Torsion in Khovanov link homology

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Summer School on Modern Knot Theory
Freiburg
7 June 2017

Torsion in Khovanov homology

Conjecture (Shumakovitch)

Let L be any prime link other than the unknot or the Hopf link. Then $Kh(L)$ contains 2-torsion.

- The conjecture implies that Khovanov homology is an unknot detector.
- Kronheimer and Mrowka (2010) proved Khovanov homology is an unknot detector using gauge theory.
- The conjecture is known to be true in many cases.

COMPUTATIONS

- Experimentally, $Kh(L)$ has an abundance of torsion: among all 1,701,936 prime knots with at most 16 crossings
- all non-trivial knots up to 14 crossings have only 2-torsion
- 38 knots with 15 crossings and 129 knots with 16 crossings have 4-torsion
- the first known knot with odd torsion $T(5, 6)$-torus knot.
- Infinite families of links whose Khovanov homology contains $\mathbb{Z}_n$, $2 < n < 9$ and $\mathbb{Z}_{2^n}$-torsion for $n < 24$.

Methods for proving things about torsion in $Kh(L)$

- Explicit construction
  (Asaeda,Przytycki, Silvero, Mukherjee, Wang, Yang)
- Connections with Hochschild homology of algebras
  (Khovanov, Przytycki)
- Relations with chromatic graph homology
  (Helme-Guizon, Lowrance, Pabiniak, Przytycki, Rong, S., Scofield)
- Spectral sequence arguments
  (Lowrance, S., Shumakovitch)

Hochschild homology of $\mathcal{A} = \mathbb{Z}[x]/(x^2)$ and $Kh(T_{2,n})$

- Let P_n be the polygon with n vertices.
- Let $C_n(A)$ be the space generated by labelings of the vertices of P_n with elements of A.
- Define a map $C_n(A) \to C_{n-1}(A)$ obtained by contracting edges and multiplying the labels on the identified vertices.
- Przytycki (2005) showed this complex gives the Hochschild homology $HH(A)$ and the Khovanov homology of $Kh(T_{2,n})$ in certain gradings.
- Allows for explicit computations of 2-torsion inside of $Kh(T_{2,n})$.

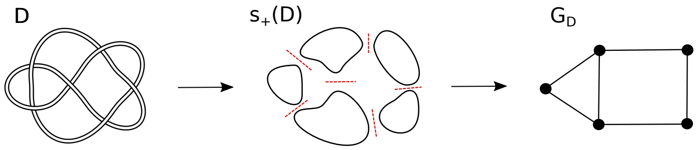
From Hochschild to chromatic graph cohomology

- Hochschild homology gives a sort of comultiplication free version of Khovanov homology for a polygon.
- Helme-Guizon and Rong (2004) define the chromatic graph cohomology $H(G)$.
- $H(G)$ a comultiplication free version of Khovanov homology for any graph or as
- $H(G)$ an extension of Hochschild homology from cycles to graphs.
- Its definition follows a similar recipe as the construction of Khovanov homology.

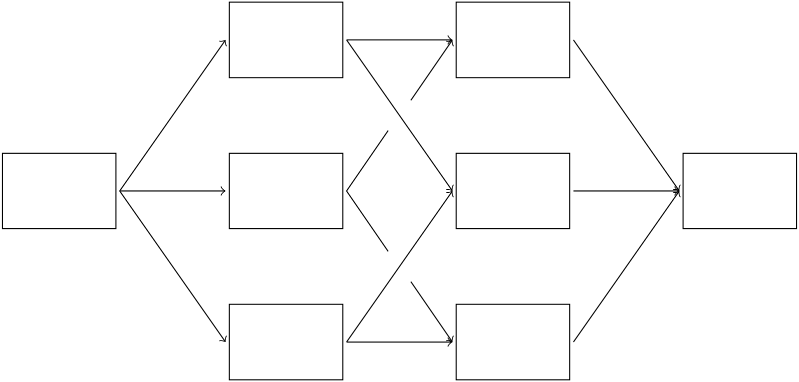
Correspondence between link diagrams and graphs

- *Kauffman state* is a collection of simple closed curves obtained by taking an A or B resolution at each crossing.

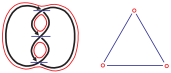
- Graph $G_s(D)$: vertices are Kauffman circles, edges are crossings shared by circles.
- $G_A(D)$ / $G_B(D)$ graphs corresponding to the all-A/ all-B state.

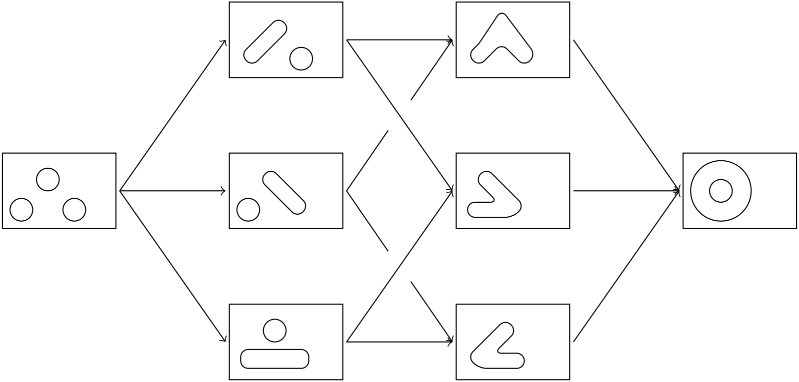


Hypercube

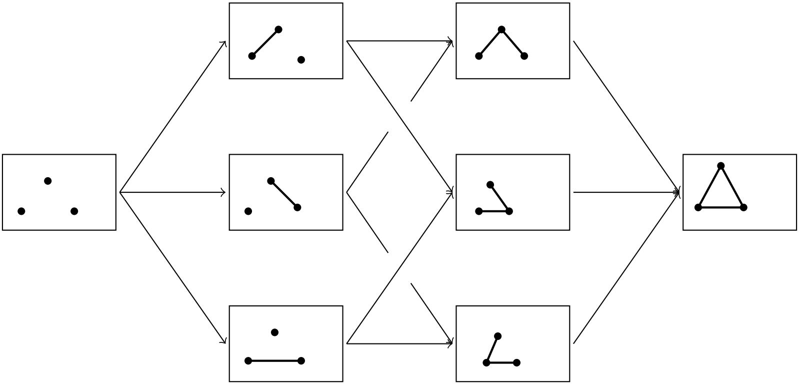


KAUFFMAN STATES

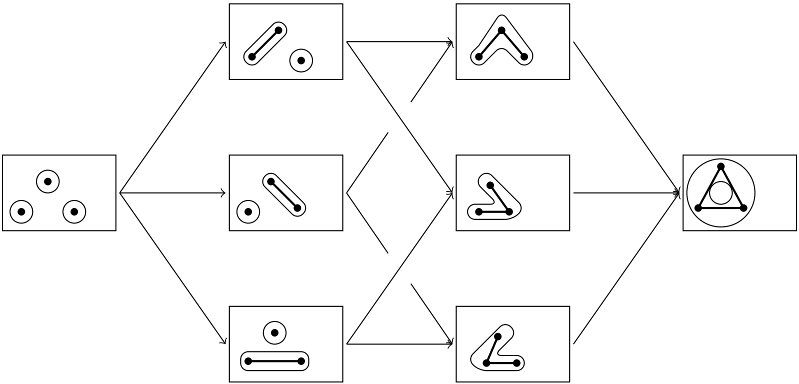




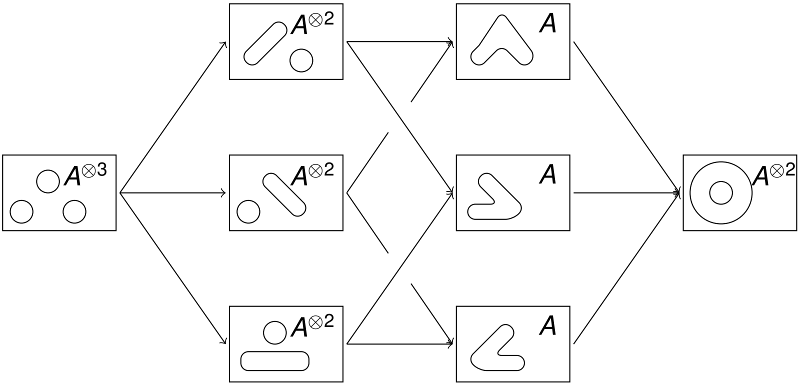
Spanning subgraphs



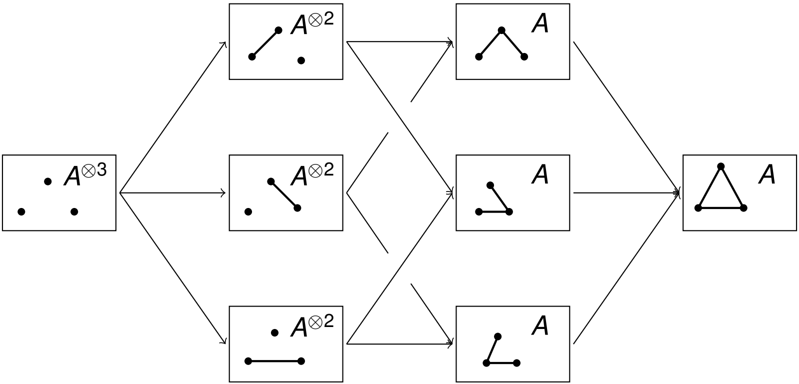
Kauffman states and spanning subgraphs



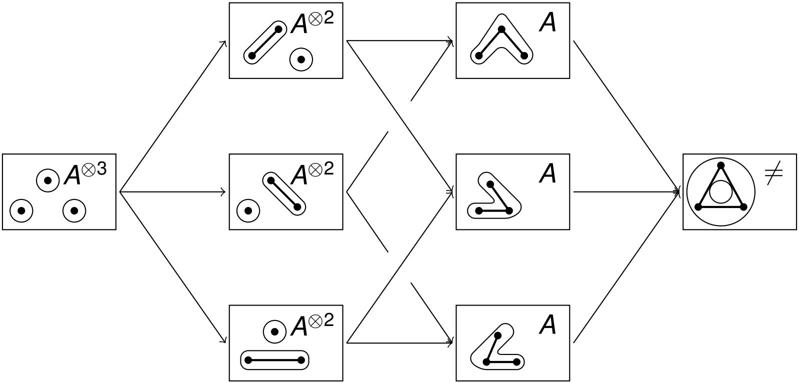
KAUFFMAN STATES AND SPACES



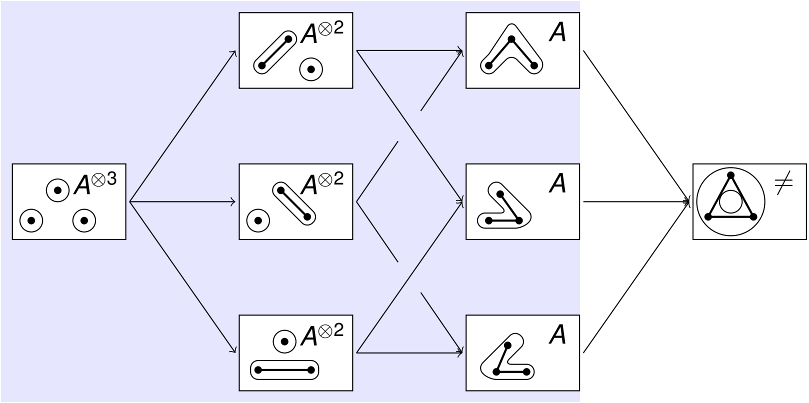
Spanning subgraphs and spaces



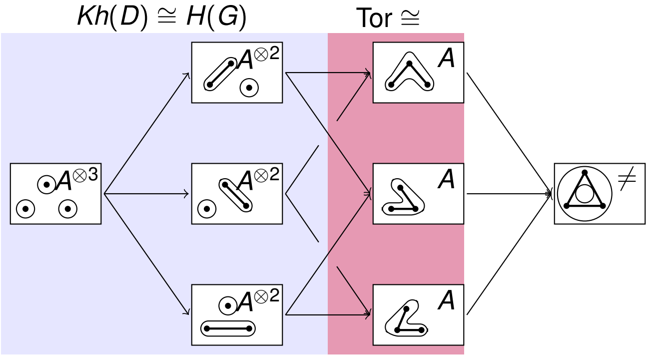
Spaces for both



Multiplication until a cycle closes



Partial Isomorphism Picture



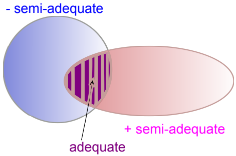
Theorem (Helme-Guizon, Przytycki, Rong - 2006)

If the length g of the shortest cycle in G is greater than one

- $Kh(D) \cong H(G_A)$ *in the first* $g - 1$ *supported i-gradings*
- Tor $Kh(D) \cong$ Tor $H(G_A)$ *in the gth i-grading.*

Adequate and semi-adequate links

- A link L is *adequate* if it has a diagram where both $G_A(D)$ and $G_B(D)$ have no loops.
- A link L is *semi-adequate* if it has a diagram D where either $G_A(D)$ or $G_B(D)$ has no loops.
- Alternating links are adequate.



- Many links are semi-adequate: at least 249,649 of the 253,293 knots with 15 crossings are semi-adequate (Stoimenow '12).

Explicit computation results

Theorem (Asaeda, Przytycki - 2004)

1. *If $G_A(D)$ is loop-less and contains a cycle of odd length, then $Kh(D)$ contains 2-torsion.*
2. *If $G_A(D)$ is loop-less and contains a cycle of even length with an edge that is not part of a bigon, then $Kh(D)$ contains 2-torsion.*
3. *If D is prime and alternating and D is not the unknot or Hopf link, then either $G_A(D)$ or $G_B(D)$ contains an edge that is not part of a bigon. Thus $Kh(D)$ contains 2-torsion.*

Remark. Shumakovitch's conjecture is true for alternating links and "many" semi-adequate links.

Corollaries on Khovanov homology

- The Khovanov homology of semi-adequate links contains 2-torison if $G_A(D)$ contains
  - an odd cycle or
  - an even cycle with an edge that is not part of a bigon.
- The Khovanov homology of any semi-adequate link where $G_A(D)$ has girth at least 3 contains 2-torsion (Przytycki, S.)
- Explicit formulas for torsion in certain bigradings of Khovanov homology of
  - positive 3-braids by Przytycki-S.
  - certain classes of pretzel links by Scofield
- Shumakovitch's conjecture is true for all semi-adequate links except possibly those where $G_A(D)$ only has 2-cycles.

Back to spectral sequences

- $(C_{i,j}(K), d, d_v)$ is a double complex
  - $dC_{i,j}(K), d$ Khovanov chain complex, $d_v : M \to M$ differentials of bidegree $(0, 1)$
  - d, d_v anti-commute: $d \cdot d_v + d_v \cdot d = 0$
- d_v induces a differential d_v^* on $H(C_{i,j}, d) = Kh(K)$
- Total complex $(Tot(C_{i,j}), D)$ is graded with $Tot(C_{i,j})^n = \bigoplus_{k+l=n} C_{k,l}^{k,l}$ and $D = d + d_v$ and $D^2 = d^2 + dd_v + dd_h + d_v^2 = 0$
- $\exists$ a spectral sequence $\{E_r, d_r\}$ with $E_0 = \{C_{i,j}(K)\}$, $E_1 = Kh(K)$, and $E_2 = H(Kh(K), d_v^*)$.
- If M has bounded support, then this spectral sequence converges to $H(Tot(M), d)$.
- Good news: One can find E_∞
- Bad: pages E_r and d_r for $r \neq 0, 1, \infty$ are often elusive.

Spectral sequences

- Khovanov homology and related invariants arise in many spectral sequences.
- These spectral sequences are often only defined over certain coefficient rings (e.g. $\mathbb{Q}$, $\mathbb{Z}_2$, or $\mathbb{Z}_p$ for odd p).
- Use the behavior of these sequences to prove or disprove the existence of torsion.
- Recall that in all known examples (over $\mathbb{Q}$) the Lee spectral sequence collapses after the bidegree $(1, 4)$ differential.
- In such cases, $Kh(D; \mathbb{Q})$ can be arranged into “knight move” pairs.

NO ODD TORSION THEOREM FOR KHOVANOV HOMOLOGY

THEOREM (SHUMAKOVITCH 2004)

If L is homologically thin, then $Kh(L)$ contains no odd torsion.

$Kh(T(2,5))$	-5	-4	-3	-2	-1	0
-3						$\mathbb{Z}$
-5						$\mathbb{Z}$
-7				$\mathbb{Z}$		
-9				$\mathbb{Z}_2$		
-11		$\mathbb{Z}$	$\mathbb{Z}$			
-13		$\mathbb{Z}_2$				
-15	$\mathbb{Z}$					

$Kh(K)$ CONTAINS NO TORSION OF ODD ORDER

				$\mathbb{Q}$
		$\mathbb{Q}$	$\mathbb{Q}$	
	$\mathbb{Q}$			

Suppose that $Kh(K;\mathbb{Q})$ is above.

$Kh(K)$ CONTAINS NO TORSION OF ODD ORDER

				$\mathbb{Q}$
		$\mathbb{Q}$	$\mathbb{Q}$	
	$\mathbb{Q}$			

				$\mathbb{Z}_p$
		$\mathbb{Z}_p$	$\mathbb{Z}_p$	
	$\mathbb{Z}_p$			

Then $Kh(K; \mathbb{Z}_p)$ has at least these summands.

$Kh(K)$ CONTAINS NO TORSION OF ODD ORDER

				$\mathbb{Q}$	
		$\mathbb{Q}$	$\mathbb{Q}$		
	$\mathbb{Q}$				

				$\mathbb{Z}_p$	
		$\mathbb{Z}_p$	$\mathbb{Z}_p$		
	$\mathbb{Z}_p$				

				$\mathbb{Z}$	
		$\mathbb{Z}$	$\mathbb{Z}$		
	$\mathbb{Z}$				

The free part of $Kh(K)$ looks as above and recall that all torsion appears only on the bottom diagonal.

$Kh(K)$ CONTAINS NO TORSION OF ODD ORDER

				$\mathbb{Q}$
		$\mathbb{Q}$	$\mathbb{Q}$	
	$\mathbb{Q}$			

				$\mathbb{Z}_p$
		$\mathbb{Z}_p$	$\mathbb{Z}_p$	
	$\mathbb{Z}_p$			

				$\mathbb{Z}$
		$\mathbb{Z}$	$\mathbb{Z}$	
		$\mathbb{Z}_{p^k}$		
	$\mathbb{Z}$			

Suppose that there exists $\mathbb{Z}_{p^k}$ in $Kh(K)$ for some odd p.
Let the pictured $\mathbb{Z}_{p^k}$ summand be in the maximum i-grading of any p^m torsion in $Kh(K)$.

$Kh(K)$ CONTAINS NO TORSION OF ODD ORDER

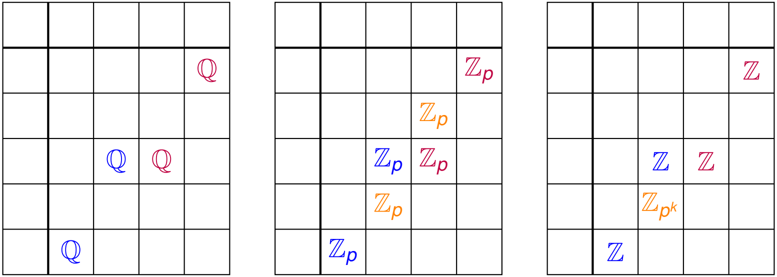
				$\mathbb{Q}$
		$\mathbb{Q}$	$\mathbb{Q}$	
	$\mathbb{Q}$			

				$\mathbb{Z}_p$
		$\mathbb{Z}_p$	$\mathbb{Z}_p$	
		$\mathbb{Z}_p$		
	$\mathbb{Z}_p$			

				$\mathbb{Z}$
		$\mathbb{Z}$	$\mathbb{Z}$	
		$\mathbb{Z}_{p^k}$		
	$\mathbb{Z}$			

Then we have the corresponding copy of $\mathbb{Z}_p$ in $Kh(K, \mathbb{Z}_p)$

$Kh(K)$ CONTAINS NO TORSION OF ODD ORDER



Theorem. $Kh(K, \mathbb{Z}_p)$ can be arranged in knight move pairs.

$Kh(K)$ CONTAINS NO TORSION OF ODD ORDER

				$\mathbb{Q}$
		$\mathbb{Q}$	$\mathbb{Q}$	
	$\mathbb{Q}$			

				$\mathbb{Z}_p$
			$\mathbb{Z}_p$	$\mathbb{Z}_p$
		$\mathbb{Z}_p$	$\mathbb{Z}_p$	
		$\mathbb{Z}_p$		
	$\mathbb{Z}_p$			

				$\mathbb{Z}$
				$\mathbb{Z}_{p^l}$
		$\mathbb{Z}$	$\mathbb{Z}$	
		$\mathbb{Z}_{p^k}$		
	$\mathbb{Z}$			

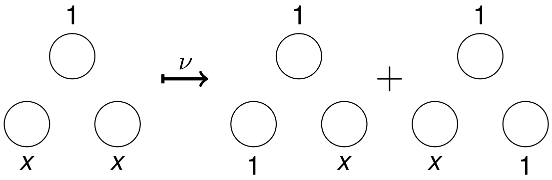
The universal coefficient theorem implies that $Kh(K;\mathbb{Z}_p)$ and $Kh(K)$ looks like above. Contradiction!

Only 2-torsion

- Khovanov homology of thin links has no odd torsion
- All torsion in $Kh(K)$ must be of order 2^k for some k.
- Next we outline A. Shumakovitch's unpublished proof that the Khovanov homology of a homologically thin knot has only 2-torsion.
- Proving that $Kh(K)$ only contains $\mathbb{Z}_2$-torsion and no Z_{2^k} amounts to showing that the Bockstein spectral sequence converges on the correct (2nd) page
- In order to prove convergence, we analyze relation between the Bockstein differentials and some new maps on Khovanov homology.

VERTICAL MAPS ν ON $Kh(D;\mathbb{Z}_2)$

Shumakovitch defines maps ν of bidegree $(0,2)$ on $C(D;\mathbb{Z}_2)$.



PROPERTIES OF ν

- ν commutes with the Khovanov differential, and thus induces a map $\nu^* : Kh(D;\mathbb{Z}_2) \to Kh(D;\mathbb{Z}_2)$.
- Homology with respect to ν is trivial, and so the induced map ν^* is an isomorphism on homology level.

Properties of the vertical map

- ν^* is an isomorphism.
- The “vertical” Euler characteristic of $Kh(D; \mathbb{Z}_2)$ is trivial.

	$\mathbb{Q}$
$\mathbb{Q}$	

knight move pair

	$\mathbb{Z}_2$
$\mathbb{Z}_2$	$\mathbb{Z}_2$
$\mathbb{Z}_2$	

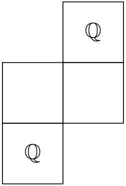
tetromino

	$\mathbb{Z}$
	$\mathbb{Z}_{2^k}$
$\mathbb{Z}$	

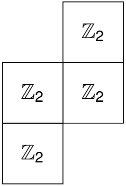
with $\mathbb{Z}$ coefficients

Knight moves

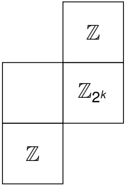
so far...



over $\mathbb{Q}$

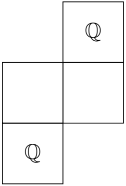


over $\mathbb{Z}_2$

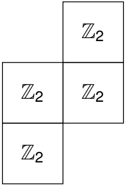


over $\mathbb{Z}$

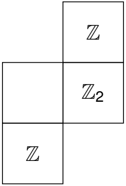
next show



over $\mathbb{Q}$



over $\mathbb{Z}_2$

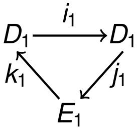


over $\mathbb{Z}$

Spectral sequence associated to an exact couple

- Given the exact triangle on the left, define $d_1 = j_1 \circ k_1 : E_1 \to E_1$.
- $d_1^2 = j_1 \circ (k_1 \circ j_1) \circ k_1 = 0$ i.e. d is a differential

  $E_2 = H(E_1, d_1)$ and $D_2 = \mathrm{Im}(i_1) = \mathrm{Ker}(j_1)$ and maps i_2, j_2, and k_2 can be defined so that the triangle on the right is exact.



- Iterating this process yields a spectral sequence $\{E_r, d_r\}_{r \geq 1}$.
- Note If D, E bigraded modules then $\mathrm{deg(i)} = (-1, 1)$, $\mathrm{deg(j)} = (0, 0)$, $\mathrm{deg(k)} = (1, 0)$, therefore $\mathrm{deg(d_r)} = (\mathrm{r}, 1 - \mathrm{r})$

Bockstein spectral sequence

Which exact couple to use?

- Given Khovanov chain complex (any complex of free Abelian groups) and a prime p
- Consider the short exact sequence of coefficient rings $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\text{mod } 2} \mathbb{Z}_2 \to 0$
- It induces a short exact sequence of Khovanov chain complexes

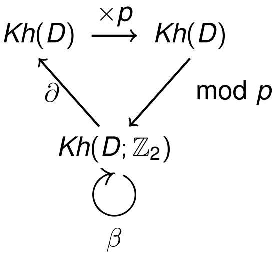
$$0 \to C(D) \xrightarrow{\times 2} C(D) \xrightarrow{\text{mod } 2} C(D; \mathbb{Z}_2) \to 0,$$

- and, in turn, a long exact sequence on homology

$$\cdots \to Kh(D) \xrightarrow{\times 2} Kh(D) \xrightarrow{\text{mod } 2} Kh(D; \mathbb{Z}_2) \xrightarrow{\partial} Kh(D) \to \cdots .$$

We have an exact couple!

Bockstein spectral sequence



- Define the Bockstein homomorphism $\beta : Kh(D; \mathbb{Z}_p) \to Kh(D; \mathbb{Z}_2)$ by $\beta = \partial$ mod p.
- β is the connecting homomorphism induced from $0 \to \mathbb{Z}_p \to \mathbb{Z}_{p^2} \to \mathbb{Z}_p \to 0$

- For a finitely generated chain complex, like Khovanov, there exists a spectral sequence $\{B_r, b_r\}$ with $B_1 = Kh(D; \mathbb{Z}_2)$ and $b_1 = \beta$ that converges to the free $Kh(D) \otimes \mathbb{Z}_p$.
- b_r is induced from $0 \to \mathbb{Z}_{p^r} \to \mathbb{Z}_{p^{2r}} \to \mathbb{Z}_{p^r} \to 0$
- $B_r^* \cong \mathrm{im}(\mathrm{H}^*(\mathrm{C}; \mathbb{Z}_{\mathrm{p^r}}) \xrightarrow{\times \mathrm{p^{r-1}}} \mathrm{H}^*(\mathrm{C}; \mathbb{Z}_{\mathrm{p^r}})) \cong \mathrm{im}(\mathrm{Kh}^*(\mathrm{C}; \mathbb{Z}_{\mathrm{p^r}}) \xrightarrow{\times \mathrm{p^{r-1}}} \mathrm{Kh}^*(\mathrm{C}; \mathbb{Z}_{\mathrm{p^r}}))$

More on the Bockstein spectral sequence

The $\mathbb{Z}_2$-Bockstein spectral sequence satisfies the following.

- The E_1 page of the Bockstein spectral sequence is $H(G;\mathbb{Z}_2)$.
- The E_∞ page of the Bockstein spectral sequence is $[Kh(K)/\operatorname{Tor} Kh(K)] \otimes \mathbb{Z}_2$.
- If the Bockstein spectral sequence converges at the 2nd page, then $Kh(K)$ has no torsion of order 2^k for $k \geq 2$.

Theorem

$Kh(D)$ has no 2^k torsion if and only if the Bockstein spectral sequence collapses at the kth page.

Bockstein example

Goal. Show that the Bockstein spectral sequence collapses at the first page for homologically thin links.

- $H(G) = \mathbb{Z}^{a_0} \oplus \mathbb{Z}_2^{a_1} \oplus \mathbb{Z}_4^{a_2} \oplus \cdots \oplus \mathbb{Z}_{2^k}^{a_k}$.
- $E_1 = \mathbb{Z}_2^{a_0} \oplus \mathbb{Z}_2^{a_1} \oplus \mathbb{Z}_2^{a_1} \oplus \mathbb{Z}_2^{a_2} \oplus \mathbb{Z}_2^{a_2} \oplus \cdots \oplus \mathbb{Z}_2^{a_k} \oplus \mathbb{Z}_2^{a_k}$.
- $E_2 = \mathbb{Z}_2^{a_0} \oplus \mathbb{Z}_2^{a_2} \oplus \mathbb{Z}_2^{a_2} \oplus \cdots \oplus \mathbb{Z}_2^{a_k} \oplus \mathbb{Z}_2^{a_k}$.
- $E_\infty = \mathbb{Z}_2^{a_0}$.

New new Goal: If β is the Bockstein map on the E_1 page, then we want to show that the rank of β is the number of tetrominoes N in $Kh(G; \mathbb{Z}_2)$.

Turner's differential with $\mathbb{Z}_2$ coefficients

- Define $\mathbb{Z}_2$-linear maps

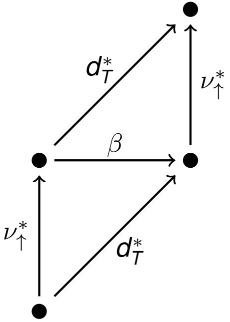
$$m_T : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \qquad m_T : \begin{cases} 1 \otimes 1 \mapsto 0 & 1 \otimes x \mapsto 0 \\ x \otimes 1 \mapsto 0 & x \otimes x \mapsto x \end{cases}$$

$$\Delta_T : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A} \qquad \Delta_T : \begin{cases} 1 \mapsto 1 \otimes 1 \\ x \mapsto 0. \end{cases}$$

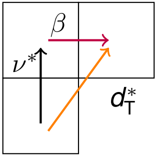
- There is a differential $d_T : C^{i,j}(G; \mathbb{Z}_2) \to C^{i+1,j+2}(G; \mathbb{Z}_2)$.
- It induces a map $d_T^* : Kh^{i,j}(G; \mathbb{Z}_2) \to H^{i+1,j+2}(G; \mathbb{Z}_2)$.

The Turner differential

- $(C(D;\mathbb{Z}_2), d, d_{\mathrm{T}})$ form a double complex, and so there is an associated spectral sequence.
- d_{T} commutes with the usual Khovanov differential, and so there is an induced map $d^*_{\mathrm{Turner}} : Kh(D;\mathbb{Z}_2) \to Kh(D;\mathbb{Z}_2)$.
- For a knot, the above spectral sequence converges to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.
- If the homology is thin, the last non-zero map in the spectral sequence is d^*_{T}.



PUTTING IT ALL TOGETHER



1 $\nu^* : Kh(D;\mathbb{Z}_2) \to Kh(D;\mathbb{Z}_2)$ is an isomorphism.

2 The Turner spectral sequence collapses at the first page.

3 $d^*_{\mathrm{T}} = \nu^* \circ \beta + \beta \circ \nu^*$.

1–3 implies that the Bockstein spectral sequence collapses after the first page.

4 $\nu^*_{\uparrow} \circ d^*_T = d^*_T \circ \nu^*_{\uparrow}$.

5 On each diagonal, rank $d^*_T = N$.

6 rank $\beta = N$.

Torsion in Khovanov homology of homologically thin knots

Theorem (Shumakovitch)

If K is homologically thin, then its Khovanov homology only has 2-torsion.

Corollary

The Khovanov homology $Kh(Kh, \mathbb{Z})$ of an alternating knot is determined by its Jones polynomial and signature.

Corollary

If K is a nontrivial, homologically thin knot, then its Khovanov homology contains 2-torsion.

Remark. The second corollary requires Kronheimer-Mrowka's result that Khovanov homology detects the unknot.

Computations of odd torsion

- Torus knots $(5,6),\ (5,7),\ (5,8)$, and $(5,9)$ have 5-torsion in their Khovanov homology.

- Przytycki and Sazdanović predicted that the closure K of

$$\sigma_1^2\sigma_2^2\sigma_1^3\sigma_2^2\sigma_1\sigma_3\sigma_2^2\sigma_4^2\sigma_3\sigma_1^2\sigma_2^2\sigma_1^3\sigma_2^3\sigma_1^2\sigma_3\sigma_2^2\sigma_4^2\sigma_3^2$$

  has 5-torsion in its Khovanov homology.

- Shumakovitch (2012) confirmed that K has 5-torsion in its Khovanov homology by showing that the difference of the Poincare polynomials of $Kh(K;\mathbb{Z}_5)$ and $Kh(K;\mathbb{Z}_7)$ is

$$(t^{12}+t^{11})q^{51}+(t^{11}+t^{10})q^{47}.$$

$Kh(K; \mathbb{Z}_5)$

$$
\begin{aligned}
KH_5(K) = {} & q^{31}t^{0} + q^{33}t^{0} + q^{35}t^{2} + q^{39}t^{3} + 2q^{37}t^{4} + q^{39}t^{4} + 2 + q^{41}t^{5} + q^{43}t^{5} + q^{39}t^{6} + \\
& 2q^{41}t^{6} + 2q^{43}t^{7} + 2q^{45}t^{7} + 4q^{41}t^{8} + 3q^{43}t^{8} + q^{47}t^{8} + 13q^{43}t^{9} + 4q^{45}t^{9} + 4q^{47}t^{9} + 2q^{43}t^{10} + \\
& 29q^{45}t^{10} + 14q^{47}t^{10} + q^{51}t^{10} + 9q^{45}t^{11} + 44q^{47}t^{11} + 31q^{49}t^{11} + q^{51}t^{11} + 2q^{45}t^{12} + 34q^{47}t^{12} + \\
& 68q^{49}t^{12} + 42q^{51}t^{12} + 2q^{53}t^{12} + 11q^{47}t^{13} + 85q^{49}t^{13} + 97q^{51}t^{13} + 59q^{53}t^{13} + 45q^{49}t^{14} + \\
& 159q^{51}t^{14} + 142q^{53}t^{14} + 63q^{55}t^{14} + 137q^{51t^{1}5} + 245q^{53}t^{15} + 202q^{55}t^{15} + 9q^{57}t^{15} + 345q^{53}t^{16} + \\
& 5376q^{55}t^{16} + 237q^{57}t^{16} + 54q^{59}t^{16} + 735q^{55}t^{17} + 589q^{57}t^{17} + 260q^{59}t^{17} + 37q^{61}t^{17} + \\
& 1328q^{57}t^{18} + 953q^{59}t^{18} + 253q^{61}t^{18} + 21q^{63}t^{18} + 2040q^{59}t^{19} + 1501q^{61}t^{19} + 220q^{63}t^{19} + \\
& 9q^{65}t^{19} + 2729q^{61}t^{20} + 2149q^{63}t^{20} + 173q^{65}t^{20} + 2q^{67}t^{20} + 2q^{61}t^{21} + 3203q^{63}t^{21} + \\
& 2779q^{65}t^{21} + 109q^{67}t^{21} + 11q^{63}t^{22} + 3344q^{65}t^{22} + 3219q^{67}t^{22} + 50q^{69}t^{22} + 36q^{65}t^{23} + \\
& 3127q^{67}t^{23} + 3345q^{69}t^{23} + 16q^{71}t^{23} + 81q^{67}t^{24} + 2608q^{69}t^{24} + 3116q^{71}t^{24} + 3q^{73}t^{24} + \\
& 137q^{69}t^{25} + 1934q^{71}t^{25} + 2572q^{73}t^{25} + 191q^{71}t^{26} + 1271q^{73}t^{26} + 1853q^{75}t^{26} + 228q^{73}t^{27} + \\
& 759q^{75}t^{27} + 1134q^{77}t^{27} + 238q^{75}t^{28} + 446q^{77}t^{28} + 568q^{79}t^{28} + 219q^{77}t^{29} + 294q^{79}t^{29} + \\
& 218q^{81}t^{29} + 175q^{79}t^{30} + 226q^{81}t^{30} + 56q^{83}t^{30} + 119q^{81}t^{31} + 175q^{83}t^{31} + 7q^{85}t^{31} + 65q^{83}t^{32} + \\
& 119q^{85}t^{32} + 26q^{85}t^{33} + 65q^{87}t^{33} + 7q^{87}t^{34} + 26q^{89}t^{34} + q^{89}t^{35} + 7q^{91}t^{35} + q^{93}t^{36}
\end{aligned}
$$

$Kh(K;\mathbb{Z}_7)$

$$
\begin{aligned}
KH_7(K) = {} & q^{31}t^0 + q^{33}t^0 + q^{35}t^2 + q^{39}t^3 + 2q^{37}t^4 + q^{39}t^4 + 2q^{41}t^5 + q^{43}t^5 + q^{39}t^6 + 2q^{41}t^6 + \\
& 2q^{43}t^7 + 2q^{45}t^7 + 4q^{41}t^8 + 3q^{43}t^8 + q^{47}t^8 + 13q^{43}t^9 + 4q^{45}t^9 + 4q^{47}t^9 + \\
& 2q^{43}t^{10} + 29q^{45}t^{10} + 13q^{47}t^{10} + q^{51}t^{10} + 9q^{45}t^{11} + 43q^{47}t^{11} + 31q^{49}t^{11} + \\
& 2q^{45}t^{12} + 34q^{47}t^{12} + 68q^{49}t^{12} + 41q^{51}t^{12} + 2q^{53}t^{12} + 11q^{47}t^{13} + 85q^{49}t^{13} + \\
& 97q^{51}t^{13} + 59q^{53}t^{13} + 45q^{49}t^{14} + 159q^{51}t^{14} + 142q^{53}t^{14} + 63q^{55}t^{14} + 137q^{51}t^{15} + \\
& 245q^{53}t^{15} + 202q^{55}t^{15} + 59q^{57}t^{15} + 345q^{53}t^{16} + 376q^{55}t^{16} + 237q^{57}t^{16} + \\
& 54q^{59}t^{16} + 735q^{55}t^{17} + 589q^{57}t^{17} + 260q^{59}t^{17} + 37q^{61}t^{17} + 1328q^{57}t^{18} + \\
& 953q^{59}t^{18} + 253q^{61}t^{18} + 21q^{63}t^{18} + 2040q^{59}t^{19} + 1501q^{61}t^{19} + 220q^{63}t^{19} + \\
& 9q^{65}t^{19} + 2729q^{61}t^{20} + 2149q^{63}t^{20} + 173q^{65}t^{20} + 2q^{67}t^{20} + 2q^{61}t^{21} + 3203q^{63}t^{21} + \\
& 2779q^{65}t^{21} + 109q^{67}t^{21} + 11q^{63}t^{22} + 3344q^{65}t^{22} + 3219q^{67}t^{22} + 50q^{69}t^{22} + \\
& 36q^{65}t^{23} + 3127q^{67}t^{23} + 3345q^{69}t^{23} + 16q^{71}t^{23} + 81q^{67}t^{24} + 2608q^{69}t^{24} + \\
& 3116q^{71}t^{24} + 3q^{73}t^{24} + 137q^{69}t^{25} + 1934q^{71}t^{25} + 2572q^{73}t^{25} + 191q^{71}t^{26} + \\
& 1271q^{73}t^{26} + 1853q^{75}t^{26} + 228q^{73}t^{27} + 1134q^{77}t^{27} + 238q^{75}t^{28} + 446q^{77}t^{28} + \\
& 568q^{79}t^{28} + 219q^{77}t^{29} + 294q^{79}t^{29} + 218q^{81}t^{29} + 759q^{75}t^{27} + 175q^{79}t^{30} + 226q^{81}t^{30} + \\
& 56q^{83}t^{30} + 119q^{81}t^{31} + 175q^{83}t^{31} + 7q^{85}t^{31} + 65q^{83}t^{32} + 119q^{85}t^{32} + 26q^{85}t^{33} + \\
& 65q^{87}t^{33} + 7q^{87}t^{34} + 26q^{89}t^{34} + q^{89}t^{35} + 7q^{91}t^{35} + q^{93}t^{36}
\end{aligned}
$$

Still mysterious

- Torsion in reduced homology
  - 2-torsion appears first for 13-crossing knots (13n3663)
  - the simplest knot having odd torsion in reduced homology is $T(5,6)$ and that is $\mathbb{Z}_3$
- the simplest knot having odd torsion in unreduced homology is T(5, 6) which has a copy of $\mathbb{Z}_3$ and a copy of $\mathbb{Z}_5$
- some knots, e.g. T(5, 6), have odd torsion in unreduced homology which is not seen in the reduced theory, but the other way around is also possible: T(7, 8) has an odd torsion group in reduced that is not seen in unreduced.

Questions

- How can we generate odd torsion in $Kh(L)$?
- Torsion of thick knots?

  Shumakovitch conjectures that there is only $\mathbb{Z}_2$ and $\mathbb{Z}_4$ in knots whose Khovanov homology is supported on 3 diagonals.
- Can we explain the difference in torsion in various versions of Khovanov homology?
- Exploiting torsion in functorality.
- What does torsion in $Kh(L)$ tell us about the link?
- Can torsion be used to distinguish links that Jones can not? Distinguish between elements families of links with trivial Jones polynomial (Eliahou, Kaufman, Thistlethwaite)

Thank you

Challenge: Find knots in Freiburg!