm{d}}(x)$ such that*
   - $\mathcal{A} \models \tau_a^{\mathrm{d}}(a)$, *and,*
   - *for all $a' \subset \mathcal{A}' \models T$, if $\mathcal{A}' \models \tau_a^{\mathrm{d}}(a')$ then $\mathrm{d}(a') \leq \mathrm{d}(a)$.*
2. *For all $n \geq 1$, for all $r \geq 0$, there is a set $\Phi_{n,r}(x)$ of existential L-formulas such that for all $\mathcal{A} \in \mathcal{C}$ and all $a \in A^n$,*
$$\delta(a) \leq r \iff \mathcal{A} \models \bigvee \Phi_{n,r}(a).$$
3. *For all $n \geq 1$, for all $r \geq 0$, there is a set $\Psi_{n,r}(x)$ of existential L-formulas such that for all $\mathcal{A} \in \mathcal{C}$ and all $a \in A^n$,*
$$\mathrm{d}(a) \leq r \iff \mathcal{A} \models \bigvee \Psi_{n,r}(a).$$

4. *For all models $\mathcal{A}_1, \mathcal{A}_2$ of T^*, every finite partial L^*-isomorphism from $\mathcal{A}_1$ to $\mathcal{A}_2$ preserves the dimension function* d.

5. *For all ω-saturated models $\mathcal{A}_1, \mathcal{A}_2$ of T^*, every finite partial L^*-isomorphism from $\mathcal{A}_1$ to $\mathcal{A}_2$ extends to a member of $\mathcal{F}(\mathcal{A}_1, \mathcal{A}_2)$.*

6. *T is a near model complete L-theory.*

Proof. 1. It is easy to see that we can take $\tau_a^{\mathrm{d}}(x)$ to be $\exists y \tau_{(a,b)}^{\delta}(x, y)$ where b is such that (a, b) is a cl_0-basis of the self-sufficient closure of a.

2. Given n, r, put

$$\Phi_{n,r}(x) = \{\tau_a^{\delta}(x) : \mathcal{A} \models T, a \in A^n, \delta(a) \leq r\}$$

3. Given n, r, put

$$\Psi_{n,r}(x) = \{\tau_a^{\mathrm{d}}(x) : \mathcal{A} \models T, a \in A^n, \mathrm{d}(a) \leq r\}$$

4. Immediate from the previous part.

5. Let $f : a_1 \mapsto a_2$ be a partial L^*-isomorphism from $\mathcal{A}_1$ to $\mathcal{A}_2$. Let b_1 be an enumeration of $\mathrm{sscl}(a_1)$. Let $\Theta(x, y)$ be the quantifier-free L-type of $a_1 b_1$. Notice that since f is an L^*-partial isomorphism, $\Theta(a_2, y)$ is finitely satisfiable in $\mathcal{A}_2$. Therefore, by the ω-saturation of $\mathcal{A}_2$, there exists a realisation b_2 in A_2 of $\Theta(a_2, y)$. We thus have an L-partial isomorphism $\hat{f} : b_1 \mapsto b_2$ extending f.

   Also,
$$\delta(b_2) = \delta(b_1) = \mathrm{d}(a_1) = \mathrm{d}(a_2),$$
   where the last equality holds by the previous part of this lemma; hence b_2 is self-sufficient in $\mathcal{A}_2$. Therefore, $\hat{f}$ is in $\mathcal{F}(\mathcal{A}_1, \mathcal{A}_2)$.

6. It follows immediately from the previous part and Lemma 2.3.8 that finite partial L^*-isomorphisms are L-elementary maps, and hence L^*-elementary maps. Thus, T^* has quantifier elimination, i.e. T is near model complete.

□

Chapter 3

Algebraic preliminaries

This chapter collects the main algebro-geometric results that will be used in the later chapters. We start by introducing the algebraic groups that we shall work with and then give precise statements and references for results relating to the Conjecture on Intersections with Tori, the Mordell-Lang property, Ax's theorem and the Thumbtack Lemma.

3.1 Algebraic groups

For us, an *algebraic variety* (often simply *variety*) $\mathbb{A}$ over a field k_0 will be given by a set of polynomial equations and inequations over k_0 defining a subset of affine or projective space. Thus, given an algebraic variety $\mathbb{A}$, for each field K extending k_0 we have a corresponding set $\mathbb{A}(K)$ consisting of all K-points of $\mathbb{A}$. In fact, we shall also call the sets of the form $\mathbb{A}(K)$ algebraic varieties, but this should not give rise to confusion.

Suppose $\mathbb{A}$ is a variety over a field k_0. A variety $\mathbb{B}$ over a field extension k_1 of k_0 such that $\mathbb{B}(K)$ is a Zariski closed subset of $\mathbb{A}(K)$ for every $K \supset k_1$ is said to be a *subvariety* of $\mathbb{A}$ defined over k_1. We also call the Zariski closed subsets of $\mathbb{A}(K)$ subvarieties of $\mathbb{A}(K)$.

An *algebraic group* $\mathbb{A}$ over k_0 is an algebraic variety together with polynomials over k_0 defining a group operation on $\mathbb{A}(K)$ and the corresponding inversion operation for all field extensions K of k_0.

A subvariety $\mathbb{B}$ of an algebraic group $\mathbb{A}$ such that $\mathbb{B}(K)$ is a subgroup of $\mathbb{A}(K)$, for every K over which both $\mathbb{A}$ and $\mathbb{B}$ are defined, is said to be an *algebraic subgroup* of $\mathbb{A}$. In this case we also say that each $\mathbb{B}(K)$ is an algebraic subgroup of $\mathbb{A}(K)$.

The most basic examples of algebraic groups are probably those of the additive and multiplicative groups. *The additive group*, denoted $\mathbb{G}_a$, consists of the variety defined by the polynomial equation in one variable $x = x$ together addition as group operation. *The multiplicative group*, denoted $\mathbb{G}_m$, has set of K-points K^* for every field K and its group operation is given by multiplication. [1]

An *elliptic curve* is a 1-dimensional abelian variety. An *abelian variety* is an irreducible, complete algebraic group. The group operation on an abelian variety is always commutative ([30, Example 4.6]), additive notation is therefore used.

For details on the definitions of algebraic groups and abelian varieties, see e.g. [30]. A general reference for elliptic curves is [36].

Over fields of characteristic zero, which is the only case that we shall deal with, all elliptic curves can be realised as subvarieties of the projective plane $\mathbb{P}^2$ given by a homogeneous Weierstrass equation

$$zy^2 = 4x^3 + \alpha xz^2 + \beta z^3,$$

where $\alpha, \beta \in k_0$ are such that the polynomial $4x^3 + \alpha x + \beta$ has distinct roots, and having the point $O := [0, 1, 0]$ at infinity as the identity element for the group operation. Sometimes, especially in the context of equations, we also use 0 to denote the point O.

Let $\mathbb{A}$ be the multiplicative group or an elliptic curve over a field k_0 of characteristic 0. We use additive notation for the group operation on $\mathbb{A}$.

The endomorphisms of $\mathbb{A}$ given by regular functions (i.e. given by polynomials on affine charts) form a ring, where addition is induced by the group operation of $\mathcal{A}$ and multiplication is given by composition. We denote this ring by $\mathrm{End}(\mathbb{A})$.

If $\mathbb{A}$ is the multiplicative group then the ring $\mathrm{End}(\mathbb{A})$ consists only of the maps $x \mapsto n{\cdot}x$ (in additive notation) for $n \in \mathbb{Z}$ and is thus isomorphic to $\mathbb{Z}$. If $\mathbb{A}$ is an elliptic curve then $\mathrm{End}(\mathbb{A})$ is isomorphic to either $\mathbb{Z}$ or $\mathbb{Z}[\theta]$, for some imaginary quadratic algebraic integer θ. In the latter case it is said that $\mathbb{A}$ has complex multiplication (CM) by $\mathbb{Z}[\theta]$.

Let $A = \mathbb{A}(K)$ with K an algebraically closed field containing k_0.

Note that A is an $\mathrm{End}(\mathbb{A})$-module. We have a dimension function on A given by the $\mathrm{End}(\mathbb{A})$-linear dimension, which we denote by $\mathrm{lin.\,d.}_{\mathrm{End}(\mathbb{A})}$, or simply by lin. d.. We use $\langle Y \rangle$ or $\mathrm{span}_{\mathrm{End}(\mathbb{A})}(Y)$ to denote the $\mathrm{End}(\mathbb{A})$-span of a subset Y of A.

[1]This description of $\mathbb{G}_m$ does not, in fact, fit our definition of algebraic groups. However, this imperfection can be ignored by a customary identification of K^* with the algebraic subvariety H of K^2 defined by the equation $xy = 1$, under which each $a \in K^*$ is identified with the point (a, a^{-1}) on H, multiplication on K^* corresponds to coordinatewise multiplication on the hyperbola H and inversion on K^* corresponds to switching the two coordinates on H.

Since K is algebraically closed, A is divisible. Also, the ring $\mathrm{End}(\mathbb{A})$ is an integral domain and $k_{\mathbb{A}} := \mathrm{End}(\mathbb{A}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is its fraction field. The quotient $A/\operatorname{Tor}(A)$ is a $k_{\mathbb{A}}$-vector space and for every $Y \subset A$, $\mathrm{lin.\,d.}_{\mathrm{End}(\mathbb{A})}(Y)$ equals the $k_{\mathbb{A}}$-linear dimension of $\phi(Z)$ in $A/\operatorname{Tor}(A)$, where $\phi : A \to A/\operatorname{Tor}(A)$ is the quotient map. The pregeometry on $A/\operatorname{Tor}(A)$ given by the $k_{\mathbb{A}}$-span induces a pregeometry on A that we shall denote by span; this means that for $Y \subset A$, $\mathrm{span}(Y) = \phi^{-1}(\mathrm{span}_{k_{\mathbb{A}}}(\phi(Y)))$.

The algebraic group structure on A induces an algebraic group structure on each cartesian power A^n, $n \geq 1$. There is the following characterisation of the algebraic subgroups of A^n; for the case of the multiplicative group a proof can be found in [8, Section 3.2], for the elliptic curve case we refer to [40, Lemma 1].

Every algebraic subgroup C of A^n is defined by a system of equations of the form

$$m \cdot y = 0,$$

where y is a n-tuple of variables and m, and m is a $k \times n$-matrix with entries in $\mathrm{End}(\mathbb{A})$, for some k.

Furthermore, if m has rank k, then C has dimension $n - k$ (as a Zariski closed set). In this case we say that C has *codimension* k.

It follows that a coset $\alpha + C$ of an algebraic subgroup C is defined by a system of equations of the form

$$m \cdot y = b,$$

with y, m as before and $b \in A^k$.

Note that for any algebraically closed intermediate field $k_0 \subset K' \subset K$ we have: the coset $\alpha + C$ is defined over K', if and only if, $\alpha + C$ has a K'-rational point, if and only if, $\alpha + C = \alpha' + C$ for some $\alpha' \in \mathbb{A}(K')^n$.

3.2 The Conjecture on Intersections with Tori (CIT)

Let $B = \mathbb{B}(K)$, with K algebraically closed, be a smooth algebraic variety. If V, W are subvarieties of B such that $V \cap W \neq \emptyset$, then every irreducible component of the intersection has dimension at least $\dim V + \dim W - \dim B$. This follows from [35, I.6 Theorem 6] and the fact that dimension is a local notion.

Definition 3.2.1. Let V, W be subvarieties of B with non-empty intersection. Let S be an irreducible component of $V \cap W$. Then, S is said to be an *atypical component of the intersection of* V *and* W if

$$\dim S > \dim V + \dim W - \dim B.$$

Otherwise, that is if $\dim S = \dim V + \dim W - \dim B$, we say that S is a *typical component of the intersection of* V *and* W.

Let B' be a smooth subvariety of B containing S. Then, S is said to be an *atypical component of the intersection of* V *and* W *with respect to* B' if

$$\dim S > \dim V \cap B' + \dim W \cap B' - \dim B'.$$

Otherwise, that is if $\dim S = \dim V \cap B' + \dim W \cap B' - \dim B'$, we say that S is a *typical component of the intersection of* V *and* W *with respect to* B'.

In order to explain the terminology introduced in the above definition, let us note that if the dimension of the intersection of V and W is larger than $\dim V + \dim W - \dim B$, this is due to the existence of algebraic dependences between the equations defining V and those defining W. Such dependences are closed conditions, therefore in generic cases the dimension of each of the irreducible components of the intersection is equal to $\dim V + \dim W - \dim B$.

We shall now state Zilber's Conjecture on Intersections with Tori (CIT). We formulate the statement in the full generality of semiabelian varieties. A *semiabelian variety* is a commutative algebraic group that is an extension of an abelian variety by an algebraic torus (i.e. a power of $\mathbb{G}_m$). We shall not actually work with semiabelian varieties (in fact we would have to use a more general notion of variety to do so). For our purposes it suffices to know that both abelian varieties and algebraic tori are semiabelian varieties. Indeed, we shall only consider the cases of powers of an elliptic curve and of algebraic tori.

We consider below a semiabelian variety $\mathbb{B}$ defined over a field k_0, which is always assumed to have characteristic zero. When $\mathbb{B}$ is the multiplicative group, k_0 is assumed to be the field of rational numbers. Throughout K denotes an algebraically closed field extending k_0.

Conjecture 3.2.2 (CIT)**.** *Let* $B = \mathbb{B}(K)$ *be a semiabelian variety defined over a field* k_0 *of characteristic zero.*

For every $k \geq 0$*, every subvariety* W *of* B *defined over* k_0*, there exists a finite collection of proper algebraic subgroups* $C_1, \dots, C_s$ *of* B *with the following property: if* S *is an atypical component of the intersection of* W *and a proper algebraic subgroup* C *of* B*, then for some* $i \in \{1, \dots, s\}$*,* S *is contained in* C_i*.*

In the multiplicative group case, the above is Conjecture 1 in [44]. There it is shown that the CIT implies the following version that allows parameters (see Theorem 1 and Proposition 1 in that paper).

Conjecture 3.2.3 (CIT with parameters). *Let $B = \mathbb{B}(K)$ be a semiabelian variety defined over a field k_0 of characteristic zero.*

For every $k \geq 0$, every subvariety $W(x, y)$ of B^{1+k} defined over k_0 and every $c \in B^k$, there exists a finite collection of proper algebraic subgroups $C_1, \dots, C_s$ of B and elements $\alpha^1, \dots, \alpha^r$ of B with the following property:
for every coset $\alpha + C$ of a proper algebraic subgroup C of B, if S is an atypical component of the intersection of $W(x, c)$ and $\alpha + C$, then for some $i \in \{1, \dots, s\}$ and some $j \in \{1, \dots, r\}$, S is contained in $\alpha^j + C_i$ and S is a typical component of the intersection of $W(x, c)$ and $\alpha + C$ with respect to $\alpha^j + C_i$.

The following theorem deals with the same situation as the CIT. Here, however, the conclusion is weaker. Following common practice, we refer to the theorem as *Weak CIT*. For the multiplicative group this is Corollary 3 in [44] and Corollaire 3.6 in [33]. In the general case of semiabelian varieties the result is due to Kirby, Theorem 4.6 in [19].

Theorem 3.2.4 (Weak CIT). *Let $B = \mathbb{B}(K)$ be a semiabelian variety defined over a field k_0 of characteristic zero.*

Let $k \geq 0$ and let $W(x, y)$ be a subvariety of B^{1+k} defined over k_0.

Then there exists a finite collection of proper algebraic subgroups $C_1, \dots, C_s$ of B with the following property:
for any $c \in B^k$ and any coset $\alpha + C$ of a proper algebraic subgroup C of B, if S is an atypical component of the intersection of $W(x, c)$ and $\alpha + C$, then for some $i \in \{1, \dots, s\}$ and some $\alpha' \in B$, S is contained in $\alpha' + C_i$ and S is a typical component of the intersection of $W(x, c)$ and $\alpha + C$ with respect to $\alpha' + C_i$.

Indeed, it is easy to see that the CIT with parameters (and hence also the CIT) implies the Weak CIT. The difference between the two consists in that the CIT with parameters provides a finite collection of cosets of algebraic subgroups that controls all atypical intersections while the Weak CIT only gives a finite collection of subgroups such that the collection of all their cosets controls all atypical intersections.

3.3 The Mordell-Lang Property

We are now interested in the content of the (absolute) Mordell-Lang conjecture, in characteristic zero. This is a theorem, after work of Laurent, Faltings, Vojta, Raynaud and McQuillan. For more precise attributions and bibliography we refer to [12].

Theorem 3.3.1 (Mordell-Lang conjecture)**.** *Let K be an algebraically closed field of characteristic 0. Let $B = \mathbb{B}(K)$ be a semiabelian variety. Let Γ be a subgroup of B of finite rank. Then for every subvariety W of B, there exist a natural number r, elements $\gamma_1, \ldots, \gamma_r$ of Γ and algebraic subgroups $B_1, \ldots, B_r$ of B such that $\gamma_i + B_i \subset W$ and*

$$W \cap \Gamma = \bigcup_{i=1}^{r} \gamma_i + (B_i \cap \Gamma).$$

As it is already in use, we shall say that a subgroup Γ of B has the *Mordell-Lang property* if it satisfies the conclusion of the above theorem.

Let us remark that in [44] it is proved that the CIT implies the Mordell-Lang conjecture in characteristic zero.

3.4 Ax's Theorem

The following theorem of Ax from differential algebra plays an important role in several aspects of the present work.

Theorem 3.4.1 (Ax, [1])**.** *Let $(F, +, \cdot, D)$ be a differential field of characteristic 0 with field of constants k. Suppose $x_1, \ldots, x_n \in F$ and $y_1, \ldots, y_n \in F^*$ are such that*

$$\frac{Dy_i}{y_i} = Dx_i, \text{ for each } i = 1, \ldots, n,$$

and $x_1, \ldots, x_n$ are $\mathbb{Q}$-linearly independent.

Then

$$\operatorname{tr.d.}(x_1, \ldots, x_n, y_1, \ldots, y_n / k) \geq n + 1.$$

In fact, the proofs of the multiplicative group case of the Weak CIT (3.2.4) in [44] and [33] show that that statement follows comparably easily from the above theorem by a model-theoretic argument. The same happens in the semiabelian case, where the Weak CIT follows from a generalisation of Ax's Theorem ([19]). We shall not discuss the more general version, but we will use the following version for elliptic curves from [18].

Theorem 3.4.2 ([18])**.** *Let $(F, +, \cdot, D)$ be a differential field of characteristic 0 with field of constants k and let $\mathbb{E}$ be an elliptic curve over F with affine part given by the equation $z^2 = f(y)$, where $f(y)$ is a cubic with distinct roots. Suppose $x_1, \ldots, x_n, y_1, \ldots, y_n \in F$ are such that $f(y_i) \neq 0$ and $\frac{(Dy_i)^2}{f(y_i)} = (Dx_i)^2$ for all $i = 1, \ldots, n$ and $x_1, \ldots, x_n$ are $\mathbb{Q}$-linearly independent.*

Then

$$\mathrm{tr.\,d.}(x_1, \ldots, x_n, y_1, \ldots, y_n/k) \geq n+1.$$

Ax's theorem, in both the above forms, will be important in proving the *countable closure property* for certain pregeometries associated to the complex exponential field and some related structures. This is explained in Chapter 5.

3.5 The Thumbtack Lemma

In order to establish the ω-stability of the theories of expansions of algebraic groups by green points that will be constructed in Chapter 4, we will need to apply a result that is usually referred to as the *Thumbtack Lemma.*

Definition 3.5.1. Let K/k be an extension of algebraically closed fields, both extending k_0. A tuple $a \subset A = \mathbb{A}(K)$ is said to be *Kummer generic over* k, if every $\mathrm{End}(\mathbb{A})$-module automorphism of $\mathrm{span}(\mathbb{A}(k) + \langle a \rangle)$ fixing $\mathbb{A}(k) + \langle a \rangle$ pointwise is induced by a field automorphism of $k(a)^{\mathrm{alg}}$ over k_0.

The above terminology is taken from [11]. In fact, the definition here differs from the one in [11], Definition 4.1, but our use of the term is legitimized by Fact 4.2 there.

To explain the concept of Kummer genericity, let a be a tuple of elements of $A = \mathbb{A}(K)$ and consider a sequence $(a^i : i \geq 1)$ of tuples in A with $a^1 = a$ and $(a^{ij})^i = a^j$ for all $i, j \geq 1$. If V is the locus of a over k, then for every i, a^i lies in the variety $\frac{1}{i}V$, that is the inverse image of V under the map $x \mapsto ix$ (with respect to the group operation on $\mathcal{A}$). Since, in general, the variety $\frac{1}{i}V$ is not irreducible, the type of a^i over k is, in general, not determined by that of a.

However, it is easy to see that a is Kummer generic over k if and only if all sequences $(a^i : i \geq 1)$, with $a^1 = a$ and $(a^{ij})^i = a^j$ for all $i, j \geq 1$, are conjugated by a field automorphism of $k(a)^{\mathrm{alg}}$ over k_0. Thus, in other words, a is Kummer generic over k if and only if the type of a over k determines the type over k of any sequence sequence $(a^i : i \geq 1)$ as above.

Theorem 3.5.2 (Thumbtack Lemma)**.** *Let* $\mathbb{A}$ *be the multiplicative group or an elliptic curve. Let* k *and* K *be algebraically closed fields extending* k_0 *with* $k \subset K$.

For every $a \in A = \mathbb{A}(K)$*, there exists a* $k_{\mathbb{A}}$*-linear basis* a' *of* $\mathrm{span}(\mathbb{A}(k) \cup a)$ *over* $\mathbb{A}(k)$ *that is Kummer generic over* k.

In the multiplicative group case, the above theorem is Theorem 2.3 in [6] (which builds upon [52]). For elliptic curves without complex multiplication and defined over a number field k_0, it is case $N = 1$ of Lemma 4.2.1.iii in [3]. The result holds for arbitrary semiiabelian varieties by a theorem of Bays, Gavrilovich and Hils ([4][Theorem 1.1]), which in fact follows from a version by the same authors in the more general context of groups of finite Morley rank ([4][Theorem 6.4]).

Chapter 4

Expansions of algebraic groups by green points

This chapter contains the construction of the theories of green points on the multiplicative group and on elliptic curves. As mentioned in the introduction, this generalises the work of Poizat in [33].

4.1 Structures

In this section, we begin by considering a class of structures with a predimension function and showing how this is an instance of the general setting of Chapter 2.

4.1.1 An explanation: Varieties as structures

Let $\mathbb{A}$ be an algebraic variety defined over a field k_0.

Let $L_{\mathbb{A}}$ be the first-order language consisting of an n-ary predicate for each subvariety of $\mathbb{A}^n$ defined over k_0, $n \geq 1$.

Let K be an algebraically closed field extending k_0. Consider the natural $L_{\mathbb{A}}$-structure on $\mathbb{A}(K)$:

$$(\mathbb{A}(K), (W(K))_{W \in L_{\mathbb{A}}}).$$

Let $T_{\mathbb{A}}$ denote the complete theory of this structure.

The following is the key fact about $T_{\mathbb{A}}$ that we shall use. Details on bi-interpretations and a proof of this fact can be found in [3, Fact A.2.1].

Fact 4.1.1. *Let ACF_{k_0} be the theory of algebraically closed fields extending k_0 in the expansion of the field language by constants for each element of k_0. Then, $T_{\mathbb{A}}$ is bi-interpretable with ACF_{k_0}.*

Since we consider $\mathbb{A}$ as embedded in affine or projective space, there is a natural interpretation of $T_{\mathbb{A}}$ in ACF_{k_0}. In fact, there is an interpretation of ACF_{k_0} in $T_{\mathbb{A}}$ that gives the bi-interpretability when paired with the natural interpretation of $T_{\mathbb{A}}$ in ACF_{k_0}.

Fact 4.1.1 has several relevant consequences: the theory $T_{\mathbb{A}}$ does not depend on the choice of K, and every model A of $T_{\mathbb{A}}$ is of the form $A = \mathbb{A}(K)$ for a corresponding algebraically closed field K extending k_0. Also, $T_{\mathbb{A}}$ has quantifier elimination.

Suppose $\mathbb{A}$ is irreducible and 1-dimensional. Then $T_{\mathbb{A}}$ is strongly minimal. Therefore $\mathrm{acl}_{T_{\mathbb{A}}}$ induces a pregeometry on every model A of $T_{\mathbb{A}}$. By the bi-interpretability, $\mathrm{acl}^{\mathrm{eq}}_{T_{\mathbb{A}}}$ equals $\mathrm{acl}^{\mathrm{eq}}_{ACF_{k_0}}$. From this we get that for any tuple of elements b of a model of $T_{\mathbb{A}}$, the $\mathrm{acl}_{T_{\mathbb{A}}}$-dimension of b equals the $\mathrm{acl}^{\mathrm{eq}}_{ACF_{k_0}}$-dimension of b when viewed as living in K^{eq}, where K is the field corresponding to A. The $\mathrm{acl}^{\mathrm{eq}}_{ACF_{k_0}}$-dimension of b equals the transcendence degree over k_0 of any normalised representation of b in homogeneous coordinates, which we shall denote by $\mathrm{tr.d.}(b/k_0)$, or, in this chapter, simply by $\mathrm{tr.d.}(b)$, since $\mathbb{A}$ and k_0 are fixed.

4.1.2 Setup

Let $\mathbb{A}$ be the multiplicative group $\mathbb{G}_m$ or an elliptic curve over a field of characteristic zero.

Let $L = L_{\mathbb{A}} \cup \{G\}$ be the expansion of the language $L_{\mathbb{A}}$ by a unary predicate G.

Let $\mathcal{C}$ be the class of all L-structures $\mathcal{A} = (A, G)$ where $A \models T_{\mathbb{A}}$ and G is a divisible $\mathrm{End}(\mathbb{A})$-submodule of A. Note that the class $\mathcal{C}$ is elementary.

Following a convention introduced by Poizat, given an L-structure $\mathcal{A} = (A, G)$ in $\mathcal{C}$, we call the elements of G *green points* and the elements of $A \setminus G$ *white points*.

Let $\mathcal{A} = (A, G)$ be a structure in $\mathcal{C}$. Let $\mathrm{cl}_0^{\mathcal{A}}$ be the pregeometry on A induced by the pregeometry given by the $k_{\mathbb{A}}$-linear span on the quotient $A/\operatorname{Tor}(A)$.

Note that if $\mathcal{A}, \mathcal{B}$ are structures in $\mathcal{C}$ with $\mathcal{A} \subset \mathcal{B}$, then A is $\mathrm{cl}_0^{\mathcal{B}}$-closed and for all $Y \subset A$, $\mathrm{cl}_0^{\mathcal{A}}(Y) = \mathrm{cl}_0^{\mathcal{B}}(Y)$. Therefore we shall write simply cl_0 for $\mathrm{cl}_0^{\mathcal{A}}$ for any $\mathcal{A}$. In the terminology of Chapter 2, cl_0 is a pregeometry for the class $\mathcal{C}$.

Note that the dimension function associated to cl_0 coincides with the $\mathrm{End}(\mathbb{A})$-linear dimension. Henceforth in this chapter, we shall refer to the cl_0-dimension simply as linear dimension and write $\mathrm{lin.d.}$ for it. Moreover, for a subset X of a structure $\mathcal{A} \in \mathcal{C}$, we also denote the set $\mathrm{cl}_0(X)$ by $\mathrm{span}(X)$.

Note that in the lattice of cl_0-closed sets we have $X \vee Y = X + Y$ and $X \wedge Y = X \cap Y$. Moreover, it is easy to see that the modularity of the pregeometry on $A/\operatorname{Tor} A$ given by the $k_{\mathbb{A}}$-linear span implies that cl_0 is also a modular pregeometry.

Consider the predimension function $\delta^{\mathcal{A}}$ defined on finite subsets Y of A by

$$\delta^{\mathcal{A}}(Y) = 2\,\mathrm{tr.\,d.}(Y) - \mathrm{lin.\,d.}(\mathrm{span}(Y) \cap G).$$

It is clear that if $\mathcal{A}, \mathcal{B}$ are structures in $\mathcal{C}$ with $\mathcal{A} \subset \mathcal{B}$, then for all $Y \subset_{\mathrm{fin}} A$, $\delta^{\mathcal{A}}(Y) = \delta^{\mathcal{B}}(Y)$. Thus, we shall henceforth drop the superindex and write simply δ.

Lemma 4.1.2. *Let $\mathcal{A}$ be a structure in $\mathcal{C}$. The function δ is a submodular predimension function on A with respect to* cl_0.

Proof. Well-definedness with respect to cl_0 is clear from the definition of δ and the inclusion $\mathrm{span}(Y) \subset \mathrm{acl}_{T_{\mathbb{A}}}(Y)$.

We now show that δ satisfies the submodularity inequality with respect to cl_0. Note that tr. d. is submodular with respect to cl_0 (for any dimension function is submodular with respect to its corresponding pregeometry and also with respect to any weaker pregeometry). Hence for all finite dimensional cl_0-closed sets X, Y,

$$\mathrm{tr.\,d.}(X + Y/Y) \leq \mathrm{tr.\,d.}(X/X \cap Y). \tag{4.1}$$

Let us look at the negative term in the definition of δ_G and consider the function $\mathrm{lin.\,d.}_G(y) := \mathrm{lin.\,d.}(\mathrm{span}(y) \cap G)$, for $y \subset_{\mathrm{fin}} A$. Clearly, $\mathrm{lin.\,d.}_G$ is well-defined with respect to cl_0. We further note that it satisfies the following *supermodularity* property: for all finite dimensional cl_0-closed sets X, Y,

$$\mathrm{lin.\,d.}_G(X + Y/Y) \geq \mathrm{lin.\,d.}_G(X/X \cap Y). \tag{4.2}$$

One can see this as follows:

$$\begin{aligned}
\mathrm{lin.\,d.}_G(X/X \cap Y) &= \mathrm{lin.\,d.}(X \cap G/X \cap Y \cap G) \\
&= \mathrm{lin.\,d.}(X \cap G/(X \cap G) \cap (Y \cap G)) \\
&= \mathrm{lin.\,d.}((X \cap G) + (Y \cap G)/Y \cap G) \\
&\quad \text{(by the modularity of lin. d. with respect to span)} \\
&\leq \mathrm{lin.\,d.}((X + Y) \cap G/Y \cap G).
\end{aligned}$$

Thus, for all finite dimensional cl_0-closed sets X, Y, we have, substracting 4.2 from 4.1,

$$\delta(X + Y/Y) \leq \delta(X/X \cap Y). \tag{4.3}$$

This means that δ is submodular with respect to cl_0. □

We have thus seen that δ is a submodular predimension function for the class $\mathcal{C}$ with respect to the modular pregeometry cl_0.

As in Chapter 2, let us denote by $\mathrm{Sub}\,\mathcal{C}$ the class of substructures of structures in $\mathcal{C}$ whose domain is a cl_0-closed set.

Let us fix a finite dimensional structure $\mathcal{X}_0 = (X_0, G_0)$ in $\mathrm{Sub}\,\mathcal{C}$.

Let $\mathcal{C}_0$ be the class of structures $\mathcal{A}$ in $\mathcal{C}$ into which there is a self-sufficient embedding of $\mathcal{X}_0$, expanded by constants for the image of such an embedding. For $\mathcal{A} \in \mathcal{C}_0$, we identify $\mathcal{X}_0$ with the self-sufficient substructure of A consisting of the interpretations of the constants; with this convention in place, we shall not be explicit about the interpretation of constants in our notation for the structures in $\mathcal{C}_0$, writing simply $\mathcal{A} = (A, G)$, instead of $\mathcal{A} = (A, G)_{X_0}$.

Note that for $A_0 := \mathrm{acl}_{T_{\mathbb{A}}}(X_0)$, the structure $\mathcal{A}_0 = (A_0, G_0)$ is in $\mathcal{C}_0$. In particular, this implies that the class $\mathcal{C}_0$ is always non-empty. Moreover, the structure $\mathcal{A}_0$ embeds self-sufficiently in all structures in $\mathcal{C}_0$, hence we may say that it is *prime in* $\mathcal{C}_0$ *with respect to self-sufficient embeddings*. Also, note that for all $\mathcal{A} = (A, G) \in \mathcal{C}_0$, we have $\mathrm{Tor}\, G = \mathrm{Tor}\, G_0$; indeed, recall that $A = \mathbb{A}(K)$ for some algebraically closed field $K \supset k_0$ and note that all the torsion points of A have coordinates in k_0^{alg}, therefore we have $\mathrm{Tor}\, A = \mathrm{Tor}\, \mathbb{A}(k_0^{\mathrm{alg}}) = \mathrm{Tor}\, A_0$, and hence also $\mathrm{Tor}\, G = \mathrm{Tor}\, A \cap G = \mathrm{Tor}\, A_0 \cap G = \mathrm{Tor}\, G_0$.

Lemma 4.1.3. *δ_{X_0} is a proper predimension function for the class $\mathcal{C}_0$ with respect to the modular pregeometry $(\mathrm{cl}_0)_{X_0}$.*

Proof. Since for all $\mathcal{A}$ in $\mathcal{C}$ the predimension function δ is submodular on A with respect to cl_0, also for all $\mathcal{A} \in \mathcal{C}_0$ the predimension function δ_{X_0} is submodular with respect to the modular pregeometry $(\mathrm{cl}_0)_{X_0}$ (see 2.2.11).

Moreover, by the very definition of $\mathcal{C}_0$, for all $\mathcal{A}$ in $\mathcal{C}_0$ the values of δ_{X_0} on finite subsets of A are non-negative, and clearly $\delta_{X_0}(\emptyset) = 0$. □

We have thus seen that all assumptions in Chapter 2 hold in the situation considered in this chapter. All definitions from Chapter 2 are thus in place and all results from Chapter 2 are at hand.

Henceforth, we shall always work over X_0 and, in order to ease the notation, we shall write simply δ for δ_{X_0} and cl_0 for $(\mathrm{cl}_0)_{X_0}$.

A simple but very useful remark, particular to the case treated in this chapter, follows.

Remark 4.1.4. Let Y be a finite dimensional cl_0-closed subset of a structure $\mathcal{A} \in \mathcal{C}_0$. Recall that, by definition, Y is self-sufficient in $\mathcal{A}$ if for every finite dimensional cl_0-closed subset X of A, $\delta(X/Y) \geq 0$. We now note that Y is self-sufficient in $\mathcal{A}$ if and only if for every finite dimensional cl_0-closed set X *contained in* G, $\delta(X/Y) \geq 0$.

This follows immediately from the inequality: $\delta(X/Y) \geq \delta((X+Y) \cap G/Y)$. We also point out that Y is self-sufficient in $\mathcal{A}$ if and only if for every $\mathrm{End}(\mathbb{A})$-linearly independent tuple $x \subset G$, $\delta(x/Y) \geq 0$.

4.1.3 Existence of rich structures

To establish the existence of rich structures in the class $\mathcal{C}_0$ we now show that the sufficient conditions found in 2.3.9 hold for the class $\mathcal{C}_0$.

Amalgamation property

Definition 4.1.5 (Free Amalgam)**.** Let $\mathcal{X}_i = (X_i, G_i)$, for $i = 1, 2, 3$, be structures in $\mathrm{Sub}\,\mathcal{C}$ and assume $\mathcal{X}_1 \subset \mathcal{X}_i$ for $i = 2, 3$. The *free amalgam* $\mathcal{X} = (X, G)$ of $\mathcal{X}_2$ and $\mathcal{X}_3$ over $\mathcal{X}_1$ is defined as follows: Replace $\mathcal{X}_3$ by an isomorphic copy over $\mathcal{X}_1$ so that X_2 and X_3 are $\mathrm{acl}_{T_{\mathbb{A}}}$-independent over X_1 inside a monster model $\bar{A}$ of $T_{\mathbb{A}}$. Then let $X := X_2 + X_3$ (in $\bar{A}$) with the induced structure from $\bar{A}$ and let $G = G_2 + G_3$.

Remark 4.1.6. Note that the above $\mathcal{X}$ is in $\mathrm{Sub}\,\mathcal{C}$. Indeed, $A := \mathrm{acl}_{T_{\mathbb{A}}}(X)$ is an infinite algebraically closed set in a model of the strongly minimal theory $T_{\mathbb{A}}$ and hence a model of $T_{\mathbb{A}}$. Also, it is easy to see that G is a divisible subgroup of X.

See that $\mathcal{X}$ comes with canonical embeddings $\mathcal{X}_i \to \mathcal{X}$ over $\mathcal{X}_1$, $i = 2, 3$. Moreover, $\mathcal{X}$ is unique up to isomorphism over these embeddings.

Lemma 4.1.7 (Asymmetric Amalgamation Lemma)**.** *Let $\mathcal{X}_i = (X_i, G_i)$, for $i = 1, 2, 3$, be structures in* $\mathrm{Sub}\,\mathcal{C}_0$ *and assume $\mathcal{X}_1 \subset \mathcal{X}_i$ for $i = 2, 3$. If $\mathcal{X}_1$ is self-sufficient in $\mathcal{X}_2$, then the free amalgam $\mathcal{X}$ of $\mathcal{X}_2$ and $\mathcal{X}_3$ over $\mathcal{X}_1$ is in* $\mathrm{Sub}\,\mathcal{C}_0$ *and, under the canonical embedding, $\mathcal{X}_3$ is self-sufficient in $\mathcal{X}$.*

Proof. Let us identify $\mathcal{X}_3$ with its image under the canonical embedding into $\mathcal{X}$. Since we know $\mathcal{X} \in \mathrm{Sub}\,\mathcal{C}$, by the transitivity of self-sufficiency it suffices to show that $\mathcal{X}_3$ is self-sufficient in $\mathcal{X}$. Let Y be a finite dimensional cl_0-closed subset of G and let us show that $\delta(Y/X_3) \geq 0$ (this is enough, by 4.1.4). Since $G = G_2 + G_3$, there exists finite dimensional cl_0-closed sets Y_2 and Y_3 of X_2 and X_3, respectively, such that $Y = Y_2 + Y_3$. Then, since $\delta(-/X_3)$ is well defined with respect to $(\mathrm{cl}_0)_{X_3}$, we have $\delta(Y/X_3) = \delta(Y_2/X_3)$. Also, by the independence of X_2 and X_3 over X_1, we know that $\delta(Y_2/X_3) = \delta(Y_2/X_1)$. Thus, $\delta(Y/X_3) = \delta(Y_2/X_1)$, and the latter is non-negative because $\mathcal{X}_1$ is self-sufficient in $\mathcal{X}_2$. $\square$

Corollary 4.1.8. $\operatorname{Sub}\mathcal{C}_0$ *has the amalgamation property with respect to self-sufficient embeddings.*

Remark 4.1.9. Since the free amalgam of finite dimensional structures in $\operatorname{Sub}\mathcal{C}_0$ is finite dimensional, we also have that the amalgamation property with respect to self-sufficient embeddings holds for the class $\operatorname{Fin}\mathcal{C}_0$ of finite dimensional structures in $\operatorname{Sub}\mathcal{C}_0$.

Unions of chains

Lemma 4.1.10. *The class $\mathcal{C}_0$ is closed under unions of self-sufficient increasing chains.*

Proof. Since $T_{\mathbb{A}}$ and the theory of divisible abelian groups are $\forall\exists$-axiomatizable, so is the class $\mathcal{C}$. Therefore $\mathcal{C}$ is closed under unions of increasing chains. Since every element of a self-sufficient increasing chain is self-sufficient in the union of the chain and self-sufficiency is transitive (2.2.16), it follows immediately that the class $\mathcal{C}_0$ is closed under unions of self-sufficient increasing chains. $\square$

Extension property

Lemma 4.1.11. *For all $\mathcal{X}$ in $\operatorname{Sub}\mathcal{C}_0$ there exists $\mathcal{A} \in \mathcal{C}_0$ with $X \leq \mathcal{A}$.*

Proof. Let $\mathcal{X} = (X, G)$ be a structure in $\operatorname{Sub}\mathcal{C}$. Let $\bar{A}$ be a model of $T_{\mathbb{A}}$ with $X \subset \bar{A}$. In $\bar{A}$, let $A = \operatorname{acl}_{T_{\mathbb{A}}}(X)$. Note that A, with the induced structure from $\bar{A}$, is a model of $T_{\mathbb{A}}$. Then $\mathcal{A} = (A, G)$ is a structure in $\mathcal{C}$. It is easy to see that, not having any new green points, $\mathcal{A}$ is a self-sufficient extension of $\mathcal{X}$. By transitivity of self-sufficiency, if $\mathcal{X}$ is in $\operatorname{Sub}\mathcal{C}_0$, then $\mathcal{A}$ is in $\mathcal{C}_0$. $\square$

Furthermore, we have the following useful fact:

Lemma 4.1.12. *Let $\mathcal{A}$ be a structure in $\mathcal{C}$. Let Y be a finite dimensional cl_0-closed subset of A and let $Z = \operatorname{acl}_{T_{\mathbb{A}}}(Y)$. Then Y is self-sufficient in $\mathcal{A}$ if and only if Z is self-sufficient in $\mathcal{A}$.*

Proof. Assume Y is self-sufficient in A. Let Y' be a finite dimensional cl_0-closed subset of Z containing Y. Let X be a finite dimensional cl_0-closed subset of A. Since $\mathrm{acl}_{T_{\mathbb{A}}}(Y) = \mathrm{acl}_{T_{\mathbb{A}}}(Y')$, we have $\mathrm{tr.\,d.}(X/Y') = \mathrm{tr.\,d.}(X + Y'/Y)$. Since Y is self-sufficient, there is no green point in $Z \setminus Y$ and hence $Y \cap G = Y' \cap G = Z \cap G$; therefore $\mathrm{lin.\,d.}_G(X/Y') = \mathrm{lin.\,d.}((X+Y')\cap G/Y'\cap G) = \mathrm{lin.\,d.}((X+Y')\cap G/Y\cap G) = \mathrm{lin.\,d.}_G(X + Y'/Y)$. It follows that $\delta_G(X/Y') = \delta_G(X + Y'/Y) \geq 0$. Thus, Y' is self-sufficient in $\mathcal{A}$. Since Z is the union of the directed system of all such Y', we get that Z is self-sufficient in A.

To prove the converse, suppose Z is self-sufficient in A. Then, by definition, it is the union of a directed system of self-sufficient finite dimensional cl_0-closed sets. Since Y is finite dimensional, using the directedness of the system we can find a finite dimensional cl_0-closed subset Y' of Z that contains Y and is self-sufficient in A. For every finite dimensional cl_0-closed subset X of A, we have $\mathrm{tr.\,d.}(X/Y) = \mathrm{tr.\,d.}(X/Y')$ and $\mathrm{lin.\,d.}_G(X/Y) \leq \mathrm{lin.\,d.}(X/Y')$, hence $\delta_G(X/Y) \geq \delta_G(X/Y') \geq 0$. Therefore Y is self-sufficient in A. □

4.2 The theories

We now turn to the task of finding axioms for the complete theory common to all rich structures in the class $\mathcal{C}_0$.

We shall henceforth assume that our choice of $\mathcal{X}_0$ is such that X_0 has a cl_0-basis consisting of green points.

4.2.1 Axiomatizing $\mathcal{C}_0$

The first step in finding axioms for the theory of rich structures in $\mathcal{C}_0$ is to axiomatize the class $\mathcal{C}_0$. The following lemma is the key element in doing this. It generalises Corollaire 3.4 of [33].

Given an algebraic variety (or, more generally, a definable set) of the form $W(x, y)$, the algebraic variety (respectively, definable set) $W(x, c)$ is also denoted by W_c.

Lemma 4.2.1. *Let $\mathcal{A} = (A, G) \in \mathcal{C}$. For every complete $L_{\mathbb{A}}$-l-type $\Theta(y)$, there exists a partial L-l-type $\Phi_\Theta(y)$ consisting of universal formulas with the following property: for every realisation c of Θ in $\mathcal{A}$ with $c \in G^l$,*

$$\mathcal{A} \vDash \Phi_\Theta(c) \text{ if and only if } \operatorname{span} c \text{ is self-sufficient in } \mathcal{A}.$$

Proof. For any type Θ with no realisations consisting purely of green points the statement is trivial, thus assume we have a realisation $c' \in G^l$ of $\Theta(y)$. Let $\Phi_\Theta(y)$ be the partial type consisting of the following formulas:

For each $n \geq 1$ and each subvariety $W(x, y)$ of A^{n+l} defined over k_0 such that $W_{c'}$ is irreducible over $k_0(c')$ and has dimension $< \frac{n}{2}$, the formula

$$\forall x \Big(\big(W(x,y) \wedge \bigwedge_{1 \leq j \leq n} G(x_j) \wedge \neg W^*(x,y) \big) \\ \rightarrow \bigvee_{1 \leq i \leq s} \bigvee_{\substack{1 \leq j \leq r_i \\ B_{ij} \text{ proper}}} N^{ij} \cdot y + B_{ij}(n_{ij}(M^i \cdot x)) \Big),$$

where:

- s, $C_1, \ldots, C_s$ are as provided by Theorem 3.2.4 (Weak CIT) for the subvariety $W_{c'}$ of A^n, and each C_i is defined by the system of equations $M^i \cdot x = 0$, $M^i \in \mathrm{Mat}_{n_i \times n}(\mathrm{End}(\mathbb{A}))$ of rank n_i;
- for each $i \in \{1, \ldots, s\}$; r_i, $\gamma'_{i1}, \ldots, \gamma'_{ir_i}$, $B_{i1}, \ldots, B_{ir_i}$ are as provided by Theorem 3.3.1 (Mordell-Lang property) for the variety W_i, which we define to be the $k_0(c')$-Zariski closure of $M^i \cdot W_{c'}$, and the finite rank subgroup $(\operatorname{span} c')^{n_i}$ of A^{n_i}; and $N^{i1}, \ldots, N^{ir} \in \mathrm{Mat}_{n_i \times l}(\mathrm{End}(\mathbb{A}))$, $n_{i1}, \ldots, n_{ir_i} \in \mathbb{N}$ are such that $n_{ij}\gamma'_{ij} = N^{ij} \cdot c'$;
- $W^*(x, y) := \bigcup_{i=1}^{s} W^{*i}(x, y)$ and, for each $i = 1, \ldots, s$, we define $W^{*i}(x, y)$ to be a variety such that $W^{*i}(x, c')$ is the $k_0(c')$-Zariski closure of the set

  $$\{x \in W_{c'} : \dim W_{c'} \cap x + C_i > \dim W_{c'} - \dim W_i\}.$$

  Note that the above set is the union of the non-generic (i.e. not of minimal dimension) fibres inside $W_{c'}$ for the map given by $x \mapsto x^{M^i}$. By a standard fact, this set is contained in a proper closed subset of $W_{c'}$. Therefore $W^*_{c'} \subsetneq W_{c'}$.

Let c be any realisation of Θ in $\mathcal{A}$ with $c \subset G$. Note that since Θ is a complete $L_{\mathbb{A}}$-type, c and c' are conjugates by an automorphism of the $L_{\mathbb{A}}$-structure A.

Suppose $\mathcal{A} \models \Phi_\Theta(c)$. To see that $\operatorname{span} c$ is then self-sufficient in $\mathcal{A}$, suppose towards a contradiction that there exists $b \in A^n$ such that $\delta(b/\operatorname{span} c) < 0$. It is easy to see that we may assume b to be in G^n and linearly independent over $\operatorname{span} c$. Let $W_c := W(x, c)$ be the algebraic locus of b over $k_0(c)$. Then, since $\delta(b/\operatorname{span} c) < 0$, we have $\dim W_c < \frac{n}{2}$. Hence also $W_{c'}$ is irreducible over $k_0(c')$ and $\dim W_{c'} < \frac{n}{2}$.

Thus, there is a formula in $\Phi_\Theta(y)$ corresponding to $W(x,y)$. If the disjunction in the formula is non-empty then we get a linear dependence on b over $\operatorname{span} c$, hence a contradiction. If the disjunction is empty, then the fact that the formula is satisfied by c means that the set $(W_c \setminus W_c^*) \cap G^n$ is empty; but our b is in this set (b is not in W_c^* because it is generic and, as noted earlier, W_c^* is contained in a proper closed subset of W_c), hence also a contradiction. This proves that $\operatorname{span} c$ is then self-sufficient in $\mathcal{A}$.

Conversely, assume $\operatorname{span} c$ is self-sufficient in A and let us see that $\mathcal{A} \models \Phi_\Theta(c)$. Let $n \geq 1$ and let $W(x,y)$ be a subvariety of A^{n+l} over k_0 such that $W(x,c)$ is irreducible over $k_0(c)$ and of dimension $< \frac{n}{2}$ and suppose b is an element of the set $(W_c \setminus W_c^*) \cap G^n$.

Since $\operatorname{tr.d.}(b/c) \leq \dim W < \frac{n}{2}$ and by assumption $\delta(b/\operatorname{span} c) \geq 0$, the tuple b must be linearly dependent over $\operatorname{span} c$. Thus, let $\alpha + C$ be a coset of a proper algebraic subgroup of A^n containing b of dimension $\operatorname{lin.d.}(b/\operatorname{span} c)$, $\alpha \in \operatorname{span} c$.

Let S be an irreducible component of $W_c \cap \alpha + C$ containing b. Then S is an atypical of the intersection of W_c and $\alpha + C$: to see this note that, on the one hand, S is defined over $k_0(c)^{\mathrm{alg}}$ and so $\dim S \geq \operatorname{tr.d.}(b/c) \geq \frac{1}{2} \operatorname{lin.d.}(b/\operatorname{span} c) = \frac{1}{2}\dim C$ and, on the other hand, since $\dim W_c < \frac{n}{2}$, we have $\dim W_c + \dim(\alpha + C) - n < \dim C - \frac{n}{2} < \frac{1}{2}\dim C$.

Applying an automorphism $\sigma \in \operatorname{Aut}(A)$ with $\sigma(c) = c'$, we have that $\sigma(S)$ is an atypical component of the intersection of $W_{c'}$ and $\sigma(\alpha) + C$. Therefore there exists $i \in \{1, \ldots, s\}$ such that $\sigma(S)$ is contained in a coset $\sigma(\alpha^*) + C_i$ and $\sigma(S)$ is a typical component of the intersection of $W_{c'}$ and $\sigma(\alpha) + C$ with respect to $\sigma(\alpha^*) + C_i$. Applying σ^{-1}, we get that S is contained in $\alpha^* + C_i$ and S is a typical component of the intersection of W_c and $\alpha + C$ with respect to $\alpha^* + C_i$.

Since S is defined over $k_0(c)^{\mathrm{alg}}$, it has $k_0(c)^{\mathrm{alg}}$-rational points. Hence we may assume $\alpha^* \in (k_0(c)^{\mathrm{alg}})^n$. Let us now look at the coefficients of the equations defining the coset $\alpha^* + C_i$, namely $\beta^* := M^i \cdot \alpha^* \in (k_0(c)^{\mathrm{alg}})^{n_i}$. Also, $\beta^* = M^i \cdot b \in G^{n_i}$. Thus, since $\operatorname{span} c$ is self-sufficient, $\beta^* \in (\operatorname{span} c)^{n_i}$. So, $M^i \cdot b \in W_i \cap (\operatorname{span} c)^{n_i}$ and, applying appropriate automorphisms as before, we have $W_i \cap (\operatorname{span} c)^{n_i} = \bigcup_{j=1}^{r_i} \gamma_{ij} + (B_{ij} \cap (\operatorname{span} c)^{n_i})$, where $\gamma_{ij} \in \operatorname{span} c$ satisfies $n_{ij}\gamma_{ij} = N^{ij} \cdot c$. Therefore we can find $j \in \{1, \ldots, r_i\}$ such that $M^i \cdot b \in \gamma_{ij} + B_{ij}$, and hence $n_{ij}(M^i \cdot b) \in (N^{ij} \cdot c) + B_{ij}$.

It now suffices to show that B_{ij} is a *proper* algebraic subgroup of A^n. This follows from the fact that W_i is a proper subvariety of A^{n_i}, as the following dimension calculations show: first, from the atypicality of S we have

$$\dim S > \dim W_c + \dim C - n.$$

Also, from the typicality of S with respect to $\alpha^* + C_i$ we have

$$\dim S = \dim W_c \cap (\alpha^* + C_i) + \dim(\alpha + C) \cap (\alpha^* + C_i) - \dim(\alpha^* + C_i).$$

Combining the last two expressions we get

$$\dim W_c + \dim C - n < \dim W_c \cap (\alpha^* + C_i) + \dim(\alpha + C) \cap (\alpha^* + C_i) - \dim(\alpha^* + C_i).$$

Reorganising terms and noting that $\alpha + C = b + C$ and $\alpha^* + C_i = b + C_i$,

$$\begin{aligned} &\dim W_c - \dim W_c \cap (b + C_i) \\ &< n - \dim(b + C_i) + \dim(b + C) \cap (b + C_i) - \dim(b + C) \\ &\leq n - \dim(b + C_i) \\ &= n_i. \end{aligned}$$

Since b is not in W_c^*, we know $\dim W_i = \dim W_c - \dim W_c \cap (b + C_i)$. Therefore $\dim W_i < n_i$. □

Remark 4.2.2. If one works under the assumption that the group G is torsion-free, then a simpler argument, using the Weak CIT but not the Mordell-Lang property, suffices to prove the above lemma. Indeed, this is the well-known argument of Poizat in Corollaire 3.4 of [33].

In [33], it is noted that in the more general situation, where the torsion of G is not necessarily trivial, the statement holds if one assumes the CIT. Without the extra assumption, however, the question of how to get the result was left open. The above proof answers this question.

Let us remark some limitations of the above lemma, in comparison with the argument in [33], which applies to the torsion-free case. Here one limitation is that we had to restrict to tuples c with coordinates in G, which is not necessary there. But also, the result there is more uniform, since for each l, it gives a type $\Phi_l(y)$ that works for all l-tuples b, independently of their algebraic types. This difference is due to the fact that the Weak CIT is uniform in families, but the same kind of uniformity is not available for the Mordell-Lang property.

Lemma 4.2.3. *There exists an L_{X_0}-theory T^0 such that for every L_{X_0}-structure $\mathcal{A} = (A, G)$, $\mathcal{A} \models T^0$ if and only if $\mathcal{A}$ is in $\mathcal{C}_0$.*

Proof. It suffices to show that the following conditions on a structure (A, G) can be expressed by a set of L_{X_0}-sentences.

1. A is a model of $T_{\mathbb{A}}$,

2. G is a divisible subgroup of A,

3. $\text{qf-tp}^{\mathcal{A}}(X_0) = \text{qf-tp}^{\mathcal{X}_0}(X_0)$,

4. X_0 is self-sufficient in $\mathcal{A}$,

It is clear that we can find a set of L_{X_0}-sentences Σ expressing conditions 1, 2, 3.

Let c^0 be a $k_{\mathbb{A}}$-linear basis of X_0 consisting of green points, let $\Theta = \text{qf-tp}_{L_{\mathbb{A}}}(c^0)$. By Lemma 4.2.1, the set of L_{X_0}-sentences $\Phi_\Theta(c^0)$ expresses 4 modulo Σ. Thus, $T^0 := \Sigma \cup \Phi_\Theta(c^0)$ is as required. □

Henceforth let T^0 denote the theory found in the proof of the above lemma.

4.2.2 Rotund varieties

The rest of Section 4.2 is dedicated to finding a theory whose ω-saturated models are precisely the rich structures in $\mathcal{C}_0$. We then show that the theory is the complete theory of every rich structure in $\mathcal{C}_0$.

We start by defining *rotund varieties*, which serve as the main technical tool in finding the required theory.

Let $A = \mathbb{A}(K)$ be a model of $T_{\mathbb{A}}$.

Definition 4.2.4. An irreducible subvariety W of A^n is said to be *rotund* if for every $k \times n$-matrix M with entries in $\text{End}(\mathbb{A})$ of rank k, the dimension of the constructible set $M \cdot W$ is at least $\frac{k}{2}$. [1]

Remark 4.2.5. For any subvariety W of A^n and any $C \subset A$ such that W is defined over $k_0(C)$, if b is a generic point of W over $k_0(C)$, then: W is rotund if and only if for every $k \times n$-matrix M with entries in $\text{End}(\mathbb{A})$ of rank k,

$$\text{tr.d.}(M \cdot b/C) \geq \frac{k}{2}.$$

Remark 4.2.6. If W is a rotund subvariety of A^n, then, in particular, for every non-zero $m \in \text{End}(\mathbb{A})^n$, $\dim m \cdot W \geq 1$. This implies that W is not contained in any coset of a proper algebraic subgroup of A^n. To refer to this property, we say that W is *free (of linear dependences)*.

[1]Recall that the dimension of a constructible subset of A^n is, by definition, the dimension of its Zariski closure.

Remark 4.2.7. Let us note that rotund varieties correspond to strong extensions of structures in $\mathrm{Sub}\,\mathcal{C}_0$ as follows:

Consider a structure $\mathcal{X} \in \mathrm{Sub}\,\mathcal{C}_0$, with $X \subset \bar{A} \models T_{\mathbb{A}}$. Let W be an irreducible subvariety of $\bar{A}^n$ defined over X and let b be a generic point of W over $k_0(X)$ in $\bar{A}$. Let Y be the substructure of $\bar{A}$ with domain $Y = X + \mathrm{span}\, b$ and set $G(Y) := G(X) + \langle b^i : i \geq 1 \rangle$, where b^i is a sequence of tuples in $(\mathrm{span}\, b)^n$ with $b^1 = b$ and for all $i, j \geq 1$, $jb^{ij} = b^i$. Then $\mathcal{Y} = (Y, G(Y))$ is a structure in $\mathrm{Sub}\,\mathcal{C}$ extending $\mathcal{X}$.

Moreover, for every $k \times n$-matrix M of rank k with entries in $\mathrm{End}(\mathbb{A})$, we have

$$2 \dim(M \cdot W) - k = 2\, \mathrm{tr.d.}(M \cdot b/k_0(X)) - \mathrm{lin.d.}(M \cdot b) = \delta(M \cdot b/X).$$

Hence, if $\mathcal{Y}$ is a strong extension of $\mathcal{X}$, then the above value is always non-negative, and hence W is a rotund variety.

Conversely, assume W is a rotund variety and let us see that then $\mathcal{Y}$ is a strong extension of $\mathcal{X}$. Indeed, for every tuple $b' \subset Y$, there exists a $k \times n$-matrix M such that $\mathrm{span}(b'X) = \mathrm{span}((M \cdot b)X)$; therefore $\delta(b'/X) = \delta(M \cdot b/X) = 2 \dim(M \cdot W) - k$ and, by the rotundity of W, this value is non-negative.

Examples of rotund varieties We shall now give some examples of rotund varieties. Besides illustrating the notion, they will be useful for some of our later arguments.

For the first two families of examples let us consider the case where $\mathbb{A}$ is the multiplicative group. We thus write the group operation on $A = K^*$ multiplicatively.

First example: "$X + Y = c$". For any $c \in A = K^*$, the subvariety of $(K^*)^2$ defined by the equation $x + y = c$, where $+$ denotes addition in K, is rotund. This follows from transcendence degree calculations. [2]

Second example: Generic hyperplanes. A *hyperplane* in K^n is a variety defined by an equation of the form $c \cdot x = d$ for some non-zero $c \in K^n$ and some $d \in K$. If C is a subset of K and $c \in K^n$ is such that $\mathrm{tr.d.}(c/C) = n$ then the hyperplane defined by the equation $c \cdot x = 1$ is said to be a *generic hyperplane over* C.

[2] Explanation: Note that it is sufficient to show that the variety is free. Let $b = (b_1, b_2)$ be a generic point of the variety defined by $x + y = c$ over $\mathbb{Q}(c)^{\mathrm{alg}}$. Note that b_1 is transcendental over $\mathbb{Q}(c)^{\mathrm{alg}}$ and $b_2 = c - b_1$.

Let $m = (m_1, m_2) \in \mathbb{Z}^2$ be non-zero. Suppose towards a contradiction that $b^m = c'$ for some c' in $\mathbb{Q}(c)^{\mathrm{alg}}$. Then $c' = b_1^{m_1} b_2^{m_2} = b_1^{m_1}(c - b_1)^{m_2}$. Since b_1 is transcendental over c, we see that $m_2 = -m_1$. Thus, $\frac{1}{c'} = \frac{(c-b_1)^{m_1}}{b_1^{m_1}} = (\frac{c}{b_1} - 1)^{m_1}$. But this contradicts that b_1 is transcendental over c.

Let $H_{n,k}(x, y^1, \dots, y^k)$ be the subvariety of K^{n+kn} defined by the system of equations $M \cdot x = 1$, where M is the $k \times n$-matrix with rows $y^1, \dots, y^k$ and 1 denotes the tuple in K^k whose entries are all equal to 1.

A variety of the form $H_{n,k}(x, c^1, \dots, c^k)$ for some $c^1, \dots, c^k \in K^n$ is the intersection of k hyperplanes. If C is a subset of K and $\operatorname{tr.d.}(c^1, \dots, c^k/C) = nk$ then we say that $H_{n,k}(x, c^1, \dots, c^k)$ is the *intersection of k (independent) generic hyperplanes over C*.

The following lemma follows Lemme 3.1 in [33] and Lemma 5.2 in [46].

Lemma 4.2.8. *If $V \subset (K^*)^n$ is a rotund variety defined over $C \subset K$ of dimension d with $d - 1 \geq \frac{n}{2}$ and $H_{n,1}(x, c)$ is a generic hyperplane over C, then $V \cap H_{n,1}(x, c)$ is a rotund variety of dimension $d - 1$.*

Proof. Let H_c denote the hyperplane $H_{n,1}(x, c)$.

Let us first show that all irreducible components of $V \cap H_c$ have dimension $d - 1$: Since V is defined over C, V has rational points in $\mathbb{Q}(C)^{\text{alg}}$ and such points cannot be in H_c, for c is assumed to be algebraically independent over C, hence $V \not\subset H_c$. Therefore $V \cap H_c$ is a proper subvariety of the irreducible variety V. Thus, $\dim V \cap H_c < \dim V = d$. But also, by the smoothness of $(K^*)^n$ and the dimension of intersection inequality, the dimension of every irreducible component of $V \cap H_c$ is at least $\dim V + \dim H_c - n = d + (n - 1) - n = d - 1$. Thus, every component has dimension $d - 1$.

Let us now see that $V \cap H_c$ is in fact irreducible: Let V_1 and V_2 be irreducible components of $V \cap H_c$ (not necessarily distinct). Let a^1 be a generic point of V_1 over Cc and let a^2 be a generic point of V_2 over Cca^1. Note $\operatorname{tr.d.}(a^1/Cc) = \operatorname{tr.d.}(a^2/Cca^1) = d - 1$. Using the additivity of the transcendence degree, we therefore obtain $\operatorname{tr.d.}(a^1a^2c/C) = n + 2d - 2$. By the independence of a^1 and a^2 over C, $\dim H_{a^1} \cap H_{a^2} = n - 2$; hence $\operatorname{tr.d.}(c/Ca^1a^2) \leq n - 2$. Using again the additivity we get $\operatorname{tr.d.}(a^1a^2/C) \geq 2d$. It easily follows that, in fact, $\operatorname{tr.d.}(a^1a^2/C) = 2d$ and $\operatorname{tr.d.}(c/Ca^1a^2) = n - 2$. Thus, a^1 and a^2 are independent generic points of the irreducible variety V over C and c is a generic point of the irreducible variety $H_{a^1} \cap H_{a^2}$ over Ca^1a^2. This means that independently of the choice of the components V_1 and V_2 the type of a^1a^2c over C is always the same. Then $V \cap H_c$ must have only one irreducible component, as otherwise we get different types by choosing $V_1 = V_2$ and $V_1 \neq V_2$.

We now turn to showing that $V \cap H_c$ is rotund. Let a be a generic point of $V \cap H_c$ over $\mathbb{Q}(Cc)^{\text{alg}}$. Note that a is then also a generic point of V over $\mathbb{Q}(C)^{\text{alg}}$ and $\operatorname{tr.d.}(c/Ca) = n - 1$.

Claim: Let $b \subset \mathbb{Q}(Ca)^{\mathrm{alg}}$. If $\operatorname{tr.d.}(b/Cc) < \operatorname{tr.d.}(b/C)$, then $a \subset \mathbb{Q}(b)^{\mathrm{alg}}$.

Proof of claim: Assume $\operatorname{tr.d.}(b/Cc) < \operatorname{tr.d.}(b/C)$. Let U be the locus of c over $\mathbb{Q}(Cb)^{\mathrm{alg}}$. From our assumption, using the exchange principle, we have $\operatorname{tr.d.}(c/Cb) < \operatorname{tr.d.}(c/C)$. Thus, $\dim U = \operatorname{tr.d.}(c/Cb) < \operatorname{tr.d.}(c/C) = n$.

The hyperplane H_a must be contained in U; otherwise, $\dim U \cap H_a < \dim U < n$, contradicting the fact that $\operatorname{tr.d.}(c/Ca) = n-1$. Since $\dim U \leq n-1 = \dim H_a$, we conclude that $U = H_a$.

Since U is defined over $\mathbb{Q}(Cb)^{\mathrm{alg}}$, we can find $b' \in U$ with coordinates in $\mathbb{Q}(Cb)^{\mathrm{alg}}$. Then a is uniquely determined by the conditions (1) $b' \cdot a = 1$ and (2) for all $z \in H_a$, $(z - b') \cdot a = 0$. Thus, $a \subset \mathbb{Q}(Cb)^{\mathrm{alg}}$.

Applying the claim: Let M be a $k \times n$-matrix with integer entries of rank k. Let $b = a^M \in (K^*)^k$. Suppose towards a contradiction that $\operatorname{tr.d.}(b/Cc) < \frac{k}{2}$. By the rotundity of V, $\operatorname{tr.d.}(b/C) \geq \frac{k}{2}$. Hence $\operatorname{tr.d.}(b/Cc) < \operatorname{tr.d.}(b/C)$. Therefore, by the claim, $a \subset \mathbb{Q}(Cb)^{\mathrm{alg}}$. Thus, $\operatorname{tr.d.}(b/Cc) = \operatorname{tr.d.}(a/Cc) = d-1 \geq \frac{n}{2} \geq \frac{k}{2}$. A contradiction. This shows that $V \cap H_c$ is rotund. $\square$

Remark 4.2.9. It follows from the above lemma, by induction, that if $V \subset (K^*)^n$ is a rotund variety defined over $C \subset K$ of dimension d and $H_{n,k}(x, c^1, \ldots, c^k)$ is the intersection of k generic hyperplanes over C with $d - k \geq \frac{n}{2}$, then $V \cap H_{n,k}(x, c^1, \ldots, c^k)$ is a rotund variety of dimension $d - k$.

In particular, the subvariety $H_{n,k}(x, c^1, \ldots, c^k)$ of $(K^*)^n$ defined by the intersection of k generic hyperplanes with $k \leq \frac{n}{2}$ is rotund.

Third example: Generic hyperplanes in the elliptic curve case. Let us now consider analogues of the rotund varieties in the previous example in the case where $\mathbb{A}$ is an elliptic curve.

As mentioned before, we consider $A = \mathbb{A}(K)$ as a subvariety of $\mathbb{P}^2(K)$ whose affine part is defined by an equation $y^2 = 4x^3 + \alpha x + \beta$, with $\alpha, \beta \in k_0$.

The above arguments about generic hyperplanes in the multiplicative group case can be easily adapted to show the following: Let V be a rotund subvariety of A^n defined over C of dimension d with $d - 1 \geq \frac{n}{2}$. If H is a hyperplane in K^{2n}, generic over C, then the intersection of V and the Zariski closure of H in $(\mathbb{P}^2)^n$ is a rotund subvariety of A^n of dimension $d - 1$.

Also, if H is the intersection of k hyperplanes in K^{2n}, generic over C, with $d - k \geq \frac{n}{2}$, then the intersection of V and the Zariski closure of H in $(\mathbb{P}^2)^n$ is a rotund subvariety of A^n of dimension $d - k$. In particular, the intersection of A^n and the Zariski closure in $(\mathbb{P}^2)^n$ of the intersection of k generic hyperplanes in K^{2n}, where $k \leq \frac{n}{2}$, is a rotund subvariety of A^n of dimension $n - k$.

Definability of rotundity

Lemma 4.2.10. *For every subvariety $W(x, y)$ of $\mathbb{A}^{n+k}$ defined over k_0, there exists a quantifier-free $L_{\mathbb{A}}$-formula $\theta(y)$ such that for all $A \models T_{\mathbb{A}}$ and all $c \in A^k$,*

$$A \vDash \theta(c) \iff W(x, c) \text{ is rotund.}$$

Proof. Let $W(x, y)$ be a subvariety of $\mathbb{A}^{n+k}$ defined over k_0. Let $C_1, \ldots, C_s$ be proper algebraic subgroups of A^n as provided by the Weak CIT (3.2.4) for the family of subvarieties of A^n defined by $W(x, y)$. For each $i = 1 \ldots, s$, let M^i be an $n_i \times n$-matrix with entries in $\mathrm{End}(\mathbb{A})$ of rank n_i such that C_i is defined by the system of equations $M^i \cdot x = 0$.

Let $\theta(y)$ be the conjunction of the following:

- a quantifier-free $L_{\mathbb{A}}$-formula $\theta_0(y)$ such that $\theta_0(c)$ holds if and only if the variety W_c is irreducible and has dimension $\geq \frac{n}{2}$,
- for each $i = 1, \ldots, s$, a quantifier-free formula $\theta_i(y)$ such that $\theta_i(c)$ holds if and only if the dimension of $M^i \cdot W_c$ is at least $\frac{n_i}{2}$.

The existence of the formulas $\theta_i(y)$, $i = 0, \ldots, s$ is given by the following facts: that the theory of algebraically closed fields of any given characteristic (in this case 0) has the *definable multiplicity property* (Lemma 3 in [13]), which transfers to the theory $T_{\mathbb{A}}$, since the bi-interpretation is rank preserving, to give that the irreducibility of W_c is a definable property on c; the definability of Morley rank in strongly minimal theories (Corollary 5.6 in [43]), which here corresponds to the definability of dimension. We also use that the theory $T_{\mathbb{A}}$ has quantifier elimination.

It is clear that for all c, if $W(x, c)$ is rotund, then $\theta(c)$ holds.

To prove the converse, suppose towards a contradiction that we have c such that $\theta(c)$ holds but $W(x, c)$ is not rotund. We can then find a $k \times n$-matrix M with entries in $\mathrm{End}(\mathbb{A})$ of rank $k \geq 1$ such that $\dim M \cdot W(x, c) < \frac{k}{2}$. Let C be the algebraic subgroup of A^n defined by the equation $M \cdot x = 0$.

Let b be a generic point of W_c over $k_0(c)^{\mathrm{alg}}$ and let S be an irreducible component of $W_c \cap b + C$ containing b.

Note that $\dim S = \dim W_c \cap b + C$. Indeed, we have

$$\dim S \geq \mathrm{tr.\,d.}(b/c(M \cdot b)) = \mathrm{tr.\,d.}(b/c) - \mathrm{tr.\,d.}(M \cdot b/c) = \dim W_c - \dim M \cdot W_c,$$

and, by the theorem on the dimension of fibres ([35, I.6.3]), $\dim W_c - \dim M \cdot W_c = \dim W_c \cap (b + C)$; hence $\dim S = \dim W_c \cap b + C$.

Since $\dim M \cdot W_c < \frac{k}{2}$; in particular, $\dim M \cdot W_c < k$. Hence

$$\dim S = \dim W_c - \dim M \cdot W_c > \dim W_c - k.$$

Therefore

$$\dim S > \dim W_c - k.$$

But

$$\dim W_c - k = \dim W_c + (n-k) - n = \dim W_c + \dim(b + C) - n.$$

Thus, $\dim S > \dim W_c + \dim C - n$, i.e. S is an atypical component of the intersection of W_c and $b + C$.

Thus, by 3.2.4 (Weak CIT), there exists $i \in \{1, \ldots, s\}$ and $b' \in A^n$ such that S is contained in $b' + C_i$ and S is a typical component of the intersection of W_c and $b + C$ with respect to $b' + C_i$, i.e. $\dim S = \dim W_c \cap (b' + C_i) + \dim(b + C) \cap (b' + C_i) - \dim(b' + C_i)$.

Note that our assumption that $\dim M \cdot W_c < \frac{k}{2}$ can be written as $\dim W_c \cap (b + C) > \frac{1}{2} \dim C$. Also, if C' is any algebraic subgroup of A^n with $C' \subset C$ and $b + C' \supset S$, then

$$\dim W_c \cap (b + C') \geq \dim S = \dim W_c \cap (b + C) > \frac{1}{2} \dim C \geq \frac{1}{2} \dim C'.$$

Hence $\dim W_c \cap (b + C') > \frac{1}{2} \dim C'$. This implies that we may assume C to be the minimal algebraic subgroup having a coset that contains S. Thus, $C \cap C_i = C$ and therefore the typicality equation becomes

$$\dim S = \dim W_c \cap (b' + C_i) + \dim C - \dim C_i.$$

One can then easily see the following

$$\begin{aligned} \dim W_c \cap (b' + C_i) &= \dim S - \dim C + \dim C_i \\ &> -\frac{1}{2} \dim C + \dim C_i \\ &\geq \frac{1}{2} \dim C_i. \end{aligned}$$

This implies that $\dim M^i \cdot W_c < \frac{n_i}{2}$, which contradicts the fact that $\theta(c)$ holds. $\square$

4.2.3 The EC-property

Definition 4.2.11. A structure (A, G) in $\mathcal{C}_0$ is said to have the *EC-property* if for every even $n \geq 1$ and every rotund subvariety W of A^n of dimension $\frac{n}{2}$, the intersection $W \cap G^n$ is Zariski dense in W; i.e. for every proper subvariety W' of W the intersection $(W \setminus W') \cap G^n$ is non-empty.

Lemma 4.2.12. *There exists a set of* $\forall\exists$*-*L*-sentences* T^1 *such that for any structure* (A, G) *in* $\mathcal{C}_0$

$$(A, G) \models T^1 \iff (A, G) \text{ has the EC-property.}$$

Proof. For each even integer $n \geq 1$, and each subvariety $W(x, y)$ of $\mathbb{A}^{n+k}$ defined over k_0, let $\theta_W(y)$ be a formula as provided by Lemma 4.2.10. Let T^1 be the theory containing, for each pair of subvarieties $W(x, y)$ and $W'(x, y)$ of $\mathbb{A}^{n+k}$, the following sentence:

$$\forall y \Big(\big(\theta_W(y) \wedge \dim W_y = \frac{n}{2} \wedge W'_y \subsetneq W_y\big) \\ \rightarrow \exists x \big(W(x, y) \wedge \neg W'(x, y) \wedge \bigwedge_{i=1}^{n} G(x_i)\big)\Big)$$

It is clear that T^1 expresses the EC-property. □

Henceforth, let T^1 denote the theory defined in the above proof. Also, let $T := T^0 \cup T^1$.

The rest of this section shows that T axiomatizes the complete theory common to all rich structures in $\mathcal{C}_0$.

Definition 4.2.13. Let $\mathcal{A}$ be a structure in $\mathcal{C}_0$.

$\mathcal{A}$ is said to be *existentially closed (with respect to self-sufficient extensions)* if for any quantifier-free formula $\phi(x)$, if there exist a self-sufficient extension $\mathcal{A}'$ of $\mathcal{A}$ and a tuple $b' \subset A'$ such that $\mathcal{A}' \models \phi(b')$, then there exists $b \subset A$ such that $\mathcal{A} \models \phi(b)$.

$\mathcal{A}$ is said to be *strongly existentially closed (with respect to self-sufficient extensions)* if for any partial type $\Phi(x)$ over a finite subset of $\mathcal{A}$ consisting of quantifier-free formulas, if there exist a self-sufficient extension $\mathcal{A}'$ of $\mathcal{A}$ and a tuple $b' \subset A'$ such that $\mathcal{A}' \models \Phi(b')$, then there exists $b \subset A$ such that $\mathcal{A} \models \Phi(b)$.

Clearly, if $\mathcal{A}$ is strongly existentially closed, then $\mathcal{A}$ is existentially closed. Also, if $\mathcal{A}$ is ω-saturated, then $\mathcal{A}$ is strongly existentially closed if and only if it is existentially closed.

Lemma 4.2.14. *If $\mathcal{A} \in \mathcal{C}_0$ is existentially closed, then $\mathcal{A}$ has the EC-property.*

Proof. Let $\mathcal{A} = (A, G) \in \mathcal{C}_0$ be existentially closed, with $A = \mathbb{A}(K)$. Let W be a rotund subvariety of of A^n of dimension $\frac{n}{2}$ and let W' be a proper subvariety of W, both defined over K. We want to see that $(W \setminus W') \cap G^n$ is non-empty. Let b be a generic point of W over K in a model $\bar{A}$ of $T_{\mathbb{A}}$ extending A. Let Y be the substructure of A' with domain $A + \operatorname{span}(b)$. Let $(b^i)_{i \geq 1}$ be a sequence of tuples in $(\operatorname{span} b)^n$ such that $b^1 = b$ and for all $i, j \geq 1$, $jb^{ij} = b^i$, and let $G(Y) := G(A) + \langle b^i : i \geq 1 \rangle$. The rotundity of W gives that $\mathcal{Y} = (Y, G(Y))$ is a strong extension of $\mathcal{A}$ in $\operatorname{Sub} \mathcal{C}_0$ (as noted in 4.2.7). By the extension property, we can find $\mathcal{A}'$ in $\mathcal{C}_0$ with $\mathcal{Y} \leq \mathcal{A}'$. Thus, $\mathcal{A} \leq \mathcal{A}'$ and b is a solution in $\mathcal{A}'$ of the quantifier-free formula $W(x) \wedge \neg W'(x) \wedge \bigwedge_{i=1}^{n} G(x_i)$. Since $\mathcal{A}$ is existentially closed, there exists a solution of the same formula in $\mathcal{A}$, hence $(W \setminus W') \cap G^n$ is non-empty. $\square$

Lemma 4.2.15. *If $\mathcal{A} \in \mathcal{C}_0$ is rich, then $\mathcal{A}$ is strongly existentially closed.*

Proof. Let $\mathcal{A} \in \mathcal{C}_0$ be rich. Let $\Phi(x)$ be a quantifier-free partial type over a finite subset c of A, $\mathcal{A}'$ be a strong extension of $\mathcal{A}$ and b be a solution of $\Phi(x)$ in $\mathcal{A}'$. By replacing c by an appropriate basis of its self-sufficient closure, we may assume that $\operatorname{span} c$ is self-sufficient in $\mathcal{A}$. Let $\mathcal{X}$ be the substructure of $\mathcal{A}$ with domain $\operatorname{span} c$. Let $\mathcal{Y}$ be the substructure of $\mathcal{A}'$ with domain $\operatorname{span} bc$. Since $\mathcal{X} \leq \mathcal{A} \leq \mathcal{A}'$, by transitivity $\mathcal{X} \leq \mathcal{A}'$. In particular, $\mathcal{X} \leq \mathcal{Y}$. Thus, by the richness of $\mathcal{A}$, we can find an embedding j of $\mathcal{Y}$ into $\mathcal{A}$ over X. Then $j(b)$ is clearly a solution of $\Phi(x)$ in $\mathcal{A}$. $\square$

4.2.4 Axiomatizing richness up to ω-saturation

It is clear from the definitions that the models of T are precisely the structures in $\mathcal{C}_0$ satisfying the EC-property. We shall now see that the ω-saturated models of T are precisely the rich structures in $\mathcal{C}_0$. It follows that T axiomatizes the complete theory common to all rich structures in $\mathcal{C}_0$.

Rich structures are models of T

Remark 4.2.16. By 4.2.14, every existentially closed structure in $\mathcal{C}_0$ is a model of T. In particular, by 4.2.15, every rich structure in $\mathcal{C}_0$ is a model of T.

ω-saturated models of T are rich

Definition 4.2.17. Let $\mathcal{X}, \mathcal{Y}$ be structures in $\operatorname{Sub} \mathcal{C}_0$. Assume $\mathcal{X} \leq \mathcal{Y}$. We say that $\mathcal{X} \leq \mathcal{Y}$ is a *minimal* strong extension if $X \neq Y$ and there exists no $\mathcal{Z} \in \operatorname{Sub} \mathcal{C}_0$ with $X \subsetneq Z \subsetneq Y$ such that $\mathcal{X} \leq \mathcal{Z} \leq \mathcal{Y}$.

Remark 4.2.18. Note that every strong extension $\mathcal{X} \leq \mathcal{Y}$ with $\mathcal{X}, \mathcal{Y}$ in $\operatorname{Fin} \mathcal{C}_0$ decomposes into a finite tower of minimal strong extensions, i.e. there exist a positive integer n and $\mathcal{X}_0, \ldots, \mathcal{X}_n \in \operatorname{Fin} \mathcal{C}_0$ such that $\mathcal{X}_0 = \mathcal{X}$, $\mathcal{X}_n = \mathcal{Y}$ and for all $i = 0, \ldots, n-1$, $\mathcal{X}_i \leq \mathcal{X}_{i+1}$ is a minimal strong extension.

Consequently, for all $\mathcal{A} \in \mathcal{C}_0$, the structure $\mathcal{A}$ is rich if and only if for all $\mathcal{X}, \mathcal{Y} \in \operatorname{Fin} \mathcal{C}_0$ such that $\mathcal{X} \leq \mathcal{A}$ and $\mathcal{X} \leq \mathcal{Y}$ is a *minimal* strong extension, there exists a self-sufficient embedding of $\mathcal{Y}$ into $\mathcal{A}$ over $\mathcal{X}$.

Remark 4.2.19. Let $\mathcal{X}, \mathcal{Y} \in \operatorname{Fin} \mathcal{C}_0$ with $\mathcal{X} \leq \mathcal{Y}$ and $\delta(X) = \delta(Y)$. It is easy to see that the extension $\mathcal{X} \leq \mathcal{Y}$ is minimal if and only if for every cl_0-closed Z with $X \subsetneq Z \subsetneq Y$, $\delta(Z/X) > 0$.

Lemma 4.2.20. *Let $\mathcal{X}, \mathcal{Y}$ be structures in $\operatorname{Fin} \mathcal{C}_0$ such that $\mathcal{X} \leq \mathcal{Y}$ is a minimal strong extension. Then exactly one of the following holds:*

1. *(*Prealgebraic extension*) There exists an even number $n \geq 2$ and a cl_0-basis $b \in G(Y)^n$ of Y over X with $\mathrm{tr.d.}(b/X) = \frac{n}{2}$, and hence $\delta(b/X) = 0$.*

2. *(*Green generic extension*) There is $b_0 \in G(Y) \setminus X$ such that $Y = X + \operatorname{span} b_0$ and b_0 is transcendental over X, hence $\delta(b_0/X) = 1$.*

3. *(*Algebraic extension*) $G(Y) = G(X)$ and there is $b_0 \in Y \setminus X$ such that $Y = X + \operatorname{span} b_0$ and b_0 is algebraic over X, hence $\delta(b_0/X) = 0$.*

4. *(*White generic extension*) $G(Y) = G(X)$ and there is $b_0 \in Y \setminus X$ such that $Y = X + \operatorname{span} b_0$ and b_0 is transcendental over X, hence $\delta(b_0/X) = 2$.*

Proof. Let $\mathcal{X}, \mathcal{Y}$ be structures in $\operatorname{Fin} \mathcal{C}_0$ such that $\mathcal{X} \leq \mathcal{Y}$ is a minimal strong extension. Then exactly one of the following cases occurs:

Case 1: $G(Y) \neq G(X)$ and $\delta(Y/X) = 0$.

First note that, by minimality, $Y = X + \operatorname{span} G(Y)$. Let n be the linear dimension of Y over X, which is also the linear dimension of $G(Y)$ over $G(X)$, by modularity. Then, since $\delta(Y/X) = 0$, we have $\mathrm{tr.d.}(Y/X) = \frac{n}{2}$. Thus, if $b \in G(Y)^n$ is a cl_0-basis of Y over X, then b is also a cl_0-basis of $G(Y)$ over $G(X)$ and $\mathrm{tr.d.}(b/X) = \frac{n}{2}$.

Case 2: $G(Y) \neq G(X)$ and $\delta(Y/X) > 0$.

As in the previous case, $Y = X + \operatorname{span} G(Y)$. Moreover,

$$\operatorname{lin.d.}(Y/X) = \operatorname{lin.d.}(G(Y)/G(X)) = 1.$$

To see this, take an element b_0 in $G(Y) \setminus G(X)$, and note that $\operatorname{tr.d.}(b_0/X) = 1$, and hence $\delta(b_0/X) = 1$. Now, if $\operatorname{lin.d.}(Y/X)$ is strictly greater than one then we get a tower of proper strong extensions: either $X \leq X + \operatorname{span} b_0 \leq Y$ is such a tower, or otherwise there exists a tuple $b \subset Y$ starting with the element b_0 such that $\operatorname{span} b_0 \subsetneq \operatorname{span} b \subsetneq Y$ and $\delta(\operatorname{span} b / \operatorname{span} b_0) < 0$, and hence $\delta(X + \operatorname{span} b) = \delta(X)$, from which it follows that $X \leq X + \operatorname{span} b \leq Y$ is a tower of (proper) strong extensions.

Thus, for any element $b_0 \in G(Y) \setminus X$, $Y = X + \operatorname{span} b_0$.

Case 3: $G(Y) = G(X)$ and $\delta(Y/X) = 0$.

By the minimality of the extension, $Y = X + \operatorname{span} b_0$ for any $b_0 \in Y \setminus X$. Also, since $\delta(Y/X) = 0$, any such b_0 is algebraic over X.

Case 4: $G(Y) = G(X)$ and $\delta(Y/X) > 0$.

As in the previous case, $Y = X + \operatorname{span} b_0$, for any $b_0 \in Y \setminus X$. Thus, $\operatorname{tr.d.}(Y/X) \leq 1$ and, since we are assuming $\delta(Y/X) > 0$, such b_0 must be transcendental over X, and hence $\delta(Y/X) = 2$. □

We shall henceforth use the names given in the above lemma to the different kinds of extensions. For the sake of brevity let us also stress the language by making the following definition:

Definition 4.2.21. Let us say that a rotund subvariety W of A^n is *prealgebraic minimal* if it has dimension $\frac{n}{2}$ and for every $1 \leq k < n$, and every $k \times n$-matrix with entries in $\operatorname{End}(\mathbb{A})$ of rank k, $\dim M \cdot W > \frac{k}{2}$.

It is easy to see that a rotund variety W is prealgebraic minimal if and only if any strong extension constructed from W as in Remark 4.2.7 is a prealgebraic minimal extension.

Using Remark 4.2.19, a minor modification of the proof of Lemma 4.2.10 (strengthening the formula $\theta_0(y)$ to require $\dim W_y = \frac{n}{2}$ and each $\theta_i(y)$ to correspond to the *strict* inequality $\dim M^i \cdot W_c > \frac{n_i}{2}$) yields the following :

Lemma 4.2.22. *For every subvariety $W(x,y)$ of $\mathbb{A}^{n+k}$ defined over k_0, there exists a quantifier-free $L_{\mathbb{A}}$-formula $\theta'(y)$ such that for all $A \models T_{\mathbb{A}}$ and all $c \in A^k$,*

$$A \vDash \theta'(c) \iff W(x,c) \textit{ is a prealgebraic minimal rotund subvariety.}$$

The following two remarks use the above definability lemma and model-theoretic arguments to find new rotund varieties from the ones coming from intersections of generic hyperplanes. These results will then be applied in the proof of Lemma 4.2.25.

Remark 4.2.23. Let us first consider the case where $\mathbb{A}$ is the multiplicative group. For each $n \geq 1$, if $H_{2n,n}(x, c^1, \ldots, c^n)$ is the intersection of n generic hyperplanes in A^{2n} (i.e. $\operatorname{tr.d.}(c^1, \ldots, c^n) = 2n^2$), then $H_{2n,n}(x, c^1, \ldots, c^n)$ is a prealgebraic minimal rotund variety.

Indeed, we already know, by 4.2.9, that $H_{2n,n}(x, c^1, \ldots, c^n)$ is a rotund variety of dimension n. Now consider a generic point b of $H_{2n,n}(x, c^1, \ldots, c^n)$ over $C := \bigcup c^i$ and let b' be a subtuple of b of length k with $1 \leq k < 2n$. Using the algebraic independence of C one sees the following: if $1 \leq k \leq n$, then $\operatorname{tr.d.}(b'/C) = \operatorname{tr.d.}(b/C) - \operatorname{tr.d.}(b/Cb') \geq n - (n-k) = k > \frac{k}{2}$; if $n < k < 2n$, then $\operatorname{tr.d.}(b'/C) = \operatorname{tr.d.}(b/C) - \operatorname{tr.d.}(b/Cb') \geq n - 0 = n > \frac{k}{2}$. This shows that $H_{2n,n}(x, c^1, \ldots, c^n)$ is prealgebraic minimal.

Then, by Lemma 4.2.22 and the model-completeness of $T_{\mathbb{A}}$, it follows that for each $n \geq 1$, there exist tuples $c^{1*}, \ldots, c^{n*} \in ((\mathbb{Q}^{\text{alg}})^*)^n$ such that $H_{2n,n}(x, c^{1*}, \ldots, c^{n*})$ is a prealgebraic minimal rotund subvariety of A^{2n}.

As noted earlier, in the elliptic curve case we have an analogue of each generic hyperplane $H_{2n,n}(x, c^1 \ldots, c^n)$ with algebraically independent $c^1 \ldots, c^n$ over k_0, namely the Zariski closure of the intersection of the generic hyperplane $H_{4n,2n}(x, c'^1 \ldots, c'^{2n})$, where $c'^1, \ldots, c'^{2n}$ are algebraically independent over k_0, and A^n. Therefore we also have an analogue of the varieties $H_{2n,n}(x, c^{1*}, \ldots, c^{n*})$ found above, i.e. prealgebraic minimal rotund subvarieties of A^{2n} defined over k_0^{alg}. To ease the notation, we shall henceforth, also in the case where $\mathbb{A}$ is an elliptic curve, use $H_{2n,n}(x, c^{1*}, \ldots, c^{n*})$ to denote such a subvariety of A^{2n}.

Remark 4.2.24. Let $\mathcal{A} = (A, G)$ be a model of T. Let us see that $A = \operatorname{acl}_{T_{\mathbb{A}}}(G)$.

If $\mathcal{A}$ is the multiplicative group, then a stronger statement is easy to prove: by the rotundity of the varieties defined by the equations $X + Y = c$ with $c \in A = K^*$, that $\mathcal{A}$ satisfies the EC-property directly implies that $G + G = A$.

Assume now that $\mathbb{A}$ is an elliptic curve.

Let $c \in K$, $c \neq 0$. Let $q_1, \ldots, q_4$ be algebraically independent over $k_0(c)$. The set $H(x, y, q, c)$ defined by the equation $q_1x_1 + q_2y_1 + q_3x_1 + q_4y_2 = c$ is a generic hyperplane in K^4. Therefore this equation defines a rotund subvariety of A^2. By the algebraic independence assumption, the algebraic type of q over $k_0(c)^{\text{alg}}$ does not fork over k_0^{alg}. An alternative way of expressing this last fact is that the algebraic type of q over $k_0(c)^{\mathrm{alg}}$ is a coheir of its restriction to k_0^{alg}, which means that it is finitely satisfiable in k_0^{alg}. Combining this finite satisfiability with the fact that the rotundity of $H(x, y, q, c)$ is definable on the parameters q, c (in the algebraic language), we see that there exists $q_1^*, \dots, q_4^* \in k_0^{\mathrm{alg}}$ such that the equation $q_1^* x_1 + q_2^* y_1 + q_3^* x_1 + q_4^* y_2 = c$ defines a rotund subvariety of A^2. Since $\mathcal{A}$ satisfies the EC-property, this subvariety has a solution in G^2. This implies that c is in $\mathrm{acl}^{\mathrm{eq}}_{\mathbb{A}}(G)$. It follows that $K \subset \mathrm{acl}^{\mathrm{eq}}_{\mathbb{A}}(G)$ and, therefore, $A = \mathrm{acl}_{\mathbb{A}}(G)$.

Additionally, let us note the following: suppose we start the above argument with $c \in K$ transcendental over k_0 and thus find $q_1^*, \dots, q_4^* \in k_0^{\mathrm{alg}}$ such that $q_1^* x_1 + q_2^* y_1 + q_3^* x_1 + q_4^* y_2 = c$ defines a rotund subvariety of A^2. Then, for any $c' \in K$ transcendental over k_0, since c and c' are conjugates over k_0^{alg}, the subvariety of A^2 defined by $q_1^* x_1 + q_2^* y_1 + q_3^* x_1 + q_4^* y_2 = c'$, with *the same* $q_1^*, \dots, q_4^*$ as before, is also rotund. This is used in the proof of the following lemma.

Lemma 4.2.25. *If $\mathcal{A}$ is an ω-saturated model of T, then $\mathcal{A}$ is a rich structure in $\mathcal{C}_0$.*

Proof. Assume $\mathcal{A} = (A, G)$ is an ω-saturated model of T. Since T contains T^0, which by definition axiomatizes the class $\mathcal{C}_0$, it is clear that $\mathcal{A}$ is in $\mathcal{C}_0$.

To show that $\mathcal{A}$ is rich, let $\mathcal{X}, \mathcal{Y}$ be structures in $\mathrm{Fin}\,\mathcal{C}_0$ such that $\mathcal{X}$ is self-sufficient in $\mathcal{A}$ and $\mathcal{X} \leq \mathcal{Y}$ is a minimal strong extension. We need to show that there is a self-sufficient embedding of $\mathcal{Y}$ into $\mathcal{A}$ over X. We consider different cases for the different kinds of minimal strong extensions, as determined in Lemma 4.2.20.

Case 1: $\mathcal{X} \leq \mathcal{Y}$ is a prealgebraic minimal extension.

Let $b \in G(Y)^n$ be a cl_0-basis of Y over X.

Let $(b^i)_{i \geq 1}$ be a sequence of tuples in $G(Y)^n$ such that $b^1 = b$ and for all $i, j \geq 1$, $jb^{ij} = b^i$. Note that $G(Y) = G(X) + \langle b^i : i \geq 1 \rangle$. It is sufficient to show that $\text{qf-tp}((b^i)_i / X)$ is realised in $\mathcal{A}$ to obtain an embedding of Y into $\mathcal{A}$ over X. Moreover, since in this case $\delta(X) = \delta(Y)$, any such embedding is necessarily self-sufficient.

Let us show that indeed $\text{qf-tp}((b^i)_i / X)$ is realised in $\mathcal{A}$. By the ω-saturation of $\mathcal{A}$, it is sufficient to show that for each $n \geq 1$, the type $\text{qf-tp}((b^i)_{i \leq n} / X)$ is realised in $\mathcal{A}$: Fix $n \geq 1$. Let $N = \prod_{i \leq n} i$. Let $V := \mathrm{locus}(b^N / k_0(X)^{\mathrm{alg}})$. By Remark 4.2.7, V is a rotund subvariety of A^n. Notice that, since $\delta(b^N / X) = \delta(Y / X) = 0$, we have $\dim V = \frac{n}{2}$. Since $\mathcal{A}$ is a model of T, for every proper subvariety V' of V over $k_0(X)^{\mathrm{alg}}$, the intersection $(V \setminus V') \cap G^n$ is non-empty. By the ω-saturation of $\mathcal{A}$, it follows that we can find a generic point b'^N of V over $k_0(X)^{\mathrm{alg}}$ in G^n. Note that b'^N is a realisation of $\text{qf-tp}(b^N / X)$. For each $i \leq n$, let $(b'^i) := \frac{N}{i} b'^N$. Then $(b'^i)_i$ is a realisation of $\text{qf-tp}((b^i)_{i \leq n} / X)$ in $\mathcal{A}$.

Case 2: $\mathcal{X} \leq \mathcal{Y}$ is a green generic extension.

This amounts to finding an element $b_0 \in \mathcal{A}$ such that $\mathrm{d}(b_0/X) = 1$, or, equivalently, $b_0 \notin \mathrm{cl_d}(X)$.

By 4.2.24, we can find a finite dimensional $X' \subset G$ such that $X \subset \mathrm{acl}_{T_{\mathbb{A}}}(X')$. Since always $\mathrm{acl}_{T_{\mathbb{A}}} \subset \mathrm{cl_d}$, we also have $\mathrm{cl_d}(X) \subset \mathrm{cl_d}(X')$. It therefore suffices to find $b_0 \in \mathcal{A}$ outside $\mathrm{cl_d}(X')$. Thus, we may assume that X is contained in G.

There is a partial type $\Phi(x_0)$ over X expressing the following: $x_0 \in G$; $x_0 \notin \mathrm{acl}_{T_{\mathbb{A}}}(X)$; for each $n \geq 1$, for all $x_1, \ldots, x_n \in G$, $\delta(x_0, \ldots, x_n/X) > 0$.

Indeed, for each $n \geq 1$, the condition above can be expressed by a set of formulas in the variables $x_1, \ldots, x_n$. To see this one can slightly modify the argument in the proof of 4.2.1, this time looking at varieties of dimension $\leq \frac{n}{2}$, instead of $< \frac{n}{2}$, which one can see makes no essential difference. Note that here we use our assumption that X is contained in G, for it is only in this case that we know how to express the predimension inequalities.

It is now sufficient to find a realisation for the type $\Phi(x_0)$ in $\mathcal{A}$. By the ω-saturation of $\mathcal{A}$, it suffices to show that $\Phi(x_0)$ is finitely satisfiable in $\mathcal{A}$. For this, it is in turn sufficient to show that for each $n \geq 1$, there is a prealgebraic *minimal* extension $\mathcal{X}_n$ of $\mathcal{X} = (X, G \cap X)$ with $\mathrm{lin.\,d.}(X_n/X) > n$ (and apply the result of Case 1).

Such extensions can be found as follows: For each $n \geq 1$, let $H_{2n,n}(x, c^{*1}, \ldots, c^{*n})$ be as in (the end of) Remark 4.2.23 and let b be a generic of $H_{2n,n}(x, c^{*1}, \ldots, c^{*n})$ over $k_0(X)^{\mathrm{alg}}$. Put $X_n := X + \mathrm{span}\, b$ and $G_n := G(X) + \langle b^i : i \geq 1 \rangle$, where $(b^i)_{i \geq 1}$ is a sequence of tuples such that $b^1 = b$ and for all $i, j \geq 1$, $j b^{ij} = b^i$. Then $\mathcal{X}_n = (X_n, G_n)$ is a prealgebraic minimal extension of $\mathcal{X}$ and $\mathrm{lin.\,d.}(X_n/X) = 2n > n$.

Case 3: $\mathcal{X} \leq \mathcal{Y}$ is an algebraic extension.

Let b_0 be any element of $Y \setminus X$. Let $b_0' \in A$ be a root of the minimal polynomial of b_0 over X. Then $\mathcal{Y}$ is isomorphic to the substructure of $\mathcal{A}$ with domain $Y' := X + \mathrm{span}\, b_0'$. Since $\delta(Y') = \delta(Y) = \delta(X)$, Y' is self-sufficient in $\mathcal{A}$.

Case 4: $\mathcal{X} \leq \mathcal{Y}$ is a white generic extension.

It suffices to find an element b_0 in $\mathcal{A}$ with $\mathrm{d}(b_0/X) = 2$. By the argument in Case 2, we can find $b_1 \in A$ with $\mathrm{d}(b_1/X) = 1$ and $b_2 \in A$ with $\mathrm{d}(b_2/Xb_1) = 1$. Note that, by additivity, $\mathrm{d}(b_1, b_2/X) = 2$,

We shall now find such a b_0.

Let us first deal with the case where $\mathbb{A}$ is the multiplicative group. Let $b_0 := b_1 + b_2$ and let us show that indeed $\mathrm{d}(b_0/X) = 2$. Let us consider two cases:

Case 1: $b_1 \in \mathrm{acl}_{T_{\mathbb{A}}}(\mathrm{sscl}(Xb_0))$. Then $\mathrm{d}(b_1/Xb_0) = 0$. And also $b_2 \in \mathrm{acl}_{T_{\mathbb{A}}}(\mathrm{sscl}(Xb_0))$, hence $\mathrm{d}(b_2/Xb_0) = 0$. It follows that $\mathrm{d}(b_0/X) = \mathrm{d}(b_1, b_2/X) = 2$.

Case 2: $b_1 \notin \mathrm{acl}_{T_{\mathbb{A}}}(\mathrm{sscl}(Xb_0))$. Then (b_1, b_2) is a generic point of the variety defined by $X + Y = b_0$ over $\mathrm{sscl}(Xb_0)$. Therefore $\delta(b_1, b_2/\,\mathrm{sscl}(Xb_0)) = 0$, and hence $\mathrm{d}(b_1, b_2/Xb_0) = 0$. Thus, $\mathrm{d}(b_0/X) = 2$.

Let us now show how to do an analogous argument in the elliptic curve case. Since b_1 and b_2 are independent in the sense of $\mathrm{cl_d}$, they are also independent in the sense of $\mathrm{acl}_{\mathbb{A}}$. In particular, b_1 and b_2 are not algebraic over the empty set in the sense of $T_{\mathbb{A}}$ and therefore lie in the affine part of the elliptic curve A (the unique point at infinity is algebraic). Let us thus write $b_1 = (b_{11}, b_{12})$ and $b_2 = (b_{21}, b_{22})$, with each b_{ij} in K.

By Remark 4.2.24, we can find $q_1^*, \ldots, q_4^* \in k_0^{\mathrm{alg}}$ such that for any $c \in K$ transcendental over k_0, the equation $q_1^*x_1 + q_2^*y_1 + q_3^*x_2 + q_4^*y_2 = c$ defines a rotund subvariety of A^2. Define $b_{01} := q_1^*b_{11} + q_2^*b_{12} + q_3^*b_{21} + q_4^*b_{22}$, and let $b_{02} \in K$ be such that the point $b_0 := (b_{01}, b_{02})$ is in A. Since b_1 and b_2 are $\mathrm{acl}_{\mathbb{A}}$-independent, b_{11} and b_{12} are algebraically independent over k_0. Therefore b_{01} is transcendental over k_0. Hence, the equation $q_1^*x_1 + q_2^*y_1 + q_3^*x_2 + q_4^*y_2 = b_{01}$ defines a rotund subvariety of A^2 having (b_1, b_2) as a solution. From here one can follow the same argument as in the multiplicative group case (considering two cases, etc.), to show that $\mathrm{d}(b_0/X) = \mathrm{d}(b_1, b_2/X) = 2$.

□

Rich structures are ω-saturated

Proposition 4.2.26. *The theory T is complete and its ω-saturated models are precisely the rich structures.*

Proof. By Remark 4.2.16, every rich structure is a model of T. Note that the existence of rich structures in $\mathcal{C}_0$ was proved in Section 4.1.3, and thus we know T is consistent.

By Lemma 4.2.25, every ω-saturated model of T is rich.

Let us now show that every rich structure is ω-saturated. Let $\mathcal{A}$ be a rich structure. Let $\mathcal{A}'$ be an ω-saturated elementary extension of $\mathcal{A}$. By Lemma 4.2.25, $\mathcal{A}'$ is rich and, by Lemma 2.3.8, $\mathcal{A}$ and $\mathcal{A}'$ are $L_{\infty\omega}$-equivalent. Since $L_{\infty\omega}$-equivalence preserves λ-saturation for all infinite cardinals λ, $\mathcal{A}$ is ω-saturated.

We have thus seen that the ω-saturated models of T are precisely the rich structures. Completeness of T follows immediately by 2.3.8. □

4.3 Model-theoretic properties of T

In this section we prove that the theory T is ω-stable and near model complete. We also calculate U-ranks and Morley ranks in the theory T. For missing definitions and basic facts from stability theory we refer to Chapter 1 of [29] and [32].

Throughout this section we work in a monster model $\bar{\mathcal{A}} = (\bar{A}, G)$ of T (sufficiently saturated and strongly homogeneous).

4.3.1 ω-stability

Here we show how the quantifier elimination for T provided by Proposition 2.3.10 implies that the theory T is ω-stable. We follow the proof of Poizat in [33], but we note that an application of the Thumbtack Lemma (3.5.2) is necessary for the argument to yield the full result.

Lemma 4.3.1. *For every subset B of $\bar{\mathcal{A}}$, $\mathrm{acl}_T(B) = \mathrm{acl}_{T_{\mathbb{A}}}(\mathrm{sscl}(B))$.*

Moreover, every algebraically closed subset of $\bar{\mathcal{A}}$ is self-sufficient.

Proof. For the first part, it is sufficient to show that the equality holds for finite B. Thus, fix a finite $B \subset \bar{\mathcal{A}}$

($\supset$) Let a be a cl_0-basis of $\mathrm{sscl}(B)$. It suffices to show that $a \subset \mathrm{acl}_T(B)$. Every automorphism of $\bar{\mathcal{A}}$ fixing B pointwise fixes $\mathrm{sscl}(B)$ set-wise. Thus, all conjugates of a over B are contained in $\mathrm{sscl}(B)$, and hence there are at most countably many of them. By the saturation of $\bar{\mathcal{A}}$, we get that in fact there must be only finitely many. This shows that every element in a is algebraic over B.

($\subset$) Let a be an element of $\bar{A} \setminus \mathrm{acl}_{\mathbb{A}}(\mathrm{sscl}(B))$ and let us show that a is not in $\mathrm{acl}_T(B)$.

Let $C = \mathrm{sscl}(B)$ and $D = \mathrm{sscl}(aB)$. For each $n \geq 1$, let D_n be the free amalgam of n isomorphic copies of D over C. By the richness of $\bar{\mathcal{A}}$, each D_n embeds strongly into $\bar{\mathcal{A}}$ over C. The different copies of a in each D_n have all the same type, this is because the different copies of D are self-sufficient in D_n and hence are self-sufficiently embedded in $\bar{\mathcal{A}}$, so the isomorphisms between them are elementary maps. It follows that the type of a over B has infinitely many realisations in $\bar{\mathcal{A}}$. Therefore a is not in $\mathrm{acl}_T(B)$.

The second part of the statement follows immediately from the first and Lemma 4.1.12.

□

Lemma 4.3.2. *For all sequences $a, a' \subset \bar{A}$, possibly infinite, the following are equivalent:*

1. $\mathrm{tp}(a) = \mathrm{tp}(a')$,
2. *The map $a \mapsto a'$ extends to an L-isomorphism from $\mathrm{acl}_T(a)$ onto $\mathrm{acl}_T(a')$.*
3. *The map $a \mapsto a'$ extends to an L-isomorphism from $\mathrm{sscl}(a)$ onto $\mathrm{sscl}(a')$.*

4. *The map $a \mapsto a'$ extends to an L-isomorphism between self-sufficient subsets of $\bar{\mathcal{A}}$,*

Proof. That (1.) implies (2.) holds for arbitrary first-order theories, by an easy argument.

To see that (2.) implies (3.), note that any isomorphism from $\mathrm{acl}_T(a)$ onto $\mathrm{acl}_T(a')$ sending a to a' has to map $\mathrm{sscl}(a)$ onto $\mathrm{sscl}(a')$. Indeed, to see this simply notice that, since $\mathrm{acl}_T(a)$ is self-sufficient in $\bar{\mathcal{A}}$ (by Lemma 4.3.1), $\mathrm{sscl}(a)$ is the intersection of all finite dimensional cl_0-closed subsets B of $\mathrm{acl}_T(a)$ that are self-sufficient in $\mathrm{acl}_T(a)$, and similarly for a'.

It is clear that (3.) implies (4.).

The implication from (4.) to (1.) follows immediately from Lemma 2.3.8. □

Theorem 4.3.3. *The theory T is ω-stable.*

Proof. Let λ be an infinite cardinal. We shall see that for all $B \subset \bar{\mathcal{A}}$ with $|B| \leq \lambda$, there are no more than λ complete 1-types over B; thus showing that T is λ-stable for all infinite λ, i.e. that it is ω-stable.

Let $B \subset \bar{A}$ be of cardinality λ. By passing to the algebraic closure $\mathrm{acl}_T(B)$ of B, we may assume that B is a self-sufficient subset of $\bar{\mathcal{A}}$ of the form $\mathbb{A}(K)$ for some algebraically closed subfield K of $\bar{K}$, where $\bar{K}$ is an algebraically closed field such that $\bar{A} = \mathbb{A}(\bar{K})$.

Consider an arbitrary element a_0 in $\bar{A}$. Recall that by Lemma 4.3.2, the type of a_0 over B is determined by the isomorphism type of $\mathrm{sscl}(Ba_0)$ over B.

Let a be a cl_0-basis of $\mathrm{sscl}(Ba_0)$ over B. Note that a is a finite tuple: working over (the *self-sufficient* set) B, the set $\mathrm{sscl}(Ba_0)$ is the self-sufficient closure of a finite set and is thus finite dimensional (over B). Suppose $(a^i)_{i\geq 1}$ is a sequence of tuples such that (1) $a^1 = a$ and for all $i, j \geq 1$, $ja^{ij} = a^i$, and such that (2) if a coordinate a_j of a is in G, then for all $i \geq 1$, a^i_j is also in G. Then the isomorphism type of $\mathrm{sscl}(Ba_0)$ over B is determined by the quantifier-free type of the sequence $(a^i)_{i\geq 1}$ over B.

Thus, there are at most as many possibilities for the type of the element a_0 over B as possible quantifier-free types over B of sequences $(a^i)_{i\geq 1}$ with the above properties. Let us now, at first, see that there are at most $\lambda \cdot 2^{\aleph_0}$ possibilities for the quantifier-free type of such sequence $(a_i)_{i\geq 1}$ over B. Indeed, if we fix the algebraic type of a over B then then there are only finitely many possibilities for the algebraic type of each a^i over B. Therefore there are at most $\lambda \cdot 2^{\aleph_0}$ possibilities for the algebraic type of the sequence $(a^i)_{i\geq 1}$ over B. Also, there are only finitely many possibilities for the colouring of the tuple a, which determines the colouring of the sequence $(a^i)_{i\geq 1}$.

Thus, there are no more than $\lambda \cdot 2^{\aleph_0}$ possibilities for the quantifier-free type of such a sequence $(a^i)_{i\geq 1}$. This shows that the theory T is λ-stable for all $\lambda \geq 2^{\aleph_0}$ and hence superstable.

Let us now improve on the above to see that, in fact, the theory T is ω-stable. For this we use the Thumbtack Lemma (3.5.2). Indeed, by the Thumbtack Lemma, we may assume without loss of generality that a is Kummer generic over B. Then the algebraic type of a over B determines the algebraic type of the whole sequence $(a^i)_{i\geq 1}$. Therefore there are at most λ possible quantifier-free types over B of sequences $(a^i)_{i\geq 1}$ as above. This shows that T is ω-stable. □

4.3.2 Near model completeness

We shall now show that the theory T is near model complete. We do this by showing that the sufficient condition found in Prop 2.3.11 holds in our case. Here we follow Lemma 10.3 in [2].

Let L^* be the expansion of the language L by a predicate for each existentially definable set in L. As with any extension of the language by definable predicates, T extends canonically to a complete L^*-theory T^*.

Lemma 4.3.4. *Let $V(x, y)$ be a subvariety of $\mathbb{A}^{n+k}$ defined over k_0. There exists a quantifier-free $L_{\mathbb{A}}$-formula $\phi(x, y)$ such that for every $b \in \bar{A}^k$ such that V_b is a prealgebraic minimal rotund subvariety of $\bar{A}^n$ the following holds:*

- *every generic point a of V_b over b satisfies $\phi(x, b)$, and*
- *for every $a \models \phi(x, b)$ and every self-sufficient cl_0-closed set B containing b, either $a \subset B$ or a is a generic point of V_b over B.*

Proof. This is Lemma 4.4 in [2]. Since the lemma deals only with the definability of relative predimension, one sees that it holds without changes in our greater generality. □

Lemma 4.3.5. *For all $a, b \subset \bar{\mathcal{A}}$ with $\mathrm{span}(b) \leq \bar{\mathcal{A}}$, there exists an existential L-formula $\tau^\delta_{a,b}(x, y)$ such that*

- *$\mathcal{A} \models \tau^\delta_{a,b}(a, b)$, and*
- *for all $a', b' \subset \bar{\mathcal{A}}$, if $\mathcal{A} \models \tau^\delta_{a,b}(a', b')$ then $\delta(a'/b') \leq \delta(a/b)$.*

Proof. Let $B := \text{span}(b)$ and $A := \text{span}(ab)$. Then $B \leq \mathcal{A}$ is a finite dimensional strong extension.

We may assume that the extension $B \leq \mathcal{A}$ is minimal. Indeed, in the general case, the extension can be decomposed into a tower of minimal strong extensions and the conjunction of the formulas obtained for each of the minimal extensions in the tower is equivalent to an existential formula with the required properties.

We deal separately with the different cases from Lemma 4.2.20. Cases 2,3,4 are easy: if the extension $B \leq \mathcal{A}$ is a green generic extension (case 2), then we can take the formula $\tau^{\delta}_{a,b}(x, y)$ to be $G(x)$; if the extension is algebraic (case 3) then we can take $\tau^{\delta}_{a,b}(x, y)$ to be any formula witnessing the algebraicity of a over b; if the extension is white generic (case 4), then $\tau^{\delta}_{a,b}(x, y)$ can be the formula $x = x$.

Let us thus assume that $B \leq \mathcal{A}$ is a prealgebraic minimal extension (case 1). Let n be the linear dimension of A over B. Let $c \subset G^n$ be a green linear basis of A over B. Let $V(z, d)$ be the locus of c over $\mathbb{Q}(b)^{\text{alg}}$ and let $\tau(w, b)$ be a formula isolating the type of d over b. Consider the conjunction $\tilde{\tau}(x, y, z)$ of the following formulas:

- $\psi(x, z)$, where $\psi(x, z)$ is such that $\mathcal{A} \models \psi(a, c)$ and for all a', c', if $\mathcal{A} \models \psi(a', c')$ then $\text{span}(a') = \text{span}(c')$;
- $\bigwedge_i G(z_i)$;
- $\exists w(\tau(w, y) \wedge \phi(z, w) \wedge \theta'(w)$, where $\theta'(w)$ expresses that V_w is prealgebraic minimal rotund (see Lemma 4.2.22) and $\phi(z, w)$ is as provided by Lemma 4.3.4 for $V(z, w)$.

Note that $\tilde{\tau}(x, y, z)$ is equivalent to an existential formula. Thus, there is also an existential formula equivalent to $\exists z \tilde{\tau}(x, y, z)$; which works as $\tau^{\delta}_{a,b}(x, y)$ as we shall now see. Indeed, one can simply note that $\mathcal{A} \models \tilde{\tau}(a, b, c)$, and, if $\mathcal{A} \models \tilde{\tau}(a', b', c')$ then

$$\delta(a'/b') = \delta(c'/b') \leq 0,$$

(the equality holds because, since $\mathcal{A} \models \psi(a', c')$, we have $\text{span}(a') = \text{span}(c')$; the inequality follows directly from the definitions of τ, ϕ, θ, etc). □

Theorem 4.3.6. *The theory T is near model complete.*

Proof. Immediate from Lemma 4.3.5 and Proposition 2.3.11. □

4.3.3 Forking and ranks

We now calculate U-ranks and Morley ranks of 1-types in the theory T. Below the symbol $\mathop{\smile\!\!\!\!\!\!|}$ denotes non-forking independence for the theory T.

Definition 4.3.7. Let A, B, C be cl_0-closed subsets of $\bar{A}$ with $B \subset A, C$.

We say that *A and C are in free amalgam over B* if A and C are $\mathrm{acl}_{\mathbb{A}}$-independent over B and $G \cap (A + C) = (G \cap A) + (G \cap C)$.

We say that *A and C are in self-sufficient free amalgam over B* if A and C are in free amalgam over B and $A + C$ is self-sufficient in $\bar{A}$.

Proposition 4.3.8. *Let $a \subset \bar{A}$ and B, C be self-sufficient cl_0-closed subsets of $\bar{A}$ with $B \subset C$. Then, $a \mathop{\smile\!\!\!\!\!\!|}_B C$ if and only if $A := \mathrm{sscl}(aB)$ and C are in self-sufficient free amalgam over B.*

Proof. Let a, B, C be as in the statement of the lemma. Let $A := \mathrm{sscl}(aB)$. Throughout the following, we use Lemma 4.3.2 without explicit mention.

Let us first remark that if A and C are in self-sufficient free amalgam over B, then this determines $\mathrm{tp}(a/C)$ uniquely among the extensions of $\mathrm{tp}(a/B)$ over C. To see this, assume A and C are in self-sufficient free amalgam over B. Let $a' \subset \bar{A}$ be such that $\mathrm{tp}(a'/B) = \mathrm{tp}(a/B)$ and $A' := \mathrm{tp}(a'B)$ and C are in self-sufficient free amalgam over B. Since $\mathrm{tp}(a'/B) = \mathrm{tp}(a/B)$, the map $a \mapsto a'$ extends to an isomorphism from A to A' over B. Since A and C are in free amalgam over B and so are A' and C, then the above isomorphism can be extended to an isomorphism from $A + C$ to $A' + C$ over C. By the self-sufficiency of $A + C$ and $A' + C$, this shows that $\mathrm{tp}(a/C) = \mathrm{tp}(a'/C)$.

In order to prove the proposition, it is sufficient to show the following:

Claim: Suppose A and C are in self-sufficient free amalgam over B. Then $\mathrm{tp}(a/C)$ *does not split* over B.

Indeed, by the stability of T, it follows from the claim that if a and C are in free amalgam over B, then $\mathrm{tp}(a/C)$ is the *unique* non-spliting, and thus non-forking, extension of $\mathrm{tp}(a/B)$ over C. [3].

Proof of claim: Suppose A and C are in self-sufficient free amalgam over B. Let $p = \mathrm{tp}(a/C)$. Let σ be an automorphism of (the induced structure on) C over B. We want to see that $\sigma(p) = p$.

[3]For the uniqueness see Corollary 12.6 in [32]. There, *son* means extension, *special* means non-splitting, *heir* means non-forking extension

Since C is self-sufficient, σ is an elementary map. Therefore we can find an automorphism $\bar{\sigma}$ of $\bar{A}$ extending σ. Since $\sigma(p) = \mathrm{tp}(\bar{\sigma}(a)/C)$, what we want to see is that $\mathrm{tp}(\bar{\sigma}(a)/C) = \mathrm{tp}(a/C)$.

Let $A' := \bar{\sigma}(A)$. It is easy to see that $A' = \mathrm{sscl}(\bar{\sigma}(a)B)$. Note that the elementary maps $\bar{\sigma}|_A$ and id_C coincide on B. Since A and C are in free amalgam, the union of $\bar{\sigma}|_A$ and id_C extends to an isomorphism from $A+C$ to $A'+C$. By assumption, $A+C$ is self-sufficient, and so is $A'+C$, for it is the image of $A+C$ under the automorphism $\bar{\sigma}$ of $\bar{A}$. We have thus found an elementary map over C sending a to $\bar{\sigma}(a)$. Hence $\mathrm{tp}(\bar{\sigma}(a)/C) = \mathrm{tp}(a/C)$, as required. □

Remark 4.3.9. The above proof shows that types over acl-closed sets (in the (real) base sort) are stationary.

Before calculating Morley ranks for the theory T, we calculate the U-rank of 1-types over finitely generated sets of parameters. Since T is superstable, for every type there is a finite set over which it does not fork; it follows that the U-rank of any type equals that of a restriction to a finitely generated set of parameters.

Lemma 4.3.10. *Let $a \in \bar{A}$ and let $B \subset \bar{A}$ be a self-sufficient finitely generated cl_0-closed set.*

1. *If* $\mathrm{d}(a/B) = 0$*, then* $\mathrm{RU}(a/B) < \omega$*.*
2. *If* $\mathrm{d}(a/B) = 1$*, then* $\mathrm{RU}(a/B) = \omega + m$ *for some* $m \in \omega$*.*
3. *If* $\mathrm{d}(a/B) = 2$*, then* $\mathrm{RU}(a/B) = \omega \cdot 2$*.*

Proof. Let $A := \mathrm{sscl}(Ba)$. Note that $\delta(A/B) = \mathrm{d}(a/B)$.

Proof of 1. Suppose $\mathrm{d}(a/B) = 0$.

If $a \in \mathrm{acl}(B)$, then $\mathrm{RU}(a/B) = 0$. Thus, assume $a \notin \mathrm{acl}(B)$.

Let us assume first that the extension $B \leq A$ is minimal. We shall see that in this case $\mathrm{RU}(a/B) = 1$.

Let C be a finitely generated self-sufficient cl_0-closed set containing B. We want to see that if $\mathrm{tp}(a/C)$ forks over B, then $\mathrm{tp}(a/C)$ is algebraic.

Suppose $\mathrm{tp}(a/C)$ is not algebraic, i.e. $a \notin \mathrm{acl}(C)$. Then, by the minimality assumption, $A \cap \mathrm{acl}(C) = B$.

It follows that A and C are $\mathrm{acl}_{\mathbb{A}}$-independent over B: To see this, let $D := \mathrm{acl}_{\mathbb{A}}(A) \cap \mathrm{acl}_{\mathbb{A}}(C)$. We want to show $D = \mathrm{acl}_{\mathbb{A}}(B)$. Note that $\mathrm{acl}(B) \subset \mathrm{acl}(D) \subset$

$\mathrm{acl}(A) = \mathrm{acl}(Ba)$, therefore either $\mathrm{acl}(D) = \mathrm{acl}(B) = \mathrm{acl}_{\mathbb{A}}(B)$ or $\mathrm{acl}(D) = \mathrm{acl}(Ba) = \mathrm{acl}(A)$. But the latter is impossible, because then $A \subset \mathrm{acl}(D)$ and hence

$$A = A \cap \mathrm{acl}(D) \subset A \cap \mathrm{acl}(\mathrm{acl}(C)) = A \cap \mathrm{acl}(C) = B,$$

a contradiction. Therefore the former holds and hence $D \subset \mathrm{acl}_{\mathbb{A}}(B)$. Since the inclusion from right to left is obvious, we get $D = \mathrm{acl}_{\mathbb{A}}(B)$.

Moreover, A and C are in free amalgam over B: Indeed, using the modularity of cl_0 and the $\mathrm{acl}_{\mathbb{A}}$-independence of A and C over B, one sees that:

$$\delta(A/C) = \delta(A/B) - \mathrm{lin.\,d.}((A + C) \cap G/A \cap G + C \cap G).$$

Since $\delta(A/B) = 0$ and $\delta(A/C) \geq 0$ (because C is self-sufficient), we get that $\mathrm{lin.\,d.}((A + C) \cap G/A \cap G + C \cap G) = 0$, which means that A and C are in free amalgam over B.

Also, we see that $\delta(A + C/C) = 0$, hence $A + C$ is self-sufficient.

Therefore, A and C are in self-sufficient free amalgam over B, which means that $\mathrm{tp}(a/C)$ does not fork over B.

We have thus seen that $\mathrm{RU}(a/B) = 1$.

Without the minimality assumption, the extension $B \leq A$ decomposes into a tower of prealgebraic minimal extensions $B = B_0 \leq B_1 \leq \ldots \leq B_n = A$. Using the additivity of finite U-ranks, one sees that $RU(a/B) = n$ (for each i, take an element b_i in $B_i \setminus B_{i-1}$; note that B_i is contained in $\mathrm{acl}(B_{i-1}b_i)$; then $RU(a/B) = \sum_i RU(b_i/B_{i-1}) = n$.)

Proof of 2. Suppose $\mathrm{d}(a/B) = 1$.

Assume the extension $B \leq A$ is minimal. We want to see that then $\mathrm{RU}(a/B) = \omega$.

Let C be a finitely generated self-sufficient cl_0-closed set containing B. We claim that if $\mathrm{tp}(a/C)$ forks over B, then $\mathrm{d}(a/C) = 0$.

Suppose $\mathrm{d}(a/C) \neq 0$. Then $\mathrm{d}(a/C) = 1$, for $\mathrm{d}(a/C) \leq \mathrm{d}(a/B) = 1$. Clearly, $a \notin \mathrm{acl}(C)$. As in the proof of part 1., it follows that A and C are $\mathrm{acl}_{\mathbb{A}}$-independent over B. Also, since

$$\delta(A/C) = \delta(A/B) - \mathrm{lin.\,d.}((A + C) \cap G/A \cap G + C \cap G),$$

we have:

$$\begin{aligned}\text{lin. d.}((A+C)\cap G/A\cap G+C\cap G) &= \delta(A/B)-\delta(A/C)\\ &= \text{d}(a/B)-\delta(a/C)\\ &\leq \text{d}(a/B)-\text{d}(a/C)\\ &= 1-1\\ &= 0.\end{aligned}$$

Therefore A and C are in free amalgam over B. From this we also get,

$$\delta(A+C/C)=\delta(A/B)=1=\text{d}(a/C),$$

which implies that $A+C$ is self-sufficient. Thus, A and C are in self-sufficient free amalgam over B, which means that $\text{tp}(a/C)$ does not fork over B

Applying the result of the previous part, we obtain that all forking extensions of $\text{tp}(a/B)$ have finite U-rank. Therefore $\text{RU}(a/B)\leq\omega$.

By considering towers of minimal prealgebraic extensions, one sees that there are elements in G of arbitrarily large U-rank over B and, in fact, $\text{RU}(a/B)=\omega$.

Without the minimality assumption, the extension $B\leq A$ decomposes into a tower of minimal extensions, of which one is a minimal green generic extension and all the other are prealgebraic. From the previous arguments and Lascar's inequalities, we conclude that $\text{RU}(a/B)=\omega+m$ for some natural number m.

Proof of 3. Suppose $\text{d}(a/B)=2$.

Note that in this case $A=B+\text{span}(a)$ and the extension $B\leq A$ is therefore minimal.

Let C be a finitely generated self-sufficient cl_0-closed set containing B. Let us see that if $\text{tp}(a/C)$ forks over B, then $\text{d}(a/C)\leq 1$.

Suppose $\text{d}(a/C)>1$, i.e. $\text{d}(a/C)=2$. Clearly, $a\notin\text{acl}(C)$. As before, it follows that A and C are $T_{\mathbb{A}}$-independent over B.

As in the previous parts, we see that

$$\text{lin. d.}((A+C)\cap G/A\cap G+C\cap G)\leq\text{d}(a/B)-\text{d}(a/C)=2-2=0.$$

Therefore A and C are in free amalgam over B. Then also $\delta(A+C/C)=\delta(A/B)=2=\text{d}(a/C)$, and hence A and C are in self-sufficient free amalgam over B.

Thus, all forking extensions of $\text{tp}(a/B)$ have U-rank smaller than $\omega\cdot 2$. Therefore $\text{RU}(a/B)\leq\omega\cdot 2$.

Looking at towers of minimal extensions, this time adding a green generic extension at the top of the towers of prealgebraic extensions, one sees that, in fact, $\mathrm{RU}(a/B) = \omega \cdot 2$. □

Lemma 4.3.11. *For T, Morley rank and U-rank coincide on all 1-types.*

Proof. It is sufficient to prove this for global types, i.e. types over $\bar{A}$.

It is well-known and easy to show that for any totally transcendental complete theory, for any (global) type p, $\mathrm{RU}(p) \leq \mathrm{RM}(p)$. In order to show that also $\mathrm{RM}(p) \leq \mathrm{RU}(p)$ for all p, it is sufficient to prove the following: for every $p \in S_1(\bar{A})$, there is a formula $\psi \in p$ such that for all $q \in S_1(\bar{A})$, if $\psi \in q$ and $\mathrm{RU}(q) \geq \mathrm{RU}(p)$ then $q = p$ (i.e. ψ isolates p among the types in $S_1(\bar{A})$ with U-rank $\geq \mathrm{RU}(p)$). [4]

Let $p \in S_1(\bar{A})$. We shall find $\psi \in p$ that isolates p among the global types of greater or equal U-rank. Let $B \subset \bar{A}$ be the self-sufficient closure of a finite set over which p does not fork. Let $a \in \bar{A}$ be a realisation of $p|_{\mathrm{acl}(B)}$. Let $A := \mathrm{sscl}(Ba)$. Note that it is sufficient to find a formula $\psi(x)$ over $\mathrm{acl}(B)$ that isolates $p|_{\mathrm{acl}(B)}$ among the types over $\mathrm{acl}(B)$ of greater or equal U-rank. This is what we do in each of the following cases.

Case 1: $\mathrm{RU}(a/B)$ *is finite, i.e.* $\mathrm{d}(a/B) = 0$.

Assume $a \notin \mathrm{acl}(B)$, otherwise the type of a over $\mathrm{acl}(B)$ is obviously isolated.

Let us assume first that the extension $B \leq A$ is minimal. Since $\mathrm{d}(a/b) = 0$, we can find a cl_0-basis a' of A over B with all coordinates in G. Then a is algebraic over a' in the language $L_{\mathbb{A}}$ and $\delta(a'/B) = 0$. Let $V(y,d)$ be the algebraic locus of a' over $\mathrm{acl}(B)$. The variety $V(y,d)$ is a minimal prealgebraic rotund variety and therefore we can find a formula $\phi(y,d)$ for $V(y,d)$ as in Lemma 4.3.4. We claim that the formula $\phi(y,d) \wedge \bigwedge_i G(y_i)$ isolates the type of a' over $\mathrm{acl}(B)$ among the types of greater or equal U-rank. Indeed, by the choice of $\phi(y,d)$ and the self-sufficiency of $\mathrm{acl}(B)$, the following holds: for every a'' satisfying the formula $\phi(y,d)$, either $a'' \subset \mathrm{acl}(B)$ or a'' is a generic of $V(y,d)$ over $\mathrm{acl}(B)$. By the Thumbtack Lemma, we may assume that our a' is Kummer generic over $\mathrm{acl}(B)$, which implies that every green generic of $V(y,d)$ over $\mathrm{acl}(B)$ has the same type as a' over $\mathrm{acl}(B)$. Thus, we

[4] Explanation for the sufficiency: Assume that for every $p \in S_1(\bar{A})$, there is $\psi \in p$ such that ψ isolates p among the types in $S_1(\bar{A})$ with U-rank $\geq \mathrm{RU}(p)$. One can then show, by induction, that for every ordinal α, if $\mathrm{RM}(p) \geq \alpha$ then $\mathrm{RU}(\alpha) \geq \alpha$: The cases where $\alpha = 0$ or α is a limit ordinal are easy and do not need the extra assumption. Suppose now that the implication holds for α and $\mathrm{RM}(p) \geq \alpha + 1$. Since Morley rank coincides with the Cantor rank on $S_1(\bar{A})$ (Proposition 17.17 in [32]), p is an accumulation point of types $p_i \in S_1(\bar{A})$ with $\mathrm{RM}(p_i) \geq \alpha$. By the induction hypothesis, $\mathrm{RU}(p_i) \geq \alpha$ and $\mathrm{RU}(p) \geq \alpha$. If $\mathrm{RU}(p) = \alpha$, the assumption yields that there is a formula that isolates p from the p_i, hence a contradiction. Thus, $\mathrm{RU}(p) \geq \alpha + 1$.

have: for every a'' satisfying $\phi(y,d)$, either $a'' \subset \mathrm{acl}(B)$ or a'' has the same type a' over $\mathrm{acl}(B)$. This directly implies that $\phi(y,d) \wedge \bigwedge_i G(y_i)$ isolates the type of a' over $\mathrm{acl}(B)$ among the types of greater or equal U-rank, as we claimed. Since a is algebraic over $\mathrm{acl}(B) \cup a'$, we can find an formula $\theta(x,y)$ over $\mathrm{acl}(B)$ such that $\theta(x,a')$ isolates the type of a over $a' \cup \mathrm{acl}(B)$. Then, using the additivity of finite U-ranks, the type of a over $\mathrm{acl}(B)$ is isolated among the types of greater of equal U-rank by the formula $\psi(x) := \exists y(\theta(x,y) \wedge \phi(y,d))$.

Without the minimality assumption, the extension $B \leq A$ decomposes into a tower of minimal prealgebraic extensions $B = A_0 \leq \ldots \leq A_n = A$ with $a \in A_n \setminus A_{n-1}$. For each $i = 1,\ldots,n$, let a^i be a green basis of A_i over A_{i-1}. Let $\theta(x,y^1,\ldots,y^n)$ be a formula over $\mathrm{acl}(B)$ such that $\theta(x,a^1,\ldots,a^n)$ isolates the type of a over $\mathrm{acl}(B) \cup a^1 \cup \cdots \cup a^n$, which is algebraic. Also, for each $i = 1,\ldots,n$, let $\phi(y^i,d^{i-1})$ be a formula with $d^{i-1} \in \mathrm{acl}(A_{n-1})$ that isolates the type of y^i over $\mathrm{acl}(A_{i-1})$ among the types of greater or equal U-rank, which we have seen above is possible to find. Finally, for each $i = 1,\ldots,n-1$, let $\theta_i(z^i,y^1,\ldots,y^{n-1})$ be a formula over $\mathrm{acl}(B)$ such that $\theta_i(z^i,a^1,\ldots,a^{n-1})$ isolates the type of d^i over $\mathrm{acl}(B) \cup a^1,\ldots,a^{n-1}$, which is algebraic. Then take $\psi(x)$ to be the following formula over $\mathrm{acl}(B)$:

$$\exists y^1 \ldots \exists y^n \Big(\theta(x,y^1,\ldots,y^n) \wedge \\ \exists z^1 \ldots \exists z^{n-1} (\bigwedge_{i=1}^{n} \phi(y^i,d^{i-1}) \wedge \bigwedge_{i=1}^{n-1} \theta_i(z^i,y^1,\ldots,y^{n-1}))\Big). \quad (4.4)$$

Again, using the additivity of finite U-ranks, one sees that the formula $\psi(x)$ isolates the type of a over $\mathrm{acl}(B)$ among the types of greater or equal U-rank.

Case 2: $\mathrm{RU}(a/B) = \omega + m$ *for some* $m \in \omega$, *i.e.* $\mathrm{d}(a/B) = 1$.

Note that if $a \in G$, then $\delta(a/B) = 1 = \mathrm{d}(a/B)$, and hence $B + \mathrm{span}(a)$ is self-sufficient and $\mathrm{RU}(a/B) = \omega$. Therefore, if $a \in G$, then $\psi(x)$ can be taken to be $G(x)$.

More generally, whenever the extension $B \leq A$ is minimal, $B + \mathrm{span}(a)$ is self-sufficient and (although a need not be in G) there is a' in G with $\mathrm{RU}(a'/B) = \omega$ such that a' is a multiple of a. In particular, a is algebraic over a' and, a fortiori, over $\mathrm{acl}(B) \cup a'$. Let $\theta(x,y)$ be a formula over $\mathrm{acl}(B)$ such that $\theta(x,a')$ isolates the type of a over $\mathrm{acl}(B) \cup a'$. Then, the formula $\psi(x)$ can be taken to be $\exists y(\theta(x,y) \wedge G(y))$.

For arbitrary a with $\mathrm{d}(a/B) = 1$, the extension $B \leq A$ decomposes into a tower of minimal extensions $B = A_0 \leq \ldots \leq A_n = B$, where one is a green generic minimal extension and all others are prealgebraic. By the arguments up to this point, each of these minimal extensions can be dealt with and it is easy to see that the different formulas can be combined as in Case 1 to obtain an appropriate $\psi(x)$.

Case 3: $\mathrm{RU}(a/B) = \omega \cdot 2$, *i.e.* $\mathrm{d}(a/B) = 2$

Since there is only one global type of U-rank $\omega \cdot 2$, the formula $\psi(x)$ can be taken to be $x = x$.

□

Theorem 4.3.12. *For T, the Morley rank of the universe is $\omega \cdot 2$ and the Morley rank of G is ω.*

Proof. From the definitions we know that the Morley rank of a definable set is the maximum of the Morley ranks of the types containing a defining formula for the set. Also, Morley rank and U-rank coincide on all 1-types, by Lemma 4.3.11, and their values are as in Lemma 4.3.10. Thus, we have:

$$\mathrm{RM}(x = x) = \max\{\mathrm{RM}(p) \ : \ p \in S_1(\bar{A}),\ x = x \in p\} = \omega \cdot 2,$$
$$\mathrm{RM}(G(x)) = \max\{\mathrm{RM}(p) \ : \ p \in S_1(\bar{A}),\ G(x) \in p\} = \omega.$$

□

Chapter 5

Related structures on the complex numbers

Here starts the second part of the thesis, in which we are interested in explicitly finding models for the theories constructed in Chapter 4 on the complex points of the algebraic group $\mathbb{A}$. This is done, under additional assumptions, in Chapter 6, in the case where $\mathbb{A}$ is the multiplicative group, and in Chapter 7, in the case where $\mathbb{A}$ is an elliptic curve without complex multiplication defined over the reals.

This chapter contains preliminaries for the subsequent ones. We consider first-order structures encompassing the complex exponential function and the analogous exponential functions of elliptic curves, as well as the associated structures of raising to powers. In each case, we define a corresponding submodular predimension function and show how the properness of the predimension function, in the sense of Chapter 2, relates to a version of the Schanuel conjecture or the Elliptic Schanuel Conjecture. We also see that in all of these instances the associated pregeometry has the *countable closure property*. Towards the end of the chapter we also include some basic results from the theory of o-minimality and facts on the o-minimality of structures on the real numbers related to the exponential functions.

5.1 Exponentiation and raising to powers

5.1.1 Exponentiation

Let $\mathbb{C}_{\exp} = (\mathbb{C}, +, \cdot, \exp)$ be the expansion of the complex field by the exponential function.

The *Schanuel Conjecture* from transcendental number theory, which we state below, can be regarded as a statement about $\mathbb{C}_{\exp}$.

Conjecture 5.1.1 (The Schanuel Conjecture (SC)). *For every* n *and every* $\mathbb{Q}$*-linearly independent tuple* $x \in \mathbb{C}^n$*,*

$$\mathrm{tr.\,d.}(x \exp x) \geq n.$$

The predimension function $\delta_{\exp}$ is defined on any tuple $x \subset \mathbb{C}$ by

$$\delta_{\exp}(x) := \mathrm{tr.\,d.}(x \exp x) - \mathrm{lin.\,d.}(x).$$

Note that $\delta_{\exp}$ is submodular with respect to the modular pregeometry given by the $\mathbb{Q}$-linear span. Moreover, the Schanuel conjecture is equivalent to the statement that for every $x \subset \mathbb{C}$, $\delta_{\exp}(x) \geq 0$. Therefore, if the SC holds, then $\delta_{\exp}$ is a proper predimension function on $\mathbb{C}_{\exp}$.

In [51], a model-theoretic study of the structure $\mathbb{C}_{exp}$ is carried out by means of a predimension construction with respect to the function $\delta_{\exp}$, as in Chapter 2, and a subsequent (non-elementary) axiomatization is found in a way similar to our Chapter 4. Here we shall only need one aspect of that work, namely Zilber's proof that the pregeometry associated to $\delta_{\exp}$ has the *countable closure property* ([51, Lemma 5.12]). Versions of this fact will be essential in our arguments in chapters 6 and 7. We include the proof, in slightly greater detail than in [51].

Let us assume the Schanuel Conjecture for the rest of this subsection. We thus have a dimension function $\mathrm{d}_{\exp}$ on $\mathbb{C}_{\exp}$ defined from $\delta_{\exp}$ as in Definition 2.2.22 and a corresponding pregeometry $\mathrm{cl}_{\exp}$, as in Remark 2.2.23.

Consider the following definitions:

Definition 5.1.2. A pregeometry cl on a set A is said to have the *Countable Closure Property (CCP)* if for every finite subset X of A, the set $\mathrm{cl}(X)$ is countable.

Equivalently, cl has the CCP if for every countable subset X of A, the set $\mathrm{cl}(X)$ is countable.

Also, let us note the trivial fact that if a pregeometry cl has the CCP, then every localisation cl_D of cl over a countable set D also has the CCP.

Definition 5.1.3. Let $n \geq 1$. A subvariety W of $\mathbb{C}^n \times (\mathbb{C}^*)^n$ is said to be ex-*rotund* if for every $k \times n$-matrix M with entries in $\mathbb{Z}$ of rank k, $\dim W' \geq k$, where W' is the image of W under the map from $\mathbb{C}^n \times (\mathbb{C}^*)^n$ to $\mathbb{C}^k \times (\mathbb{C}^*)^k$ given by $(x, y) \mapsto (M \cdot x, y^M)$

Definition 5.1.4. Let $W \subset \mathbb{C}^n \times (\mathbb{C}^*)^n$ be an ex-rotund variety and let $B \subset \mathbb{C}$ be such that W is defined over $B \cup \exp(B)$. Let us say that $a \in \mathbb{C}^n$ is a *generic realisation* of W over B, if $(a, \exp(a))$ is a generic point of W over $B \cup \exp(B)$.

We can now give the statement and proof of Lemma 5.12 from [51].

Lemma 5.1.5. *Assume SC holds. Then the pregeometry* $\mathrm{cl}_{\exp}$ *has the CCP.*

Proof. Let B be a finite subset of $\mathbb{C}$. We shall prove that $\mathrm{cl}_{\exp}(B)$ is countable. By passing to its self-sufficient closure, we may assume that B is self-sufficient with respect to $\delta_{\exp}$. Note that for any element $x_0 \in \mathbb{C}$, x_0 is in $\mathrm{cl}_{\exp}(B)$ if and only if $x_0 \in \mathrm{span}_{\mathbb{Q}}(B)$ or there exists $x \supset x_0$, $\mathbb{Q}$-linearly independent over B, such that $\delta_{\exp}(x/B) = 0$.

It is clear that $\mathrm{span}_{\mathbb{Q}}(B)$ is countable, it therefore suffices to show that the set

$$\{x \subset \mathbb{C} : x \text{is } \mathbb{Q}\text{-linearly independent over } B \text{ and } \delta_{\exp}(x/B) = 0\}$$

is also countable.

Suppose x is $\mathbb{Q}$-linearly independent over B and let W be the algebraic locus of $(x, \exp(x))$ over $B \cup \exp(B)$. Since B is self-sufficient, the variety W is ex-rotund and, clearly, x is a generic realisation of W over B. Also, note that $\delta_{\exp}(x/B) = 0$ if and only if $\dim W = n$. Thus, it is sufficient to prove that for every n and for every ex-rotund variety $W \subset \mathbb{C}^{2n}$ defined over $B \cup \exp(B)$ of dimension n, the set of generic realisations of W over B is countable (clearly, there are only countably many such varieties W.) This is done below.

Let $W \subset \mathbb{C}^{2n}$ be an ex-rotund variety defined over $B \cup \exp(B)$ of dimension n.

The proof of the following claim completes the proof of the lemma.

Claim: Consider the (analytic) set

$$\mathcal{S} = \{x \in \mathbb{C}^n : (x, \exp x) \in W\}.$$

There is an analytic set $\mathcal{S}_0$ of dimension zero contained in $\mathcal{S}$ such that every generic realisation of W over B either is in $\mathcal{S}_0$ or is an isolated point of $\mathcal{S}$.[1]

Indeed, the claim implies that the set of generic realisations of W over B is countable: Since $\mathcal{S}_0$ is an analytic set of dimension zero, it consists of isolated points, it is therefore discrete and hence countable (for every discrete subset of Euclidean space is countable). Also, the set of isolated points of $\mathcal{S}$ is clearly discrete and hence countable.

Proof of Claim: Being analytic, the set $\mathcal{S}$ can be written as a union $\bigcup_{0 \le i \le d} \mathcal{S}_i$ where, for each i, the set $\mathcal{S}_i$ is a complex manifold of dimension d (possibly empty)

[1]An *analytic subset* of a domain U in $\mathbb{C}^n$ is a set that locally, around every point in U, is defined as the zero set of some complex analytic functions. We call analytic subsets of $\mathbb{C}^n$ simply *analytic sets.* For precise definitions see [10, Section 2.1])

and the union $\bigcup_{0\le j\le i}\mathcal{S}_j$ is an analytic set ([10, Section 5.5]). In particular, the set $\mathcal{S}_0$ is an analytic set of dimension 0.

Let us now show that any generic realisation of W over B in $\mathcal{S}\setminus\mathcal{S}_0$ is an isolated point of $\mathcal{S}$.

Suppose not. Then there exists a generic realisation a of W over B in $\mathcal{S}\setminus\mathcal{S}_0$ that is not an isolated point of the analytic set $\mathcal{S}$.

Since a is in some $\mathcal{S}_i$ with $i>0$, there exists an analytic isomorphism $x : t\mapsto x(t)$ from an open disc D around 0 in $\mathbb{C}$ onto a subset of $\mathcal{S}$ mapping 0 to a.

Set $y(t) := \exp(x(t))$. Then for every $t\in D$, $(x(t), y(t))$ is in W.

We can consider (the germ of) each coordinate function of x and y as an element of the differential ring $\mathcal{R}$ of germs near 0 of functions which are analytic on a neighbourhood of 0. [2] Note that the ring of constants of $\mathcal{R}$ is (isomorphic to) $\mathbb{C}$. Using the fact that the zero set of an analytic function in one variable consists of isolated points, it is easy to see that $\mathcal{R}$ is an integral domain. Thus, $\mathcal{R}$ embeds into its field of fractions, $\mathcal{F}$. The derivation on $\mathcal{R}$ extends to a derivation on $\mathcal{F}$ (by the usual differentiation rule) with field of constants $\mathcal{C}\supset\mathbb{C}$.

Since $(x,y)\in W(\mathcal{F})$, we get that

$$\operatorname{tr.d.}(x, y/B\cup\exp(B))\le n.$$

In fact, since $x(0)=a$ is a generic realisation over B, $\operatorname{tr.d.}(x(0), y(0)/B\cup\exp(B)) = n$, and hence

$$\operatorname{tr.d.}(x, y/B\cup\exp(B)) = n.$$

Let $k\in\{0,\dots,n\}$ be the number of independent $\mathbb{Q}$-linear dependences among $Dx_1,\dots,Dx_n$. After a $\mathbb{Q}$-linear change of coordinates we can assume that $Dx_1,\dots,Dx_k$ are all identically zero and $Dx_{k+1},\dots,Dx_n$ are $\mathbb{Q}$-linearly independent. Thus, $x_1,\dots,x_k$ are all constant, with values $a_1,\dots,a_k$ respectively. Since W is ex-rotund, we have

$$\operatorname{tr.d.}(a_1,\dots,a_k,\exp(a_1),\dots,\exp(a_k)/B\cup\exp(B))\ge k.$$

[2]The equivalence relation defining the germs being given by $f\sim g$, if f and g coincide on a punctured neighbourhood of 0.

Hence

$$
\begin{aligned}
&\operatorname{tr.d.}(x_{k+1},\ldots,x_n,y_{k+1},\ldots,y_n/\mathcal{C})\\
\le&\operatorname{tr.d.}(x_{k+1},\ldots,x_n,y_{k+1},\ldots,y_n/\mathbb{C})\\
\le&\operatorname{tr.d.}(x_{k+1},\ldots,x_n,y_{k+1},\ldots,y_n/B\cup\{a_1,\ldots,a_k\}\cup\exp(B\cup\{a_1,\ldots,a_k\}))\\
=&\operatorname{tr.d.}(x_{k+1},\ldots,x_n,y_1,\ldots,y_n/B\cup\exp(B))\\
&-\operatorname{tr.d.}(x_1,\ldots,x_k,y_1,\ldots,y_k/B\cup\exp(B))\\
\le& n-k.
\end{aligned}
$$

Ax's Theorem then implies that $Dx_{k+1},\ldots,Dx_n$ must be $\mathbb{Q}$-linearly dependent. This gives a contradiction. □

Remark 5.1.6. The only use of the Schanuel Conjecture in Lemma 5.1.5 is in the assertion that $\mathrm{cl}_{\exp}$ is a pregeometry. By results of Kirby (Theorem 1.1 and Theorem 1.2 in [20]), without assuming the Schanuel conjecture, there is a countable self-sufficient subset D of $\mathbb{C}$ with respect to the predimension function $\delta_{\exp}$. Thus, unconditionally, for such D, the localisation $(\mathrm{cl}_{\exp})_D$ of $\mathrm{cl}_{\exp}$ is a pregeometry and, by precisely the same argument as in the proof of Lemma 5.1.5, has the CCP.

5.1.2 Raising to powers

Let K be a subfield of $\mathbb{C}$. The structure $\mathbb{C}_K$ of *raising to powers in K* is the following two-sorted structure:

$$(\mathbb{C},+,(\lambda\cdot)_{\lambda\in K})\xrightarrow{\exp}(\mathbb{C},+,\cdot),$$

where the structure on the first-sort is the natural K-vector space structure, the structure on the second sort is the usual field structure and exp is the complex exponential function.

A model-theoretic study of the above structures, in analogy with the case of $\mathbb{C}_{\exp}$, has been done by Zilber in [45], with additions in [53] and [44]. As in the previous section, we are interested in a CCP result and give only a brief account of the necessary material.

Consider the predimension function δ_K defined on tuples $x\subset\mathbb{C}$ by

$$\delta_K(x):=\operatorname{lin.d.}_K(x)+\operatorname{tr.d.}(\exp(x))-\operatorname{lin.d.}(x).$$

Note that δ_K is a submodular predimension function with respect to the pregeometry given by the $\mathbb{Q}$-linear span.

Assume K has finite transcendence degree. Then, the Schanuel Conjecture implies that $\delta_K(x) \geq -\operatorname{tr.d.}(K)$ for all x. Indeed, $\operatorname{lin.d.}_K(x) \geq \operatorname{tr.d.}(x/K) \geq \operatorname{tr.d.}(x) - \operatorname{tr.d.}(K)$, therefore:

$$\begin{aligned}\delta_K(x) &= \operatorname{lin.d.}_K(x) + \operatorname{tr.d.}(\exp(x)) - \operatorname{lin.d.}(x)\\ &\geq \operatorname{tr.d.}(x) - \operatorname{tr.d.}(K) + \operatorname{tr.d.}(\exp(x)) - \operatorname{lin.d.}(x)\\ &\geq -\operatorname{tr.d.}(K)\end{aligned}$$

where the last inequality follows from the Schanuel conjecture.

Thus, SC implies the following conjecture:

Conjecture 5.1.7 (Schanuel Conjecture for raising to powers in K (SC_K))**.** *Let K be a subfield of $\mathbb{C}$ of finite transcendence degree. Then,*

(SC_K) *For all $x \subset \mathbb{C}$,*

$$\delta_K(x) \geq -\operatorname{tr.d.}(K).$$

The following theorem shows that a stronger version of the Schanuel Conjecture for raising to powers in K is satisfied in the case where K is generated by powers that are *exponentially algebraically independent*. This result is due to Bays, Kirby and Wilkie; in the form below, it follows easily from their Theorem 1.3 in [5].

Theorem 5.1.8 (Strong Schanuel Condition for K (SC^*_K))**.** *Suppose $K = \mathbb{Q}(\lambda)$ where λ is an exponentially algebraically independent tuple of complex numbers. Then the following holds:*

(SC^*_K) *For all $x \subset \mathbb{C}$,*

$$\delta_K(x) \geq 0.$$

For the definition of exponential algebraic independence we refer to [5]; for our purposes it suffices to know the following: exponential algebraic independence implies algebraic independence, and, if β is a real number that is *generic* in the o-minimal structure $\mathbb{R}_{\exp}$ (this is defined in Section 5.3 below) then β is exponentially transcendental (i.e. the singleton $\{\beta\}$ is exponentially algebraically independent). In particular, these two facts imply that if $\beta \in \mathbb{R}$ is generic in $\mathbb{R}_{\exp}$, then the complex number βi is exponentially transcendental.

Assume SC_K. Then the values of the submodular predimension function δ_K are bounded from below in $\mathbb{Z}$. Therefore there exists a smallest self-sufficient set for δ_K, namely the self-sufficient closure of the empty set (cf. Section 2.2.3). By localising δ_K over the self-sufficient closure of the empty set, we obtain a proper predimension function. Let us denote by d_K the associated dimension function and by cl_K the corresponding pregeometry (without explicit mention of the localisation).

Definition 5.1.9. A subset L of $\mathbb{C}^n$ defined by an equation of the form

$$M \cdot x = c,$$

where M is a $k \times n$-matrix with entries in K and $c \in \mathbb{C}^n$, is said to be a *K-affine subspace* of $\mathbb{C}^n$. If $C \subset \mathbb{C}$ contains all the coordinates of c, then we say that L is defined over C. Note that if the matrix M has rank r over K, then the dimension of L, denoted $\dim L$, is $n - r$.

In analogy with Definition 4.2.4, in the case of green points, and Definition 5.1.3, in the case of exponentiation, we have the following definition, which will be essential in our arguments in Chapter 6, Section 6.3.

Definition 5.1.10. A pair (L, W) of a K-affine subspace L of $\mathbb{C}^n$ and a subvariety W of $(\mathbb{C}^*)^n$ is said to be *K-rotund* if for any $k \times n$-matrix m with entries in $\mathbb{Z}$ of rank k we have

$$\dim m \cdot L + \dim W^m \geq k.$$

Minor modifications of the proof of the CCP for $\mathbb{C}_{\exp}$ yield a proof of the CCP in the powers case under the assumption that the SC_K holds. Thus, we have:

Lemma 5.1.11. *Assume SC_K. Then the pregeometry cl_K on $\mathbb{C}$ has the CCP.*

Remark 5.1.12. Notice that if D is a self-sufficient subset of $\mathbb{C}$ with respect to $\delta_{\exp}$ containing K, then D is also self-sufficient with respect to δ_K. This is due to the fact that for every set $D \subset \mathbb{C}$ containing K, the inequality $\delta_K(x/D) \geq \delta_{\exp}(x/D)$ holds for all $x \subset \mathbb{C}$. Indeed, this can be seen as follows:

$$\begin{aligned}
\delta_K(x/D) &= \mathrm{lin.\,d.}_K(x/D) + \mathrm{tr.\,d.}(\exp(x)/\exp(D)) - \mathrm{lin.\,d.}(x/D) \\
&\geq \mathrm{tr.\,d.}(x/D) + \mathrm{tr.\,d.}(\exp(x)/\exp(D)) - \mathrm{lin.\,d.}(x/D) \\
&\geq \mathrm{tr.\,d.}(x\exp(x)/D\exp(D)) - \mathrm{lin.\,d.}(x/D) \\
&= \delta_{\exp}(x/D)
\end{aligned}$$

Thus, the result of Kirby mentioned in Remark 5.1.6, which provides a countable self-sufficient set D with respect to $\delta_{\exp}$, also gives a countable self-sufficient subset of $\mathbb{C}$ with respect to δ_K for any countable K, namely the self-sufficient closure of K with respect to $(\delta_{\exp})_D$.

Also, the proof of the CCP works to prove that for any countable self-sufficient set D with respect to δ_K, the localisation $(\mathrm{cl}_K)_D$ is a pregeometry with the CCP.

5.2 Exponentiation and raising to powers on an elliptic curve

5.2.1 Basic setting and exponentiation

Let $\mathbb{E}$ be an elliptic curve defined over a subfield k_0 of $\mathbb{C}$. Put $E := \mathbb{E}(\mathbb{C}) \subset \mathbb{P}^2(\mathbb{C})$. As noted in Section 3.1, $\mathbb{E}$ has an algebraic group structure with identity element $[0, 1, 0]$ and is defined by a homogeneous equation of the form:

$$zy^2 = 4(x - e_1)(x - e_2)(x - e_3),$$

where e_1, e_2 and e_3 are distinct complex numbers.

Associated to $\mathbb{E}$ there is a lattice $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ in $\mathbb{C}$ and a corresponding Weierstrass function $\wp$, defined for x in $\mathbb{C} \setminus \Lambda$ by

$$\wp(x) := \frac{1}{x^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left(\frac{1}{(x - \omega)^2} - \frac{1}{\omega^2}\right). \tag{5.1}$$

For all $x \in \mathbb{C} \setminus \Lambda$, $\wp$ satisfies the differential relation

$$(\wp'(x))^2 = 4(\wp(x) - e_1)(\wp(x) - e_2)(\wp(x) - e_3), \tag{5.2}$$

and

$$e_1 = \wp(\frac{\omega_1}{2}), \quad e_2 = \wp(\frac{\omega_2}{2}), \quad e_3 = \wp(\frac{\omega_1 + \omega_2}{2}). \tag{5.3}$$

As in Section 3.1, we denote by $\operatorname{End}(\mathbb{E})$ the ring of regular endomorphisms of $\mathbb{E}$ and by $k_{\mathbb{E}}$ its field of fractions. Also, E is an $\operatorname{End}(\mathbb{E})$-module and we denote by lin. d. the corresponding linear dimension. Here we identify $\operatorname{End}(\mathbb{E})$ with the subring of $\mathbb{C}$ consisting of all $\alpha \in \mathbb{C}$ such that $\alpha\Lambda \subset \Lambda$. With this convention in place, for all $x \subset \mathbb{C}$ we have $\text{lin. d.}_{\operatorname{End}(\mathbb{E})}(x/\Lambda) = \text{lin. d.}(\exp_{\mathbb{E}}(x))$.

The map $\exp_{\mathbb{E}} : \mathbb{C} \to E$ given by

$$z \mapsto \begin{cases} [\wp(z) : \wp'(z) : 1], & \text{if } z \notin \Lambda, \\ O, & \text{if } z \in \Lambda, \end{cases}$$

is a group homomorphism from the additive group of $\mathbb{C}$ onto E. It is called the exponential map of E.

Finally, the j-invariant of $\mathbb{E}$ will be denoted by $j(\mathbb{E})$.

Let us also consider the action of complex conjugation on the above setting. Throughout, we denote by z^c the complex conjugate of a complex number z. The lattice Λ^c obtained from Λ by applying complex conjugation has an associated Weierstrass function $\wp^c$ satisfying the relation $\wp^c(z^c) = (\wp(z))^c$ for all $z \notin \Lambda$. Let us denote by $\mathbb{E}^c$ the corresponding elliptic curve. By 5.3, the affine part of $\mathbb{E}^c$ is defined by the equation

$$y^2 = 4(x - e_1^c)(x - e_2^c)(x - e_3^c). \tag{5.4}$$

Also, since j is the value of a rational function on e_1, e_2, e_3 (see the proof of [37, I.4.5]), $j(\mathbb{E}^c) = j(\mathbb{E})^c$.

5.2.2 The Elliptic Schanuel Conjecture

The following is the Elliptic Conjecture from [7]. There it is shown to be an instance of more general conjectures of Grothendieck and André. We will refer to it as the *Elliptic Schanuel Conjecture (ESC)*.

Let us start by introducing some conventions related to the theory of elliptic integrals. Given an element $y \in E$, an *integral of the first kind* is a preimage of y under the exponential map $\exp_{\mathbb{E}}$. A *period* of E is an integral of the first kind of the point O, i.e. an element of Λ. *Integrals of the second kind* are more difficult to describe and, although they appear in the statement of the ESC below, we will not need to use their definition. *Quasiperiods* are integrals of the second kind of the point O. For complete definitions we refer to Section I.5 in [37].

We assume that the generators ω_1 and ω_2 of the lattice Λ of periods satisfy $\Im(\omega_2/\omega_1) > 0$, and let η_1 and η_2 be corresponding quasiperiods, so that the Legendre relation $\omega_2\eta_1 - \omega_1\eta_2 = 2\pi i$ holds ([37, I.5.2]).

In the rest of this section, given a tuple $y = (y_1, \ldots, y_r)$ of points on the curve E, let us denote by $x = (x_1, \ldots, x_r)$ and $z = (z_1, \ldots, z_r)$ corresponding integrals of the first and the second kind, respectively.

Conjecture 5.2.1 (Elliptic Schanuel Conjecture (ESC))**.** *Let $\mathbb{E}^1, \ldots, \mathbb{E}^n$ be pairwise non-isogenous elliptic curves. For any tuples $y^\nu = (y_1^\nu, \ldots, y_{r^\nu}^\nu)$ of points of $\mathbb{E}^\nu$, $\nu = 1, \ldots, n$, we have:*

$$\begin{aligned}\operatorname{tr.d.}&(j(\mathbb{E}^\nu), \omega_1^\nu, \omega_2^\nu, \eta_1^\nu, \eta_2^\nu, y^\nu, x^\nu, z^\nu)_\nu \\ &\geq 2\sum_\nu \operatorname{lin.d.}_{k_{\mathbb{E}_\nu}}(x^\nu/\Lambda^\nu) + 4\sum_\nu (\operatorname{lin.d.}_{\mathbb{Q}} k_{\mathbb{E}_\nu})^{-1} - n + 1\end{aligned} \tag{5.5}$$

In fact, we do not need to deal directly with the quasiperiods or the integrals of the second kind for our purposes, for we can use a consequence of the conjecture that ignores the precise contribution of these points to the transcendence degree on the left hand side of inequality (5.5) by using obvious upper bounds. Let us therefore show that the above conjecture implies the following simpler statement:

Conjecture 5.2.2 (Weak Elliptic Schanuel Conjecture (wESC))**.** *Let* $\mathbb{E}^1, \dots, \mathbb{E}^n$ *be pairwise non-isogenous elliptic curves. For any tuples* $x^\nu \in \mathbb{C}^{r^\nu}$, $k_{\mathbb{E}_\nu}$*-linearly independent over* Λ^ν, $\nu = 1, \dots, n$, *we have:*

$$\operatorname{tr.d.}(j(\mathbb{E}^\nu), x^\nu, \exp_{\mathbb{E}}^\nu(x^\nu))_\nu \geq \sum_\nu r^\nu. \tag{5.6}$$

Proof of ESC (5.2.1) $\Rightarrow$ *wESC (5.2.2).* Let $\mathbb{E}^1, \dots, \mathbb{E}^n$ be pairwise non-isogenous elliptic curves. For $\nu = 1, \dots, n$, let $x^\nu \in \mathbb{C}^{r^\nu}$ be $k_{\mathbb{E}_\nu}$-linearly independent over Λ^ν. Set $y^\nu = \exp_{\mathbb{E}}^\nu(x^\nu)$. Then, by 5.2.1,

$$\begin{aligned}\operatorname{tr.d.}(j(\mathbb{E}^\nu), \omega_1^\nu, \omega_2^\nu, \eta_1^\nu, \eta_2^\nu, y^\nu, x^\nu, z^\nu)_\nu \\ \geq 2\sum_\nu r^\nu + 4\sum_\nu (\operatorname{lin.d.}_{\mathbb{Q}} k_{\mathbb{E}_\nu})^{-1} - n + 1.\end{aligned} \tag{5.7}$$

Without loss of generality let us assume that $\mathbb{E}^1, \dots, \mathbb{E}^l$ have no CM and $\mathbb{E}^{l+1}, \dots, \mathbb{E}^n$ have CM, $0 \leq l \leq n$. Then $\sum_\nu (\operatorname{lin.d.}_{\mathbb{Q}} k_{\mathbb{E}_\nu})^{-1} = l + \frac{1}{2}(n - l)$.

Thus,

$$\operatorname{tr.d.}(j(\mathbb{E}^\nu), \omega_1^\nu, \omega_2^\nu, \eta_1^\nu, \eta_2^\nu, y^\nu, x^\nu, z^\nu)_\nu \geq 2\sum_\nu r^\nu + 4l + 2(n-l) - n + 1. \tag{5.8}$$

For each ν, the Legendre relation $\omega_2^\nu \eta_1^\nu - \omega_1^\nu \eta_2^\nu = 2\pi i$ holds. In particular, restricting our attention to $\mathbb{E}_1, \dots, \mathbb{E}_l$, this gives

$$\operatorname{tr.d.}(\omega_1^\nu, \omega_2^\nu, \eta_1^\nu, \eta_2^\nu / 2\pi i)_{\nu=1,\dots,l} \leq 3l.$$

In the CM case, hence for $\nu = l+1, \dots, n$, there are further algebraic dependences. Indeed, it is clear that in this case ω_1^ν and ω_2^ν are $\mathbb{Q}^{\text{alg}}$-linearly dependent and, in fact, by a theorem of Masser ([24][3.1 Theorem III]), $1, \omega_1^\nu, \eta_1^\nu, 2\pi i$ form a $\mathbb{Q}^{\text{alg}}$-linear basis of the $\mathbb{Q}^{\text{alg}}$-linear span of $1, \omega_1^\nu, \omega_2^\nu, \eta_1^\nu, \eta_2^\nu, 2\pi i$. Therefore

$$\operatorname{tr.d.}(\omega_1^\nu, \omega_2^\nu, \eta_1^\nu, \eta_2^\nu / 2\pi i)_{\nu=l+1,\dots,n} \leq n - l.$$

Combining the last two inequalities we get

$$\operatorname{tr.d.}(\omega_1^\nu, \omega_2^\nu, \eta_1^\nu, \eta_2^\nu)_{\nu=1,\dots,n} \leq 3l + (n-l) + 1.$$

Thus, inequality 5.8 implies the following:

$$\operatorname{tr.d.}(j(\mathbb{E}^\nu), y^\nu, x^\nu, z^\nu)_\nu \geq \big(2\sum_\nu r^\nu + 4l + 2(n-l) - n + 1\big) - \big(3l + (n-l) + 1\big).$$

Therefore

$$\operatorname{tr.d.}(j(\mathbb{E}^\nu), y^\nu, x^\nu)_\nu \geq \sum_\nu r^\nu.$$

□

Consider the case of a single elliptic curve $\mathbb{E}$ defined over $k_0 \subset \mathbb{C}$. Let $E = \mathbb{E}(\mathbb{C})$. Let us define a predimension function $\delta_{\exp_{\mathbb{E}}}$ on $\mathbb{C}$ as follows: for all $x \subset \mathbb{C}$, let

$$\delta_{\exp_{\mathbb{E}}}(x) := \operatorname{tr.d.}(j(\mathbb{E}), x, \exp_{\mathbb{E}}(x)) - \operatorname{lin.d.}_{k_{\mathbb{E}}}(x/\Lambda).$$

This predimension function is submodular with respect to the pregeometry given by the $k_{\mathbb{E}}$-linear span. Also, the wESC is clearly equivalent to the statement that for all $x \subset \mathbb{C}$, $\delta_{\exp_{\mathbb{E}}}(x) \geq 0$; which means that if the wESC holds, then the predimension function $\delta_{\exp_{\mathbb{E}}}$ is also proper. Thus, assuming the wESC, one obtains an associated dimension function $\mathrm{d}_{\exp_{\mathbb{E}}}$ and corresponding pregeometry $\mathrm{cl}_{\exp_{\mathbb{E}}}$. Using the same argument as in Zilber's proof of the CCP (5.1.5), this time applying the version of Ax's theorem for the Weierstrass $\wp$-functions from [18], one can see that the pregeometry $\mathrm{cl}_{\exp_{\mathbb{E}}}$ has the CCP.

It may be worth noting that a comprehensive study of the model theory of the exponential functions of elliptic curves or more general abelian varieties, in analogy with Zilber's program for the exponential function, is not yet available.

5.2.3 Raising to powers on $\mathbb{E}$

Fix a subfield K of $\mathbb{C}$ extending $k_{\mathbb{E}}$.

The two-sorted structure $\mathbb{E}_K$ of *raising to powers in* K *on* $\mathbb{E}$ is given by:

$$(\mathbb{C}, +, (\lambda\cdot)_{\lambda \in K}) \xrightarrow{\exp_{\mathbb{E}}} (E, (W(\mathbb{C}))_{W \in L_{\mathbb{E}}}).$$

where the first sort has the natural K-vector field structure, the second sort has the algebraic structure on E, and the map $\exp_{\mathbb{E}}$ is the exponential map of E.

Consider the predimension function $\delta_{E,K}$ defined on tuples $x \subset \mathbb{C}$ by

$$\delta_{E,K}(x) = \operatorname{lin.d.}_K(x) + \operatorname{tr.d.}(j(\mathbb{E}), \wp(x)) - \operatorname{lin.d.}_{k_{\mathbb{E}}}(x/\Lambda).$$

If K has finite transcendence degree, then the Weak Elliptic Schanuel Conjecture (5.2.2) implies that the inequality $\delta_{E,K}(x) \geq -\operatorname{tr.d.}(K)$ holds for all $x \subset \mathbb{C}$.

Let us state this consequence of the wESC for a single elliptic curve $\mathbb{E}$ as an independent conjecture.

Conjecture 5.2.3 (Weak ESC for raising to powers in K on E (wESC$_K$))**.** *Let $\mathbb{E}$ be an elliptic curve. Let K is a subfield of $\mathbb{C}$ extending $k_{\mathbb{E}}$ of finite transcendence degree. Then the following holds:*

(wESC$_K$) *For all $x \subset \mathbb{C}$,*

$$\delta_{E,K}(x) \geq -\operatorname{tr.d.}(K).$$

Assume wESC_K holds. Then we can obtain a proper predimension function from $\delta_{E,K}$ by localising over the self-sufficient closure of the empty set, for which we have an associated dimension function, which we shall denote $\mathrm{d}_{E,K}$, and pregeometry, which will be denoted by $\mathrm{cl}_{E,K}$. The same argument as in the proof of 5.1.5, using the version of Ax's theorem for Weierstrass $\wp$-functions from [18], shows that for any countable K, $\mathrm{cl}_{E,K}$ has the CCP.

Let us extend Definition 5.1.10 from the multiplicative case to include the elliptic curve case. Since there is no space for confusion, we keep the same terminology.

Definition 5.2.4. A pair (L, W) of a K-affine subspace L of $\mathbb{C}^n$ and an algebraic subvariety W of $\mathbb{E}^n$ is said to be *K-rotund* if for any $k \times n$-matrix m with entries in $\mathrm{End}(\mathbb{E})$ of rank k we have we have

$$\dim m \cdot L + \dim m \cdot W \geq k.$$

5.3 o-minimality

We now review some basic facts of the theory of o-minimality, of which we shall make use in Chapter 6 and 7. Of particular interest to us is the property of *definable choice* of o-minimal expansions of groups and some facts about the dimension theory in o-minimal expansions of the real ordered field. We formulate the statements on dimension for locally definable sets (see definition below), instead of definable sets, because this allows us to apply them directly in situations involving real analytic sets. Since dimension is a local property, the reduction to the analogous results for definable sets is straightforward.

The standard reference for the basics of o-minimality is [39]. For a very brief introduction, we also suggest Section 3 of [34].

Definition 5.3.1. A first-order structure $\mathcal{R} = (R, <, \dots)$ where $<$ is a linear order on R is said to be *o-minimal* if every subset of R definable in $\mathcal{R}$ is a finite union of intervals of the form (a, b) with $a, b \in R \cup \{-\infty, +\infty\}$ and singletons.

The most fundamental example of an o-minimal structure is the real ordered field, that is the structure $\mathcal{R} = (\mathbb{R}, <, +, -, 0, \cdot, 1)$. That the structure $\mathcal{R}$ is o-minimal follows easily from the Tarski-Seidenberg theorem, which states that the structure $\mathcal{R}$ has quantifier elimination. Later in this section we give examples of expansions of this structure that are also o-minimal, and which will be later be used in our arguments. But first we review some basic results about o-minimal structures.

We start with the following fact about algebraic closure and definable closure.

Fact 5.3.2. *Let $\mathcal{R}$ be an o-minimal structure. Then the following hold:*

1. $\mathrm{acl}_{\mathcal{R}} = \mathrm{dcl}_{\mathcal{R}}$,
2. $\mathrm{dcl}_{\mathcal{R}}$ *is a pregeometry.*

The first statement in the above fact is rather easy to prove. For the second statement, the non-trivial part is proving that $\mathrm{dcl}_{\mathcal{R}}$ satisfies the exchange principle. This follows from the Monotonicity Theorem ([39, 3.1.2]), which states that, for any o-minimal structure $\mathcal{R}$, for any definable function $f : (a, b) \to R$ defined on the interval $(a, b) \subset R$, there exist points $a = a_0 < a_1 < \cdots < a_k = b$ such that on each interval (a_i, a_{i+1}) the function f is either constant, or strictly monotone and continuous.

Let $\dim_{\mathcal{R}}$ denote the dimension function associated to the pregeometry $\mathrm{acl}_{\mathcal{R}}$. We shall now state some facts about the the dimension theory in o-minimal structures developed around the dimension function $\dim_{\mathcal{R}}$, for a more detailed treatment see [28, Section 1].

Definition 5.3.3. Let $\mathcal{R}$ be an expansion of the real ordered field. We say that $X \subset \mathbb{R}^n$ is *locally definable* over A in $\mathcal{R}$ if for any box U with rational end-points, the intersection $X \cap U$ is definable over A in $\mathcal{R}$. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be locally definable over A in $\mathcal{R}$ if its graph is.

Fact 5.3.4. *Suppose $\mathcal{R}$ is an expansion of the real ordered field in a countable language. Then for any $X \subset \mathbb{R}^n$ that is locally definable in $\mathcal{R}$ over a countable set A we have*

$$\max_{x \in X} \dim_{\mathcal{R}}(x/A) = \dim_{\mathbb{R}} X,$$

where $\dim_{\mathbb{R}} X$ *is the* topological dimension *of X, i.e. the maximum $k \leq n$ such that for some coordinate projection π from $\mathbb{R}^n$ to $\mathbb{R}^k$, the set $\pi(X)$ has interior.*

If X is a real analytic set, then $\dim_{\mathbb{R}} X$ *is also its* real analytic dimension*, i.e. the maximum k such that for some $x \in X$ and open neighbourhood V_x of x, $X \cap V_x$ is a real analytic submanifold of $\mathbb{R}^n$ of dimension k.*

The first part of the fact above follows easily from the Baire Category Theorem, it is also explicitly proved in [16, Lemma 2.17]. The second part is a standard fact in real analytic geometry.

Definition 5.3.5. Let $\mathcal{R}$ be an o-minimal structure. Let $X \subset R^n$ be a locally definable set in $\mathcal{R}$ over a $A \subset R$. An element b in X is said to be *generic in X over A* if

$$\dim_{\mathcal{R}}(b/A) = \max_{x \in X} \dim_{\mathcal{R}}(x/A).$$

As a convention, if x is a tuple in R^n, we say that x is *generic in* $\mathcal{R}$ if it is generic in R^n over the empty set.

The following fact is a consequence of the C^1-Cell Decomposition Theorem ([39, 7.3.2]), together with the fact that the boundary of any subset of $\mathbb{R}^n$ has dimension strictly less than n ([39, 4.1.10]).

Fact 5.3.6. *Let* $\mathcal{R} = (R, < .+, 0, -, \cdot, 1, \dots)$ *be an o-minimal expansion of an ordered field. Let* U *be an open subset of* R^n *and let* $f : U \to R^m$ *be a definable function. Then the dimension of the set of points in* U *at which* f *is discontinuous is strictly less than* n*.*

The following is the Definable Choice Property of o-minimal expansions of ordered abelian groups. For a proof see [39, 6.1.2] or [34, 3.7].

Fact 5.3.7 (Definable choice property)**.** *Let* $\mathcal{R} = (R, <, +, 0, -, 1, \dots)$ *be an o-minimal expansion of an ordered abelian group with a constant* 1 *for a non-zero element. Let* $S \subset R^{m+n}$ *be a definable set and let* $\pi : R^{m+n} \to R^n$ *be the projection on the first* m *coordinates. Then there is a definable map* $f : \pi(S) \to R^n$ *such that for all* $a \in \pi(S)$*,* $(a, f(a))$ *is in* S*.*

Let us introduce two important examples of o-minimal structures. The expansion of the real ordered field by all restricted analytic functions, denoted $\mathbb{R}_{an}$, that is the expansion having a function symbol for each restriction $f|_B$ where B is a bounded box in $\mathbb{R}^n$ and f is an analytic function on an open set containing B, is o-minimal. This was noted by van den Dries to be a consequence of results of Gabrielov and Łowasjewicz ([38]). The expansion of the real ordered field by the real exponential function, $\mathbb{R}_{\exp} = (\mathbb{R}, < .+, 0, -, \cdot, 1, e^x)$, is also o-minimal, by a theorem of Wilkie ([42]).

In Chapter 6, we shall consider the expansion of the real ordered field by the restrictions of the $\sin x$ and e^x functions to intervals with rational endpoints. This structure is o-minimal, for it is a reduct of $\mathbb{R}_{an}$. Note that the complex exponential function is locally definable in the structure. This follows from the formula

$$\exp(x + iy) = e^x(\cos y + i \sin y),$$

together with the fact that the restrictions to bounded intervals with rational endpoints of the exponential and the sine functions are definable, by the very definition of $\mathcal{R}$, and the restrictions of the cosine function to the same intervals are definable in $\mathcal{R}$ as the derivatives of the restrictions of the sine function.

Analogously, in Chapter 7 we shall need an o-minimal expansion of the real ordered field in a countable language in which the function $\wp$ is locally definable. The existence of such a structure $\mathcal{R}$, as a reduct of $\mathbb{R}_{an}$, follows from the fact that the *addition formula*,

$$\wp(z_1 + z_2) = -\wp(z_1) - \wp(z_2) + \frac{1}{4}(\frac{\wp'(z_1) - \wp'(z_2)}{\wp(z_1) - \wp(z_2)})^2,$$

allows to locally define $\wp$ in terms of its restriction to a closed parallelogram contained in the interior of the fundamental parallelogram of vertices $0, 1, \tau, 1 + \tau$ (e.g. the one with vertices $\frac{1+\tau}{8}$, $\frac{3+\tau}{8}$, $\frac{1+3\tau}{8}$, $\frac{3+3\tau}{8}$), around which it is analytic ([21]). Indeed, this corresponds to the fact that $\exp_{\mathbb{E}}$ is a homomorphism and its values can therefore be calculated from those of any restriction to an open subset of the fundamental parallelogram.

Chapter 6

Models on the complex numbers: the multiplicative group case

In this chapter, we find models for the theories of green points constructed in Chapter 4 in the multiplicative group case, under the assumption that an instance of the Schanuel Conjecture for raising to powers holds. In generic cases, our assumption is known to be true, by Theorem 5.1.8, and the result is therefore unconditional.

6.1 The Models

Throughout this Chapter, let $\mathbb{A} = \mathbb{G}_m$ and $A = \mathbb{A}(\mathbb{C}) = \mathbb{C}^*$. Since we work in the multiplicative group, we shall use multiplicative notation. We also use the expressions *multiplicatively (in)dependent* instead of $\mathrm{End}(\mathbb{A})$-linearly (in)dependent.

Let $\epsilon \in \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R})$ and let Q be a non-trivial divisible subgroup of $(\mathbb{R}, +)$ of finite rank. Put

$$G = \exp(\epsilon\mathbb{R} + Q).$$

Note that G is a divisible subgroup of $\mathbb{C}^*$.

We assume henceforth that ϵ is of the form $1 + \beta i$ with β a non-zero real number, for we can always replace any $\epsilon \in \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R})$ for one of this form giving rise to the same G.

Consider the L-structure $(\mathbb{C}^*, G)$.

As noted in Chapter 4, the predimension function δ_G on $\mathbb{C}^*$, defined by $\delta_G(y) := 2\,\mathrm{tr.\,d.}(y) - \mathrm{mult.\,d.}(\mathrm{cl}_0(y) \cap G)$, is submodular with respect to the pregeometry cl_0 on $\mathbb{C}^*$ given by the multiplicative divisible hull.

The following theorem is the main result of this chapter.

Theorem 6.1.1. *Let $\epsilon = 1 + \beta i$, with β a non-zero real number, and let Q be a non-trivial divisible subgroup of $(\mathbb{R}, +)$ of finite rank. Let*

$$G = \exp(\epsilon\mathbb{R} + Q).$$

Assume SC_K holds for $K = \mathbb{Q}(\beta i)$. Then:

1. *For every tuple $c \subset \mathbb{C}^*$, there exists a tuple $c' \subset \mathbb{C}^*$ extending c, such that c' is self-sufficient with respect to the predimension function $(\delta_G)_{c'}$. If $c \subset G$, then we can find such a c' also contained in G.*

2. *The structure $(\mathbb{C}^*, G)$ has the EC-property. Therefore, for every tuple $c \subset G$, self-sufficient with respect to $(\delta_G)_c$, the structure $(\mathbb{C}^*, G)_{X_0}$ is a model of the theory T_{X_0}, where $X_0 = \mathrm{span}(c)$ with the structure induced from $(\mathbb{C}^*, G)$.*

The above theorem follows immediately from Propositions 6.2.2 and 6.3.1 below. Sections 6.2 and 6.3 are devoted to the corresponding proofs.

Note that indeed the two parts of the theorem correspond to the two sets of sentences in $T := T^0 \cup T^1$. In the second part of the theorem we restrict to tuples c with coordinates in G because it is only for those that the theory T^0 has been defined.

6.2 The Predimension Inequality

In this section we prove the first part of Theorem 6.1.1. The proof here improves upon the corresponding one in [46].

Lemma 6.2.1. *Let $K = \mathbb{Q}(\beta i)$ and assume SC_K holds.*

Then for all $y \in (\mathbb{C}^*)^n$, we have $\delta_G(y) \geq -3\,\mathrm{lin.\,d.}\,Q - \mathrm{tr.\,d.}(K)$.

Proof. We may assume $y \in G^n$ and is multiplicatively independent. Let $x \in \mathbb{C}^n$ be such that $\exp(x) = y$ with $x = \epsilon t + q$, $t \in \mathbb{R}^n$, $q \in Q^n$. Note that x is $\mathbb{Q}$-linearly independent over the kernel of exp.

Since complex conjugation is a field automorphism of $\mathbb{C}$, we have

$$2\,\mathrm{tr.\,d.}(y) = \mathrm{tr.\,d.}(y) + \mathrm{tr.\,d.}(y^c). \tag{6.1}$$

Also,

$$\mathrm{tr.\,d.}(y) + \mathrm{tr.\,d.}(y^c) \geq \mathrm{tr.\,d.}(yy^c) = \mathrm{tr.\,d.}(\exp(x)\exp(x^c)). \tag{6.2}$$

By the SC_K,

$$\mathrm{lin.\,d.}_K(xx^c) + \mathrm{tr.\,d.}(yy^c) - \mathrm{lin.\,d.}(xx^c) \geq -\,\mathrm{tr.\,d.}(K)$$

and therefore

$$\operatorname{tr.d.}(yy^c) \geq \operatorname{lin.d.}(xx^c) - \operatorname{lin.d.}_K(xx^c) - \operatorname{tr.d.}(K) \tag{6.3}$$

Combining 6.1,6.2 and 6.3, we obtain

$$2 \operatorname{tr.d.}(y) \geq \operatorname{lin.d.}(xx^c) - \operatorname{lin.d.}_K(xx^c) - \operatorname{tr.d.}(K).$$

Thus, in order to prove the lemma, it is sufficient to show that the difference $\operatorname{lin.d.}(xx^c) - \operatorname{lin.d.}_K(xx^c)$ is always at least $n - 3 \operatorname{lin.d.} Q$.

Since $x = \epsilon t + q$, we have $x^c = \epsilon^c t + q$. We also note the following:

$$\frac{\epsilon^c}{\epsilon} = \frac{1 - \beta i}{1 + \beta i} \in \mathbb{Q}(\beta i) = K.$$

From this we obtain the following upper bound for $\operatorname{lin.d.}_K(xx^c)$:

$$\operatorname{lin.d.}_K(xx^c) \leq \operatorname{lin.d.}_K(\epsilon t, q) \leq n + \operatorname{lin.d.} Q. \tag{6.4}$$

We now need to bound $\operatorname{lin.d.}(xx^c)$ from below. Note that the values $\operatorname{lin.d.}(xx^c)$ and $\operatorname{lin.d.}_K(xx^c)$ do not change if we replace x by any x' with the same $\mathbb{Q}$-linear span (and x^c by x'^c accordingly). It follows that we can assume that for every $i \in \{\operatorname{lin.d.} Q + 1, \dots, n\}$, $q_i = 0$. Indeed, one can apply appropriate regular $\mathbb{Q}$-linear transformations to x (and accordingly to x^c) to reduce to this case.

Since x is linearly independent, in particular we have $\operatorname{lin.d.}(x_{\operatorname{lin.d.} Q+1}, \dots, x_n) = n - \operatorname{lin.d.} Q$, i.e. $\operatorname{lin.d.}(\epsilon t_{\operatorname{lin.d.} Q+1}, \dots, \epsilon t_n) = n - \operatorname{lin.d.} Q$. Moreover, since $\epsilon \notin \mathbb{R} \cup i\mathbb{R}$, ϵ and ϵ^c are $\mathbb{R}$-linearly independent. Therefore

$$\operatorname{lin.d.}(\epsilon t_{\operatorname{lin.d.} Q+1}, \dots, \epsilon t_n, \epsilon^c t_{\operatorname{lin.d.} Q+1}, \dots, \epsilon^c t_n) = 2(n - \operatorname{lin.d.} Q).$$

Thus,

$$\operatorname{lin.d.}(xx^c) \geq \operatorname{lin.d.}(x_{\operatorname{lin.d.} Q+1}, \dots, x_n, x^c_{\operatorname{lin.d.} Q+1}, \dots, x^c_n) \geq 2n - 2 \operatorname{lin.d.} Q. \tag{6.5}$$

From 6.4 and 6.5 we conclude

$$\operatorname{lin.d.}(xx^c) - \operatorname{tr.d.}(xx^c) \geq (2n - 2 \operatorname{lin.d.} Q) - (n + \operatorname{lin.d.} Q) = n - 3 \operatorname{lin.d.} Q.$$

□

Proposition 6.2.2. *Assume SC_K holds for $K = \mathbb{Q}(\beta i)$.*

Then for every tuple $c \subset \mathbb{C}^*$, there exists a tuple $c' \subset \mathbb{C}^*$, extending c, such that c' is self-sufficient with respect to the predimension function $(\delta_G)_{c'}$. If $c \subset G$, then we can find such a c' also contained in G.

Proof. By Lemma 6.2.1, the set of values of δ_G on $(\mathbb{C}^*, G)$ is bounded from below in $\mathbb{Z}$. We can therefore find a tuple c^0 such that $\delta_G(c^0)$ is minimal. Since for every cl_0-closed set X, $\delta_G(X) \geq \delta_G(X \cap G)$, we can find such c_0 with all its coordinates in G. Clearly, c^0 is self-sufficient for δ_G, hence the localisation $(\delta_G)_{c_0}$ is a proper predimension function on $\mathbb{C}^*$.

For every $c \subset \mathbb{C}^*$, let $c' \subset \mathbb{C}^*$ be a tuple containing both c and c_0 that generates the self-sufficient closure of c with respect to $(\delta_G)_{c_0}$. The tuple c' is, by definition, self-sufficient for $(\delta_G)_{c_0}$. Since $c_0 \supset c'$, it follows that c' is self-sufficient for $(\delta_G)_{c'}$. It is easy to see, that if c is contained in G, then c' can be taken to be contained in G. □

6.3 Existential Closedness

This section is devoted to the proof of the following proposition:

Proposition 6.3.1. *The structure $(\mathbb{C}^*, G)$ has the EC-property. Therefore, for every tuple $c \subset G$, self-sufficient with respect to $(\delta_G)_c$, the structure $(\mathbb{C}^*, G)_{X_0}$ is a model of the theory T_{X_0}, where $X_0 = \mathrm{span}(c)$ with the structure induced from $(\mathbb{C}^*, G)$.*

For the rest of Section 6.3, let us fix an even number $n \geq 1$ and a rotund variety $V \subset (\mathbb{C}^*)^n$ of dimension $\frac{n}{2}$ defined over $k_0(C)$ for some finite $C \subset \mathbb{C}$. We need to show that the intersection $V \cap G^n$ is Zariski dense in V.

Let us define the set

$$\mathcal{X} = \{(s, t) \in \mathbb{R}^{2n} : \exp(\epsilon t + s) \in V\}.$$

Note that if (s, t) is in $\mathcal{X} \cap (Q^n \times \mathbb{R}^n)$, then the corresponding point $y := \exp(\epsilon t + s)$ is in $V \cap G^n$. Thus, in order to find points in the intersection $V \cap G^n$, we shall look for points (s, t) in $\mathcal{X}$ with $s \in Q^n$.

Our strategy for this is to find an implicit function for $\mathcal{X}$ defined on an open set $S \subset \mathbb{R}^n$, assigning to every $s \in S$ a point $t(s) \in \mathbb{R}^n$ such that $(s, t(s)) \in \mathcal{X}$. Since Q^n is dense in $\mathbb{R}^n$, the intersection $S \cap Q^n$ is non-empty, and therefore we can find points $(s, t(s))$ in $\mathcal{X}$ with $s \in Q^n$.

Let $\mathcal{R}$ be the expansion of the real ordered field by the restrictions of the real exponential function and the sine function to all bounded intervals with rational end-points, and by constants for the real and imaginary parts of the elements of $\mathbb{Q}(C)$.

Since $\mathcal{R}$ is an expansion by constants of a reduct of $\mathbb{R}_{an}$, $\mathcal{R}$ is o-minimal. Note that the complex exponential function and the set $\mathcal{X}$ are locally definable in $\mathcal{R}$.

Our proof of Proposition 6.3.1 relies on the following lemma, whose proof we postpone until the next subsection.

Lemma 6.3.2 (Main Lemma). *Suppose (s^0, t^0) is an $\mathcal{R}$-generic point of $\mathcal{X}$, i.e. $\dim_{\mathcal{R}}(s^0, t^0) = \dim_{\mathbb{R}} \mathcal{X} = n$. Then $\dim_{\mathcal{R}}(s^0) = n$.*

Let us now continue with the proof of the existential closedness, using the Main Lemma.

Lemma 6.3.3. *Suppose (s^0, t^0) is an $\mathcal{R}$-generic point of $\mathcal{X}$. There is a continuous $\mathcal{R}$-definable function $s \mapsto t(s)$ defined on a neighbourhood $S \subset \mathbb{R}^n$ of s^0 and taking values in $\mathbb{R}^n$ such that for all $s \in S$, the point $y(s) := \exp(\epsilon t(s) + s)$ is in V.*

Proof. Let $\pi : \mathbb{R}^{2n} \to \mathbb{R}^n$ be the projection onto the first n coordinates.

Let $\mathcal{X}^0$ be the intersection of $\mathcal{X}$ and a box with rational end-points containing (s^0, t^0). By the Main Lemma (6.3.2), we have $\dim_{\mathcal{R}}(s^0) = n$. Since $\pi(\mathcal{X}^0)$ is definable in $\mathcal{R}$, $\dim_{\mathbb{R}} \pi(\mathcal{X}^0) = \max_{s \in \pi(\mathcal{X}^0)} \dim_{\mathcal{R}}(s) \geq \dim_{\mathcal{R}}(s^0) = n$. Therefore $\dim_{\mathbb{R}} \pi(\mathcal{X}^0) = n$; and hence the set $\pi(\mathcal{X}^0)$ contains an open neighbourhood S of s^0.

By the definable choice property of $\mathcal{R}$ (Fact 5.3.7), there is an $\mathcal{R}$-definable map $t : \pi(\mathcal{X}^0) \to \mathbb{R}^n$ such that for all $s \in \pi(\mathcal{X}^0)$, $(s, t(s))$ is in $\mathcal{X}^0$. In particular, for all $s \in S$, $(s, t(s)) \in \mathcal{X}$, i.e. $y(s) := \exp(\epsilon t(s) + s)$ is in V.

The o-minimality of $\mathcal{R}$ also gives that the set of points where the $\mathcal{R}$-definable function t is discontinuous is $\mathcal{R}$-definable and of dimension strictly lower than n (Fact 5.3.6). Thus, by making S smaller if necessary, we may assume t is continuous on S. □

Proof Proposition 6.3.1:

Proof. Let V' be a proper subvariety of V. We need to see that the intersection $(V \setminus V') \cap G^n$ is non-empty.

Extending C if necessary, we may assume that V and V' are defined over C.

Take an element y^0 of $V \setminus V'$ with $\dim_{\mathcal{R}}(y^0) = \dim_{\mathbb{R}} V = n$. Let $x^0 \in \mathbb{C}^n$ be such that $\exp(x^0) = y^0$ and let $t^0, s^0 \in \mathbb{R}^n$ be such that $x^0 = \epsilon t^0 + s^0$.

Note that $\dim_{\mathcal{R}}(s^0, t^0) = \dim_{\mathcal{R}}(x^0) = \dim_{\mathcal{R}}(y^0) = n$. Hence (s^0, t^0) is $\mathcal{R}$-generic in $\mathcal{X}$.

Let S and the map $s \mapsto t(s)$ be as provided by Lemma 6.3.3 for $\mathcal{R}$ and (s^0, t^0). Consider the map $s \mapsto y(s) := \exp(\epsilon t(s) + s)$ defined on S. This map is continuous, hence $y^{-1}(V')$ is a closed subset of S not containing s^0. Thus, $S' = S \setminus y^{-1}(V')$ is an open neighbourhood of s^0.

Since Q^n is dense in $\mathbb{R}^n$, we can take a point q in $S' \cap Q^n$, and thus obtain a corresponding point $y(q)$ in $(V \setminus V') \cap G^n$. □

6.3.1 Proof of the Main Lemma

The image of V under complex conjugation, V^c, plays an important role in our proof of the Main Lemma. Since complex conjugation is a field isomorphism, V^c is also an irreducible algebraic variety defined over the set C^c. Extending C if necessary, we may assume that V^c is also defined over C.

Notation 6.3.4. For a tuple x of variables or complex numbers, the expression $\bar{x}$ will denote another tuple of variables or complex numbers, respectively, of the same length, bearing no formal relation to the former. This notation is meant to imply that we are particularly interested in the case where x is a complex number and $\bar{x}$ equals x^c.

Throughout this section, let $K = \mathbb{Q}(\beta i)$.

Definition 6.3.5. For $s \in \mathbb{C}^n$, we define the set

$$L_s = \{(x, \bar{x}) \in \mathbb{C}^{2n} : (x + \bar{x}) + \beta^{-1} i(x - \bar{x}) = 2s\}.$$

Remark 6.3.6. Note that $\beta^{-1} i = -(\beta i)^{-1} \in K$, hence L_s is a K-affine subspace.

Remark 6.3.7. Suppose s is in $\mathbb{R}^n$. Then, for all $x \in \mathbb{C}^n$, the point (x, x^c) belongs to L_s if and only if $x = \epsilon t + s$ for some $t \in \mathbb{R}^n$.

To see this, let $x \in \mathbb{C}^n$ be given, and let $t \in \mathbb{C}^n$ be such that $x = \epsilon t + s$. Then:

$$\begin{aligned}
(x, x^c) \in L_s &\iff (x + x^c) + \beta^{-1} i(x - x^c) = 2s \\
&\iff 2\operatorname{Re}(x) + \beta^{-1} i(2i\operatorname{Im}(x)) = 2s \\
&\iff (\operatorname{Re}(\epsilon t) + s) - \beta^{-1}\operatorname{Im}(\epsilon t) = s \\
&\iff \operatorname{Re}(\epsilon t) - \beta^{-1}\operatorname{Im}(\epsilon t) = 0 \\
&\iff (\operatorname{Re}(t) - \beta\operatorname{Im}(t)) - \beta^{-1}(\operatorname{Im}(t) + \beta\operatorname{Re}(t)) = 0 \\
&\iff (\beta + \beta^{-1})\operatorname{Im}(t) = 0 \\
&\iff \operatorname{Im}(t) = 0 \\
&\iff t \in \mathbb{R}^n.
\end{aligned}$$

Lemma 6.3.8. *Suppose $s \in \mathbb{R}^n$. Then for all linearly independent $(m^1, n^1), \ldots, (m^k, n^k) \in \mathbb{Z}^{2n}$ ($m^i, n^i \in \mathbb{Z}^n$), we have*

$$\dim(m, n) \cdot L_s \geq \frac{k}{2}.$$

Proof. Suppose $(m^1, n^1), \dots, (m^k, n^k) \in \mathbb{Z}^{2n}$ are linearly independent $(m^i, n^i \in \mathbb{Z}^n)$.

Let D be any countable set over which L_s is defined and let $t \in \mathbb{R}^n$ be such that $\operatorname{lin.d.}_K(t/D) = n$. For $x = \epsilon t + s$ and $\bar{x} = x^c = \epsilon^c t + s$, the tuple $(x, \bar{x})$ is in L_s, as 6.3.7 shows. Then, we have:

$$\begin{aligned}\dim(m,n)\cdot L_s &\geq \operatorname{lin.d.}_K((m,n)\cdot(x,\bar{x})/D)\\ &= \operatorname{lin.d.}_K((m^1,n^1)\cdot(x,\bar{x}),\dots,(m^k,n^k)\cdot(x,\bar{x})/D)\\ &= \operatorname{lin.d.}_K((m'^1,n'^1)\cdot(t,\beta i t),\dots,(m'^k,n'^k)\cdot(t,\beta i t)/D)\end{aligned}$$

where $m'^i = m^i + n^i$ and $n'^i = m^i - n^i$, for all $i = 1, ..., k$. Since $m^i = \frac{1}{2}(m'^i + n'^i)$ and $n^i = \frac{1}{2}(m'^i - n'^i)$, the matrix (m', n') has the same rank as (m, n), that is k. Therefore we can take a matrix $M \in \mathrm{GL}_k(\mathbb{Z})$ and $t' = (t_{j_1}, \dots, t_{j_l}, \beta i t_{j_{l+1}}, \dots, \beta i t_{j_k})$, with $1 \leq l \leq k$, such that

$$\operatorname{lin.d.}_K((m'^1,n'^1)\cdot(t,\beta i t),\dots,(m'^k,n'^k)\cdot(t,\beta i t)/D) = \operatorname{lin.d.}_K(M\cdot t'/D).$$

Thus,

$$\dim(m,n)\cdot L_s \geq \operatorname{lin.d.}_K(M\cdot t'/D).$$

But note that $\operatorname{lin.d.}_K(M\cdot t'/D)$ is at least $\frac{k}{2}$, for we have

$$\begin{aligned}\operatorname{lin.d.}_K(M\cdot t'/D) &= \operatorname{lin.d.}_K(t'/D)\\ &\geq \max\{\operatorname{lin.d.}_K(t_{j_1},\dots,t_{j_l}/D), \operatorname{lin.d.}_K(\beta i t_{j_{l+1}},\dots,\beta i t_{j_k}/D)\}\\ &= \max\{l, k-l\} \geq \frac{k}{2}.\end{aligned}$$

Therefore, $\dim(m,n)\cdot L_s \geq \frac{k}{2}$ □

Lemma 6.3.9. *Let $s \in \mathbb{R}^n$. Then the pair $(L_s, V \times V^c)$ is K-rotund.*

Proof. Suppose $(m^1, n^1), \dots, (m^k, n^k) \in \mathbb{Z}^{2n}$ are linearly independent $(m^i, n^i \in \mathbb{Z}^n)$.

The rotundity of V implies that the variety $V \times V^c$ is also rotund. Hence $\dim(V \times V^c)^{(m,n)} \geq \frac{k}{2}$.

Also, by Lemma 6.3.8, $\dim(m,n)\cdot L_s \geq \frac{k}{2}$.

Therefore, we have:

$$\dim(m,n)\cdot L_s + \dim(V\times V^c)^{(m,n)} \geq \frac{k}{2} + \frac{k}{2} = k.$$

Thus, the pair $(L_s, V \times V^c)$ is K-rotund. □

Proof of the Main Lemma:

Proof. Consider the set

$$L_{s^0} \cap \log(V \times V^c).$$

It is an analytic subset of $\mathbb{C}^{2n}$ containing the point $(x^0, (x^0)^c)$. Since every analytic set can be written as the union of its irreducible components and this union is locally finite ([10, Section 5.4]), there exist a neighbourhood B of $(x^0, (x^0)^c)$, a positive integer l and irreducible analytic subsets $S_1, \ldots, S_l$ of B containing $(x^0, (x^0)^c)$ such that

$$L_{s^0} \cap \log(V \times V^c) \cap B = S_1 \cup \cdots \cup S_l.$$

We may assume B is a box with rational end-points.

Claim. *Every S_i has complex analytic dimension* 0*.*

Before proving the claim, let us show how the lemma follows. The claim implies that each S_i is a closed discrete subset of B; since B is bounded, each S_i must then be finite. Being the union of the S_i, the set $L_{s^0} \cap \log(V \times V^c) \cap B$ is therefore finite, and it is clearly $\mathcal{R}$-definable over s^0. Thus, the singleton $\{(x^0, (x^0)^c)\}$ is $\mathcal{R}$-definable over s^0 as the intersection of $L_{s^0} \cap \log(V \times V^c) \cap B$ and a sufficiently small $\mathcal{R}$-definable open box around $(x^0, (x^0)^c)$. Therefore $\dim_{\mathcal{R}}(s^0) = \dim_{\mathcal{R}}(x^0) = n$.

Proof of the claim. Suppose towards a contradiction that there exists i such that the set $S := S_i$ is of positive dimension.

Let us show that there are uncountably many points in S whose image under exponentiation is a generic point of $V \times V^c$ over C.

To see this, suppose V' is a proper subvariety of $V \times V^c$ over C. Note that $(y^0, (y^0)^c)$ is a generic point of $V \times V^c$ over C, for we have

$$\operatorname{tr.d.}(y^0, (y^0)^c/C) = \operatorname{tr.d.}(\operatorname{Re}(y^0), \operatorname{Im}(y^0)/C) \geq \dim_{\mathcal{R}}(\operatorname{Re}(y^0), \operatorname{Im}(y^0)) = n = \dim V \times V^c.$$

Hence $(x^0, (x^0)^c)$ does not belong to $\log V'$, and therefore $S \cap \log V'$ is an analytic subset of B properly contained in S. Then, by the irreducibility of S, for any such V', $S \cap \log V'$ is nowhere-dense in S. Since S has positive dimension we can apply the Baire Category Theorem to conclude that there exist uncountably many $(x, \bar{x})$ in S that do not belong to $\log V'$ for any such V', i.e. their images under exponentiation are generic points of $V \times V^c$ over C.

Let D be a countable self-sufficient subset of $\mathbb{C}$ with respect to δ_K (provided by Remark 5.1.12). Let D' be the self-sufficient closure of $\log C \cup s^0$ with respect to $(\delta_K)_D$.

For any tuple $z \subset \mathbb{C}^*$, if $\delta_K(z/D') \leq 0$ then all the coordinates of z lie in $\mathrm{cl}_K(D')$. But $\mathrm{cl}_K(D')$ is countable, because D' is countable and $(\mathrm{cl}_K)_D$ has the Countable Closure Property, so there can be no more than countably many tuples z with $\delta_K(z/D') \leq 0$. Thus, we can find $(x, \bar{x}) \in L_{s^0}$ such that $(\exp(x), \exp(\bar{x}))$ is a generic point of $V \times V^c$ over C and $\delta_K(x, \bar{x}/D') > 0$.

Then:

$$0 < \delta_K(x\bar{x}/D') \leq \dim L_{s^0} \cap N + \dim(V \times V^c) \cap \exp N - \dim N, \tag{6.6}$$

where N is the minimal $\mathbb{Q}$-affine subspace over D' containing the point $(x, \bar{x})$.

Since $\dim L_{s^0} = \dim(V \times V^c) = n$, it immediately follows from the inequality above that N cannot be the whole of $\mathbb{C}^{2n}$, as in that case the right hand side would be 0. Therefore $\dim N < 2n$.

Thus, there exist $k \geq 1$ and linearly independent $m^1, \ldots, m^k \in \mathbb{Z}^{2n}$ such that N is a translate of the subspace of $\mathbb{C}^{2n}$ defined by the system of equations $m^i \cdot (z, \bar{z}) = 0$ $(i = 1, \ldots, k)$. Note that $\dim N = 2n - k$.

Note that $(V \times V^c) \cap \exp N$ is a generic fibre of the map $(\ )^m$ on $(V \times V^c)$, for it contains the generic point $(y, \bar{y})$ of $V \times V^c$ over C. The addition formula for the dimension of fibres of algebraic varieties then gives

$$\dim(V \times V^c)^m = \dim V \times V^c - \dim(V \times V^c) \cap \exp N.$$

Also, by the addition formula for the dimension of K-affine subspaces,

$$\dim m \cdot L_{s^0} = \dim L_{s^0} - \dim L_{s^0} \cap N.$$

Adding up the two equations,

$$\dim m \cdot L_{s^0} + \dim(V \times V^c)^m = 2n - (\dim L_{s^0} \cap N + \dim(V \times V^c) \cap \exp N)$$

Using (6.6) we get

$$\dim m \cdot L_{s^0} + \dim(V \times V^c)^m < 2n - \dim N = k.$$

This implies that the pair $(L_{s^0}, V \times V^c)$ is not K-rotund, contradicting 6.3.9. $\square$

$\square$

6.4 The question of ω-saturation

A natural question for which we are not able to give an answer is whether the model $(\mathbb{C}^*, G)$ is ω-saturated. Here we present two remarks on the issue.

First we show that, assuming the CIT with parameters (Conjecture 3.2.3), we can prove an a priori stronger version of Proposition 6.3.1 that is implied by ω-saturation. In fact, if the CIT with parameters holds, then every model of T satisfies the stronger EC-property.

Proposition 6.4.1. *Assume the CIT with parameters holds. Then every model of T satisfies the following* strong EC-property*:*

For any rotund variety $V \subset (K^*)^n$ of dimension $\frac{n}{2}$ defined over a finite set C, there exists a generic of V over C in G^n.

Proof. Let (K^*, G) be a model of T. Let V be a rotund variety of dimension $\frac{n}{2}$ defined over a finite set C.

It is sufficient to find a proper subvariety V' of V such that for any $y \in V \cap G^n$, if y does not lie in V' then y is a generic point of V over C. Indeed, that (K^*, G) satisfies the EC-property guarantees that we can find a point $y \in (V \setminus V') \cap G^n$, and y would then be a generic point of V over C.

Without loss of generality we assume that C is self-sufficient in $\mathcal{A}$. Then for any $y \in G^n$ $\delta_G(y/C) \geq 0$. In particular, for any y in $V \cap G^n$, if y is not a generic point of V over C then y has to be multiplicatively dependent over C. Thus, it is enough to find a proper subvariety V' of V over C such that for every $y \in G^n \cap V$, if y is multiplicatively dependent over C then y is in V'.

By the CIT with parameters, there exist cosets $H_1, \ldots, H_l$ of proper algebraic subgroups of $(K^*)^n$ such that any atypical irreducible component of the intersection of V and a coset of a proper algebraic subgroup of $(K^*)^n$ is contained in some H_i.

Let $V' = V \cap \bigcup_i H_i$. We shall now show that V' has the required property. Suppose $y \in G^n \cap V$ is multiplicatively dependent over C, and let us see that y then belongs to V'. Let H be the smallest coset of a proper tori that is defined over C and contains y. Let c_H denote the codimension of H, then $c_H \geq 1$.

Let Y be an irreducible component of $V \cap H$ containing y. Since $V \cap H$ is defined over C, Y is defined over $\mathbb{Q}(C)^{\mathrm{alg}}$. Then, by the predimension inequality over C, $2 \dim Y - \dim H \geq 0$. Therefore we have

$$2 \dim Y \geq \dim H = n - c_H = 2 \dim V - c_H,$$

and consequently,

$$\dim Y \geq \dim V - \frac{1}{2}c_H > \dim V - c_H.$$

Hence $\dim Y > \dim V - c_H$, which means that Y is an atypical irreducible component of the intersection $V \cap H$.

Indeed, the CIT with parameters then tells us that y must belong to one of the H_i, and thus to V'. □

Proposition 6.4.2. *Assume the CIT with parameters holds. Suppose (K^*, G) is a model of T where G has infinite dimension for the dimension function associated to δ. Then (K^*, G) is ω-saturated.*

Proof. In the light of Proposition 4.2.26, it is sufficient to show that $(\mathcal{K}^*, G)$ is *rich.*

By the previous lemma, the CIT assumption implies that (K^*, G) satisfies the the strong EC-property, which means that the richness property holds for *prealgebraic minimal extensions.*

The assumption on the dimension of G implies that that the richness property also holds for *green generic minimal extensions.* This amounts to proving that for any finite self-sufficient subset C of K^*, there exists $b \in K^*$ with $\mathrm{d}_G(b/C) = 1$. But this is clear since $\mathrm{d}_G(G)$ is infinite and C is finite.

For *minimal white generic extensions* we need to find, for any C as before, an element $b \in K^*$ with $\mathrm{d}_G(b/C) = 2$. We proceed by taking b_1 and b_2 with $\mathrm{d}_G(b_1/C) = \mathrm{d}_G(b_2/Cb_2) = 1$ and setting $b = b_1 + b_2$.

It is sufficient to show that $\delta_G(b_1, b_2/C, b_1 + b_2) = 0$. Indeed, we then get

$$0 \leq \mathrm{d}_G(b_1, b_2/C, b_1 + b_2) \leq \delta_G(b_1, b_2/C, b_1 + b_2) = 0,$$

so $\mathrm{d}_G(b_1, b_2/C, b_1 + b_2) = 0$, and hence $\mathrm{d}_G(b_1 + b_2/C) = \mathrm{d}_G(b_1, b_2/C) = 2$.

Now the calculation of $\delta_G(b_1, b_2/C, b_1 + b_2)$: By definition,

$$\delta_G(b_1, b_2/C, b_1 + b_2) = 2\,\mathrm{tr.\,d.}(b_1, b_2/C, b_1 + b_2) - \mathrm{mult.\,d.}(b_1, b_2/C, b_1 + b_2).$$

It is easy to see that

$$\mathrm{tr.\,d.}(b_1, b_2/C, b_1 + b_2) = 1.$$

Also,

$$\mathrm{mult.\,d.}(b_1, b_2/C, b_1 + b_2) = 2,$$

because the variety defined by the equation $X + Y = b_1 + b_2$ is rotund. Thus, $\delta_G(b_1, b_2/C, b_1 + b_2) = 2(1) - 2 = 0$. □

Unfortunately, it is not clear that in our model $(\mathbb{C}^*, G)$ the dimension $\mathrm{d}_G(G)$ is infinite. Note that this would immediately follow if one could show that the corresponding pregeometry on the uncountable set G has the CCP.

6.5 A variant: Emerald points

In this section we present a variation of the theories of green points in the multiplicative group case and exhibit models on the complex numbers for the new theories.

The models of the new theories are expansions of the algebraic structure on the multiplicative group by a subgroup H that is elementarily equivalent to the additive group of the integers [1]. This contrasts with the case of green points where G is divisible. In the new structures we call the elements of the subgroup H *emerald points*; as before, we call the elements outside the distinguished subgroup *white points*. The theories of emerald points are constructed are obtain by modifying the construction of the theories of green points presented in Chapter 4. As in Chapter 4, the theories are shown to be near model complete and superstable, they are not, however, ω-stable.

The motivation for our interest in these structures comes from Zilber's investigations on connections between model theory and noncommutative geometry; in particular, the content of [49] and the survey [50]. In [49], a connection is established between the construction of noncommutative tori, which are basic examples of noncommutative spaces, and the model theory of the expansions of the complex field by a multiplicative subgroup of the form

$$H = \exp(\epsilon\mathbb{R} + q\mathbb{Z}),$$

where $\epsilon = 1 + i\beta$ and β and q are non-zero real numbers such that βq and π are $\mathbb{Q}$-linearly independent (this guarantees that H is torsion-free).

In the last part of this section we show that, assuming the Schanuel Conjecture for raising to powers in $K = \mathbb{Q}(\beta i)$ holds, these structures are indeed models of the theories of emerald points, and are therefore superstable.

Let us now review some basic facts about the model theory of the theory of the additive group of the integers, whose interpretability in the theories of emerald points is responsible for the differences with respect to the green case. The theory of the additive group of the integers eliminates quantifiers after adding to the semigroup

[1] More generally, one can allow H to have torsion and rather require $H/\operatorname{Tor} H$ to be elementarily equivalent to the additive group of the integers. For the sake of simplicity of notation we shall only work in the case where H is torsion-free, but the more general case should not be any more difficult.

language $\{+, 0\}$ a predicate P_m for each subgroup of the form $m\mathbb{Z}$, $m \geq 2$, and a constant for 1. In this expanded language, a structure $(H, \cdot, 1, (P_m), e)$[2] is elementarily equivalent to $(\mathbb{Z}, +, 0, (m\mathbb{Z}), 1)$ if and only if the following conditions are met:

(i) $(H, \cdot, 1)$ is a torsion-free abelian group,

(ii) for every $m \geq 2$, P_m is the set H^m of all m-powers in H,

(iii) for every $m \geq 2$, eH^m generates the quotient group H/H^m.

If $(H, \cdot, 1, (P_m), e)$ is elementarily equivalent to $(\mathbb{Z}, +, 0, (m\mathbb{Z}), 1)$, then it is said to be a *$\mathbb{Z}$-group*. We also call this expanded language the *language of $\mathbb{Z}$-groups*. Note that every congruence equation in the integers of the form

$$x \equiv k \pmod{m},$$

where x is a variable, m is a positive integer and $k \in \{0, \ldots, m-1\}$, has a corresponding congruence equation in any $\mathbb{Z}$-group $(H, \cdot, 1, (P_m), e)$, that we write multiplicatively as

$$x \equiv e^k \pmod{m},$$

and is expressed by the quantifier-free formula $P_m(xe^{m-k})$. Also, every congruence equation in the integers of the form

$$t \equiv t' \pmod{m},$$

where t and t' are terms in the language of $\mathbb{Z}$-groups and m is a positive integer, is equivalent to a Boolean combination of congruence equations of the above simpler form. Moreover, the complete (quantifier-free) type of an element in a $\mathbb{Z}$-group is determined by the set of congruence equations, of the simple form above, that it satisfies. With this, it is easy to see that the theory of $\mathbb{Z}$-groups is λ-stable if and only if $\lambda \geq 2^{\aleph_0}$. The theory is thus superstable, non-ω-stable.

With the above remarks in mind and in order to obtain the same quantifier elimination results for the theories of emerald points as in the green case, we work in a language L which in addition to $L_{\mathbb{A}} \cup \{H\}$ also contains predicates P_m, for $m \geq 2$, and a constant e; and require the expansion $(H, \cdot, 1, (P_m), e)$ of the subgroup $(H, \cdot, 1)$ of A to be a $\mathbb{Z}$-group. The obvious limitation with respect to the green case is that our theories of emerald points cannot be λ-stable for any $\lambda < 2^{\aleph_0}$; in particular, they cannot be ω-stable. This is in fact the the only limitation in terms of stability, for the theories are in fact λ-stable for all $\lambda \geq 2^{\aleph_0}$, and hence superstable.

[2]Note the multiplicative notation.

6.5.1 Construction of the theories

Let $\mathbb{A}$ be the multiplicative group.

Let L be the language $L_{\mathbb{A}} \cup \{H, (P_m)_{m\geq 2}, e\}$, where H and each P_m are unary predicates and e is a constant.

Let $\mathcal{C}$ be the class of structures $\mathcal{A} = (A, H, (P_m)_{m\geq 2}, e)$ where

- A is a model of $T_{\mathbb{A}}$,
- H is a subgroup of A,
- $(H, \cdot, 1, (P_m)_{m\geq 2}, e)$ is a $\mathbb{Z}$-group, i.e. it is elementarily equivalent to $(\mathbb{Z}, +, 0, (m\mathbb{Z})_{m\geq 2}, 1)$, i.e. is a $\mathbb{Z}$-group.

If $\mathcal{A}$ is a structure in $\mathcal{C}$, then we call the elements of H *emerald points*. The elements of $A \setminus H$ are called *white points*.

For each $\mathcal{A}$ in $\mathcal{C}$ we have:

- a pregeometry $\mathrm{cl}_0^{\mathcal{A}}$ on A induced by the $\mathbb{Q}$-linear span pregeometry on the $\mathbb{Q}$-vector space $A/\operatorname{Tor} A$, and
- a submodular predimension function $\delta_H^{\mathcal{A}}$ with respect to $\mathrm{cl}_0^{\mathcal{A}}$ defined by: for all finite $Y \subset A$,

$$\delta_H^{\mathcal{A}}(Y) = 2\,\mathrm{tr.\,d.}(Y) - \mathrm{lin.\,d.}(\mathrm{cl}_0(Y) \cap \mathrm{cl}_0(H)).$$

Moreover, the required compatibility conditions hold and we therefore have a pregeometry cl_0 and a submodular predimension function δ_H with respect to cl_0 for the class $\mathcal{C}$.

Let $\mathrm{Sub}\,\mathcal{C}$ be the class of substructures of structures in $\mathcal{C}$ whose domain is a cl_0-closed set and let $\mathrm{Fin}\,\mathcal{C}$ be the class of structures in $\mathrm{Sub}\,\mathcal{C}$ whose domain has finite cl_0-dimension.

Remark 6.5.1. It useful to note that if H is a $\mathbb{Z}$-group and D is a divisible torsion-free abelian group, then the direct sum of H and D is a $\mathbb{Z}$-group (with the obvious interpretations for the symbols of the expanded language). Also, if $H = (H, \cdot, 1, (P_m)_M, e)$ is a $\mathbb{Z}$-group and D and D_0 are divisible torsion-free abelian groups such that H and D_0 are subgroups of D and $e \in D_0$, then the intersection of H and D_0 is a $\mathbb{Z}$-group.

It follows from the latter observation that for every $\mathcal{X} = (X, H^{\mathcal{X}})$ in $\mathrm{Sub}\,\mathcal{C}$, $H^{\mathcal{X}}$ is a $\mathbb{Z}$-group.

Let us fix $\mathcal{X}_0 \in \operatorname{Fin}\mathcal{C}$ and let $\mathcal{C}_0$ be the class of structures in $\mathcal{C}$ in which $\mathcal{X}_0$ embeds self-sufficiently. After an identification, every structure in $\mathcal{C}_0$ is assumed to have $\mathcal{X}_0$ as a self-sufficient substructure.

Also, let $\operatorname{Sub}\mathcal{C}_0$ be the class of substructures of structures in $\mathcal{C}_0$ whose domain is a cl_0-closed set containing X_0. Equivalently, $\operatorname{Sub}\mathcal{C}_0$ is the class of structures in $\operatorname{Sub}\mathcal{C}$ in which $\mathcal{X}_0$ embeds self-sufficiently, again identifying $\mathcal{X}_0$ with its image under one such embedding. Finally, let $\operatorname{Fin}\mathcal{C}_0$ be the class of structures in $\operatorname{Sub}\mathcal{C}_0$ whose domain has finite cl_0-dimension.

As in Chapter 4, we work in the class $\operatorname{Sub}\mathcal{C}_0$ and to simplify the notation we use cl_0 to denote the localisation $(\mathrm{cl}_0)_{X_0}$ and similarly δ_H for $(\delta_H)_{X_0}$. We also use the notation $\mathrm{span}(X)$ for $\mathrm{cl}_0(X)$.

Note that the L-structure $\mathcal{A}_0$ with domain $A_0 = \mathrm{acl}_{\mathbb{A}}(X_0)$, $H^{\mathcal{A}_0} = H^{\mathcal{X}_0}$ and the obvious interpretations for the other symbols is in $\mathcal{C}_0$ and is prime in $\mathcal{C}_0$ with respect to self-sufficient embeddings.

Remark 6.5.2. Suppose $\mathcal{X} = (X, H)$ is a structure in $\operatorname{Sub}\mathcal{C}$. Consider the structure $\mathcal{X}' = (X, G)$, in the expansion of the language $L_{\mathbb{A}}$ by a unary predicate, where G is a divisible subgroup of X with $H \subset G \subset \mathrm{cl}_0(H)$. It is clear that then $\mathcal{X}'$ is a structure in $\operatorname{Sub}\mathcal{C}$ in the sense of the green points construction. Also, if $\mathcal{X}$ is in $\mathcal{C}$ or $\operatorname{Fin}\mathcal{C}$, then $\mathcal{X}'$ is in the corresponding class in the sense of the green points construction.

Furthermore, note that for all finite-dimensional cl_0-closed subset Y of X, we have

$$\delta_H^{\mathcal{X}}(Y) = \delta_G^{\mathcal{X}'}(Y).$$

It follows that for all $\mathcal{X}, \mathcal{Y} \in \operatorname{Sub}\mathcal{C}$ with $\mathcal{X} \subset \mathcal{Y}$, if $\mathcal{X}'$ and $\mathcal{Y}'$ are such that $G^{\mathcal{X}'} \subset G^{\mathcal{Y}'}$ and $\operatorname{Tor} G^{\mathcal{X}'} = \operatorname{Tor} G^{\mathcal{Y}'}$, then: $\mathcal{X} \leq \mathcal{Y}$ if and only if $\mathcal{X}' \leq \mathcal{Y}'$. In simple words, this shows that the new predimension function δ_H is in a strong sense the same as the predimension function δ_G, which implies that many results transfer effortlessly to the new setting.

Remark 6.5.3. Using the previous remark, it is easy to see that the following statements follow from their analogues in the green points construction: the (Asymmetric) Amalgamation Lemma for the class $\operatorname{Sub}\mathcal{C}_0$ (Lemma 4.1.7 and Corollary 4.1.8), the closure of the class $\mathcal{C}_0$ under unions of self-sufficient chains (Lemma 4.1.10), and the extension property from structures in $\operatorname{Sub}\mathcal{C}_0$ to structures in $\mathcal{C}_0$ (Lemma 4.1.11).

By the results of Chapter 2, this implies the existence of rich structures in the class $\mathcal{C}_0$. Also, as shown in Chapter 2, all rich structures are models of the same complete first order theory and the usual quantifier elimination result holds.

The following two lemmas are directly implied by the corresponding ones in the green case.

Lemma 6.5.4. *Let $\mathcal{A} = (A, H) \in \mathcal{C}$. For every complete $L_{\mathbb{A}}$-l-type $\Theta(y)$, there exists a partial $L_{\mathbb{A}} \cup \{H\}$-l-type $\Phi_\Theta(y)$ consisting of universal formulas such that for every realisation c of Θ in $\mathcal{A}$,*

$$\mathcal{A} \vDash \Phi_\Theta(c) \text{ if and only if } \operatorname{span} c \text{ is self-sufficient in } \mathcal{A}.$$

Lemma 6.5.5. *There exists an L_{X_0}-theory T^0 such that for every L_{X_0}-structure $\mathcal{A} = (A, H)$ in $\mathcal{C}$, $(A, H) \models T^0$ if and only if (A, H) is in $\mathcal{C}_0$.*

Henceforth let T^0 be a theory as in the above lemma.

The following two definitions and the two subsequent lemmas are identical to their analogues in the green case.

Definition 6.5.6. An irreducible subvariety W of A^n is said to be *rotund* if for every $k \times n$-matrix M with entries in $\operatorname{End}(\mathbb{A})$ of rank k, the dimension of the constructible set $M \cdot W$ is at least $\frac{k}{2}$.

Definition 6.5.7. A structure (A, H) in $\mathcal{C}_0$ is said to have the *EC-property* if for every even $n \geq 1$, for every rotund subvariety W of A^n of dimension $\frac{n}{2}$, the intersection $W \cap H^n$ is Zariski dense in W.

Lemma 6.5.8. *For every subvariety $W(x, y)$ of $\mathbb{A}^{n+k}$ defined over k_0, there exists a quantifier-free $L_{\mathbb{A}}$-formula $\theta(y)$ such that for all $A \models T_{\mathbb{A}}$ and all $c \in A^k$,*

$$A \vDash \theta(c) \iff W(x, c) \text{ is rotund.}$$

Lemma 6.5.9. *There exists a set of $\forall\exists$-L-sentences T^1 such that for any structure (A, H) in $\mathcal{C}_0$*

$$(A, H) \vDash T^1 \iff (A, H) \text{ has the EC-property.}$$

Henceforth, let T^1 denote the theory defined in the above proof. Also, let $T := T^0 \cup T^1$.

The next step is to show that the theory T axiomatizes richness up to ω-saturation. Here, the difference with the green case lies in the following: in proving that ω-saturated models of T are rich, it is necessary to show that the following statement holds: *for any minimal prealgebraic strong extension $\mathcal{X} \leq \mathcal{Y}$ of structures in* $\operatorname{Fin} \mathcal{C}_0$, *the type of a coloured* cl_0*-basis of Y over X is finitely satisfiable in models of T.* The difference with the green case is that, unlike there, here said type includes non-trivial information about the divisibility of the elements in the basis inside the coloured group H. Lemmas 6.5.10 and 6.5.11 provide the missing step in the argument by showing that the EC-property, and hence the axioms in T, are sufficiently strong to imply the above statement.

Lemma 6.5.10. *Let $\mathcal{A}$ be a structure in $\mathcal{C}_0$. The following are equivalent:*

1. *$\mathcal{A}$ has the EC-property*
2. *for every even $n \geq 1$, for every rotund subvariety W of A^n of dimension $\frac{n}{2}$ and every consistent system of congruence equations in the integers on variables $x_1, \ldots, x_n$ of the form:*

$$\{x_i \equiv k_i \pmod{m_i} : i \in \{1, \ldots, n\}\},$$

with $m_i \geq 2$ and $k_i \in \{0, \ldots, m_i\}$, the set of solutions in H^n of the corresponding system in H is Zariski dense in W.

3. *for every even $n \geq 1$, for every rotund subvariety W of A^n of dimension $\frac{n}{2}$ and every consistent system of congruence equations in the integers on variables $x_1, \ldots, x_n$ of the form:*

$$\{x_i \equiv k_{im} \pmod{m} : i \in \{1, \ldots, n\}, m \in \{2, \ldots, N\}\},$$

with $N \in \mathbb{N}$ and $k_{im} \in \{0, \ldots, m\}$, the set of solutions in H^n of the corresponding system in H is Zariski dense in W.

Proof. Clearly, (2.) implies (1.) and (3.) implies (2.).

Using the Chinese Remainder Theorem, it is easy to see that (2.) implies (3.).

We now show that (1.) implies (2.): Let W be a rotund subvariety of A^n of dimension $\frac{n}{2}$ and consider the system of congruence equations in the integers

$$\{x_i \equiv k_i \pmod{m_i} : i \in \{1, \ldots, n\}\},$$

where $m_i \geq 2$ and $k_i \in \{0, \ldots, m_i - 1\}$. Let C be an $\mathrm{acl}_{\mathbb{A}}$-closed set over which W is defined and let $a \in A^n$ be a generic point of W over C. Let $a' \in A^n$ be such that $(a_i')^{m_i} e^{k_i} = a_i$, for $i = 1, \ldots, n$ and let W' be the locus of a' over C. It is easy to see that W' is also a rotund subvariety of A^n of dimension $\frac{n}{2}$. Assuming 1 holds, the set $W' \cap H^n$ is Zariski dense in W', which implies that the set of solutions of the corresponding system of congruence equations in H is Zariski dense in W. Thus, indeed, 1 implies 2 □

Lemma 6.5.11. *Let $\mathcal{A}$ be a structure in $\mathcal{C}_0$. Assume $\mathcal{A}$ has the EC-property and is ω-saturated. Then for every even $n \geq 1$, for every rotund subvariety W of A^n of dimension $\frac{n}{2}$ and every consistent system of congruence equations in the integers on variables $x_1, \ldots, x_n$ of the form:*

$$\{x_i \equiv k_{im} \pmod{m} : i \in \{1, \ldots, n\}, m \geq 2\},$$

where $k_{im} \in \{0, \ldots, m\}$, the set of solutions in H^n of the corresponding system in H is Zariski dense in W.

Proof. Follows immediately from the previous lemma. □

As noted before, with the above lemmas at hand, one can give the same proof as in Chapter 4 for the following proposition.

Proposition 6.5.12. *The theory T is complete and its ω-saturated models are precisely the rich structures.*

The next theorem gathers the main model-theoretic properties of the theory T.

Theorem 6.5.13. *1. T is near model complete*

2. T is λ-stable if and only if $\lambda \geq 2^{\aleph_0}$. The theory T is therefore superstable, non-ω-stable.

3. In T, $\mathrm{RU}(x = x) = \omega \cdot 2$ and $\mathrm{RU}(H(x)) = \omega$.

Proof. For this proof, let $\bar{\mathcal{A}} = (\bar{A}, H)$ be a monster model of T.

1. Remark 6.5.2 implies that with the same formulas $\tau_{a,b}$ as in the green case (see Lemma 4.3.5), the sufficient condition for near model completeness found in Proposition 2.3.11 is satisfied.

2. Let λ be an infinite cardinal and let $B \subset \bar{A}$ be set of cardinality λ. Let us show that there are at most $2^{\aleph_0} \cdot \lambda$ many 1-types over B. This clearly implies that T is λ-stable for all $\lambda \geq 2^{\aleph_0}$. Note that T is not λ-stable for any $\lambda < 2^{\aleph_0}$, for there are $2^{\aleph_0}$ 1-types over the empty set, as there are already in the theory of $\mathbb{Z}$-groups.

   By passing to the algebraic closure, we may assume that B is algebraically closed, hence cl_0-closed and self-sufficient. Let a_0 be an element of $\bar{A}$. Let $a \in \bar{A}^n$ be a cl_0-basis of the set $A = \mathrm{sscl}(Ba_0)$ over B. For each coordinate a_i

of a ($i = 1, \ldots, n$), define a sequence $(r_i^m)_{m\geq 1}$ as follows: $r_i^m := 0$ for all m, if $a_i \notin H$, and, $r^m :=$ the remainder of a_i modulo m in H, if $a_i \in H$. Also, for each $m \geq 1$, let a_i^m be a choice of m^{th}-root of $a_i(r_i^m)^{-1}$ in H, satisfying the compatibility condition that for all $m, m' \geq 1$, $(a_i^{mm'})^m = a_i^{m'}$.

By Proposition 2.3.10, the type of a_0 over B is determined by the L-isomorphism type of the set A over B. The L-isomorphism type of A over B is itself determined by the following data, collectively: the set of indices $i \in \{1, \ldots, n\}$ for which a_i is in H, the sequence $(r_m)_{m\geq 1}$ and the algebraic type of the sequence $(a^m)_{m\geq 1}$ over B (where, as usual, r_m is the tuple with coordinates r_i^m and a^m the tuple with coordinates a_i^m). There are finitely many possibilities for the set of indices, $2^{\aleph_0}$ possibilities for the sequence $(r_m)_{m\geq 1}$ and at most $\lambda \cdot 2^{\aleph_0}$ possibilities for the algebraic type of the sequence $(a^m)_{m\geq 1}$ over B. Thus, there are at most $\lambda \cdot 2^{\aleph_0}$ possibilities for the type of a_0 over B.

3. This is because the U-rank calculations work in the same way as in the green case. However, since the theory is not ω-stable, Morley rank is not bounded; in particular, it does not coincide with U-rank. This is due to the interpretability of the theory of $\mathbb{Z}$-groups, where the same occurs. Note that the isolation of types among those of greater or equal U-rank, which we established in the green case to prove that Morley and U- ranks coincide, does indeed fail for the theory of $\mathbb{Z}$-groups.

□

6.5.2 Models on the complex numbers

The following theorem provides models for the theories of emerald points on the complex numbers.

Theorem 6.5.14. *Let β and q non-zero real numbers such that βq and π are $\mathbb{Q}$-linearly independent. Let $\epsilon = 1 + \beta i$ and*

$$H = \exp(\epsilon\mathbb{R} + q\mathbb{Z}).$$

Assume SC_K holds for $K = \mathbb{Q}(\beta i)$. Then:

1. *For every tuple $c \subset \mathbb{C}^*$, there exists a tuple $c' \subset \mathbb{C}^*$ extending c, such that c' is self-sufficient with respect to the predimension function $(\delta_H)_{c'}$.*

2. *The structure* $(\mathbb{C}^*, H)$ *has the EC-property. Therefore, for every tuple* $c \subset \mathbb{C}^*$*, self-sufficient with respect to* $(\delta_H)_c$*, the structure* $(\mathbb{C}^*, H)_{X_0}$ *is a model of the theory* T_{X_0}*, where* $X_0 = \mathrm{span}(c)$ *with the structure induced from* $(\mathbb{C}^*, H)$*.*

The first part of the theorem follows directly from the analogous statement in the green case, by Remark 6.5.2. For the second part of the theorem, the proof of the analogous statement in the green case applies, simply using the density of $q\mathbb{Z} + \frac{2\pi}{\beta}\mathbb{Z}$ in $\mathbb{R}$, instead of that of the subgroup Q, at the very end of the proof.

Chapter 7

Models on the complex numbers: the elliptic curve case

For elliptic curves without complex multiplication and whose lattice of periods is invariant under complex conjugation, we find models for the theories of green points, under the assumption that the Weak Schanuel Conjecture for raising to powers on the elliptic curve holds.

7.1 The Models

Let us fix an elliptic curve $\mathbb{E}$ without complex multiplication. Let $E = \mathbb{E}(\mathbb{C})$. We use the conventions introduced in Section 5.2.

Let $\epsilon \in \mathbb{C}^*$ be such that $\epsilon\mathbb{R} \cap \Lambda = \{0\}$. Put $G = \exp_{\mathbb{E}}(\epsilon\mathbb{R})$.

Remark 7.1.1. Note that G is a divisible subgroup of E. From Chapter 4 we know that the predimension function δ on E, defined by $\delta(y) := 2\,\mathrm{tr.\,d.}_{k_0}(y) - \mathrm{lin.\,d.}(\mathrm{cl}_0(y) \cap G)$, is submodular with respect to the pregeometry cl_0 on E induced by the $k_{\mathbb{A}}$-linear span on $E/\operatorname{Tor} E$. Since $\mathbb{E}$ has no CM, $k_{\mathbb{A}} = \mathbb{Q}$ and, for any $y \subset E$, $\mathrm{cl}_0(y)$ is the divisible hull of the subgroup generated by y.

Also, G is dense in E in the Euclidean topology. To see this notice the following:

$$G = \exp_{\mathbb{E}}(\epsilon\mathbb{R}) = \exp_{\mathbb{E}}(\epsilon\mathbb{R} + \Lambda) = \exp_{\mathbb{E}}(\Gamma + \mathbb{Z} + \alpha\mathbb{Z}),$$

for $\alpha = \mathrm{Re}(\tau) - \frac{\mathrm{Re}(\epsilon)}{\mathrm{Im}(\epsilon)}\mathrm{Im}(\tau) \in \mathbb{R}$. Since $\Gamma \cap \Lambda = \{0\}$, α is irrational. It follows that $\mathbb{Z} + \alpha\mathbb{Z}$ is dense in $\mathbb{R}$. Therefore the set $G = \exp_{\mathbb{E}}(\epsilon + \mathbb{Z} + \alpha\mathbb{Z})$ is dense in E in the Euclidean topology.

We now state the main theorem of this chapter.

Theorem 7.1.2. *Let $\mathbb{E}$ be an elliptic curve without complex multiplication and let $E = \mathbb{E}(\mathbb{C})$. Assume the corresponding lattice Λ has the form $\mathbb{Z} + \tau\mathbb{Z}$ and $\Lambda = \Lambda^c$.*

Let $\epsilon = 1 + \beta i$, with β a non-zero real, be such that $\epsilon\mathbb{R} \cap \Lambda = \{0\}$. Put $G = \exp_{\mathbb{E}}(\epsilon\mathbb{R})$.

Let $K = \mathbb{Q}(\beta i)$ and assume the Weak Elliptic Schanuel Conjecture for raising to powers in K ($wESC_K$) holds for $\mathbb{E}$.

Then,

1. *For every tuple $c \subset E$, there exists a tuple $c' \subset E$ extending c, such that c' is self-sufficient with respect to the predimension function $(\delta_G)_{c'}$. If $c \subset G$, then we can find such a c' also contained in G.*

2. *The structure (E, G) has the EC-property. Therefore, for every tuple $c \subset G$, self-sufficient with respect to $(\delta_G)_c$, the structure $(E, G)_{X_0}$ is a model of the theory T, where $X_0 = \operatorname{span}(c)$ with the structure induced from (E, G).*

Let us make some remarks about the hypotheses of the theorem.

Firstly, assuming that the lattice Λ has generators $\omega_1 = 1$ and $\omega_2 = \tau$ is not truly restrictive, for this can always be achieved by passing to an isomorphic elliptic curve.

Secondly, the assumptions of $\mathbb{E}$ having no CM and Λ being invariant under complex conjugation are real restrictions on the generality of the result. The first assumption is essential, since we do not have an appropriate $\operatorname{End}(\mathbb{A})$-submodule of E that serves as analogue of the subgroup G defined above in the CM case. The second is necessary in our arguments for proving both the predimension inequality and the existential closedness for the structure (E, G). Let us remark that the two conditions hold for any non-CM elliptic curve defined over $\mathbb{R}$.

Also note that, by the remarks at the end of subsection 5.2.1, the assumption that $\Lambda = \Lambda^c$ implies that $\mathbb{E} = \mathbb{E}^c$ and $j(\mathbb{E})^c = j(\mathbb{E})$.

Finally, let us recall that our assumption that the wESC_K (5.2.3) holds for the single elliptic curve $\mathbb{E}$ means the following:

- For any tuple x of complex numbers,

$$\operatorname{lin.d.}_K(x) + \operatorname{tr.d.}(j(\mathbb{E}), \wp(x)) - \operatorname{lin.d.}(x/\Lambda) \geq -\operatorname{tr.d.}(K). \qquad (7.1)$$

For the rest of this chapter, we assume work under the hypothesis of the theorem:

Let $\mathbb{E}$ is an elliptic curve for which the hypotheses of the theorem are satisfied. That is: $\mathbb{E}$ has no CM, $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$, $\Lambda = \Lambda^c$

Let $\epsilon = 1 + \beta i$, with β a non-zero real, be such that $\epsilon\mathbb{R} \cap \Lambda = \{0\}$. Put $G = \exp_{\mathbb{E}}(\epsilon\mathbb{R})$ and $K = \mathbb{Q}(\beta i)$.

Assume the Weak Elliptic Schanuel Conjecture for raising to powers in K holds for $\mathbb{E}$.

As in the previous chapter, we divide the proof of the theorem into the proofs of two propositions, Propositions 7.2.2 and 7.3.1.

7.2 The Predimension Inequality

In this section we prove the first part of Theorem 7.1.2.

Lemma 7.2.1. *For any tuple $y \subset E$, $\delta_G(y) \geq -4 - \operatorname{tr.d.}(K) - 2\operatorname{tr.d.}(k_0)$.*

Proof. It is sufficient to show that for any n and any $y \in G^n$ with $\operatorname{lin.d.}(y) = n$, we have

$$2\operatorname{tr.d.}_{k_0}(y) \geq n - 4 - \operatorname{tr.d.}(K) - 2\operatorname{tr.d.}(k_0).$$

Fix such n and y. Let $x \in (\epsilon\mathbb{R})^n$ be such that $\exp_{\mathbb{E}}(x) = y$. Notice that x is $\mathbb{Q}$-linearly independent over Λ.

Note the following

$$\begin{aligned}
2\operatorname{tr.d.}(j(\mathbb{E}), \wp(x)) &\geq \operatorname{tr.d.}(j(\mathbb{E}), \wp(x), (\wp(x))^c) \\
&= \operatorname{tr.d.}(j(\mathbb{E}), \wp(x), \wp^c(x^c)) \\
&= \operatorname{tr.d.}(j(\mathbb{E}), \wp(xx^c)).
\end{aligned}$$

By the wESC$_K$,

$$\operatorname{lin.d.}_K(xx^c) + \operatorname{tr.d.}(j(\mathbb{E}), \wp(x), \wp(x^c)) - \operatorname{lin.d.}(xx^c/\Lambda) \geq -\operatorname{tr.d.}(K).$$

Combining the above inequalities we obtain,

$$2\operatorname{tr.d.}(j(\mathbb{E}), \wp(x)) \geq \operatorname{lin.d.}(xx^c/\Lambda) - \operatorname{lin.d.}_K(xx^c) - \operatorname{tr.d.}(K).$$

Now, on the one hand, since ϵ is not in $\mathbb{R} \cup i\mathbb{R}$, we know ϵ and ϵ^c are $\mathbb{R}$-linearly independent and hence $\operatorname{lin.d.}(xx^c) = \operatorname{lin.d.}(x) + \operatorname{lin.d.}(x^c) = 2n$. Therefore

$$\operatorname{lin.d.}(xx^c/\Lambda) \geq \operatorname{lin.d.}(xx^c) - \operatorname{lin.d.}(\Lambda) = 2n - 2.$$

On the other hand, since $x^c = \frac{\epsilon^c}{\epsilon}x$ and $\frac{\epsilon^c}{\epsilon} \in K$, we have

$$\operatorname{lin.d.}_K(xx^c) \leq \operatorname{lin.d.}_K(x) \leq n.$$

Thus,

$$2\,\mathrm{tr.\,d.}(j(\mathbb{E}), \wp(x)) \geq (2n-2) - n - \mathrm{tr.\,d.}(K) = n - 2 - \mathrm{tr.\,d.}(K).$$

Hence, using the additivity properties of the transcendence degree, we see that

$$\begin{aligned} 2\,\mathrm{tr.\,d.}(\wp(x)) &= 2\,\mathrm{tr.\,d.}(j(\mathbb{E}), \wp(x)) - 2\,\mathrm{tr.\,d.}(j(\mathbb{E})/\wp(x)) \\ &\geq 2\,\mathrm{tr.\,d.}(j(\mathbb{E}), \wp(x)) - 2 \\ &\geq n - 4 - \mathrm{tr.\,d.}(K) \end{aligned}$$

and, similarly,

$$\begin{aligned} 2\,\mathrm{tr.\,d.}_{k_0}(\wp(x)) &= 2\,\mathrm{tr.\,d.}(\wp(x)) - 2\,\mathrm{tr.\,d.}(k_0/\wp(x)) \\ &\geq n - 4 - \mathrm{tr.\,d.}(K) - 2\,\mathrm{tr.\,d.}(k_0). \end{aligned}$$

Because $\wp(x)$ and $y = \exp_{\mathbb{E}}(x)$ are interalgebraic over k_0, we obtain the inequality $2\,\mathrm{tr.\,d.}_{k_0}(y) \geq n - 4 - \mathrm{tr.\,d.}(K) - 2\,\mathrm{tr.\,d.}(k_0)$. □

By the same argument as in Chapter 6, one derives the following proposition.

Proposition 7.2.2. *For every tuple $c \subset E$, there exists a tuple $c' \subset E$, extending c, such that c' is self-sufficient with respect to the predimension function $(\delta_G)_{c'}$. If $c \subset G$, then we can find such a c' also contained in G.*

7.3 Existential Closedness

The following proposition completes the proof of Theorem 7.1.2.

Proposition 7.3.1. *The structure (A, G) has the EC-property. Therefore, for every tuple $c \subset G$, self-sufficient with respect to $(\delta_G)_c$, the structure $(A, G)_{X_0}$ is a model of the theory T, where $X_0 = \mathrm{span}(c)$ with the structure induced from (A, G).*

The proof of the above proposition is the same as in Chapter 6, with only very small differences. In order to be explicit about the differences, we review the different steps of the proof.

For the rest of Section 7.3, let us fix an even number $n \geq 1$ and a rotund variety $V \subset (\mathbb{C}^*)^n$ of dimension $\frac{n}{2}$ defined over $k_0(C)$ for some finite subset C of E. We need to show that the intersection $V \cap G^n$ is Zariski dense in V.

Let us define the set

$$\mathcal{X} = \{(s, t) \in \mathbb{R}^{2n} : \exp_{\mathbb{E}}(\epsilon t + s) \in V\}.$$

Note that if (s,t) is in $\mathcal{X} \cap ((\mathbb{Z}+\alpha\mathbb{Z})^n \times \mathbb{R}^n)$, where $\alpha = \mathrm{Re}(\tau) - \frac{\mathrm{Re}(\epsilon)}{\mathrm{Im}(\epsilon)} \mathrm{Im}(\tau)$, then the corresponding point $y := \exp_{\mathbb{E}}(\epsilon t + s)$ is in $V \cap G^n$ (see 7.1.1). Thus, in order to find points in the intersection $V \cap G^n$, we shall look for points (s,t) in $\mathcal{X}$ with $s \in (\mathbb{Z}+\alpha\mathbb{Z})^n$.

As in the previous chapter, our strategy is to find an implicit function for $\mathcal{X}$ defined on an open set $S \subset \mathbb{R}^n$, assigning to every $s \in S$ a point $t(s) \in \mathbb{R}^n$ such that $(s,t(s)) \in \mathcal{X}$. Since $(\mathbb{Z}+\alpha\mathbb{Z})^n$ is dense in $\mathbb{R}^n$, the intersection $S \cap (\mathbb{Z}+\alpha\mathbb{Z})^n$ is non-empty, and therefore we can find points $(s,t(s))$ in $\mathcal{X}$ with $s \in (\mathbb{Z}+\alpha\mathbb{Z})^n$.

Let $\mathcal{R}$ be an o-minimal expansion of the real ordered field in a countable language in which the function $\wp$ is *locally definable* and having constants for the real and imaginary parts of each element of $k_0(C)$. The existence of such a structure $\mathcal{R}$ is explained in Section 5.3.

Note that the set $\mathcal{X}$ is locally definable in $\mathcal{R}$.

The proof of Proposition 7.3.1 relies on the following main lemma:

Lemma 7.3.2 (Main Lemma). *Suppose (s^0,t^0) is an $\mathcal{R}$-generic point of $\mathcal{X}$, i.e. $\dim_{\mathcal{R}}(s^0,t^0) = \dim_{\mathbb{R}} \mathcal{X} = n$. Then $\dim_{\mathcal{R}}(s^0) = n$.*

To prove the Main Lemma, define the following set:

Definition 7.3.3. For $s \in \mathbb{C}^n$ we define the set

$$L_s = \{(x,\bar{x}) \in \mathbb{C}^{2n} : (x+\bar{x}) + \beta^{-1} i (x-\bar{x}) = 2s\}.$$

With the same proof as in Chapter 6, we have:

Lemma 7.3.4. *Suppose $s \in \mathbb{R}^n$. Then for all linearly independent $(m^1,n^1),\dots,(m^k,n^k) \in \mathbb{Z}^{2n}$ ($m^i,n^i \in \mathbb{Z}^n$), we have*

$$\dim(m,n)\cdot L_s \geq \frac{k}{2}.$$

Lemma 7.3.5. *Let $s \in \mathbb{R}^n$. Then the pair $(L_s, V \times V^c)$ is K-rotund.*

The proof of the Main Lemma of Chapter 6 from the analogous lemmas (see the end of Section 6.3.1) also works word by word in the new case.

The next lemma follows from the Main Lemma by the same argument as in Chapter 6.

Lemma 7.3.6. *Suppose (s^0,t^0) is an $\mathcal{R}$-generic point of $\mathcal{X}$. There is a continuous $\mathcal{R}$-definable function $s \mapsto t(s)$ defined on a neighbourhood $S \subset \mathbb{R}^n$ of s^0 and taking values in $\mathbb{R}^n$ such that for all $s \in S$, the point $y(s) := \exp_{\mathbb{E}}(\epsilon t(s) + s)$ is in V.*

Finally, also the proof of Proposition 7.3.1 from the lemma above is the same as the corresponding proof in Chapter 6, this time using the density of $\mathbb{Z} + \alpha\mathbb{Z}$ in $\mathbb{R}$, instead of that of the subgroup Q.

The question of whether the model on E is ω-saturated is open. Let us simply note that the remarks in Section 6.4 can be easily adapted to this case.

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