TORSION IN KHOVANOV LINK HOMOLOGY

Radmila Sazdanović
NC State

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TORSION IN KHOVANOV HOMOLOGY

CONJECTURE (SHUMAKOVITCH)

Let $L$ be any prime link other than the unknot or the Hopf link. Then $Kh(L)$ contains 2-torsion.

- The conjecture implies that Khovanov homology is an unknot detector.
- Kronheimer and Mrowka (2010) proved Khovanov homology is an unknot detector using gauge theory.
- The conjecture is known to be true in many cases.
Computations

- Experimentally, $Kh(L)$ has an abundance of torsion: among all 1,701,936 prime knots with at most 16 crossings.
- All non-trivial knots up to 14 crossings have only 2-torsion.
- 38 knots with 15 crossings and 129 knots with 16 crossings have 4-torsion.
- The first known knot with odd torsion is $T(5, 6)$-torus knot.
- Infinite families of links whose Khovanov homology contains $\mathbb{Z}_n$, $2 < n < 9$ and $\mathbb{Z}_{2n}$-torsion for $n < 24$. 
METHODS FOR PROVING THINGS ABOUT TORSION IN $\text{Kh}(L)$

- Explicit construction
  (Asaeda, Przytycki, Silvero, Mukherjee, Wang, Yang)

- Connections with Hochschild homology of algebras
  (Khovanov, Przytycki)

- Relations with chromatic graph homology
  (Helme-Guizon, Lowrance, Pabiniak, Przytycki, Rong, S., Scofield)

- Spectral sequence arguments
  (Lowrance, S., Shumakovitch)
Let $P_n$ be the polygon with $n$ vertices.

Let $C_n(A)$ be the space generated by labelings of the vertices of $P_n$ with elements of $A$.

Define a map $C_n(A) \to C_{n-1}(A)$ obtained by contracting edges and multiplying the labels on the identified vertices.

Przytycki (2005) showed this complex gives the Hochschild homology $HH(A)$ and the Khovanov homology of $Kh(T_2,n)$ in certain gradings.

Allows for explicit computations of 2-torsion inside of $Kh(T_2,n)$. 
Hochschild homology gives a sort of comultiplication free version of Khovanov homology for a polygon.

Helme-Guizon and Rong (2004) define the chromatic graph cohomology $H(G)$.

$H(G)$ a comultiplication free version of Khovanov homology for any graph or as

$H(G)$ an extension of Hochschild homology from cycles to graphs.

Its definition follows a similar recipe as the construction of Khovanov homology.
CORRESPONDENCE BETWEEN LINK DIAGRAMS AND GRAPHS

- **Kauffman state** is a collection of simple closed curves obtained by taking an A or B resolution at each crossing.

- Graph $G_s(D)$: vertices are Kauffman circles, edges are crossings shared by circles.

- $G_A(D) / G_B(D)$ graphs corresponding to the all-A/ all-B state.
KAUFFMAN STATES
Spanning subgraphs
KAUFFMAN STATES AND SPANNING SUBGRAPHS
KAUFFMAN STATES AND SPACES
SPANNING SUBGRAPHS AND SPACES
Spaces for both
MULTIPLICATION UNTIL A CYCLE CLOSES
**Theorem (Helme-Guizon, Przytycki, Rong - 2006)**

*If the length \( g \) of the shortest cycle in \( G \) is greater than one*

- \( Kh(D) \cong H(G_A) \) in the first \( g - 1 \) supported \( i \)-gradings
- \( \text{Tor } Kh(D) \cong \text{Tor } H(G_A) \) in the \( g \)th \( i \)-grading.
Adequate and semi-adequate links

- A link $L$ is *adequate* if it has a diagram where both $G_A(D)$ and $G_B(D)$ have no loops.

- A link $L$ is *semi-adequate* if it has a diagram $D$ where either $G_A(D)$ or $G_B(D)$ has no loops.

- Alternating links are adequate.

- Many links are semi-adequate: at least 249,649 of the 253,293 knots with 15 crossings are semi-adequate (Stoimenow ’12).
Explicit computation results

Theorem (Asaeda, Przytycki - 2004)

1. If $G_A(D)$ is loop-less and contains a cycle of odd length, then $Kh(D)$ contains 2-torsion.

2. If $G_A(D)$ is loop-less and contains a cycle of even length with an edge that is not part of a bigon, then $Kh(D)$ contains 2-torsion.

3. If $D$ is prime and alternating and $D$ is not the unknot or Hopf link, then either $G_A(D)$ or $G_B(D)$ contains an edge that is not part of a bigon. Thus $Kh(D)$ contains 2-torsion.

Remark. Shumakovitch’s conjecture is true for alternating links and “many” semi-adequate links.
Corollaries on Khovanov Homology

- The Khovanov homology of semi-adequate links contains 2-torison if $G_A(D)$ contains
  - an odd cycle or
  - an even cycle with an edge that is not part of a bigon.

- The Khovanov homology of any semi-adequate link where $G_A(D)$ has girth at least 3 contains 2-torsion (Przytycki, S.)

- Explicit formulas for torsion in certain bigradings of Khovanov homology of
  - positive 3-braids by Przytycki-S.
  - certain classes of pretzel links by Scofield

- Shumakovitch’s conjecture is true for all semi-adequate links except possibly those where $G_A(D)$ only has 2-cycles.
BACK TO SPECTRAL SEQUENCES

• \((C_{i,j}(K), d, d_v)\) is a double complex
  • \(dC_{i,j}(K), d\) Khovanov chain complex, \(d_v : M \to M\) differentials of bidegree \((0, 1)\)
  • \(d, d_v\) anti-commute: \(d \cdot d_v + d_v \cdot d = 0\)

• \(d_v\) induces a differential \(d_v^*\) on \(H(C_{i,j}, d) = Kh(K)\)

• Total complex \((\text{Tot}(C_{i,j}), D)\) is graded with \(\text{Tot}(C_{i,j})^n = \bigoplus_{k+l=n} C_{k,l}^{i,j}\)
  
  and \(D = d + d_v\) and \(D^2 = d^2 + dd_v + dd_h + d_v^2 = 0\)

• \(\exists\) a spectral sequence \(\{E_r, d_r\}\) with \(E_0 = \{C_{i,j}(K)\}, E_1 = Kh(K),\)
  and \(E_2 = H(Kh(K), d_v^*)\).

• If \(M\) has bounded support, then this spectral sequence converges to \(H(\text{Tot}(M), d)\).

• Good news: One can find \(E_\infty\)

• Bad: pages \(E_r\) and \(d_r\) for \(r \neq 0, 1, \infty\) are often elusive.
SPECTRAL SEQUENCES

- Khovanov homology and related invariants arise in many spectral sequences.

- These spectral sequences are often only defined over certain coefficient rings (e.g. \(\mathbb{Q}\), \(\mathbb{Z}_2\), or \(\mathbb{Z}_p\) for odd \(p\)).

- Use the behavior of these sequences to prove or disprove the existence of torsion.

- Recall that in all known examples (over \(\mathbb{Q}\)) the Lee spectral sequence collapses after the bidegree \((1, 4)\) differential.

- In such cases, \(Kh(D; \mathbb{Q})\) can be arranged into “knight move” pairs.
No odd torsion theorem for Khovanov homology

Theorem (Shumakovitch 2004)

If $L$ is homologically thin, then $\text{Kh}(L)$ contains no odd torsion.

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<th>$\text{Kh}(T(2,5))$</th>
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\( Kh(K) \) CONTAINS NO TORSION OF ODD ORDER

Suppose that \( Kh(K; \mathbb{Q}) \) is above.
\( Kh(K) \) CONTAINS NO TORSION OF ODD ORDER

Then \( Kh(K; \mathbb{Z}_p) \) has at least these summands.
\( \text{Kh}(K) \) CONTAINS NO TORSION OF ODD ORDER

The free part of \( \text{Kh}(K) \) looks as above and recall that all torsion appears only on the bottom diagonal.
Suppose that there exists $\mathbb{Z}_{p^k}$ in $Kh(K)$ for some odd $p$. Let the pictured $\mathbb{Z}_{p^k}$ summand be in the maximum $i$-grading of any $p^m$ torsion in $Kh(K)$.
$Kh(K)$ CONTAINS NO TORSION OF ODD ORDER

Then we have the corresponding copy of $\mathbb{Z}_p$ in $Kh(K, \mathbb{Z}_p)$
**Theorem.** \( Kh(K, \mathbb{Z}_p) \) can be arranged in knight move pairs.
**Kh(K) CONTAINS NO TORSION OF ODD ORDER**

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The universal coefficient theorem implies that $Kh(K; \mathbb{Z}_p)$ and $Kh(K)$ looks like above. **Contradiction!**
Only $2$-torsion

- Khovanov homology of thin links has no odd torsion
- All torsion in $Kh(K)$ must be of order $2^k$ for some $k$.
- Next we outline A. Shumakovitch’s unpublished proof that the Khovanov homology of a homologically thin knot has only 2-torsion.
- Proving that $Kh(K)$ only contains $\mathbb{Z}_2$-torsion and no $\mathbb{Z}_{2^k}$ amounts to showing that the Bockstein spectral sequence converges on the correct (2nd) page
- In order to prove convergence, we analyze relation between the Bockstein differentials and some new maps on Khovanov homology.
Vertical maps $\nu$ on $Kh(D; \mathbb{Z}_2)$

Shumakovitch defines maps $\nu$ of bidegree $(0, 2)$ on $C(D; \mathbb{Z}_2)$.

Properties of $\nu$

- $\nu$ commutes with the Khovanov differential, and thus induces a map $\nu^* : Kh(D; \mathbb{Z}_2) \to Kh(D; \mathbb{Z}_2)$.

- Homology with respect to $\nu$ is trivial, and so the induced map $\nu^*$ is an isomorphism on homology level.
Properties of the vertical map

- $\nu^*$ is an isomorphism.
- The “vertical” Euler characteristic of $Kh(D; \mathbb{Z}_2)$ is trivial.

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- knight move pair
- tetromino
- with $\mathbb{Z}$ coefficients
Knights moves

so far...

next show

over $\mathbb{Q}$

over $\mathbb{Z}_2$

over $\mathbb{Z}$
Spectral sequence associated to an exact couple

- Given the exact triangle on the left, define $d_1 = j_1 \circ k_1 : E_1 \to E_1$.

- $d_1^2 = j_1 \circ (k_1 \circ j_1) \circ k_1 = 0$ i.e. $d$ is a differential

$$E_2 = H(E_1, d_1) \text{ and } D_2 = \text{Im}(i_1) = \text{Ker}(j_1)$$

and maps $i_2, j_2,$ and $k_2$ can be defined so that the triangle on the right is exact.

- Iterating this process yields a spectral sequence $\{E_r, d_r\}_{r \geq 1}$.

- Note If $D, E$ bigraded modules then $\text{deg}(i) = (-1, 1)$,
  $\text{deg}(j) = (0, 0)$, $\text{deg}(k) = (1, 0)$, therefore $\text{deg}(d_r) = (r, 1 - r)$
Which exact couple to use?

- Given Khovanov chain complex (any complex of free Abelian groups) and a prime $p$
- Consider the short exact sequence of coefficient rings
  \[ 0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\text{mod } 2} \mathbb{Z}_2 \to 0 \]
- It induces a short exact sequence of Khovanov chain complexes
  \[ 0 \to C(D) \xrightarrow{\times 2} C(D) \xrightarrow{\text{mod } 2} C(D; \mathbb{Z}_2) \to 0, \]
- and, in turn, a long exact sequence on homology
  \[ \cdots \to Kh(D) \xrightarrow{\times 2} Kh(D) \xrightarrow{\text{mod } 2} Kh(D; \mathbb{Z}_2) \xrightarrow{\partial} Kh(D) \to \cdots. \]

We have an exact couple!
Define the Bockstein homomorphism
\( \beta : Kh(D; \mathbb{Z}_p) \to Kh(D; \mathbb{Z}_2) \) by \( \beta = \partial \mod p \).

\( \beta \) is the connecting homomorphism induced from
\( 0 \to \mathbb{Z}_p \to \mathbb{Z}_p^2 \to \mathbb{Z}_p \to 0 \)

For a finitely generated chain complex, like Khovanov, there exists a spectral sequence \( \{ B_r, b_r \} \) with \( B_1 = Kh(D; \mathbb{Z}_2) \) and \( b_1 = \beta \) that converges to the free \( Kh(D) \otimes \mathbb{Z}_p \).

\( b_r \) is induced from \( 0 \to \mathbb{Z}_{p^r} \to \mathbb{Z}_{p^{2r}} \to \mathbb{Z}_{p^r} \to 0 \)

\( B_r^* \cong \text{im}(H^*(C; \mathbb{Z}_{p^r}) \xrightarrow{\times p^{r-1}} H^*(C; \mathbb{Z}_{p^r})) \cong \text{im}(Kh^*(C; \mathbb{Z}_{p^r}) \xrightarrow{\times p^{r-1}} Kh^*(C; \mathbb{Z}_{p^r})) \)
More on the Bockstein Spectral Sequence

The $\mathbb{Z}_2$-Bockstein spectral sequence satisfies the following.

- The $E_1$ page of the Bockstein spectral sequence is $H(G; \mathbb{Z}_2)$.

- The $E_\infty$ page of the Bockstein spectral sequence is $[Kh(K)/\text{Tor }Kh(K)] \otimes \mathbb{Z}_2$.

- If the Bockstein spectral sequence converges at the 2nd page, then $Kh(K)$ has no torsion of order $2^k$ for $k \geq 2$.

**Theorem**

$Kh(D)$ has no $2^k$ torsion if and only if the Bockstein spectral sequence collapses at the $k$th page.
**Goal.** Show that the Bockstein spectral sequence collapses at the first page for homologically thin links.

- \( H(G) = \mathbb{Z}^{a_0} \oplus \mathbb{Z}_2^{a_1} \oplus \mathbb{Z}_4^{a_2} \oplus \cdots \oplus \mathbb{Z}_2^{a_k} \).

- \( E_1 = \mathbb{Z}_2^{a_0} \oplus \mathbb{Z}_2^{a_1} \oplus \mathbb{Z}_2^{a_1} \oplus \mathbb{Z}_2^{a_2} \oplus \mathbb{Z}_2^{a_2} \oplus \cdots \oplus \mathbb{Z}_2^{a_k} \oplus \mathbb{Z}_2^{a_k} \).

- \( E_2 = \mathbb{Z}_2^{a_0} \oplus \mathbb{Z}_2^{a_2} \oplus \mathbb{Z}_2^{a_2} \oplus \cdots \oplus \mathbb{Z}_2^{a_k} \oplus \mathbb{Z}_2^{a_k} \).

- \( E_\infty = \mathbb{Z}_2^{a_0} \).

**New new Goal:** If \( \beta \) is the Bockstein map on the \( E_1 \) page, then we want to show that the rank of \( \beta \) is the number of tetrominoes \( N \) in \( Kh(G; \mathbb{Z}_2) \).
Turner’s differential with $\mathbb{Z}_2$ coefficients

- Define $\mathbb{Z}_2$-linear maps

$$m_T : A \otimes A \to A$$

$$\Delta_T : A \to A \otimes A$$

- There is a differential $d_T : C^{i,j}(G; \mathbb{Z}_2) \to C^{i+1,j+2}(G; \mathbb{Z}_2)$.

- It induces a map $d_T^* : Kh^{i,j}(G; \mathbb{Z}_2) \to H^{i+1,j+2}(G; \mathbb{Z}_2)$. 

\[
\begin{align*}
\Delta_T : & \begin{cases}
1 \otimes 1 \mapsto 0 & 1 \otimes x \mapsto 0 \\
x \otimes 1 \mapsto 0 & x \otimes x \mapsto x
\end{cases} \\
\end{align*}
\]
The Turner differential

- $(C(D; \mathbb{Z}_2), d, d_T)$ form a double complex, and so there is an associated spectral sequence.
- $d_T$ commutes with the usual Khovanov differential, and so there is an induced map $d^*_\text{Turner}: Kh(D; \mathbb{Z}_2) \to Kh(D; \mathbb{Z}_2)$.
- For a knot, the above spectral sequence converges to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.
- If the homology is thin, the last non-zero map in the spectral sequence is $d^*_T$.  

\[ \nu^* \quad \beta \quad d^*_T \quad \nu^* \]
Putting it all together

1. \( \nu^* : \text{Kh}(D; \mathbb{Z}_2) \rightarrow \text{Kh}(D; \mathbb{Z}_2) \) is an isomorphism.
2. The Turner spectral sequence collapses at the first page.
3. \( d_T^* = \nu^* \circ \beta + \beta \circ \nu^* \).
4. \( \nu^*_\uparrow \circ d_T^* = d_T^* \circ \nu^*_\uparrow \).
5. On each diagonal, \( \text{rank } d_T^* = N \).
6. \( \text{rank } \beta = N \).
TORSION IN KHOVANOV HOMOLOGY OF HOMOLOGICALLY THIN KNOTS

THEOREM (SHUMAKOVITCH)
If $K$ is homologically thin, then its Khovanov homology only has 2-torsion.

COROLLARY
The Khovanov homology $Kh(Kh, \mathbb{Z})$ of an alternating knot is determined by its Jones polynomial and signature.

COROLLARY
If $K$ is a nontrivial, homologically thin knot, then its Khovanov homology contains 2-torsion.

Remark. The second corollary requires Kronheimer-Mrowka’s result that Khovanov homology detects the unknot.
Computations of odd torsion

- Torus knots (5, 6), (5, 7), (5, 8), and (5, 9) have 5-torsion in their Khovanov homology.

- Przytycki and Sazdanović predicted that the closure $K$ of

  \[
  \sigma_1^2 \sigma_2^2 \sigma_1^3 \sigma_2 \sigma_1 \sigma_3 \sigma_2^2 \sigma_3 \sigma_1^2 \sigma_3 \sigma_2 \sigma_1^2 \sigma_3 \sigma_2^2 \sigma_4 \sigma_3^2
  \]

  has 5-torsion in its Khovanov homology.

- Shumakovich (2012) confirmed that $K$ has 5-torsion in its Khovanov homology by showing that the difference of the Poincare polynomials of $Kh(K; \mathbb{Z}_5)$ and $Kh(K; \mathbb{Z}_7)$ is

  \[
  (t^{12} + t^{11})q^{51} + (t^{11} + t^{10})q^{47}.
  \]
$K\text{H}_5(K) = q^{31}t^0 + q^{33}t^0 + q^{35}t^2 + q^{39}t^3 + 2q^{37}t^4 + q^{39}t^4 + 2 + q^{41}t^5 + q^{43}t^5 + q^{39}t^6 + 2q^{41}t^6 + 2q^{43}t^7 + 2q^{45}t^7 + 4q^{41}t^8 + 3q^{43}t^8 + q^{47}t^8 + 13q^{43}t^9 + 4q^{45}t^9 + 4q^{47}t^9 + 2q^{43}t^{10} + 29q^{45}t^{10} + 14q^{47}t^{10} + q^{51}t^{10} + 9q^{45}t^{11} + 44q^{47}t^{11} + 31q^{49}t^{11} + q^{51}t^{11} + 2q^{45}t^{12} + 34q^{47}t^{12} + 68q^{49}t^{12} + 42q^{51}t^{12} + 2q^{53}t^{12} + 11q^{47}t^{13} + 85q^{49}t^{13} + 97q^{51}t^{13} + 59q^{53}t^{13} + 45q^{49}t^{14} + 159q^{51}t^{14} + 142q^{53}t^{14} + 63q^{55}t^{14} + 137q^{51}t^{15} + 245q^{53}t^{15} + 202q^{55}t^{15} + 9q^{57}t^{15} + 345q^{53}t^{16} + 5376q^{55}t^{16} + 237q^{57}t^{16} + 54q^{59}t^{16} + 735q^{55}t^{17} + 589q^{57}t^{17} + 260q^{59}t^{17} + 37q^{61}t^{17} + 1328q^{57}t^{18} + 953q^{59}t^{18} + 253q^{61}t^{18} + 21q^{63}t^{18} + 2040q^{59}t^{19} + 1501q^{61}t^{19} + 220q^{63}t^{19} + 9q^{65}t^{19} + 2729q^{61}t^{20} + 2149q^{63}t^{20} + 173q^{65}t^{20} + 2q^{67}t^{20} + 2q^{61}t^{21} + 3203q^{63}t^{21} + 2779q^{65}t^{21} + 109q^{67}t^{21} + 11q^{63}t^{22} + 3344q^{65}t^{22} + 3219q^{67}t^{22} + 50q^{69}t^{22} + 36q^{65}t^{23} + 3127q^{67}t^{23} + 3345q^{69}t^{23} + 16q^{71}t^{23} + 81q^{67}t^{24} + 2608q^{69}t^{24} + 3116q^{71}t^{24} + 3q^{73}t^{24} + 137q^{69}t^{25} + 1934q^{71}t^{25} + 2572q^{73}t^{25} + 191q^{71}t^{26} + 1271q^{73}t^{26} + 1853q^{75}t^{26} + 228q^{73}t^{27} + 759q^{75}t^{27} + 1134q^{77}t^{27} + 238q^{75}t^{28} + 446q^{77}t^{28} + 568q^{79}t^{28} + 219q^{77}t^{29} + 294q^{79}t^{29} + 218q^{81}t^{29} + 175q^{79}t^{30} + 226q^{81}t^{30} + 56q^{83}t^{30} + 119q^{81}t^{31} + 175q^{83}t^{31} + 7q^{85}t^{31} + 65q^{83}t^{32} + 119q^{85}t^{32} + 26q^{85}t^{33} + 65q^{87}t^{33} + 7q^{87}t^{34} + 26q^{89}t^{34} + q^{89}t^{35} + 7q^{91}t^{35} + q^{93}t^{36}$
\[ KH_7(K) = q^{31}t^0 + q^{33}t^0 + q^{35}t^2 + q^{39}t^3 + 2q^{37}t^4 + q^{39}t^4 + 2q^{41}t^5 + q^{43}t^5 + q^{39}t^6 + 2q^{41}t^6 + 2q^{43}t^7 + 2q^{45}t^7 + 4q^{41}t^8 + 3q^{43}t^8 + q^{47}t^8 + 13q^{43}t^9 + 4q^{45}t^9 + 4q^{47}t^9 + 2q^{43}t^{10} + 29q^{45}t^{10} + 13q^{47}t^{10} + q^{51}t^{10} + 9q^{45}t^{11} + 43q^{47}t^{11} + 31q^{49}t^{11} + 2q^{45}t^{12} + 34q^{47}t^{12} + 68q^{49}t^{12} + 41q^{51}t^{12} + 2q^{53}t^{12} + 11q^{47}t^{13} + 85q^{49}t^{13} + 97q^{51}t^{13} + 59q^{53}t^{13} + 45q^{49}t^{14} + 159q^{51}t^{14} + 142q^{53}t^{14} + 63q^{55}t^{14} + 137q^{51}t^{15} + 245q^{53}t^{15} + 202q^{55}t^{15} + 59q^{57}t^{15} + 345q^{53}t^{16} + 376q^{55}t^{16} + 237q^{57}t^{16} + 54q^{59}t^{16} + 735q^{55}t^{17} + 589q^{57}t^{17} + 260q^{59}t^{17} + 37q^{61}t^{17} + 1328q^{57}t^{18} + 953q^{59}t^{18} + 253q^{61}t^{18} + 21q^{63}t^{18} + 2040q^{59}t^{19} + 1501q^{61}t^{19} + 220q^{63}t^{19} + 9q^{65}t^{19} + 2729q^{61}t^{20} + 2149q^{63}t^{20} + 173q^{65}t^{20} + 2q^{67}t^{20} + 2q^{61}t^{21} + 3203q^{63}t^{21} + 2779q^{65}t^{21} + 109q^{67}t^{21} + 11q^{63}t^{22} + 3344q^{65}t^{22} + 3219q^{67}t^{22} + 50q^{69}t^{22} + 36q^{65}t^{23} + 3127q^{67}t^{23} + 3345q^{69}t^{23} + 16q^{71}t^{23} + 81q^{67}t^{24} + 2608q^{69}t^{24} + 3116q^{71}t^{24} + 3q^{73}t^{24} + 137q^{69}t^{25} + 1934q^{71}t^{25} + 2572q^{73}t^{25} + 191q^{71}t^{26} + 1271q^{73}t^{26} + 1853q^{75}t^{26} + 228q^{73}t^{27} + 1134q^{77}t^{27} + 238q^{75}t^{28} + 446q^{77}t^{28} + 568q^{79}t^{28} + 219q^{77}t^{29} + 294q^{79}t^{29} + 218q^{81}t^{29} + 759q^{75}t^{27} + 175q^{79}t^{30} + 226q^{81}t^{30} + 56q^{83}t^{30} + 119q^{81}t^{31} + 175q^{83}t^{31} + 7q^{85}t^{31} + 65q^{83}t^{32} + 119q^{85}t^{32} + 26q^{85}t^{33} + 65q^{87}t^{33} + 7q^{87}t^{34} + 26q^{89}t^{34} + q^{89}t^{35} + 7q^{91}t^{35} + q^{93}t^{36} \]
STILL MYSTERIOUS

- Torsion in reduced homology
  - 2-torsion appears first for 13-crossing knots (13n3663)
  - the simplest knot having odd torsion in reduced homology is $T(5, 6)$ and that is $\mathbb{Z}_3$

- the simplest knot having odd torsion in unreduced homology is $T(5, 6)$ which has a copy of $\mathbb{Z}_3$ and a copy of $\mathbb{Z}_5$

- some knots, e.g. $T(5, 6)$, have odd torsion in unreduced homology which is not seen in the reduced theory, but the other way around is also possible: $T(7, 8)$ has an odd torsion group in reduced that is not seen in unreduced.
Questions

• How can we generate odd torsion in $Kh(L)$?

• Torsion of thick knots?

  Shumakovitch conjectures that there is only $\mathbb{Z}_2$ and $\mathbb{Z}_4$ in knots whose Khovanov homology is supported on 3 diagonals.

• Can we explain the difference in torsion in various versions of Khovanov homology?

• Exploiting torsion in functorality.

• What does torsion in $Kh(L)$ tell us about the link?

• Can torsion be used to distinguish links that Jones can not? Distinguish between elements families of links with trivial Jones polynomial (Eliahou, Kaufman, Thistlethwaite)
Thank you

Challenge: Find knots in Freiburg!