INTRODUCTION TO KHOVANOV LINK HOMOLOGY

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History of Knot Theory

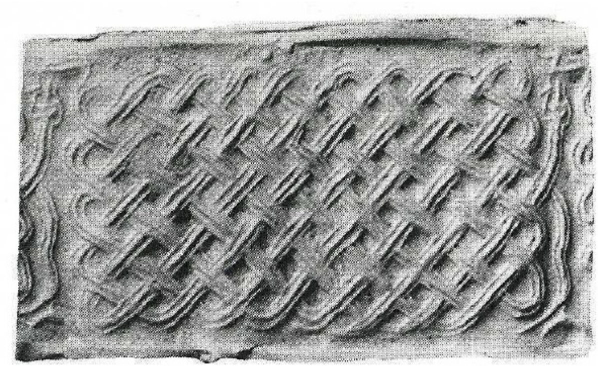
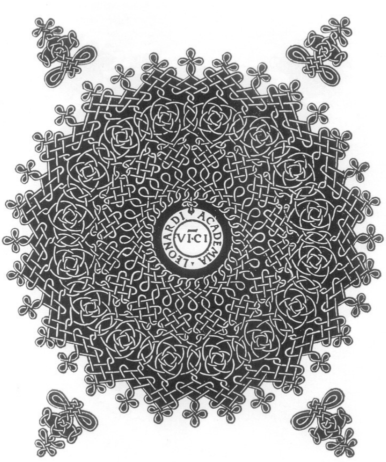


FIGURE: Snake with interlacing coil, Cylinder seal, Ur, Mesopotamia, 2600-2500 B.C.

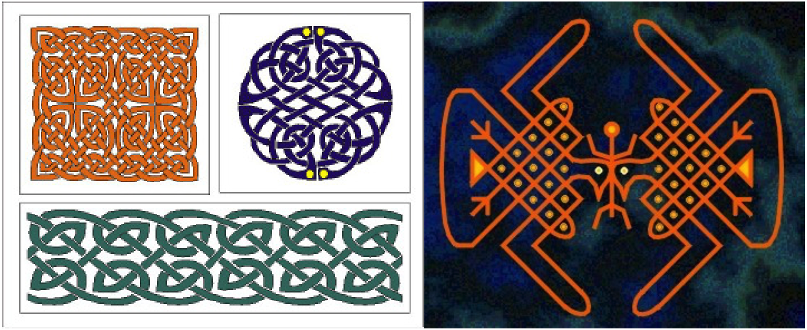
History of Knot Theory



"Leonardo spent much time in making a regular design of a series of knots so that the cord may be traced from one end to the other, the whole filling a round space..."
Bain, G.: Celtic Art - the Methods of Construction, Dover, New York, 1973.

A Mathematical Analysis of Knotting and Linking in Leonardo da Vinci's Cartelle of the Accademia Vinciana

History of Knot Theory

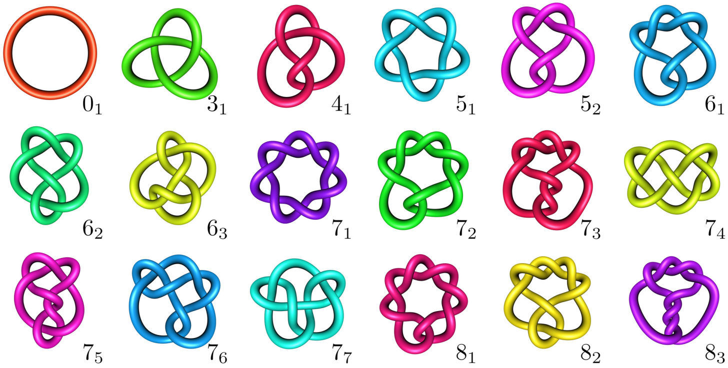


1833 Gauss developed the Gauss linking integral for computing linking numbers of two knots

1867 Lord Klevin: theory of vortex atoms

1870-99 P.G. Tait, T.P. Kirkman, C.N. Little: tables of knots up to 10 crossings

Knots and links



Def.

A knot is a smooth embedding $f : S^1 \to \mathbb{R}^3$. A link of k-components is an embedding of a disjoint union of k S^1's.

20th century Knot Theory

1923 J. W. Alexander discovered the first knot polynomial (generator of the Alexander ideal)

1928 K. Reidemeister: diagrammatic knot theory

1938 M. Dehn: developed a method, ”surgery on knots” for constructing 3-dimensional manifolds

1960-62 W.B.R. Lickorish, A.H. Wallace proved that

Theorem (Lickorish-Wallace)

Every closed, connected, oriented 3-manifold can be obtained by doing surgery on a link in S^3

1969 J. Conway: discovered diagrammatics for the Alexander polynomial

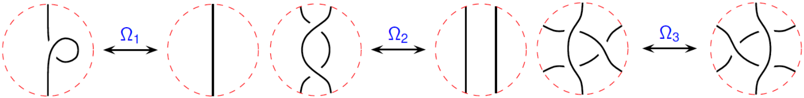
1984 V. Jones related knot theory to physics: Jones polynomial

When are two knots the same?

Two knots are isotopic if one can be smoothly deformed to the other.

How to show that knots are equal?

- Construct an isotopy
- Use Reidemeister's theorem



Theorem (Reidemeister)

Two knot diagrams represent isotopic knots iff they are related by a finite sequence of Reidemeister moves.

How to distinguish knots?

Show that there is no isotopy... or come up with something better!

How can we distinguish knots?

Knot invariants

A property of knots that is the same for isotopic knots.

Catch-22

- If invariant takes different values knots must be different.
- If invariant is the same knots are not necessarily isotopic.

Examples

- numerical: number of components, linking number, *p*-colorings
- polynomial: Alexander, Jones, Kauffman
- groups: *p*-colorings, knot group (Alexander module, quandles)
- homology theories: Khovanov, knot Floer, etc.

”In these days the angel of topology and the devil of abstract algebra fight for the soul of each individual mathematical domain.”

H. Weyl

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H. Weyl

Why algebra?

- Topological objects: easy to visualize, hard to compare
- Algebraic objects: easier to compare, sometimes diagrammatic
- Idea: Map topology to algebra in such a way to retain lots of information
- Construct a functor from category of knots to some algebraic category
- Invariants provide relations between algebraic and geometric properties of knots, and a combinatorial approach to studying low-dimensional manifolds.

Find the missing sign :)

$J(L)$ OF A LINK L IS DEFINED BY THE SKEIN RELATION:

$$\begin{aligned}[L_+] &= q^{1/2}[L_0] + q^{-1/2}[L_\infty] \quad \text{Skein relation} \\ J(L) &= q^{\text{c(writhe)}}[L] \\ J(\bigcirc) &= (q + \frac{1}{q}) \\ J(L \sqcup \bigcirc) &= (q + \frac{1}{q})J(L)\end{aligned}$$

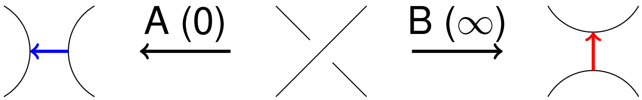
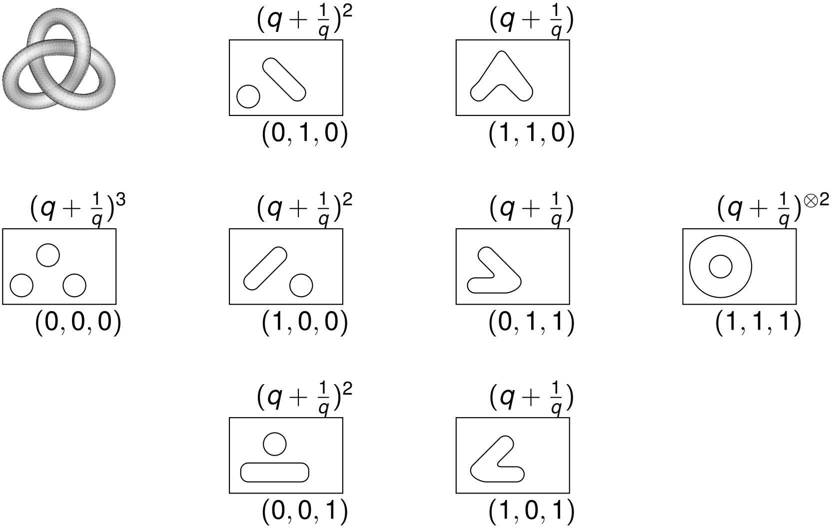


FIGURE: The resolutions of a crossing and their traces in a link diagram.

KAUFMAN'S STATE SUM FOR THE JONES



MOTIVATION

CATEGORY OF COBORDISMS Cob_n

- Objects: Closed oriented $(n-1)$-dim manifolds
- Morphisms: oriented n-manifolds with boundaries

n-DIM TQFT

tensor functor from Cob_n to additive symmetric monoidal category.

UNORIENTED 1-DIM TQFT OVER $\mathbf{k}$

$F(\bullet) = V$ $\quad$ $F($  $) = \mathbf{k} \to V \otimes V$ $\quad$ $F($  $) = V \otimes V \to \mathbf{k}$

THEOREM

$$\{\textit{Unoriented 1-d TQFTs over a field } \mathbf{k}\}$$
$$\Updownarrow$$
$$\{\textit{Finite dimensional } \mathbf{k} \textit{ vector spaces with a non-degenerate symmetric bilinear form}\}$$

Frobenius algebras

Def.1 A finite-dimensional, unital, associative algebra A equipped with a nondegenerate bilinear form $\sigma : A \times A \to \mathbf{k}$ such that $\sigma(a \cdot b, c) = \sigma(a, b \cdot c)$

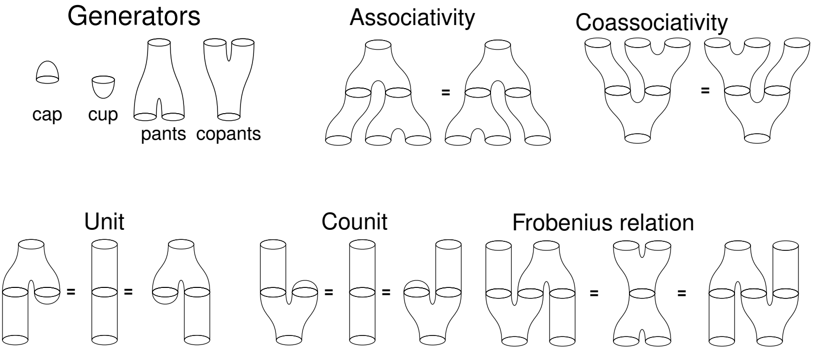
Def.2 $(A, \mu, \eta, \Delta, \epsilon)$ such that

- (A, μ, η) is a unital associative algebra,
- (A, Δ, ϵ) is a counital coassociative coalgebra, and
- Frobenius relation $(\mu \otimes 1_A) \circ (1_A \otimes \Delta) = \Delta \circ \mu = (1_A \otimes \mu) \circ (\Delta \otimes 1_A)$

Examples

- Matrix algebra with a trace
- Group ring of a finite group with $\sigma(a, b)$ a coefficient of unit in ab
- $A = \mathbf{k}[x, y]/(x^2, y^2)$

Topology to algebra



Theorem

{*2DTQFT over* **k**} $\leftrightarrow$ {*Commutative Frobenius algebras over* **k**}

KHOVANOV FOR TRIVIAL LINKS

- $Kh(\emptyset) = R$ commutative ring, say $\mathbb{Z}$
- $Kh(unknot) = H(\bigcirc) = A$ commutative Frobenius R-algebra
- $Kh(\bigcirc\ \bigcirc\ \bigcirc \ldots \bigcirc) = A^{\otimes k}$

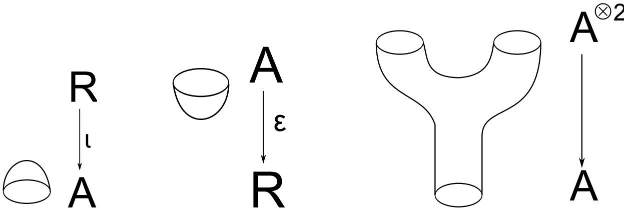


FIGURE: Unit, trace map and commutative multiplication.

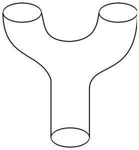
- How can we get the homology theory related to the Jones polynomial $J(K) \in \mathbb{Z}[q^{\pm 1}]$

Khovanov for trivial links

lifting the Jones polynomial

- $J(\bigcirc) = q + \frac{1}{q} = qdimA$
- $Kh(\bigcirc) = \mathbb{Z}[x]/x^2 = A$ commutative graded Frobenius algebra with $deg(x) = -1$ and $deg(1) = 1$
- unit $\eta : \mathbb{Z} \to A$, counit $\varepsilon : A \to \mathbb{Z}$ $\varepsilon(x) = 1$ and $\varepsilon(1) = 0$

$$\begin{array}{ccc} A^{\otimes 2} & \stackrel{m}{\to} & A \\ 1 \otimes 1 & \mapsto & 1 \\ 1 \otimes x & \mapsto & x \\ x \otimes x & \mapsto & 0 \end{array}$$

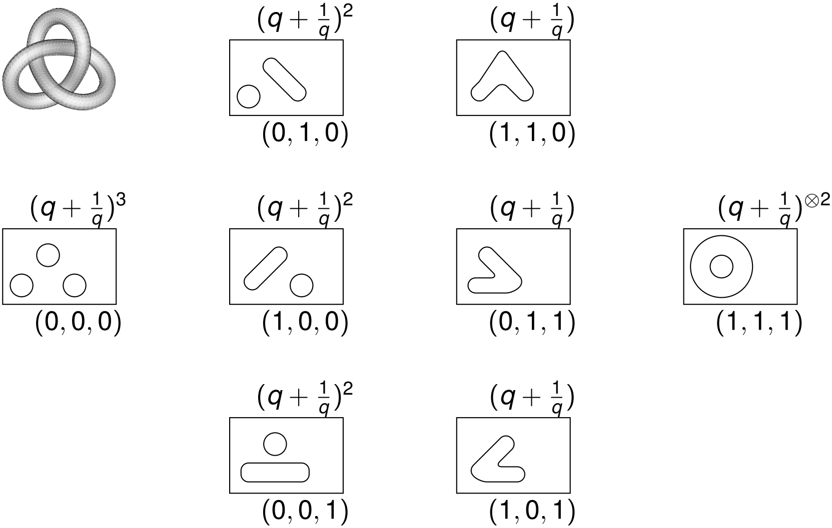


deg m =1

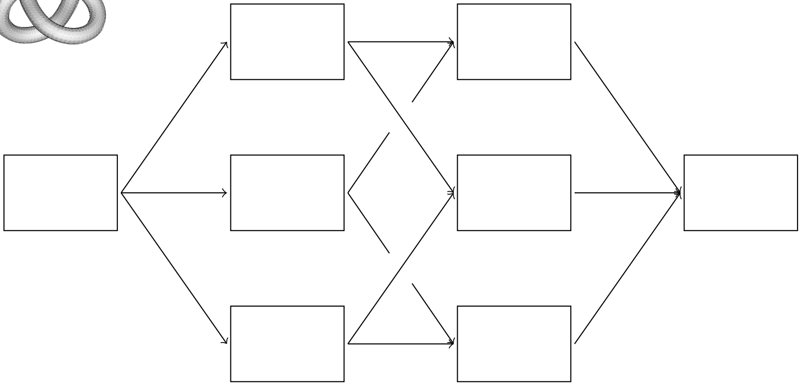
$$\begin{array}{ccc} A & \stackrel{\Delta}{\to} & A^{\otimes 2} \\ 1 & \mapsto & 1 \otimes x + x \otimes 1 \\ x & \mapsto & x \otimes x \end{array}$$

- Degree of the algebra map is equal to $-\chi(S)$ where S is the corresponding cobordism

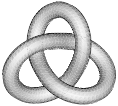
KAUFMAN'S STATE SUM FOR THE JONES

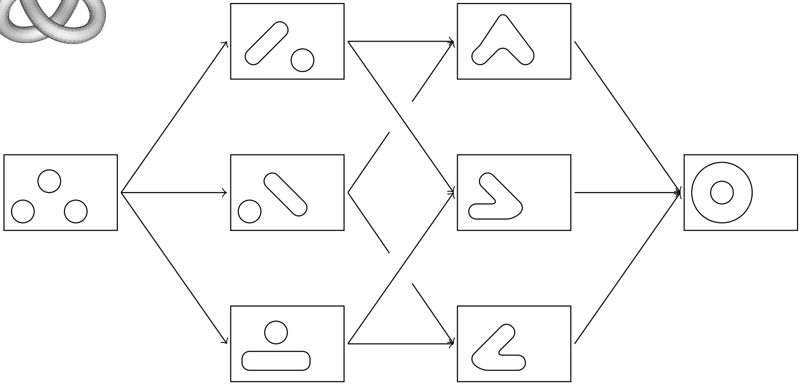


HYPERCUBE

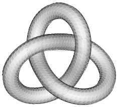


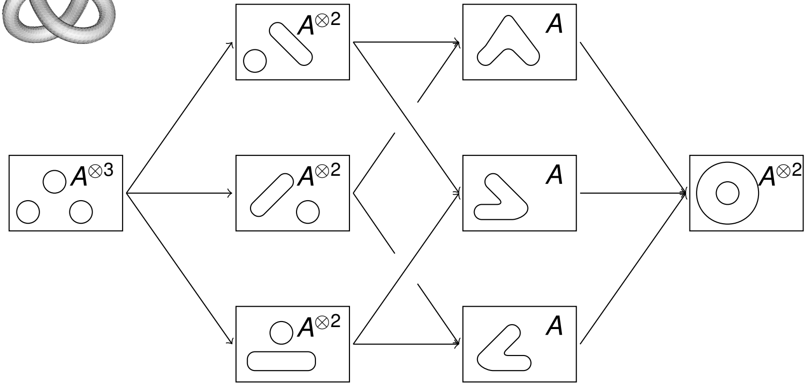
KAUFFMAN STATES



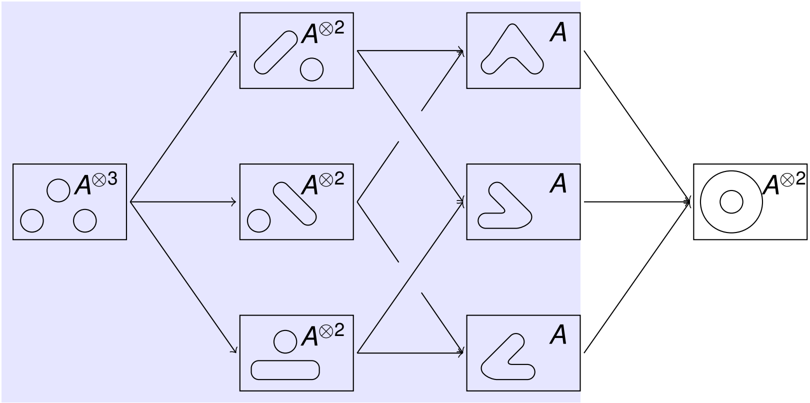


Kauffman states and spaces





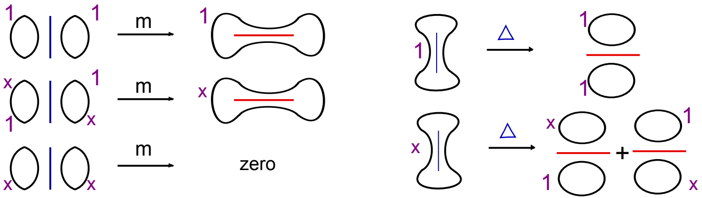
MULTIPLICATION: THE NO. OF CIRCLES DECREASES



Khovanov chain complex $C_{i,j}(D)$

- Enhanced Kauffman state S
- Khovanov chain group $C_i(D) = \bigoplus_j C_{i,j}(D)$ is a sum of free group freely generated by all enhanced Kauffman states.
- The differential $d_{i,j}(S) : C_{i,j}(D) \to C_{i-1,j}(D)$ is def. by

$$d_{i,j}(S) = \sum_{S', [S:S']=1} (-1)^{t(S:S')} S'$$



Khovanov homology
$Kh(D) = \bigoplus_{i,j} Kh^{i,j}(D)$

$Kh(D)$ is the homology of the total complex $C(D)$

If we ignore the grading, for a link diagram D with n crossings:

- there are 2^n resolutions
- they are arranged into an n-dimensional cube of resolutions
- each edge of a cube is assigned either m or $\triangle$
- Each square is commutative
- Sprinkle signs to make each square anticommutative and get $d^2 = 0$
- the resulting homology does not depend on the choice of signs.

LINK HOMOLOGY A LA BAR NATAN

Jones polynomial	Khovanov link homology
$J(\varnothing) = 1$	$Kh(\varnothing) = (0 \to \mathbb{Z} \to 0)$
$J(\bigcirc) = (q + q^{-1})$	$Kh(\bigcirc) = (0 \to A \to 0)$
$J(\bigcirc \sqcup L) = (q + q^{-1})J(L)$	$Kh(\bigcirc \sqcup L) = A \otimes Kh(L)$
$J(L_+) = J(L_0) - q^{-1}J(L_\infty)$	$Kh(L_+) =$ $\mathrm{Tot}(0 \to Kh(L_0) \xrightarrow{d} Kh(L_\infty)\{1\} \to 0)$

NOTE 1. Map d will not be defined, but it is induced by a saddle cobordism and the construction picture is called *mapping cone.*

NOTE 2. *Dror has another beautiful construction using "geometric complexes." Check it out!*

Khovanov homology

Theorem

Khovanov homology $Kh(L)$ does not depend on the choice of diagrams, i.e. it is a link invariant.

Proof.

We need to show the invariance of $Kh(L)$ under Reidemeister moves, which is usually done by taking the more complicated diagram and simplifying its invariant to get to the invariant of the ”simpler” one.

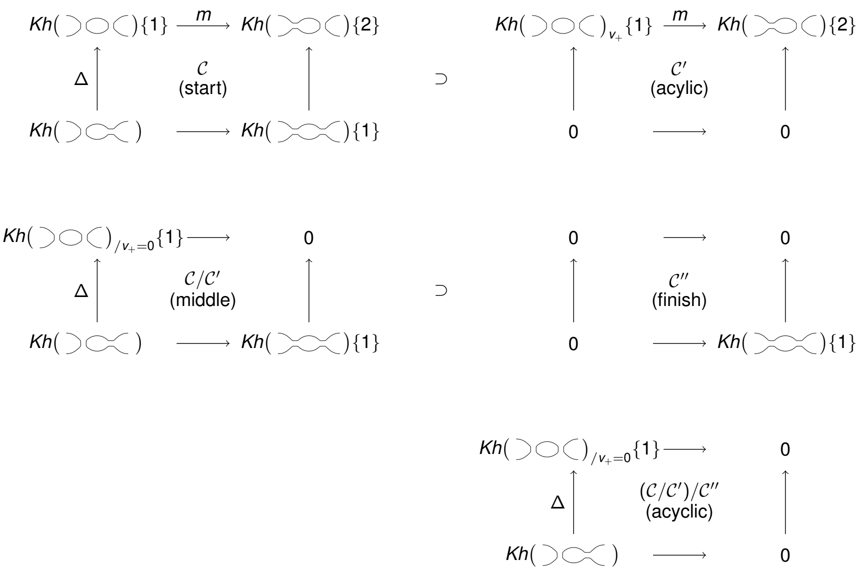
Lemma

Let C be a chain complex and let $C' \subset C$ be a sub chain complex.

- *If C' is acyclic then $H(C) \cong H(C/C')$*
- *Likewise, if C/C' is acyclic $H(C) \cong H(C')$*

□

INVARIANCE OF $Kh(L)$ UNDER R_2

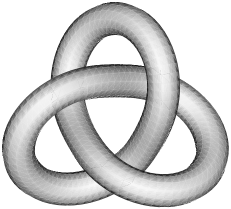


Khovanov homology categorifies the Jones polynomial

Theorem ($\chi(Kh(L)) = J(L)$)

The Khovanov polynomial is a link invariant, whose Euler characteristic is equal to the $J(L)$ $(t = -1)$.

	0	1	2	3
9				**1**
7				
5			**1**	
3	**1**			
1	**1**			



$$Kh(3_1)(t,q) = q + q^3 + t^2q^5 + t^3q^9$$
$$Kh(3_1)(-1,q) = J(3_1) = q + q^3 + q^5 - q^9$$

Khovanov homology vs. Jones polynomial

For every statement about the Jones polynomial there is a ”lifting”-corresponding statement in terms of Khovanov homology.

Theorem

Span of the Jones polynomial/Khovanov homology gives the crossing number of alternating knots.

Theorem

The same is true for adequate knots.

Theorem

Skein relation is categorified by the long exact sequence in homology:

$$\to Kh_{i-1}(L_\infty) \to Kh_i(L_0) \to Kh_i(L_+) \to Kh_i(L_\infty) \to$$

Proof.

For a diagram D of L: $0 \to C(D_0) \to C(D) \to C(D_\infty) \to 0$ □

- Alternating links: $Kh(L)$ is determined by $J(L)$ and signature of L
- Non-alternating: $Kh(L)$ is stronger than $J(L)$

$Kh(5_1)$		**i**					
		-5	**-4**	**-3**	**-2**	**-1**	**0**
	-3						$\mathbb{Z}$
	-5						$\mathbb{Z}$
	-7				$\mathbb{Z}$		
	-9				$\mathbb{Z}_2$		
j	**-11**		$\mathbb{Z}$	$\mathbb{Z}$			
	-13		$\mathbb{Z}_2$				
	-15	$\mathbb{Z}$					

$Kh(10_{132})$		**i**							
		-7	**-6**	**-5**	**-4**	**-3**	**-2**	**-1**	**0**
	-1							$\mathbb{Z}$	$\mathbb{Z}$
	-3							$\mathbb{Z}_2$	$\mathbb{Z}$
	-5					$\mathbb{Z}$	$\mathbb{Z}^2$		
j	**-7**				$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$		
	-9				$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z}$			
	-11		$\mathbb{Z}$	$\mathbb{Z}$					
	-13		$\mathbb{Z}_2$						
	-15	$\mathbb{Z}$							

- $J(5_1) = J(10_{132}) = q^{-3} - q^{-5} + q^{-7} - q^{-15}$

Khovanov homology

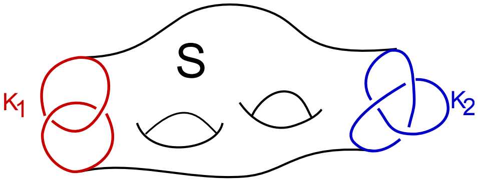
- Alternating links: $Kh(L)$ is determined by $J(L)$ and signature of L
- Non-alternating: $Kh(L)$ is stronger than $J(L)$; e.g., 5_1 and 10_{132}

$Kh(5_1)$		i					
		-5	-4	-3	-2	-1	0
j	-3						$\mathbb{Z}$
	-5						$\mathbb{Z}$
	-7				$\mathbb{Z}$		
	-9				$\mathbb{Z}_2$		
	-11		$\mathbb{Z}$	$\mathbb{Z}$			
	-13		$\mathbb{Z}_2$				
	-15	$\mathbb{Z}$					

$Kh(10_{132})$		i							
		-7	-6	-5	-4	-3	-2	-1	0
j	-1							$\mathbb{Z}$	$\mathbb{Z}$
	-3							$\mathbb{Z}_2$	$\mathbb{Z}$
	-5					$\mathbb{Z}$	$\mathbb{Z}^2$		
	-7				$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$		
	-9				$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z}$			
	-11		$\mathbb{Z}$	$\mathbb{Z}$					
	-13		$\mathbb{Z}_2$						
	-15	$\mathbb{Z}$							

- $J(5_1) = J(10_{132}) = q^{-3} - q^{-5} + q^{-7} - q^{-15}$

KHOVANOV HOMOLOGY



Objects	Classical	Categorified
Knots and links $K \subset \mathbb{R}^3$	Jones polynomial $V(K)$	Khovanov homology groups $Kh(K)$
Link cobordisms $S \subset \mathbb{R}^3 \times [0,1]$		Homomorphisms $Kh(S) : Kh(K_1) \to Kh(K_2)$

KHOVANOV HOMOLOGY KH IS A FUNCTOR

Category of link cobordisms $\overset{Kh}{\to}$ Category of abelian groups

FUNCTORIALITY

- A link cobordism $S : K_1 \to K_2$ induces a map on Khovanov homology $Kh(S) : Kh(K_1) \to Kh(K_2)$ of bidegree $(0, \chi(S))$
- Let us work over $\mathbb{Q}$, $\mathbb{Q}[x]/x^2$ for a slide

THEOREM

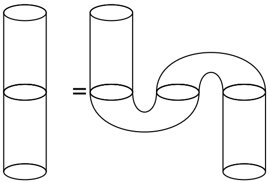
Khovanov homology of a mirror L' of a link L is isomorphic to $Kh(L)$.

PROOF.

Take cobordism $S : L \sqcup L' \to \emptyset$ which induces a map of bidegree $(0, \chi(S)) = (0, 0)$:

$$S_* : Kh(L) \otimes Kh(L') \to Kh(\emptyset)$$

the bilinear form is non-degenerate



$$(Kh(L) \otimes Kh(L'))^{(0,0)} = \oplus_{i,j} Kh^{i,j}(L') \otimes Kh^{-i,-j}(L)$$

Applications of Khovanov homology

- Khovanov homology is fully combinatorial, with its roots in representation theory, yet it is very powerful.
- It contains lots of information about the geometry of the knot.
- Khovanov homology is a stronger invariant than the Jones polynomial.
- Khovanov homology detects the unknot
- Open question: Can the Jones polynomial distinguish the unknot.
- Provides a lower bound for the slice genus.

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Thank you

Challenge: Find knots in Freiburg!