More on Khovanov homology

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Where were we?

- Classification of knots- using knot invariants!
- Q How can we get better invariants?
  A By upgrading/categorifying old ones!
- Khovanov link homology lifts the Jones polynomial.

$$J(L)(q) \quad = \quad \sum_j \left(\sum_i (-1)^i \mathrm{rkKh(L)} \right) q^j = \chi(Kh(L))$$

$$\text{skein relation} \quad \Longleftrightarrow \quad \text{long exact sequence of } \mathrm{Kh(L)}$$

- Q Why does the choice of algebra A make sense?
  A1 It's graded dimension is $J(\bigcirc)$
  A2 Frobenius algebra structure is an algebraic counterpart of the topology of 2d TQFT.

Khovanov homology is stronger than the Jones polynomial

$Kh(5_1)$	i					
j	-5	-4	-3	-2	-1	0
-3						$\mathbb{Z}$
-5						$\mathbb{Z}$
-7				$\mathbb{Z}$		
-9				$\mathbb{Z}_2$		
-11		$\mathbb{Z}$	$\mathbb{Z}$			
-13		$\mathbb{Z}_2$				
-15	$\mathbb{Z}$					

$Kh(10_{132})$	i							
j	-7	-6	-5	-4	-3	-2	-1	0
-1							$\mathbb{Z}$	$\mathbb{Z}$
-3							$\mathbb{Z}_2$	$\mathbb{Z}$
-5					$\mathbb{Z}$	$\mathbb{Z}^2$		
-7				$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$		
-9				$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z}$			
-11		$\mathbb{Z}$	$\mathbb{Z}$					
-13		$\mathbb{Z}_2$						
-15	$\mathbb{Z}$							

- Torsion is invisible for the Jones!
- $Kh(K)$ can have non-trivial groups of the same rank in gradings of different parity! They cancel out in the Euler characteristic.

Khovanov homology different coefficients

- So far: $Kh(K, \mathbb{Z})$ and $Kh(K, \mathbb{Q})$
- Next: $Kh(K, \mathbb{Z}_p)$ and $Kh(K, \mathbb{Z}_2)$ in particular.
- How do these homologies relate?

Theorem (Universal Coefficient Theorem)

$$H^n(C, \mathbb{Z}_p) \cong H^n(C, \mathbb{Z}) \otimes \mathbb{Z}_p \oplus Tor(H^{n+1}(C), \mathbb{Z}_p)$$

Or in the language of Khovanov link homology:

$$Kh^{i,j}(K, \mathbb{Z}_p) \cong Kh^{i,j}(K, \mathbb{Z}) \otimes \mathbb{Z}_p \oplus Tor(Kh^{i+1,j}(K, \mathbb{Z}_p))$$

How to compute?

$\mathrm{Tor}(A, B) = 0$ if A or B is free or torsion-free, $\mathrm{Tor}(\mathbb{Z}_p, \mathbb{Z}_q) = \mathbb{Z}_{\mathrm{GCD}(p,q)}$
$\mathbb{Z} \otimes \mathbb{Z}_p = \mathbb{Z}_p$, $\mathbb{Z}_p \otimes \mathbb{Z}_q = \mathbb{Z}_{GCD(p,q)}$, $\mathbb{Z}_2 \otimes \mathbb{Z}_3 = 0$, $\mathbb{Z}_6 \otimes \mathbb{Z}_4 = \mathbb{Z}_2$

Odd, even, unified Khovanov

- *Kh* Even (standard) Khovanov homology
- Kh_o Odd (Ozsváth, Rasmussen, Szabó 2007)
- Kh_u Unified theory over $R = \mathbb{Z}[\varepsilon]/(\varepsilon^2 - 1)$ (Putyra, 2010)
- mod$\mathbb{Z}_2$ even or odd theory computed over $\mathbb{Z}_2$

Relations between theories

- $Kh(L, \mathbb{Z}_2) \cong Kh_o(L, \mathbb{Z}_2)$
- If D' is the mirror of D then
  - Both Khovanov chain complexes and homologies are dual.
  - In odd Khovanov this is true only for homology.
- $C_u^{i,j}/(\varepsilon - 1) \cong C^{i,j}$ and $C_u^{i,j}/(\varepsilon + 1) \cong C_o^{i,j}$ (Putyra, 2010)
- Computing KH_u requires classification of R-modules...

Invariants in form of homological operations

- Algebraic operations on Kh and Kh_o arising from the action of Kh_u are link invariants!
- Trivial for alternating knots.
- Computations for all for all prime non-alternating knots with at most 16 crossings by A. Shumakovitch
- Among 201,702 knots with at most 15 crossings (not counting mirror images) there are 99 pairs with the same Kh and Kh_o but different homological operations!
- Among 33,672 knots with at most 14 crossings there are 9 pairs with the above property.
- Nothing could distinguish mirror images of these pairs!
- Kh_u is stronger than Kh and Kh_o combined!

Reduced Khovanov homology

$\widetilde{Kh}(L), \widetilde{Kh_o}(L)$

- Construction follows that of the corresponding Khovanov homology with one modification
  - Pick a basepoint on the knot diagram away from any crossings
  - Kauffman circle that contains the basepoint is labeled by x
- Reduced Khovanov homologies of a knot do not depend on the position of the basepoint.
- If L is a link, $\widetilde{Kh}(L, \mathbb{Z})$ depends on the choice of component the basepoint belongs to, but $\widetilde{Kh}(L, \mathbb{Z}_2)$ and $\widetilde{Kh_o}(L, \mathbb{Z})$ do not!
- $Kh(L, \mathbb{Z}_2) \cong \widetilde{Kh}(L, \mathbb{Z}_2) \otimes \mathbb{Z}_2[x]/x^2$ (Shumakovitch, 2004)
- $Kh_o(L, \mathbb{Z}) \cong \widetilde{Kh_0}(L, \mathbb{Z})\{-1\} \oplus \widetilde{Kh_0}(L, \mathbb{Z})\{+1\}$

$Kh(16n197566, \mathbb{Z})$

	-2	-1	0	1	2	3	4	5	6	7	8	9	10
29													**1**
27												**4**	$\mathbf{1_2}$
25											**7**	**1,** $\mathbf{3_2}$ $\mathbf{1_4}$	
23										**12**	**4,** $\mathbf{8_2}$		
21									**15**	**7,** $\mathbf{12_2}$	$\mathbf{1_2}$		
19								**17**	**12,** $\mathbf{16_2}$				
17							**16**	**15,** $\mathbf{18_2}$	$\mathbf{1_2}$				
15						**15**	**17,** $\mathbf{16_2}$	$\mathbf{1_2}$					
13					**10**	**16,** $\mathbf{15_2}$							
11				**6**	**15,** $\mathbf{10_2}$								
9			**3**	**10,** $\mathbf{6_2}$									
7		**1**	**7,** $\mathbf{2_2}$										
5		**2,** $\mathbf{1_2}$											
3	**1**												

$Kh(16n197566', \mathbb{Z})$

	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2
-3													**1**
-5												**2**	$\mathbf{1_2}$
-7											**7**	$\mathbf{1,\ 2_2}$	
-9										**10**	$\mathbf{3,\ 6_2}$		
-11									**15**	$\mathbf{6,\ 10_2}$			
-13								**16**	$\mathbf{10,\ 15_2}$				
-15							$\mathbf{17,\ 1_2}$	$\mathbf{15,\ 16_2}$					
-17						$\mathbf{15,\ 1_2}$	$\mathbf{16,\ 18_2}$						
-19					**12**	$\mathbf{17,\ 16_2}$							
-21				$\mathbf{7,\ 1_2}$	$\mathbf{15,\ 12_2}$								
-23			**4**	$\mathbf{12,\ 8_2}$									
-25		**1**	$\mathbf{7,\ 3_2}$ $\mathbf{1_4}$										
-27		$\mathbf{4,\ 1_2}$											
-29	**1**												

HOMOLOGICAL THICKNESS

- $hw_R(K)$ homological width of K is the minimal number of adjacent diagonals with $2i - j = const$ that support $Kh(K, R)$
- Homological width $hw_R(K) \geq 2$ is at least two.
- A knot is *thin if and only if it has homological width 2, otherwise it is thick.*

	-4	-3	-2	-1	0	1	2	3	4
9									1
7								3	1_2
5							3	1, 3_2	
3						4	3, 3_2		
1					5	3, 4_2			
-1				3	5, 4_2				
-3			3	4, 3_2					
-5		1	3, 3_2						
-7		3, 1_2							
-9	1								

	0	1	2	3	4	5
17						1
15						1
13				1	1	
11				1_2	1	
9			1			
7	1					
5	1					

Thickness of the Khovanov homology

	-2	-1	0	1	2	3	4	5	6	7	8	9	10
29													1
27												4	1_2
25											7	1, 3_2 1_4	
23										12	4, 8_2		
21									15	7, 12_2	1_2		
19								17	12, 16_2				
17							16	15, 18_2	1_2				
15						15	17, 16_2	1_2					
13					10	16, 15_2							
11				6	15, 10_2								
9			3	10, 6_2									
7		1	7, 2_2										
5		2, 1_2											
3	1												

	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2
-3													1
-5												2	1_2
-7											7	1, 2_2	
-9										10	3, 6_2		
-11									15	6, 10_2			
-13								16	10, 15_2				
-15							17, 1_2	15, 16_2					
-17						15, 1_2	16, 18_2						
-19					12	17, 16_2							
-21				7, 1_2	15, 12_2								
-23			4	12, 8_2									
-25		1	7, 3_2 1_4										
-27		4, 1_2											
-29	1												

FIGURE: *Kh*(16*n*197566) is thick over $\mathbb{Z}$ and $\mathbb{Z}_2$, while Khovanov homology of the mirror image is thin over $\mathbb{Z}$ but thick over $\mathbb{Z}_2$

Does thickness change with respect to R?

Sometimes. There are examples of knots such that $hw_{\mathbb{Q}} < hw_{\mathbb{Z}}, hw_{\mathbb{Z}_2}$ and $hw_{\mathbb{Z}} < hw_{\mathbb{Z}_2}$

Thickness of the Khovanov homology

- $hw_{\mathbb{Q}}(K_1 \sharp K_2) = hw_{\mathbb{Q}}(K_1) + hw_{\mathbb{Q}}(K_2) - 2$.
- $K_1 \sharp K_2$ is $\mathbb{Q}$-thin if and only if both K_1 and K_2 are $\mathbb{Q}$-thin.
- Turaev genus gives an upper bound on homological width of Khovanov homology (Champanerkar-Kofman-Stoltzfus, Manturov)

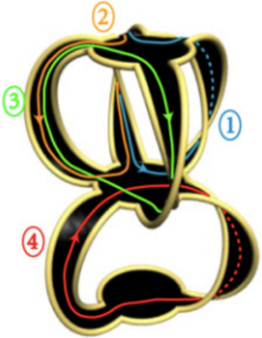
$$hw(L) - 2 \leq g_T(L)$$

- Question: Given a knot do we know if its Khovanov homology is thin or thick?

SIGNATURE OF A KNOT $\sigma(K)$

is a knot invariant that can be computed from the Seifert surface S.

- The Seifert form of S is the pairing $\phi : H_1(S) \times H_1(S) \to \mathbb{Z}$ given by taking the linking number $lk(a^+, b^-)$ where and a^+, b^- indicate the pushoffs in positive and negative directions.
- Given a basis $\{b_1, ..., b_{2g}\}$ for $H_1(S)$ the Seifert form can be represented as a Seifert matrix $V_{ij} = \phi(b_i, b_j)$
- $\sigma(K)$ is the signature of the matrix $V + V^\perp$



- $\sigma(K) = \mathrm{signature(G)} - \mu(\mathrm{D})$ (Gordon, Litherland)
- Determinant of K: $\det K = |\det G|$

Thickness of the Khovanov homology

Quasi-alternating knots

QA is the smallest set of links that contains the unknot such that if L is a link which admits a projection with a crossing such that L is in *QA* if

1. both resolutions L_0 and L_∞ at that crossing are in *QA*, and
2. $\det(L) = \det(L_0) + \det(L_\infty)$

All alternating knots are in *QA*

Theorem (E.S. Lee 2002)

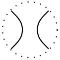
Every non-split alternating link is R-thin for every R.

Theorem (Manolescu-Ozsváth,2007)

Quasi-alternating links are R-thin for every R.

Reduced Khovanov homology $\widetilde{Kh}^{i,j}(L)$ *is trivial unless* $2i - j = \sigma(K)$

QA-KNOTS ARE THIN

Let $L =$  , $L_0 =$  and $L_\infty =$ 

LEMMA

If $det(L_0), det(L_\infty) > 0$ and $det(L_+) = det(L_0) + det(L_\infty)$, then for $e = n_-(D_h) - n_+(D_+)$: $\sigma(L_0) - \sigma(L_+) = 1$ and $\sigma(L_\infty) - \sigma(L_+) = e$.

PROOF.

Theorem QA knots satisfy the above lemma, so we have the standard Khovanov LES with grading $\delta = 2i - j$

$$\cdots \to \widetilde{Kh}^{*+\sigma(L_\infty)}(L_\infty) \to \widetilde{Kh}^{*+\sigma(L)}(L) \to \widetilde{Kh}^{*+\sigma(L_0)}(L_0) \to \widetilde{Kh}^{*+\sigma(L_\infty)-1}(L_\infty) \to \dots$$

- Therefore L is thin if L_0 and L_1 are thin.
- Apply the same argument to $L = L_0$ and $L = L_1$, apply the Lemma, and repeat iteratively.
- This process ends when we reach the unlink, whose Khovanov homology is thin.

□

Thickness of Khovanov homology

Theorem (E.S. Lee 2002)

Jones polynomial and signature determine the free part of the Khovanov homology of non-split alternating links.

Theorem (Khovanov, 2002)

Adequate non-alternating knots are known to be R-thick.

Proposition

$\widetilde{Kh}^{i,j}(L)$ *where L is an alternating link L has no 2-torsion.*

Proof.

Universal coefficient theorem gives us short exact sequences:

$$0 \to \widetilde{Kh}^{i,j}(L,\mathbb{Z}) \otimes \mathbb{Z}_2 \to \widetilde{Kh}^{i,j}(L,\mathbb{Z}_2) \to \mathrm{Tor}(\widetilde{\mathrm{Kh}}^{i+1,j}(L,\mathbb{Z}),\mathbb{Z}_2) \to 0$$

$$0 \to \widetilde{Kh}^{i-1,j}(L,\mathbb{Z}) \otimes \mathbb{Z}_2 \to \widetilde{Kh}^{i-1,j}(L,\mathbb{Z}_2) \to \mathrm{Tor}(\widetilde{\mathrm{Kh}}^{i,j}(L,\mathbb{Z}),\mathbb{Z}_2) \to 0$$

Existence of $\mathbb{Z}_2$ in $\widetilde{Kh}^{i,j}(L)$ implies that $\widetilde{Kh}(L,\mathbb{Z}_2)$ is not thin! □

Patterns in Khovanov homology

	-4	-3	-2	-1	0	1	2	3	4
9									1
7								3	1_2
5							3	1, 3_2	
3						4	3, 3_2		
1					4+1	3, 4_2			
-1				3	1+4, 4_2				
-3			3	4, 3_2					
-5		1	3, 3_2						
-7		3, 1_2							
-9	1								

	0	1	2	3	4	5
17						1
15						1
13				1	1	
11				1_2	1	
9			1			
7	1					
5	1					

Most entries are arranged in ”knight move” pairs except two entries in the 0th homological degree. Torsion fits nicely into this pattern...

Knight move

Conjecture (Bar-Natan, Garoufalidis, Khovanov)

For every knot L, the ranks of its Khovanov homology $Kh(L)$ are arranged in "knight-move" pairs, except for two adjacent entries in the $i = 0$ column, which have to be decreased by 1.

Status

- proved for QH-thin knots (Lee, 2002), therefore true for all non-split alternating links and quasi-alternating.
- also true for knots with homology supported on 3 diagonals;
- no counter-example is known

Lee's differential

- Work over $\mathbb{Q}$ (and $\mathbb{Z}_p$ for odd p).
- Define $\mathbb{Q}$-linear maps

$$m_{Lee} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \qquad m_{Lee} : \begin{cases} 1 \otimes 1 \mapsto 0 & 1 \otimes x \mapsto 0 \\ x \otimes 1 \mapsto 0 & x \otimes x \mapsto 1 \end{cases}$$

$$\Delta_{Lee} : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A} \qquad \Delta_{Lee} : \begin{cases} 1 \mapsto 0 \\ x \mapsto 1 \otimes 1. \end{cases}$$

- Lee's differential d_{Lee} is defined using the maps before and has bidegree $(1, 4)$ i.e. it does not preserve q-grading $\ldots$ but it is a filtered map.
- $(C(L), d, d_{Lee})$ form a double complex...

Bicomplexes

- (M, d_h, d_v) is a double complex if M is a bigraded module over some ring R and
  - $d_h, d_v : M \to M$ are differentials of bidegree $(1, 0)$ and $(0, 1)$
  - d_h, d_v anti-commute: $d_h \cdot d_v + d_v \cdot d_h = 0$
- d_h induces a differential d_h^* on $H(M, d_v)$ and so does d_v.
- Total complex$(Tot(M), d)$ with M a graded module over R with a homogeneous degree n part of $Tot(M)$ equal to $\oplus_{k+l=n} M^{k,l}$ and $d = d_h + d_v$
- $d^2 = d_h^2 + d_h d_v + d_v d_h + d_v^2 = 0$
- There exists a spectral sequence $\{E_r, d_r\}$ with $E_0 = M$, $E_1 = H(M, d_h)$, and $E_2 = H((H, d_h), d_v^*)$.
- If M has bounded support, then this spectral sequence converges to $H(Tot(M), d)$.

LEE'S SPECTRAL SEQUENCE

- $(C(L), d, d_{Lee})$ form a double complex so there is an associated spectral sequence.
- The bidegree of the map on the r-th page of the spectral sequence is $(1, 4r)$.
- In all known examples (over $\mathbb{Q}$) the spectral sequence collapses after the bidegree $(1, 4)$ differential.
- In such cases, $Kh(D; \mathbb{Q})$ can be arranged into "knight move" pairs.

THEOREM (E.S. LEE)

Let L be an oriented n component link, $\{S_i\}_{i=1}^n$. $dimKh_{Lee}(L) = 2^n$ and $dimKh^i_{Lee} = 2|E \subset \{2, \ldots, n\}|(\sum_{j \in E, k \notin E} 2\mathrm{lk}(\mathrm{S_j}, \mathrm{S_k}) = \mathrm{i})|$

Lee-Rasmussen spectral sequence

Theorem (Lee, Rasmussen)

There is a spectral sequence whose E_2-term is $Kh(L)$ which converges to $Kh_{Lee}(L)$.
For a knot, the spectral sequence converges to $\mathbb{Q} \oplus \mathbb{Q}$.

	$\mathbb{Q}$
$\mathbb{Q}$	

knight move pair

	$\mathbb{Z}_2$
$\mathbb{Z}_2$	$\mathbb{Z}_2$
$\mathbb{Z}_2$	

tetromino

Shumakovitch (2004) showed that this spectral sequence also exists over $\mathbb{Z}_p$ for p an odd prime.

Rasmussen's s-invariant

"In a beautiful article Eun Soo Lee introduced a second differential on the Khovanov complex of a knot (or link) and showed that the resulting (double) complex has non-interesting homology. This is a very interesting result."

D. Bar Natan

- Important: the entire spectral sequence, starting with the E^2-page is a link invariant
- Only two generators survive in the limit and they are in q-gradings s_{max} and s_{min} which differ by 2
- Rasmussen's invariant $s(K) := \frac{1}{2}(s_{max} - s_{min})$

Rasmussen's s-invariant

- $s(K)$ is additive
- Taking a mirror of a knot changes the sign of the s-invariant.
- $s(\bigcirc) = 0$
- $s(K)$ is a concordance invariant (i.e. all concordant knots have the same s-invariant)
- Slice genus of K (4-ball genus) $g^*(K)$ is the minimum genus of a smooth surface smoothly embedded in B^4 whose boundary is K.

Theorem (Rasmussen)

Rasmussen's s-invariant provides a lower bound on the slice genus

$$|s(K)| \leq 2g^*(K)$$

MILNOR CONJECTURE

THEOREM (RASMUSSEN)

Given a positive knot K with

- *n crossings and*
- *r circles in the canonical homogeneous smoothing*

then $s(K) = -r + n + 1$

COROLLARY

Torus knot $T(p, q)$ is a positive knot and $s(T(p, q)) = (p-1)(q-1)$

THEOREM (KRONHEIMER, MROWKA; RASMUSSEN)

Slice genus of a torus knot $T(p, q)$ is $\frac{1}{2}(p-1)(q-1)$.

Khovanov homology applications

Theorem (Mrowka, Kronheimer 2010)

Khovanov homology detects the unknot: if the rank of the reduced Khovanov homology of L is equal to 1 then L is the unknot.

Theorem (Hedden, Ni 2012)

Khovanov homology detects the 2-component unlink.
Note: the Jones polynomial does not. There are infinite families of links whose Jones polynomial is the same as that of an n–component unlink.

Theorem (Batson, Seed 2013)

$Kh(L)$ gives lower bound for splitting number of L.

Thank you

Challenge: Find knots in Freiburg!