Summer School on Modern Knot Theory: Aspects in Algebra, Analysis, Biology, and Physics

Khovanov homology

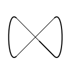
June 7, 2017

Exercises

1. Compute Khovanov homology of a trefoil, of the 1 and 3-crossing unknot (do one and three Reidemeister 1-moves on the standard unknot), and of the Hopf link.

   Solution: See [BN02] for the trefoil knot.

   Unknots with R1 moves:

   First we compute Kh().

   We start with the 1-dimensional cube of smoothings:

   0 1

   Apply a TQFT including degree shifts:

   $$V \xrightarrow{\Delta} V \otimes V\{1\}$$

   We compute

   $$\mathrm{Kh}^0 = \ker(\Delta) = 0$$

and

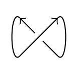
$$\begin{aligned}
\mathrm{Kh}^1 &= \frac{V \otimes V\{1\}}{\mathrm{im}(\Delta)} \\
&= \frac{\langle 1 \otimes 1, 1 \otimes x, x \otimes 1, x \otimes x \rangle}{\langle x \otimes x, 1 \otimes x + x \otimes 1 \rangle}\{1\} \\
&= \langle 1 \otimes 1, 1 \otimes x \rangle \{1\} \\
&= \mathbb{Z}_{(1)} \oplus \mathbb{Z}_{(3)}
\end{aligned}$$

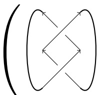
we make an overall shift by $[-n_-]\{n_+ - 2n_-\} = [-1]\{-2\}$ (where $[*]$ denotes homological shift and $\{*\}$ denotes degree shift) and we find the following:

Kh() =

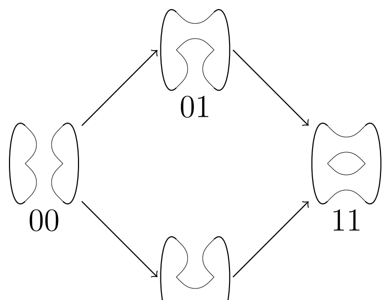
j \ i	0	1	2
-1	$\mathbb{Z}$		
0			
1	$\mathbb{Z}$		

so in the end, as desired,

Kh() $= V$

Next we compute Kh().

Again start with the cube of smoothings:



and apply our TQFT with shifts:

$$\begin{array}{ccccc} & & V\{1\} & & \\ & \overset{m}{\nearrow} & & \overset{\Delta}{\searrow} & \\ V \otimes V & & & & V \otimes V\{2\} \\ & \underset{m}{\searrow} & & \underset{-\Delta}{\nearrow} & \\ & & V\{1\} & & \end{array}$$

giving us the chain complex

$$0 \to V \otimes V \xrightarrow{m \oplus m} (V \oplus V)\{1\} \xrightarrow{\Delta \oplus -\Delta} V \otimes V\{2\} \to 0$$

we find that

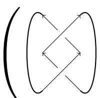
$$\mathrm{Kh}^0 = \ker(m \oplus m) = \langle x \otimes x, x \otimes 1 - 1 \otimes x \rangle = \mathbb{Z}_{(-2)} \oplus \mathbb{Z}_{(0)},$$

$$\begin{aligned} \mathrm{Kh}^1 &= \frac{\ker(\Delta \oplus -\Delta)}{\mathrm{im}(m \oplus m)}\{1\} \\ &= \frac{\langle (1,1), (x,x) \rangle}{\langle (1,1), (x,x) \rangle}\{1\} \\ &= 0 \end{aligned}$$

and

$$\begin{aligned}\mathrm{Kh}^2 &= \frac{\langle 1\otimes 1, 1\otimes x, x\otimes 1, x\otimes x\rangle}{\langle x\otimes x, 1\otimes x + x\otimes 1\rangle}\{2\}\\ &= \langle 1\otimes 1 x\otimes 1\rangle\{2\}\\ &= \mathbb{Z}_{(2)}\oplus\mathbb{Z}_{(4)}\end{aligned}$$

applying the overall shift [0]{2}, we find that

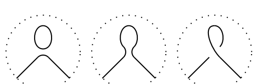
Kh() =

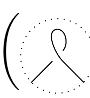
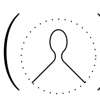
j \ i	0	1	2
0	$\mathbb{Z}$		
2	$\mathbb{Z}$		
4			$\mathbb{Z}$
6			$\mathbb{Z}$

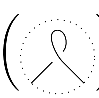
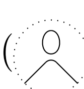
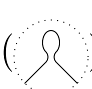
2. Prove that Khovanov link homology is indeed a link invariant: independent on the choice of knot diagram.

Solution:

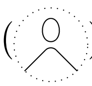
Invariance of R1:

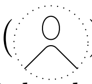
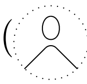


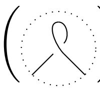
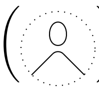
First we consider the Reidemeister move 1. We want to show that Kh() is the same as Kh(). We will use the cancellation principle for chain complexes: If $\mathcal{C}'$ is a subcomplex of $\mathcal{C}$ then the homology of $\mathcal{C}$ is the same as that of $\frac{\mathcal{C}}{\mathcal{C}'}$. Or, if the quotient $\frac{\mathcal{C}}{\mathcal{C}'}$ is acyclic, then the homology of $\mathcal{C}$ is the same as that of $\mathcal{C}'$. Now, using the skein relation for Kh, we have

$\mathcal{C}$() $= \mathcal{F}\Big($ $\mathcal{C}$() $\xrightarrow{m}$ $\mathcal{C}$() $\Big)$

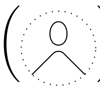
where $\mathcal{F}$ flattens a double complex to a single complex by summing down diagonals. Consider the subcomplex

$$\mathcal{C}' = \mathcal{F}\Big(\mathcal{C}($$  $$)_1 \xrightarrow{m} \mathcal{C}($$  $$)\Big)$$

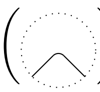
where $\mathcal{C}($  $)_1$ denotes the subcomplex of $\mathcal{C}($  $)$ where the cycle shown inside of the dotted circle always has label 1.

Since 1 is the multiplicative unit, m is an isomorphism and so $\mathcal{C}'$ is acyclic. Thus $\mathcal{C} = \mathcal{C}\Big($  $\Big)$ has the same homology as $\frac{\mathcal{C}}{\mathcal{C}'}$ which is just $\mathcal{C}\Big($  $\Big)\Big/_{1=0 \text{ in special cycle}}$.

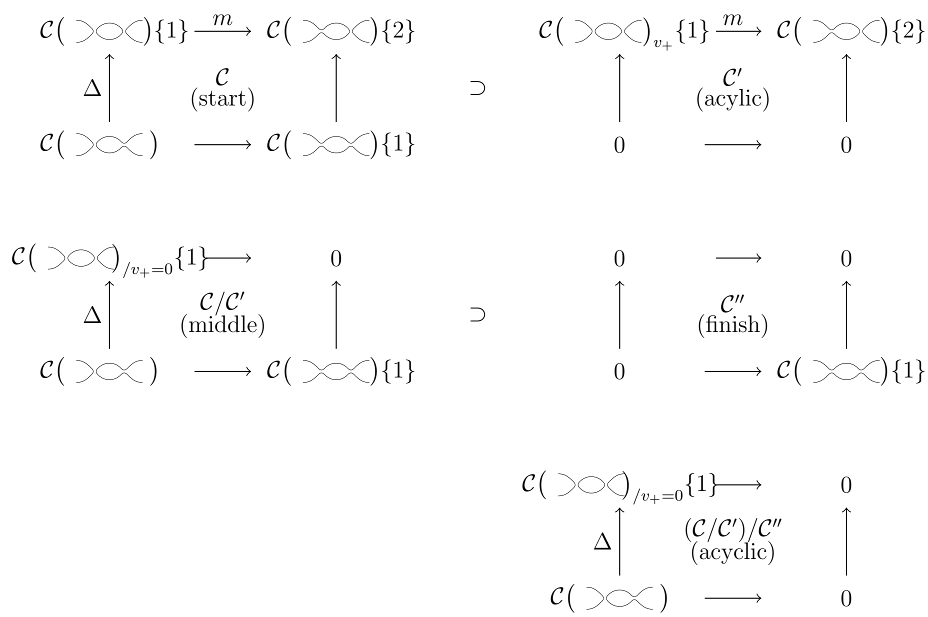
But setting 1=0 in one copy of V turns that copy into $\langle x \rangle \simeq \mathbb{Z}$ and tensoring with $\mathbb{Z}$ does nothing, so $\mathcal{C}$ has the same homology as

$$\mathcal{C}\Big($$  $$\Big)\Big/_{1=0 \text{ in special cycle}}$$

which has the same homology as

$$\mathcal{C}\Big($$  $$\Big).$$

Invariance of the 2nd Reidemeister move follows from the following picture proof:



3. Prove that the span of Khovanov homology of an adequate knot is equal to the crossing number.

Solution: Recall, a knot K is *adequate* if there exists a diagram D with the following properties:

- For each crossing of D, the strands in D_0 (the all 0 smoothing) replacing the crossing are in different state circles
- For each crossing of D, the strands in D_1 (the all 1 smoothing) replacing the crossing are in different state circles

Fact: Any adequate diagram has the minimal number of crossings over all diagrams.

Let K be an adequate knot and D an adequate diagram with crossings $1, 2, ..., n$. Then we only have nonzero chain groups in homological degree 1,2,...,n. Thus the homology can only be nonzero in degrees between 1 and n (we apply a homological shift at the end of the construction but this does not change the span), so the homological span of $\mathrm{Kh}(K)$ is less than or equal to n. But D is an adequate diagram, so n is the crossing number of K.

It remains to show that the homology is nontrivial in the extreme gradings. In the lowest homological grading, the Khovanov chain group is generated by the circles in the all 0 smoothing. Since the two strands of each smoothed crossing lie in different state circles, every time we change a smoothing from 0 to 1 we are joining circles. Thus all of the maps from homological degree 0 to 1 are multiplication maps. Thus $x \otimes x \otimes ... \otimes x$ represents a nontrivial homology class in degree 0.

Similarly in the all 1 smoothing, since the two strands of each smoothed crossing lie in different state circles, each map from degree $n-1$ to degree n is comultiplication. Thus $1 \otimes 1 \otimes \cdots \otimes 1$ is not in the image of the differential from C^{n-1} to C^n, so Kh^n is nontrivial.

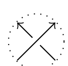
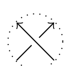
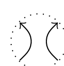
4. Compute the Khovanov homology of the connect sum of two knots.

Solution: There is a Kunneth formula for computing $Kh^{*,*}(K_1 \sqcup K_2)$ and a long exact sequence relating the Khovanov homology of $K_1 \# K_2$ and $K_1 \sqcup K_2$ (see [Kho00], section 7.4).

5. Show that alternating knots have thin Khovanov homology.

6. Show that quasi-alternating knots have thin Khovanov homology.

Recall the set of quasi-alternating links is defined to be the smallest set $\mathcal{Q}$ of links containing the unknot with the property that if a link L has a diagram D and a crossing c for which both resolutions D_0 and D_1 of D at c are in $\mathcal{Q}$ and $\det(D) = \det(D_0) + \det(D_1)$, then L is in $\mathcal{Q}$.

Let $D_+ =$  $D_- =$  $D_v =$  $D_h =$  be link diagrams agreeing outside of the dotted circles and let L_+, L_-, L_v, L_h be the corresponding links.

Lemma 1. *If det*(L_v), *det*$(L_h) > 0$ *and det*$(L_+) =$ *det*$(L_v) +$ *det*(L_h), *then*

$$\sigma(L_v) - \sigma(L_+) = 1$$

and

$$\sigma(L_h) - \sigma(L_+) = e$$

where

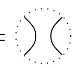
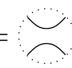
$$e = \#\textit{negative crossings in } D_h - \#\textit{negative crossings in } D_+$$

Proof. See `https://arxiv.org/abs/0708.3249`. □

The existence of the following long exact sequences follows from the definition of Khovanov homology and lifts the skein relation for the Jones polynomial. Here we use the grading $\delta = j - i$ (Note: that this differs from the convention used in the lecture where we had $\delta = 2j - i$).

$$\cdots \to \widetilde{Kh}^{*-\frac{e}{2}}(L_h) \to \widetilde{Kh}^{*}(L_+) \to \widetilde{Kh}^{*-\frac{1}{2}}(L_v) \to \widetilde{Kh}^{*-\frac{e}{2}-1}(L_h) \to \dots$$

$$\cdots \to \widetilde{Kh}^{*+\frac{1}{2}}(L_v) \to \widetilde{Kh}^{*}(L_-) \to \widetilde{Kh}^{*-\frac{e}{2}}(L_h) \to \widetilde{Kh}^{*-\frac{1}{2}}(L_v) \to \dots$$

Putting all of this together, letting $L =$  , $L_0 =$  and $L_1 =$  , we get the following:

Proposition 2. *If det(L_0), det(L_1) > 0 and det(L) $=$ det(L_0) $+$ det(L_1), then we have a long exact sequence*

$$\cdots \to \widetilde{Kh}^{*-\frac{\sigma(L_1)}{2}}(L_1) \to \widetilde{Kh}^{*-\frac{\sigma(L)}{2}}(L) \to \widetilde{Kh}^{*-\frac{\sigma(L_0)}{2}}(L_0) \to \widetilde{Kh}^{*-\frac{\sigma(L_1)}{2}-1}(L_1) \to \dots$$

Proof of Theorem. Any quasi-alternating link satisfies the assumptions of Prop 2. Using the long exact sequence in Prop 2, we can see that L is thin if L_0 and L_1 are thin. Now replace $L = L_0$ and $L = L_1$ and continue using Prop 2. Eventually we reduce the problem to the question of whether the unlink is thin (it is). □

7. Compute the s-invariant of positive knots.

   Solution: Positive knots have a canonical smoothing in degree 0 with r circles. The state with minimal q-grading in degree zero is the one in which every circle is labeled by x. (This is a generator of H^0 and uniquely represented since C^{-1} is trivial.) So $s_{min} = -r+n$ and $s = -r+n+1$, where n is the number of crossings.

8. Compute s-invariant of amphichiral knots.

   Solution: An amphichiral knot K has $s(K) = -s(K)$ (see [Ras10], Prop. 3.9) and so $s(K) = 0$.

9. Compute the s-invariant for $T(p, q)$ torus knots.

   Solution: For positive knots, $s(K) = 2g(K)$ (see [Ras10]). Draw $T(p, q)$ as a braid closure and describe the Seifert surface of this diagram. Use the Euler characteristic to find the genus (see `https://hal.archives-ouvertes.fr/hal-01208217/document`).

References

[BN02] Dror Bar-Natan. On Khovanov's categorification of the Jones polynomial. *Alg. and Geom. Topology*, 2:337–370, 2002. URL: `https://arxiv.org/abs/math/0201043`.

[Kho00] Mikhail Khovanov. A categorification of the Jones polynomial. *Duke Math. J.*, 101(3):359–426, 2000. URL: `https://arxiv.org/abs/math/9908171`.

[Ras10] Jacob Rasmussen. Khovanov homology and the slice genus. *Inventiones Math.*, 182(2):419–447, 2010. URL: `https://arxiv.org/abs/math/0402131`.