Eta-Invariants and Molien Series for Unimodular Groups

by

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Abstract

We look at the singularity $\mathbb{C}^n/\Gamma$, for Γ finite subgroup of $SU(n)$, from two perspectives. From a geometrical point of view, $\mathbb{C}^n/\Gamma$ is an orbifold with boundary S^{2n-1}/Γ. We define and compute the corresponding orbifold η-invariant. From an algebraic point of view, we look at the algebraic variety $\mathbb{C}^n/\Gamma$ and we analyze the associated Molien series. The main result is a formula which relates the two notions: η-invariant and Molien series. Along the way computations of the spectrum of the Dirac operator on the sphere are performed.

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Contents

Chapter 1

Introduction

The work in this dissertation is part of a project, in which we want to study the geometry and topology of crepant resolutions of singularities for Calabi-Yau orbifolds. The main result stated here is a relation between the η-invariant of the boundary (an analytical object) and the Hilbert-Molien series of the singularity (an algebraic object).

A Calabi-Yau orbifold in complex dimension n is locally modeled on $\mathbb{C}^n/\Gamma$ where Γ is a finite subgroup of $SL(n, \mathbb{C})$. From a geometrical perspective, we view this as an orbifold with boundary S^{2n-1}/Γ. The η-invariant is an invariant of the geometry of the boundary, measuring the spectral asymmetry of the Dirac operator. With the formula we proved, it turns out that this η-invariant comes in the algebraic package describing the variety $\mathbb{C}^n/\Gamma$. Let R be an irreducible representation of Γ. If η_R denotes the η-invariant of the R-twisted Dirac operator on the boundary, then

$$\eta_R = \operatorname{Res}_1 \frac{\Phi_R(t)}{1-t}. \tag{1.1}$$

Here Φ_R is the Hilbert-Molien series (the Hilbert series up to a factor involving $\dim R$) of the module of R-relative invariants of $\mathbb{C}[X_1, \ldots, X_n]$ under the action of Γ.

A *resolution* (X, π) of $\mathbb{C}^n/\Gamma$ is a non-singular complex n-fold X with a proper birational morphism $\pi : X \to \mathbb{C}^n/G$. If $K_X \cong \pi^* K_{\mathbb{C}^n/G}$, then X is called a *crepant resolution*[1]. Since Calabi-Yau manifolds have trivial canonical bundles, to get a Calabi-Yau structure on X one must choose a crepant resolution. Using methods from string theory, physicists were able to make predictions about the topology of the crepant resolutions (the *stringy Euler* and *Betti numbers*) in terms of the representation theory of the finite group, [8]. The aim of the project is to obtain an explicit description of the ring structure in cohomology, in terms, if possible, of the finite group. This is of interest for algebraic geometers working on minimal models of n-folds. It is also of interest for physicists because the ring structure in cohomology gives the correlation functions, which are the main output information of a quantum field theory.

The idea of the project is to exploit the Atiyah-Patodi-Singer index theorem for studying the crepant resolutions of $\mathbb{C}^n/G$, when such resolutions exist. Kronheimer and Nakajima [13] first use this approach to give a geometrical interpretation of the classical McKay Correspondence which establishes a bijection between the set of isomorphism classes of

[1]Etymology: For a resolution of singularities we can define a notion of *discrepancy*. A crepant resolution is a resolution without discrepancy.

irreducible representations of a finite subgroup of $SL(2,\mathbb{C})$ and the set of vertices of the extended Dynkin diagram of a simple Lie algebra of type ADE. As a consequence they obtain the multiplicative structure in cohomology in terms of the representation theory of the finite group.

The Atiyah-Patodi-Singer index theorem is an index theorem for manifolds with boundary or with cylindrical ends. The setup is the following. Let X be a non-compact even dimensional manifold and choose a Riemannian metric on X which is cylindrical at infinity. Moreover, assume that X is a spin manifold and choose a spin structure on it. Suppose E is a Hermitian vector bundle on X with a unitary connection and that at infinity the connection is constant in the cylinder direction. Then

$$\text{index} D_E = \int_X ch(E)\hat{A}(p) + \eta_E,$$

where indexD_E is the index (in the L^2-completion) of the Dirac operator D_E acting on spinors with coefficients in E. The symbol η_E is the η-invariant of the Dirac operator which is induced from D_E on a slice of the cylinder, Y. In the integrand, $ch(E)$ denotes the Chern character of E (as a differential form), and $\hat{A}(p)$ is the Hirzebruch $\hat{A}$-polynomial applied to the Pontrjagin forms p_i of the Riemannian metric on X and we pick out the form in the product of top dimension. The formula is still valid when the metric at infinity can be conformally changed into a cylindrical metric. For a more general metric, there is an extra integral over Y involving the second fundamental form.

We want to apply the index theorem to a crepant resolution X of $\mathbb{C}^n/\Gamma$, when such a crepant resolution exist. (In the case $n = 2$ and $n = 3$ a crepant resolution always exists, but for $n \geq 4$ such resolutions might not exist because the singularity can be terminal.) When $\mathbb{C}^n/\Gamma$ has an isolated singularity at the origin (which is also Kronheimer and Nakajima's situation in the case of a surface singularity), its geometry at infinity is very similar to the geometry of the crepant resolution away from the exceptional locus. The boundary is the smooth manifold S^{2n-1}/Γ, and the metric is an ALE (asymptotically locally euclidean) metric, which can be conformally changed into the cylindrical metric. Therefore, the Atiyah-Patodi-Singer formula is valid in this case.

But, for almost all the finite subgroups of $SL(n,\mathbb{C})$, the singularities of $\mathbb{C}^n/\Gamma$ are not isolated, i.e. there are singularities propagating to infinity, and, therefore the boundary S^{2n-1}/Γ is an orbifold. In this case the geometry of the crepant resolution is more complicated because of the new topology we introduce at infinity. The Atiyah-Patodi-Singer formula does not hold in the form above. One of the goals is to state a generalization of the Atiyah-Patodi-Singer index theorem in the case of non-isolated singularities. The program for this project is to figure out the correct interpretation for each term in the index formula.

The purpose of the work enclosed here has been to study the degenerate version of the η-invariant. In the isolated singularities case this is the honest η-invariant of the crepant resolution. The degenerate version of the boundary of the crepant resolution at infinity is S^{2n-1}/Γ endowed with the orbifold metric coming from the round metric on S^{2n-1}. In order to perform computations we work in a Γ-equivariant setup on S^{2n-1}. For each irreducible representation R of Γ, we consider the Γ-invariant Dirac operator on S^{2n-1} twisted by the vector bundle corresponding to R. Using methods from the representation theory of Lie groups we determine the spectrum of the Dirac operator on S^{2n-1} and then derive the expression for the η-invariant.

Even if the computations are just on the boundary of the orbifold, it turns out that the

η-invariant is encoded in the algebraic information which comes with the singularity. The ring of polynomials in n variables, $A = \mathbb{C}[X_1, \dots, X_n]$, can be decomposed under the action of Γ into $A = \oplus A_R^\Gamma$, where A_R^Γ is the isotopic component corresponding to the irreducible representation R, and the sum is over all the irreducible representations of Γ. We consider a version of the Hilbert series of the graded A^Γ-module A_R^Γ, which we call the *Molien series of the module of R-relative invariants*. A classical theorem of Molien, [15], gives an explicit expression for the rational function $\Phi_R(t)$ and thereby ties together invariant theory with generating functions:

$$\Phi_R(t) = \frac{1}{\dim R} \sum_{n \geq 0} \dim (A_R^\Gamma)_n \, t^n = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \frac{\chi_R(\gamma)}{\det (I - t\gamma)}.$$

The formula we obtained, (1.1), expresses the η-invariant η_R as the residue at 1 of $\Phi_R(t)/(1-t)$. Moreover, we obtain a description of the entire meromorphic part of the Laurent expansion about $t = 1$ in terms of η-invariants associated to the singularity, Proposition 5.3.3 and Proposition5.3.4.

In Chapter 2 of this dissertation we set up a general context for studying the spectrum of the Dirac operator on an odd dimensional homogeneous space. In Chapter 3 we proceed to give an explicit description of the spectrum on the odd dimensional sphere viewed as the homogeneous space $\widetilde{U(n)}/\widetilde{U(n-1)}$, where $\widetilde{U(N)}$ denotes the double cover of $U(N)$. We perform this computation for all the $\widetilde{U(n)}$-invariant metrics on S^{2n-1}. In the next chapter we compute the η-invariant for the round metric, the metric with constant sectional curvature. Finally, in Chapter 5 we introduce the Molien series, describe their properties and state the main results.

Descriptions of the spectrum of the Dirac operator on the sphere where previously done. Bär, [4], computed the spectrum of the Dirac operator for the sphere S^{2n-1} endowed with a $\widetilde{U(n)}$-invariant metric using a completely different method. We learned about his result after finishing the computations in the present work.

Chapter 2

Homogeneous Spaces

2.1 Crash course in Representation Theory

In this section we gather together elements of the representation theory of Lie groups which we are going to use. The main source of reference is [12].

Proposition 2.1.1. *Let G be a compact Lie group, with Lie algebra $\mathfrak{g}_0$. Then the real vector space $\mathfrak{g}_0$ admits an $Ad(G)$-invariant inner product. Relative to this inner-product, $Ad(g)$, for $g \in G$, acts by orthogonal transformations. At the level of the Lie algebra ad_X, for $X \in \mathfrak{g}_0$, acts by skew-symmetric transformations.*

Corollary 2.1.2. *Let G be a compact Lie group, with Lie algebra $\mathfrak{g}_0$. Then $\mathfrak{g}_0 = \mathfrak{z}_{\mathfrak{g}_0} \oplus [\mathfrak{g}_0, \mathfrak{g}_0]$, where $\mathfrak{z}_{\mathfrak{g}_0}$ is the center, and $[\mathfrak{g}_0, \mathfrak{g}_0]$ is semi-simple.*

Remark. In the language of the representation theory this claims that the Lie algebra $\mathfrak{g}_0$ is *reductive*, meaning that to each ideal $\mathfrak{a}$ in $\mathfrak{g}_0$ corresponds an ideal $\mathfrak{b}$ in $\mathfrak{g}_0$ such that $\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{b}$.

Consider the complexification of the Lie algebra of G, $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}$. We fix an $Ad(G)$-invariant inner-product on G, and we write B for its negative on $\mathfrak{g}_0$, and extend it by linearity to $\mathfrak{g}$. Since B is negative definite on $\mathfrak{g}_0$, it is a valid substitute for the Killing form. It has the properties of the Killing form:

Lemma 2.1.3.

1. *$B(X, Y) = B(Y, X)$.*
2. *$B([X, Y], Z) = B(X, [Y, Z])$.*

2'. *$B(ad_X(Y), Z) = -B(Y, ad_X(Z))$.*

Proposition 2.1.4. *The maximal tori in G are exactly the analytical subgroups corresponding to the maximal abelian algebras of $\mathfrak{g}_0$.*

From now on we fix T a maximal torus in G, and let $\mathfrak{t}_0$ be its Lie algebra. Since $\mathfrak{g}_0 = \mathfrak{z}_{\mathfrak{g}_0} \oplus [\mathfrak{g}_0, \mathfrak{g}_0]$, with $[\mathfrak{g}_0, \mathfrak{g}_0]$ semi-simple, and since $\mathfrak{t}_0$ is a maximal abelian subalgebra in $\mathfrak{g}_0$, it follows that $\mathfrak{t}_0 = \mathfrak{z}_{\mathfrak{g}_0} \oplus \mathfrak{t}'_0$, where $\mathfrak{t}'_0$ is a maximal abelian subalgebra in $[\mathfrak{g}_0, \mathfrak{g}_0]$. This holds also at the level of the complex Lie algebras: $\mathfrak{g} = \mathfrak{z} \oplus [\mathfrak{g}, \mathfrak{g}]$, and $\mathfrak{t} = \mathfrak{z} \oplus \mathfrak{t}'$, where $\mathfrak{t}'$ is a maximal abelian algebra in $[\mathfrak{g}, \mathfrak{g}]$. The subalgebra $\mathfrak{t}'$ is called a *Cartan subalgebra of* $[\mathfrak{g}, \mathfrak{g}]$.

Let α be a linear functional on $\mathfrak{t}$ and define

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} | \quad [H, X] = \alpha(H)X \quad \text{for all } H \in \mathfrak{t}\}.$$

Definition 2.1.5. We say that $\alpha \in \mathfrak{t}^* \setminus \{0\}$ is a *root*, if $\mathfrak{g}_\alpha \neq 0$. The elements of $\mathfrak{g}_\alpha$ are called *root vectors* for the root α, and we denote the set of all roots by $\Delta(\mathfrak{g}, \mathfrak{t})$.

Lemma 2.1.6. *We have the following* root space decomposition *for the Lie algebra* $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{g},\mathfrak{t})} \mathfrak{g}_\alpha. \tag{2.1}$$

Remarks. The roots of $\mathfrak{g}$ are actually roots of the semi-simple part of the Lie algebra, $[\mathfrak{g}, \mathfrak{g}]$, extended to $\mathfrak{t}$ by defining them to be zero on $\mathfrak{z}_{\mathfrak{g}}$. The set $\Delta(\mathfrak{g}, \mathfrak{t})$ has all the usual properties of the set of roots of a semi-simple Lie algebra except that they do not span the whole $\mathfrak{t}^*$, they span just $(\mathfrak{t}')^*$.

Proposition 2.1.7. *(Properties of the root space decomposition)*

1. *If α and β are in $\Delta \cup \{0\}$, and $\alpha + \beta \neq 0$, then $B(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$*
2. *If α is in $\Delta \cup \{0\}$, then B is non-singular on $\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha}$.*
3. *If $\alpha \in \Delta$, then $-\alpha \in \Delta$.*
4. *$B|_{\mathfrak{t}\times\mathfrak{t}}$ is non-degenerate. Consequently, to each root α corresponds $H_\alpha \in \mathfrak{t}$ with $\alpha(H) = B(H, H_\alpha)$, for all $H \in \mathfrak{t}$.*
5. *Δ spans $(\mathfrak{t}')^*$.*

Since T is a subgroup of G, $Ad(T)$ acts by orthogonal transformations on $\mathfrak{g}_0$ relative to our fixed inner-product. We extend the inner-product on $\mathfrak{g}_0$ to a Hermitian inner-product on $\mathfrak{g}$, and $Ad(T)$ acts on $\mathfrak{g}$ via a commuting family of unitary transformations. Such a family must have simultaneously eigenspace decomposition, and this decomposition is the root space decomposition (2.1). The action of $Ad(T)$ on the z1-dimensional $\mathfrak{g}_\alpha$ has to have the form

$$Ad_H(X) = \xi_\alpha(H)X, \text{for } H \in T.$$

Here $\xi_\alpha : T \to S^1$ is a continuous homomorphism and its differential is $\alpha|_{\mathfrak{t}_0}$. It follows, in particular, that $\alpha|_{\mathfrak{t}_0}$ is imaginary-valued, and the roots are real-valued on $i\mathfrak{t}_0$. In the language of representation theory ξ_α is called a *multiplicative character.*

We define $\mathfrak{t}_{\mathbb{R}} = i\mathfrak{t}_0$; this is the real part of $\mathfrak{t}$ on which all the roots are real. Therefore, we may regard the roots as elements of $\mathfrak{t}^*_{\mathbb{R}}$. In the language from Knapp's book, $\Delta(\mathfrak{g}, \mathfrak{t})$ form an abstract root system in the subspace of $\mathfrak{t}^*_{\mathbb{R}}$ coming from the semi-simple Lie algebra $[\mathfrak{g}, \mathfrak{g}]$.

The negative definite form B on $\mathfrak{t}_0$ gives, by complexification, a positive definite form on $\mathfrak{t}_{\mathbb{R}}$. For each $\lambda \in \mathfrak{t}^*_{\mathbb{R}}$, we choose $H_\lambda \in \mathfrak{t}_{\mathbb{R}}$ such that

$$\lambda(H) = B(H, H_\lambda), \quad \text{for } H \in \mathfrak{t}_{\mathbb{R}}.$$

The resulting linear map from $\mathfrak{t}^*_{\mathbb{R}} \to \mathfrak{t}_{\mathbb{R}}$ given by $\lambda \mapsto H_\lambda$ is an isomorphism of vector spaces. Under this isomorphism, let $(i\mathfrak{z}_{\mathfrak{g}_0})^*$ be the subspace of $\mathfrak{t}^*_{\mathbb{R}}$ corresponding to $i\mathfrak{z}_{\mathfrak{g}_0}$.

The inner-product on $\mathfrak{t}_{\mathbb{R}}$ induces an inner-product on $\mathfrak{t}_{\mathbb{R}}^*$, denoted by $\langle \cdot, \cdot \rangle$. Relative to this inner-product, the elements of $\Delta(\mathfrak{g}, \mathfrak{t})$ span the orthogonal complement to $(i\mathfrak{z}_{\mathfrak{g}_0})^*$, and $\Delta(\mathfrak{g}, \mathfrak{t})$ is an abstract reduced root system in this orthogonal complement. Also, we have

$$\langle \lambda, \mu \rangle = \mu(H_\lambda) = B(H_\lambda, H_\mu).$$

Lemma 2.1.8. *Let $\alpha \in \Delta(\mathfrak{g}, \mathfrak{t})$, and let $\beta \in \Delta \cup \{0\}$. Then the α-*string containing β *has the form $\beta + n\alpha$ for $-p \leq n \leq q$, with $p \geq 0$ and $q \geq 0$. There are no gaps. Furthermore*

$$p - q = \frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}.$$

Corollary 2.1.9. *We have*

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta},$$

for α and β in $\Delta \cup \{0\}$, and $\alpha + \beta \neq 0$.

For each root, α, choose a vector $E_\alpha \neq 0$ in $\mathfrak{g}_\alpha$. This means that $[H, E_\alpha] = \alpha(H)E_\alpha$. Then we have a few more properties of the root decomposition:

Lemma 2.1.10.

1. *If $\alpha \in \Delta(\mathfrak{g}, \mathfrak{t})$, and $X \in \mathfrak{g}_{-\alpha}$, then $[E_\alpha, X] = B(E_\alpha, X)H_\alpha$.*
2. *If α and β are in $\Delta(\mathfrak{g}, \mathfrak{t})$, then $\beta(H_\alpha)$ is a rational multiple of $\alpha(H_\alpha)$.*
3. *If $\alpha \in \Delta(\mathfrak{g}, \mathfrak{t})$, then $\alpha(H_\alpha) \neq 0$.*
4. *If $\alpha \in \Delta(\mathfrak{g}, \mathfrak{t})$, then $\mathfrak{g}_\alpha$ is 1-dimensional. Also, $n\alpha \notin \Delta(\mathfrak{g}, \mathfrak{t})$, for any integer $n \geq 2$.*
5. *The action of $ad(\mathfrak{t})$ on $\mathfrak{g}$ is simultaneously diagonable.*
6. *On $\mathfrak{t} \times \mathfrak{t}$, B is given by*

$$B(H, H') = \sum_{\alpha \in \Delta(\mathfrak{g}, \mathfrak{t})} \alpha(H)\alpha(H').$$

7. *The pair of vectors E_α and $E_{-\alpha}$ can be normalized such that $B(E_\alpha, E_{-\alpha}) = 1$*

From now on, for each root $\alpha \in \Delta(\mathfrak{g}, \mathfrak{t})$ we choose and fix a vector root vector E_α satisfying the properties in the Lemma above. This basis is known as the *Cartan-Weyl basis.*

Definition 2.1.11. For $\alpha \in \Delta(\mathfrak{g}, \mathfrak{t})$, the *root reflection* s_α is given by

$$s_\alpha(\lambda) = \lambda - \frac{2\langle \lambda, \alpha \rangle}{|\alpha|^2}\alpha.$$

We notice that the linear transformation s_α is the identity on $(i\mathfrak{z}_{\mathfrak{g}_0})^*$ (and it is the usual root reflection in the orthogonal complement).

Definition 2.1.12. The *Weyl group*, $W(\Delta(\mathfrak{g}, \mathfrak{t}))$, is the group generated by all the s_α's, for $\alpha \in \Delta(\mathfrak{g}, \mathfrak{t})$.

Remark. The Weyl group consists of the members of the usual Weyl group of the abstract root system, with each member extended to be the identity on $(i\mathfrak{z}_{\mathfrak{g}_0})^*$.

We now want to study the elements of $\mathfrak{t}$ more systematically. We saw that each root has the property that they lift from imaginary-valued linear functional on $\mathfrak{t}_0$ to multiplicative characters of T, the chosen maximal torus in G.

Proposition-Definition 2.1.13. *A linear functional $\lambda \in \mathfrak{t}_0$ is said to be* analytically integral *if it satisfies one of the following equivalent conditions:*

(i) If $H \in \mathfrak{t}_0$ satisfies $exp(H) = 1$, then $\lambda(H) \in 2\pi i\mathbb{Z}$.

(ii) There exists a multiplicative character ξ_λ of T with $\xi_\lambda(expH) = e^{\lambda(H)}$ for all $H \in \mathfrak{t}_0$.

Remarks. All the roots are analytically integral. The set of analytically integral forms for G may be regarded as an additive group in $\mathfrak{t}^*_{\mathbb{R}}$.

Proposition 2.1.14. *If G is a compact connected Lie group and $\tilde{G}$ is a finite covering of G, then the index of the group of analytically integral forms for G in the group of analytically integral forms for $\tilde{G}$ equals to the order of the kernel of the covering homomorphism $\tilde{G} \to G$.*

We now take up to study the irreducible representations of a compact connected Lie group G. First, we need to introduce a little bit more notation. We introduce a notion of positivity in the dual of the maximal abelian algebra, $\mathfrak{t}^*$. The intention is to single out a subset of nonzero elements of $\mathfrak{t}^*_{\mathbb{R}}$ as *positive*, writing $\psi > 0$ if ψ is a positive element. The only properties of positivity that we need are that

1. for any nonzero $\psi \in \mathfrak{t}^*_{\mathbb{R}}$, exactly one of ψ and $-\psi$ is positive
2. the sum of positive elements is positive, and any positive multiple of a positive element is positive.

The way in which such a notion of positivity is introduced is not important. One way to define positivity is by means of a *lexicographic ordering.* Fix a spanning set $\psi_1, \dots \psi_m$ of $\mathfrak{t}^*_{\mathbb{R}}$, and define positivity as follows: We say that $\psi > 0$ if there exists an index k such that $\langle \psi, \psi_i \rangle = 0$ for $1 \le i \le k-1$ and $\langle \psi, \psi_k \rangle > 0$. We say that $\psi \ge \psi'$ or $\psi' < \psi$ if $\psi - \psi'$ is positive. Then $>$ defines a simple ordering on $\mathfrak{t}^*_{\mathbb{R}}$ that it is preserved under addition and under multiplication by positive scalars.

Definition 2.1.15. We say that a root α is *simple* if $\alpha > 0$ and if α does not decompose as $\alpha = \beta_1 + \beta_2$ with β_1 and β_2 both positive roots.

Definition 2.1.16. Let G be a compact Lie group with maximal torus T, of rank $r + r_{\mathfrak{z}}$, where $r_{\mathfrak{z}}$ is the dimension of the center of G. Let $\Pi = \{\alpha_1, \dots, \alpha_r\}$ be a subset of independent simple roots in $\Delta(\mathfrak{g}, \mathfrak{t})$. We call Π a *simple system.*

Definition 2.1.17. We say that a member λ of $\mathfrak{t}^*_{\mathbb{R}}$ is *dominant* if $\langle \lambda, \alpha \rangle \ge 0$ for all $\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{t})$.

Remark. It is enough that $\langle \lambda, \alpha_i \rangle \ge 0$ for all $\alpha_i \in \Pi$.

Given a simple system we can define $\Delta^+(\mathfrak{g}, \mathfrak{t})$ to be all the roots of the form $\sum_i c_i\alpha_i$ with all $c_i \ge 0$. We also introduce the notation $\Delta^-(\mathfrak{g}, \mathfrak{t}) = -\Delta^+(\mathfrak{g}, \mathfrak{t})$. From the properties of the root space decomposition, we have

$$\Delta(\mathfrak{g}, \mathfrak{t}) = \Delta^+(\mathfrak{g}, \mathfrak{t}) \bigcup \Delta^-(\mathfrak{g}, \mathfrak{t}).$$

Lemma 2.1.18. *Let Φ be a finite-dimensional irreducible representation of G, compact connected Lie group. If λ is a weight of ϕ, the differential of Φ, then λ is analytically integral.*

If Φ is a representation of G on a finite-dimensional complex vector space V, and ϕ is the induced representation of the complex Lie algebra $\mathfrak{g}$, then the weights of (V, ϕ) are in $\mathfrak{t}_{\mathbb{R}}^*$. The largest weight in the ordering is called the *highest weight* of ϕ.

Now we are ready to state one of the fundamental theorems in representation theory:

Theorem 2.1.19. *(**Theorem of the Highest Weight**) Let G be a compact Lie group with complexified Lie algebra $\mathfrak{g}$, and let T be a maximal torus with complexified Lie algebra $\mathfrak{t}$, and let $\Delta^+(\mathfrak{g}, \mathfrak{t})$ be a positive system for the roots. Then there is a one-to-one correspondence between the irreducible finite dimensional representations Φ of G and dominant analytically integral linear functionals λ on $\mathfrak{t}$, the correspondence being given by sending Φ into λ, the highest weight of Φ. Moreover, the highest weight λ of Φ have the following properties*

1. *λ depends only on the simple system Π and not on the ordering used to define Π.*

2. *The weight space V_λ is 1-dimensional.*

3. *Each root vector E_α for arbitrary $\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{t})$, annihilates the members of V_λ, and the members of V_λ are the only vectors with this property.*

4. *Every weight of Φ is of the form*

$$\lambda - \sum_{\alpha \in \Pi} n_\alpha \alpha,$$

with the integers $n_\alpha \geq 0$.

5. *Each weight space V_μ for Φ has $\dim V_{w\mu} = \dim V_\mu$ for all $w \in W(\Delta)$, and each weight μ has $|\mu| \leq |\lambda|$ with the equality only if μ is in the orbit $W(\Delta)\lambda$.*

2.2 Homogeneous Spaces and Homogeneous Structures on Them

Let G be a compact connected Lie group and H be a closed connected subgroup of G. The space of cosets G/H with the natural differential structure is called *homogeneous space*. G is said to act *effectively* on G/H if $L_g = id$ implies $g = e$. There is a natural left action of G on G/H defined by $g_1[g_2H] = [g_1g_2H]$. This action is transitive since $g_1g_2^{-1}[g_2H] = [g_1H]$.

2.2.1 Homogeneous Riemannian structures

Definition 2.2.1. A Riemannian metric on G/H for which G acts by isometries is called a *G-invariant metric.*

In what follows we denote by $\mathfrak{g}_0$ and $\mathfrak{h}_0$ the Lie algebras of G, respectively H. The following result gives a complete description of the set of G-invariant metrics on the homogeneous space G/H.

Proposition 2.2.2. *([7])*

(a) *The set of G-invariant metrics on G/H is naturally isomorphic to the set of scalar products $(\cdot,\cdot)$ on $\mathfrak{g}/\mathfrak{h}$ which are invariant under the action of $Ad(H)$ on $\mathfrak{g}/\mathfrak{h}$.*

(b) *If H is connected, a scalar product $(\cdot,\cdot)$ is invariant under $Ad(H)$ if and only if for each $X \in \mathfrak{h}$, ad_X is skew symmetric with respect to $(\cdot,\cdot)$.*

(c) *If $\mathfrak{g}$ admits a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ with $Ad(H)(\mathfrak{p}) \subset \mathfrak{p}$, then G-invariant metrics on G/H are in $1-1$ correspondence with $Ad(H)$-invariant scalar products on $\mathfrak{p}$. Conversely, if G/H admits a G-invariant metric, then G admits a left invariant metric which is right invariant under H, and the restriction of the metric to H is $Ad(H)$-invariant. Setting $\mathfrak{p} = \mathfrak{h}^{\perp}$ gives the decomposition above.*

(d) *If H is connected, the condition $Ad_H(\mathfrak{p}) \subset \mathfrak{p}$ is equivalent to $[\mathfrak{h}, \mathfrak{p}] \subset \mathfrak{p}$.*

(e) *If G is compact, then G admits a left and right invariant metric.*

It follows from Proposition 2.1.1 that $\mathfrak{g}_0$ admits an $Ad(G)$-invariant inner product. In particular, such an inner-product gives a left-invariant metric on G which is right-invariant under H. The restriction of this metric to H is $Ad(H)$-invariant. According to (d) in the above Proposition, the orthogonal complement, $\mathfrak{p}_0$, of $\mathfrak{h}_0$ in $\mathfrak{g}_0$ verifies

$$\mathfrak{g}_0 = \mathfrak{h}_0 \oplus \mathfrak{p}_0, \quad Ad(H)(\mathfrak{p}_0) \subset \mathfrak{p}_0.$$

Therefore, in order to understand the G-invariant metrics on G/H, we need to have a description of the $Ad(H)$-invariant inner-products on $\mathfrak{p}_0$. Going to the complexification, this is equivalent to having a description of the $Ad(H)$-invariant Hermitian inner-products on $\mathfrak{p}$. Via the adjoint action of H, $\mathfrak{p}$ becomes a representation of H. It follows that we need to understand Hermitian structures on a representation of H. For an irreducible representation we have:

Lemma 2.2.3. *If Φ is a representation of G on a finite-dimensional complex vector space V, then V admits a Hermitian inner-product such that Φ is unitary.*

From this statement and Schur's Lemma, it follows that if we have two Hermitian structures on V such that the action of G is unitary with respect each of them, then the two Hermitian metrics differ by a non-zero scalar.

From all this discussion, we conclude with the following result which gives a description of the set of Hermitian-inner products on $\mathfrak{p}$.

Lemma 2.2.4. *Let*

$$\mathfrak{p} = m_1\mathfrak{p}_1 \oplus \ldots \oplus m_n\mathfrak{p}_n,$$

be the decomposition of $\mathfrak{p}$ into irreducible representations under the action of H, all $\mathfrak{p}_i$ being inequivalent irreducible representations of H. Then the set of Hermitian inner-products on $\mathfrak{p}$ which are invariant under H form a $\sum_{i=1}^{n} m_i^2$-parameter family.

Proof. It follows easily from the discussion above. □

2.2.2 Riemannian connections on homogeneous spaces

The map $\pi : G \to G/H$ is a fibration, therefore a submersion. Let $\mathcal{X}(G/H)$ be the space of all vector fields on G/H. We define the map $X \mapsto \widetilde{X}$ from $\mathfrak{g}$ into $\mathcal{X}(G/H)$ by

$$\widetilde{X}_x f = \frac{d}{dt} f(exp(tX) \cdot x)|_{t=0}.$$

Remarks. We have $[\widetilde{X}, \widetilde{Y}] = \widetilde{[X, Y]}$, and if we consider the projection of the Lie bracket $[X, Y]$ onto $\mathfrak{p}_0$, we have

$$[\widetilde{X}, \widetilde{Y}] = \widetilde{[X, Y]} = \widetilde{[X, Y]_{\mathfrak{p}_0}}.$$

As a consequence, we can identify $\widetilde{X}$ with X, and the Lie bracket on G/H with the projection of the Lie bracket on $\mathfrak{g}_0$ to $\mathfrak{p}_0$.

In the language of submersions, the decomposition $\mathfrak{g}_0 = \mathfrak{h}_0 \oplus \mathfrak{p}_0$ corresponds to the decomposition of T_eG into the horizontal vector space, and the vertical vector space. According to Proposition 2.1.1 (c), if G/H admits a G-invariant metric, $(\cdot,\cdot)_{G/H}$, then G admits a left-invariant metric, $(\cdot,\cdot)_G$, which is right-invariant under H. The restriction of $(\cdot,\cdot)_G$ to $\mathfrak{h}_0$ is bi-invariant, and its restriction to $\mathfrak{p}_0$ induces $(\cdot,\cdot)_{G/H}$. (When there is no danger of confusion we are going to drop the subscripts.) Then $\pi : G \to G/H$ is a Riemannian submersion and in order to understand the connection on G/H we have to understand it on G first.

Lemma 2.2.5. (Riemannian connection on G) *Let X, Y and Z be left invariant vector fields on G, endowed with the left-invariant metric $(\cdot,\cdot)_G$. Then:*

$$\begin{aligned}
\nabla_X Y = & \ \frac{1}{2}\left([X,Y] - (ad_X)^*(Y) - (ad_Y)^*(X)\right); \\
(R(X,Y)Z, W)_G = & \ \langle \nabla_X Y, \nabla_Y W \rangle - \langle \nabla_Y Z, \nabla_X W \rangle - \langle \nabla_{[X,Y]} Z, W \rangle; \\
(R(X,Y)Y, X)_G = & \ ||(ad_X)^*Y + (ad_Y)^*X||^2 - (ad_X^*(X), ad_Y^*(Y))_G \\
& -\frac{1}{2}([[X,Y],Y],X)_G - \frac{1}{2}([[Y,X],X],Y)_G.
\end{aligned}$$

Lemma 2.2.6. (Riemannian connection on G/H) *Let X, Y and Z be vector fields on G/H, i.e. elements in $\mathfrak{p}$. Then*

$$\begin{aligned}
([X,Y]_{\mathfrak{p}}, Z)_{G/H} = & \ ([X,Y], Z)_G; \\
(\nabla_X Y, Z)_{G/H} = & \ (\nabla_X Y, Z)_G \\
(R(X,Y)Z, W)_{G/H} = & \ (R(X,Y)Z, W)_G - \frac{1}{4}([X,Z]_{\mathfrak{h}}, [Y,W]_{\mathfrak{h}})_G \\
& +\frac{1}{4}([Y,Z]_{\mathfrak{h}}, [X,W]_{\mathfrak{h}})_G - \frac{1}{2}([Z,W]_{\mathfrak{h}}, [X,Y]_{\mathfrak{h}})_G
\end{aligned}$$

For a proof of these statements we refer to [7].

Lemma 2.2.7. *The sectional curvature on G/H with the G-invariant metric $(\cdot,\cdot)_{G/H}$ is*

$$\begin{aligned}
K(X,Y) = & ||(ad_X)^*Y + (ad_Y)^*X||^2 - (ad_X^*(X), ad_Y^*(Y))_G \\
& - \frac{1}{2}([[X,Y],Y],X)_G - \frac{1}{2}([[Y,X],X],Y)_G - \frac{3}{4}||[X,Y]_{\mathfrak{p}}||^2
\end{aligned} \tag{2.2}$$

Proof. In the above Lemma set $Z = Y$ and $W = X$. □

Remark. Another way to express the sectional curvature is the following

$$K(X,Y) = (\nabla_X \nabla_Y Y, X) - (\nabla_Y \nabla_X Y, X) - (\nabla_{[X,Y]} Y, X) + \frac{3}{4}([X,Y]_{\mathfrak{h}}, [X,Y]_{\mathfrak{h}}).$$

2.2.3 Homogeneous vector bundles

Definition 2.2.8. A vector bundle $\mathbb{E}$ on G/H is called a *homogeneous vector bundle* if G acts on $\mathbb{E}$ on the left and the action of G satisfies

(1) $g\mathbb{E}_x = \mathbb{E}_{gx}$ for $x \in G/H$ and $g \in G$.

(2) The mapping from $\mathbb{E}_x$ to $\mathbb{E}_{gx}$ induced by g is linear for $x \in G/H$ and $g \in G$.

To any representation (E, ρ) of H we can associate the homogeneous vector bundle $\mathbb{E} = G \times_\rho E$, constructed via the action of H on the right on $G \times E$, $(g, v)h = (gh, h^{-1}v)$ for $g \in G$, $h \in H$ and $v \in E$.

Lemma 2.2.9. *All the homogeneous vector bundles on G/H arrive via this construction.*

Proof. Let $\mathbb{E}$ be a homogeneous vector bundle on G/H. Let $E = \mathbb{E}_{[e]}$ be the fiber over the coset $[e] = eH$. From the definition of a homogeneous vector bundle, we have an action of H on E. It is easy to see that $\mathbb{E} = G \times_H E$. □

Let $\Gamma(G/H; \mathbb{E})$ be the space a continuous sections of $\mathbb{E}$. G acts on this space, via

$$(g \cdot s)(x) = gs(g^{-1}x),$$

for $g \in G$, $s \in \Gamma(G/H; \mathbb{E})$ and $x \in G/H$. When E is a unitary representation of H, this action extends to a unitary representation of G on the space of L^2-sections of the bundle $\mathbb{E}$. An equivalent way to think of this representation is given by the following lemma.

Lemma 2.2.10. *Consider the space*

$$I(E) = \{f : G \to E \,\text{continuous}\, | f(gh) = h^{-1}f(g)\}.$$

There is an isomorphism $A : \Gamma(X; \mathbb{E}) \to I(E)$, which extends to a unitary equivalence between the corresponding L^2-completions.

Proof. Given a section s of $\mathbb{E}$, we associate to it the function $f_s(g) = g^{-1}s([gH])$, where $g^{-1} : E_{[gH]} \to E_{[eH]}$. It is clear that $f_s(gh) = (gh)^{-1}s([ghH]) = h^{-1}f_s(g)$. Conversely, to any element $f \in I(E)$, we associate the section $s_f([gH]) = [g, f(g)]$. It is easy to see that this is well-defined, and that the maps $f \to s_f$ and $s \to f_s$ are inverse to each other. □

This representation, *the induced representation*, is an infinite dimensional representation. The way it decomposes into irreducibles under the action of G is embodied in Frobenius Reciprocity.

Proposition 2.2.11. (Frobenius Reciprocity) *([22]) We have*

$$\mathrm{Hom}^G(W, I(E)) \cong \mathrm{Hom}^H(\mathrm{Res}_H^G(W), E)$$

and

$$L^2(G/H; \mathbb{E}) \cong \bigoplus_W W \otimes \mathrm{Hom}^H(W, E),$$

as unitary representations of G, the sum being taken over the irreducible representations of G. Furthermore the algebraic sum

$$\bigoplus_W A^{-1}(A_W(W \otimes \mathrm{Hom}^H(W, E)))$$

is dense in $\Gamma(X;\mathbb{E})$ *relative to the uniform topology. Here* A_W *is the induced map*

$$A_W : W\otimes\mathrm{Hom}^H(W,E) \to I(E),$$

defined by $A_W(w\otimes P)(g) = P(g^{-1}w)$.

Note that the action of G on $I(E)$, translates at the level of A_W into $g\left(A_W(w\otimes P)\right) = A_W(\pi_W(g)w\otimes P)$. Here, $\mathrm{Res}^G_H(W,\pi)$ denotes the representation of H induced via restriction from the representation of G, (W,π).

2.2.4 Homogeneous Differential operators

Definition 2.2.12. Let $\mathbb{E}$ and $\mathbb{F}$ be homogeneous vector bundles on G/H. A differential operator $D : \Gamma^\infty(G/H;\mathbb{E}) \to \Gamma^\infty(G/H;\mathbb{F})$ is called a *homogeneous differential operator* if $g\cdot(Ds) = D(g\cdot s)$ for $g \in G$ and $s \in \Gamma^\infty(G/H;\mathbb{E})$.

On homogeneous spaces we had special kind of vector bundles, the homogeneous vector bundles which are in $1-1$ correspondence to the representations of H. It turns out that the homogeneous differential operators have also a description in terms of the representation theory.

Lemma 2.2.13. *Any homogeneous differential operator of order* m *corresponds to an element in*

$$(Hom(E,F)\otimes U^m(\mathfrak{g}))^H.$$

2.3 *Spin* structures and Dirac operators on Homogeneous manifolds

We recall that $Spin(n)$ is a double cover of $SO(n)$, which for $n \geq 3$ is the universal cover group. Therefore, all the irreducible representations of $SO(N)$ give us irreducible representations of $Spin(n)$. On the other hand there is a special representation, S, the *Spin-representation* of $Spin(n)$ which does not arrive via a lifting from an irreducible representation of $SO(n)$. In the following Proposition we gather together a few facts about the *Spin*-group and its *Spin*-representations.

Proposition 2.3.1. *(Properties of the Spin-representation)*

(a) The generator of the kernel of $\pi : Spin(N) \to So(N)$ *acts as* -1 *on* S.

(b) If V *denotes the standard representation of* $Spin(N)$ *i.e. the one determined by* π *on the* N*-dimensional vector space* V*, then there is a pairing of Spin-modules*

$$V\otimes S \xrightarrow{\gamma} S,$$

has the property that the map $\phi \to \gamma(v)\phi$ *is an isomorphism for any* $v \in V$, $v \neq 0$.

(c) If N *is even,* S *splits into two irreducible representations* S^+ *and* S^-*, and the above pairing induces parings*

$$V\otimes S^+ \to S^-,$$
$$V\otimes S^- \to S^+.$$

(d) If $N = 2n+1$, the Spin-representation S has dimension 2^n and the weights are

$$\frac{1}{2}(\pm\lambda_1 \pm \lambda_2 \pm \ldots \pm \lambda_n).$$

If $N = 2n$, then the weights of S^+, are

$$\frac{1}{2}(\pm\lambda_1 \pm \lambda_2 \pm \ldots \pm \lambda_n),$$

with an even number of minuses, and the weights for S^- are

$$\frac{1}{2}(\pm\lambda_1 \pm \lambda_2 \pm \ldots \pm \lambda_n),$$

with an odd number of minuses. The dimension of S^+ and S^- is 2^{n-1}.

(e) Assume that we are in the odd dimensional case, $N = 2n+1$, and let $\{e_1, e_2, \ldots, e_{2n+1}\}$ be an orthonormal basis for V. Then S is $\underbrace{\mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2}_{n\ times}$, with the kth copy of $\mathbb{C}^2$ corresponding to the weight λ_k. Clifford multiplication by $\gamma(e_{2k-1})$ and $\gamma(e_{2k})$ is trivial on all the copies of $\mathbb{C}^2$ but the kth one, where the action is given explicitly by the following matrices

$$\gamma(e_{2k-1}) \quad acts\ via \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and

$$\gamma(e_{2k}) \quad acts\ via \quad \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

From here it also follows that

$$\gamma(e_{2k-1})\gamma(e_{2k}) \quad acts\ via \quad \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$

For the last element e_{2n+1} we define the action to agree with the action of the complex volume of $\mathbb{C}l(V)$. We define the volume element

$$\omega = (i)^n \gamma(e_1)\gamma(e_2)\ldots\gamma(e_{2n-1})\gamma(e_{2n})\gamma(e_{2n+1}).$$

According to the usual convention in spin geometry, [16], we want to choose the action of $\gamma(e_{2n+1})$ such that $\omega = -i$. Since $\gamma(e_{2k-1})\gamma(e_{2k})$ acts by multiplication by $i\epsilon_k$, it follows that the action of $\gamma(e_{2n+1})$ is

$$\gamma(e_{2n+1})\phi = (-1)^{n+1} i\epsilon_1\epsilon_2\ldots\epsilon_m\phi,$$

for Ψ in the weight space $\frac{1}{2}(\epsilon_1\lambda_1 + \epsilon_2\lambda_2 + \ldots + \epsilon_n\lambda_n)$.

(f) The Lie algebra of Spin(n) sits in $\mathbb{C}l(V)$ and it is generated by

$$\{e_i e_j | i < j\}.$$

Assume now that X is an oriented Riemannian manifold of dimension N and denote by $P_{SO}(X)$ the bundle of orthonormal frames in the tangent bundle.

Definition 2.3.2. A *Spin-structure* on X is a principal $Spin(N)$-bundle $P_{Spin}(X)$ which is a double cover of $P_{SO}(X)$ such that the fiberwise restriction $\pi_x : Spin(N)_x \to SO(N)_x$ is isomorphic to the standard map $\pi : Spin(N) \to SO(N)$ $\pi : Spin(N) \to SO(N)$.

The property of a manifold to admit a *Spin*-structure is a topological property, and we have

Proposition 2.3.3. *The orientable Riemannian manifold admits a Spin-structure in and only if the second Stiefel-Whitney class of X, $w_2(X)$, is zero. Furthermore, if $w_2(X) = 0$, then the distinct Spin-structures on X are in one-to-one correspondence with the elements of $H^1(X; \mathbb{Z}_2)$.*

In the case X is the homogeneous manifold G/H, equipped with a G-invariant Riemannian metric, we have an action of G on $P_{SO}(X)$ which makes this bundle G-equivariant. We look at the possible *Spin*-structures which are compatible with the G-action. We will call such a *Spin*-structure a *G-equivariant Spin-structure*.

Lemma 2.3.4. *The homogeneous manifold G/H endowed with the G-invariant metric $(\cdot,\cdot)$, admits a G-invariant Spin-structure if and only if the adjoint representation $Ad : H \to SO(\mathfrak{p}_0)$, admits a lifting $\widetilde{Ad}$ to $Spin(\mathfrak{p}_)$*

$$\begin{array}{ccc} & & Spin(N)\ . \\ & \overset{\widetilde{Ad}}{\nearrow} & \downarrow \pi \\ H & \xrightarrow{Ad} & SO(N) \end{array}$$

Remark. First of all, for a homogeneous manifold G/H to admit a G-equivariant *Spin*-structure, we need it to be *Spin*. This is a topological condition: $w_2(G/H) = 0$. Then, we need a G-equivariant *Spin*-structure. This might exist or might not exists. For example, the odd-dimensional sphere S^{2n-1} can be viewed as the homogeneous space $SU(n)/SU(n-1)$, which admits an $SU(n)$-equivariant *Spin*-structure since $SU(n-1)$ is simply-connected and the lifting of the adjoint representation is possible. But, if we view it as the homogeneous space $U(n)/U(n-1)$, then it does not admit a $U(n)$-equivariant *Spin*-structure.

For the time being, we assume that the homogeneous space G/H admits a G-equivariant *Spin*-structure.

Since $\text{rank}H + r = \text{rank}G$, the dimension of $\mathfrak{p}_0$ is either odd or even, depending on the parity of r. In this paper we are concerned with the case $\dim(\mathfrak{p}_0)$ is odd, which means that there is only one irreducible representation of $Spin(\mathfrak{p}_0)$, $\sigma : Spin(\mathfrak{p}_0) \to End(S)$. Therefore, we must assume that r is odd-dimensional, which translates into $\dim(\mathfrak{t}_0 \cap \mathfrak{p}_0)$ is odd-dimensional. Denote the dimension of $\mathfrak{p}_0$ by $2n+1$, and fix $\{e_1, e_2, \ldots, e_{2n+1}\}$ an orthonormal basis of $\mathfrak{p}_0$ with respect our chosen G-invariant inner-product.

We choose a maximal torus in $SO(\mathfrak{p}_0)$ such that it extends the torus, $ad(T_K)$, induced by the adjoint representation. The weights of the standard representation of $SO(\mathfrak{p})$ on $\mathfrak{p}^{\mathbb{C}}$ are of the form $\pm\lambda_1, \pm\lambda_2, \ldots, \pm\lambda_n, 0$, where $\{\lambda_i\}$ are the simple roots of $SO(\mathfrak{p})$. In these terms, the weights of (S, σ) are given by

$$\frac{1}{2}\left(\pm\lambda_1 \pm \lambda_2 \pm \ldots \pm \lambda_n\right).$$

Via the lifting $\widetilde{Ad}$, S become the H-module $(S, \sigma \circ \widetilde{Ad})$.

Lemma 2.3.5. *Let $Y \in \mathfrak{h}$. Then*

$$\widetilde{ad}(Y) = \sum_{a,b=1}^{2n+1} \frac{([Y, e_a], e_b)}{4} e_a e_b,$$

where $\widetilde{ad}$ is the differential of $\widetilde{Ad}$ at $e \in H$, the identity element.

Proof. For any element $z \in \mathfrak{p}$ let $X(z) : Cl(\mathfrak{p}_0) \to Cl(\mathfrak{p}_0)$ be the linear map

$$x \to zx - xz, \quad , \quad \text{for } x \in Cl(\mathfrak{p}_0).$$

It is clear that

$$X(e_a e_b)(e_c) = \begin{cases} \quad 0 & \text{if } c \neq a, b \\ \quad 2e_b & \text{if } c = a \\ - \quad 2e_a & \text{if } c = b \end{cases} \tag{2.3}$$

It follows that for $z \in \mathfrak{spin}(\mathfrak{p}_0)$, $X(z)(\mathfrak{p}_0) \subset \mathfrak{p}_0$. Thus $X(z)$ restricts to an endomorphism of $\mathfrak{p}$, which we will also denote by $X(z)$. For $Y \in \mathfrak{h}$, we have

$$X(\widetilde{ad}(Y)) = adY,$$

when acting on $\mathfrak{p}_0$. The standard basis for $\mathfrak{spin}(\mathfrak{p}_0)$ is $\{e_a \cdot e_b | 1 \leq a < b \leq 2n+1\}$. In this basis

$$\widetilde{ad}(Y) = \sum_{1 \leq a < b \leq 2n+1} A_{ab}\, e_a e_b,$$

and our job is to compute the coefficients A_{ab}. We have

$$\begin{aligned} [Y, e_c] &= \sum_{a<b} A_{ab}\, X(e_a e_b)\, e_c \\ &= \sum_{c<b} 2A_{cb}\, e_b - \sum_{a<c} 2A_{ab}\, e_a, \end{aligned} \tag{2.4}$$

and therefore

$$([Y, e_a], e_b) = 2A_{ab} \quad \text{for } a < b.$$

It follows that

$$\widetilde{ad}(Y) = \sum_{a<b} \frac{([Y, e_a], e_b)}{2}\, e_a e_b.$$

Using the fact that for $a = b$ we have

$$([Y, e_a], e_a) = 0,$$

and that

$$([Y, e_a], e_b)\, e_a e_b = ([Y, e_b], e_a)\, e_b e_a,$$

the conclusion of the Lemma follows. □

2.3.1 The Dirac operator

We assume that there is a lifting of $Ad : H \to SO(\mathfrak{p}_0)$ to $\widetilde{Ad} : H \to Spin(\mathfrak{p}_0)$, so that S becomes an H-module. Since the Dirac operator is canonically associated to the metric and the *Spin*-structure, it follows that it will be G-invariant.

Because of Frobenius Reciprocity, the space of spinors has the following decomposition under the action of G

$$\Omega^0(G/H, \mathbb{S}) = \bigoplus_V V \otimes Hom^H(V, S). \tag{2.5}$$

The Dirac operator is a homogeneous differential operator.

Lemma 2.3.6. *Via Frobenius Reciprocity, the Dirac operator becomes a morphism in the category of G-modules. In particular, restricted to each of the isotopic components, it has the form*

$$D^V(w \otimes P) = w \otimes \gamma(e_a) \nabla^{Spin}_{e_a} \otimes 1 - w \otimes \gamma(e_a) P(\pi_V(e_a)). \tag{2.6}$$

Proof. Denote by $\Phi_{w\otimes P}$ the spinor defined via $\Phi_{w\otimes P}(g) = A_V(w\otimes P)(g) = P(g^{-1}w)$.

$$\begin{aligned} D^V(w\otimes P)(g) &= D\Phi_{w\otimes P}(g) \\ &= \sum \gamma(e_a)\nabla^{spin}_{e_a}\Phi_{w\otimes P}(g) \\ &= \sum \gamma(e_a)\nabla^{spin}_{e_a}(f_\alpha \Psi_\alpha)(g) \\ &= \sum \gamma(e_a)\left((e_a(f_\alpha)\Psi_{alpha}) + f_\alpha \nabla^{spin}_{e_a}\Psi_{alpha}\right)(g) \\ &= -\sum \gamma(e_a)P(\pi_V(e_a)g^{-1}w) + w\otimes(\sum \gamma(e_a)\nabla^{spin}_{e_a}P)(g) \\ &= -\sum w\otimes\gamma(e_a)P\pi_V(e_a) + w\otimes(\sum \gamma(e_j)\nabla^{spin}_{e_a}P)(g) \end{aligned}$$

□

An alternative way to think of this (which is going to be used when we actually do computations) is to consider the space of spinors as being

$$\Omega^0(G/H, \mathbb{S}) = (S \otimes \mathcal{C}^\infty(G))^H, \tag{2.7}$$

which can be spelled out as taking the space of smooth maps from G to S, and considering the H-invariant ones. The correspondence between the two interpretations is pretty obvious.

There is an action of $\mathfrak{g}^{\mathbb{C}}$ on $\mathcal{C}^\infty(G)$, the space of smooth complex valued functions on G, defined as

$$\pi(X)f = X(f),$$

for any $X \in \mathfrak{g}^{\mathbb{C}}$ and $f \in \mathcal{C}^\infty(G)$. The action of $\mathfrak{g}$ on $\mathcal{C}^\infty$ induces an action of $\mathfrak{g}$ on the space of spinors. In the description of the space of spinors given by (2.7), the action is

$$\pi(X)(\Phi\otimes f) = \Phi\otimes\pi(X)f,$$

where $\Phi \in S$ and $f \in \mathcal{C}^\infty(G)$ such that $\Phi\otimes f$ is a spinor on G/H, i.e. $\Phi\otimes f(S\otimes\mathcal{C}^\infty(G))^H$. Under the action of G the space of spinors decomposes as

$$\begin{aligned} \Omega^0(G/H, \mathbb{S}) &= (\sum_V V\otimes V^* \otimes S)^H \\ &= \sum_V V\otimes(V^*\otimes S)^H, \end{aligned} \tag{2.8}$$

where V is an irreducible representation of G and $(V^*\otimes S)^H$ represents the multiplicity of the irreducible representation V in the decomposition of the space of spinors under the action of G. The Dirac operator is a G-equivariant operator and therefore it preserves the decomposition (2.8). On each of the isotopic components, we can look at

$$D_V : V \otimes (V^* \otimes S)^H \to V \otimes (V^* \otimes S)^H,$$

which is an endomorphism of finite dimensional vector spaces. It turns out that the Dirac operator really acts on the $(V^* \otimes S)^H$ part, and the way it acts is given by the following lemma.

Lemma 2.3.7. *The map*

$$D_V(w \otimes \Psi) = w \otimes \left(\sum_a (\gamma(e_a) \nabla^{Spin}_{e_a} \otimes 1) \Psi + \sum_a (\gamma(e_a) \otimes \pi(e_a)) \Psi \right) \tag{2.9}$$

where $\Psi \in (V^* \otimes S)^H$, $w \in V$ *and* $\nabla^{spin}_{e_a}$ *acts on* S *via* $-\frac{1}{4}\gamma(\nabla_{e_a} e_b)\gamma(e_b)$.

Therefore, in order to compute the spectrum of the Dirac operator, we need to compute it for each D_V. From the expression of D_V, it turns out that the eigenspaces are all going to be sums of copies of V. The number of copies and the eigenvalue is a hard computation, and we are able to perform it in the next chapter in the case of the odd-dimensional sphere, viewed as a homogeneous space.

Chapter 3

The case of the odd-dimensional sphere

In this chapter we analyze the spectrum of the Dirac operator on S^{2n-1}. If we view S^{2n-1} as the homogeneous space $U(n)/U(n-1)$, there is no $U(n)$-invariant $Spin$-structure on it. Rather we have to consider the double covers $\widetilde{U(n)}$ and $\widetilde{U(n-1)}$.

3.1 Invariant $Spin$-structures on S^{2n-1}

If we consider the Lie algebra $\mathfrak{u}(N)$, the simply-connected Lie group with this Lie algebra is $SU(N) \times S^1$. The smallest Lie group which has this Lie algebra is $U(N)$. $U(N)$ arises as a quotient of $SU(N) \times S^1$ modulo a discrete group (of the center of $SU(N) \times S^1$):

$$1 \to Ker(\rho) \to SU(N) \times S^1 \xrightarrow{\rho} U(N) \to 1,$$

where $\rho(g, e^{i\theta}) = e^{i\theta} \cdot g$. Therefore $Ker(\rho)$ is generated by elements of the form

$$(e^{\frac{2\pi i}{N}} \cdot I, e^{-\frac{2\pi i}{N}}).$$

The fundamental group of $U(N)$ is $\mathbb{Z}$.

If we consider the sphere as the homogeneous space $U(n)/U(n-1)$, then the restriction of the adjoint map $Ad : U(n-1) \to SO(2n-11)$ does not lift to $Spin(2n-1)$, because the image of the fundamental group of $U(n-1)$, $Ad_*\pi_1(U(n-1), I)$ into the fundamental group of $SO(2n-1)$ is actually $\mathbb{Z}_2$. Therefore, in order to satisfy the conditions of the Lifting Theorem [18], we have to kill the 2-torsion in the fundamental group of $U(n-1)$. For this, we are led to consider the double covers of $U(n)$ and $U(n-1)$, $\widetilde{U(n)}$ and $\widetilde{U(n-1)}$. The construction is illustrated by the following diagram:

$$\begin{array}{ccccc}
\widetilde{U(n-1)} & \hookrightarrow & \widetilde{U(n)} & \longrightarrow & Spin(2n) \\
\downarrow & & \downarrow & & \downarrow \pi \\
U(n-1) & \hookrightarrow & U(n) & \hookrightarrow & SO(2n)
\end{array},$$

where $\widetilde{U(n)}$ and $\widetilde{U(n-1)}$ denote the pull-back of $Spin(2n)$.

We consider the sphere as the homogeneous space $\widetilde{U(n)}/\widetilde{U(n-1)}$.

Lemma 3.1.1. *There is an $\widetilde{U(n)}$-invariant Spin-structure on S^{2n-1}.*

We start by analyzing the $\widetilde{U(n)}$-invariant metrics on S^{2n-1}. Denote the basis for the Cartan algebra of $\mathfrak{u}(n)$, $\mathfrak{t}_n$, by $x_1, x_2, \ldots, x_n$. We think of x_j as the matrix with entry 1 on the (j,j)-position. With respect this basis, the roots are $x_j - x_k$. We choose the fundamental Weyl chamber to be $x_1 > x_2 > \ldots > x_n >$. With respect this notion of positivity, the positive roots are $\Delta_n^+ = \{x_j - x_k | 1 \leq j < k \leq n\}$, and we choose the simple roots to be $\Pi_n = \{x_1 - x_2, \ldots, x_{n-2} - x_{n-1}, x_{n-1} - x_n\}$. We are going to view the Cartan algebra of $\mathfrak{u}(n-1)$, $\mathfrak{t}_{n-1}$, as a subalgebra of $\mathfrak{t}_n$ generated by $x_1, \ldots, x_{n-1}$. Everything we discussed about above restricts to $\mathfrak{u}(n-1)$:

$$\begin{gathered}\text{the roots are } x_j - x_k, \text{ with } 1 \leq j, k \leq n-1, \\ \text{positive roots are } \Delta_{n-1}^+ = \{x_j - x_k | 1 \leq j < k \leq n-1\}, \\ \text{the simple roots are } \Pi_{n-1} = \{x_1 - x_2, \ldots, x_{n-2} - x_{n-1}\}.\end{gathered}$$

Following the notations we introduced in Chapter 2, an $Ad(\widetilde{U(n-1)})$-invariant complement of $\mathfrak{u}(n-1)$ in $\mathfrak{u}(n)$ is

$$\mathfrak{p} = \mathfrak{t}_{\mathfrak{p}} \oplus \bigoplus_{i=1}^{n-1} (\mathfrak{g}_{x_i - x_n} \oplus \mathfrak{g}_{-x_i + x_n}), \tag{3.1}$$

where $\mathfrak{t}_{\mathfrak{p}}$ is 1-dimensional, generated by H_λ, for $\lambda = \sum_{i=1}^{n-1} (x_i - x_n)$.

Remark. The only freedom in the choice of $\mathfrak{p}$ is given by the choice of $\mathfrak{t}_{\mathfrak{p}}$. Our present choice is suitable with the the decomposition of $\mathfrak{u}(n)$ into the center and the semi-simple part, and it has the advantage that it will also give an answer to the same question for the sphere as the homogeneous space $SU(n)/SU(n-1)$. Another choice is H_{x_n}, for example.

From Lemma 2.2.4, it follows that the decomposition of $\mathfrak{p}$ into irreducible representations of $\widetilde{U(n-1)}$ is

$$\mathfrak{p} = \mathfrak{t}_{\mathfrak{p}} \oplus \mathfrak{p}_{x_{n-1} - x_n},$$

where $\mathfrak{p}_{x_{n-1} - x_n} = \oplus_{j=1}^{n-1} (\mathfrak{g}_{x_j - x_n} + \mathfrak{g}_{-x_j + x_n})$.

Remark. Note that $\mathfrak{t}_{\mathfrak{p}}$ is the trivial representation of $\widetilde{U(n-1)}$, and the highest weight of the irreducible representation $\mathfrak{p}_{x_{n-1} - x_n}$ is x_1.

According to Lemma 2.2.4 there is a 2-parameter family of $\widetilde{U(n)}$-invariant metrics on S^{2n-1}. Up to homothethy, there is just one parameter family, and we are going to take the constants $C_1 = C$ and $C_{x_{n-1} - x_n} = 1$. Note that the case $C = 1$ corresponds to the case of the $\widetilde{U(n)}$-invariant metric on S^{2n-1} induced by the $Ad(\widetilde{U(n)})$-invariant metric on $\widetilde{U(n)}$.

Lemma 3.1.2. *An orthonormal basis for $\mathfrak{p}_0$ is*

$$\begin{aligned} Z &= -\frac{i}{\sqrt{(n-1)(n)\, C}} H_\lambda, \\ X_j &= \frac{1}{\sqrt{2}} (E_{x_j - x_n} - E_{-x_j + x_n}), \\ Y_j &= \frac{1}{\sqrt{2}} (i E_{x_j - x_n} + i E_{-x_j + x_n}). \end{aligned}$$

Proof. It is easy to verify that that they form an orthonormal basis with respect to the $\widetilde{U(n)}$-invariant metric of parameter C. □

Lemma 3.1.3. *The cosets for the orthonormal basis are*

$$\begin{aligned}
[Z, X_j] &= -\frac{1}{C}\sqrt{\frac{n}{n-1}}\, Y_j, \\
[Z, Y_j] &= \frac{1}{C}\sqrt{\frac{n}{n-1}}\, X_j, \\
[X_j, Y_j] &= iH_{x_j - x_n}, \\
[X_j, X_k] &= \frac{1}{2}(-E_{x_j-x_k} - E_{-x_j+x_k}), \quad \text{for } j \neq k, \\
[X_j, Y_k] &= \frac{1}{2}(iE_{x_j-x_k} - iE_{-x_j+x_k}), \quad \text{for } j \neq k, \\
[Y_j, Y_k] &= \frac{1}{2}(-E_{x_j-x_k} - E_{-x_j+x_k}), \quad \text{for } j \neq k.
\end{aligned}$$

Moreover, $[X_j, X_k]$, $[X_j, Y_k]$ *and* $[Y_j, Y_k]$ *belong to* $\mathfrak{h}$ *for* $j \neq k$. *When projected onto* $\mathfrak{p}_0$, $[X_j, Y_j]$ *is* $-C\sqrt{\frac{n}{n-1}}\, Z$.

Proof. We have

$$\begin{aligned}
[Z, X_j] &= -\frac{1}{\sqrt{2}}\,\frac{i}{\sqrt{(n-1)n}\,C}\,[H_\lambda, E_{x_j-x_n} - E_{-x_j+x_n}] \\
&= -\frac{1}{\sqrt{2}}\,\frac{i}{\sqrt{(n-1)n}\,C}\,\big(\lambda(E_{x_j-x_n})E_{x_j-x_n} - \lambda(E_{-x_j+x_n})E_{-x_j+x_n}\big) \\
&= -\frac{1}{\sqrt{2}}\,\frac{i}{\sqrt{(n-1)n}\,C}\, n\,\big(E_{x_j-x_n} + E_{-x_j+x_n}\big) \\
&= -\frac{1}{C}\sqrt{\frac{n}{n-1}}\, Y_j.
\end{aligned}$$

Also,

$$\begin{aligned}
[X_j, Y_j] &= \frac{1}{2}\,\big[E_{x_j-x_n} - E_{-x_j+x_n}, iE_{x_j-x_n} + iE_{-x_j+x_n}\big] \\
&= \frac{1}{2}\,2i\,[E_{x_j-x_n}, E_{-x_j+x_n}] \\
&= i\,H_{x_j-x_n}.
\end{aligned}$$

The vector $[X_j, Y_j]$ has a component in $\mathfrak{h}$ and a component in $\mathfrak{p}$. Since,

$$\begin{aligned}([X_j, Y_j], Z) &= (iH_{x_j - x_n}, -\frac{i}{\sqrt{(n-1)n}\,C} H_\lambda) \\ &= \frac{1}{(n-1)\sqrt{n}\,C}(H_{x_j-x_n}, H_\lambda) \\ &= -\frac{1}{\sqrt{(n-1)n}\,C} C^2 B(H_{x_j-x_n}, H_\lambda) \\ &= -\frac{C}{\sqrt{(n-1)n}} \lambda(H_{x_j-x_n}) \\ &= -C\sqrt{\frac{n}{n-1}},\end{aligned}$$

it follows that the projection onto $\mathfrak{p}$ is $-C\sqrt{\frac{n}{n-1}}\,Z$. □

Lemma 3.1.4. *The Riemannian connection associated to the chosen $\widetilde{U(n)}$-invariant metric is*

$$\begin{aligned}\nabla_Z Z &= 0, \\ \nabla_Z X_j &= -\frac{2-C^2}{2C}\sqrt{\frac{n}{n-1}}\,Y_j, \\ \nabla_Z Y_j &= \frac{2-C^2}{2C}\sqrt{\frac{n}{n-1}}\,X_j, \\ \nabla_{X_j} Z &= \frac{C}{2}\sqrt{\frac{n}{n-1}}\,Y_j, \\ \nabla_{X_j} X_j &= 0, \\ \nabla_{X_j} Y_j &= -\frac{C}{2}\sqrt{\frac{n}{n-1}}\,Z, \\ \nabla_{Y_j} Z &= -\frac{C}{2}\sqrt{\frac{n}{n-1}}\,X_j, \\ \nabla_{Y_j} X_j &= \frac{C}{2}\sqrt{\frac{n}{n-1}}\,Z, \\ \nabla_{Y_j} Y_j &= 0.\end{aligned}$$

All the other combinations are 0.

Proof. We sketch the computation for $\nabla_Z X_j$. From the general formula for a Riemannian connection we have

$$(\nabla_Z X_j, \xi) = \frac{1}{2}\left(([Z, X_j], \xi) - ([X_j, \xi], Z) - ([Z, \xi], Z)\right),$$

where $\xi \in \mathfrak{p}$. Using Lemma 3.1.3, it follows that $\nabla_Z X_j$ has components just in the direction

Y_j. And,

$$\begin{aligned}
(\nabla_Z X_j, Y_j) &= \frac{1}{2}\left(([Z,X_j],Y_j) - ([X_j,Y_j],Z) - ([Z,Y_j],X_j).\right) \\
&= \frac{1}{2}\left((-\frac{1}{C}\sqrt{\frac{n}{n-1}}Y_j,Y_j) - (-C\sqrt{\frac{n}{n-1}}Z,Z) - (\frac{1}{C}\sqrt{\frac{n}{n-1}}X_j,X_j)\right) \\
&= \frac{1}{2}\sqrt{\frac{n}{n-1}}\left(-\frac{1}{C} + C - \frac{1}{C}\right) \\
&= -\frac{2-C^2}{2\,C}\sqrt{\frac{n}{n-1}}.
\end{aligned}$$

All the other computations follow the same recipe. □

Lemma 3.1.5. *The sectional curvature is*

$$\begin{aligned}
K(Z,X_j) &= \frac{C^2}{4}\frac{n}{n-1}, \\
K(Z,Y_j) &= \frac{C^2}{4}\frac{n}{n-1}, \\
K(X_j,Y_j) &= -\frac{3C^2}{2}\frac{n}{n-1} + 2 \\
K(X_j,X_k) &= \frac{1}{2}, \quad \textit{for } j \neq k, \\
K(X_j,Y_k) &= \frac{1}{2}, \quad \textit{for } j \neq k, \\
K(Y_j,Y_k) &= \frac{1}{2}, \quad \textit{for } j \neq k.
\end{aligned}$$

Proof. We do the computations for $K(Z,X_j)$:

$$\begin{aligned}
K(Z,X_j) &= (\nabla_Z\nabla_{X_j}X_j,Z) - (\nabla_{X_j}\nabla_Z X_j,Z) - (\nabla_{[Z,X_j]}X_j,Z) \\
&= \frac{2-C^2}{2C}\sqrt{\frac{n}{n-1}}\,(\nabla_{Y_j}X_j,Z) + \frac{1}{C}\sqrt{\frac{n}{n-1}}\,(\nabla_{Y_j}X_j,Z) \\
&= -\frac{2-C^2}{4}\,\frac{n}{n-1} + \frac{1}{2}\,\frac{n}{n-1} \\
&= \frac{C^2}{4}\,\frac{n}{n-1}.
\end{aligned}$$

For $K(X_j, Y_j)$ we have

$$
\begin{aligned}
K(X_j, Y_j) &= (\nabla_{X_j}\nabla_{Y_j}Y_j, X_j) - (\nabla_{Y_j}\nabla_{X_j}Y_j, X_j) - (\nabla_{[X_j,Y_j]_{\mathfrak{p}}}Y_j, X_j) - (ad_{[X_j,Y_j]_{\mathfrak{h}}}Y_j, X_j) \\
&= \frac{C}{2}\sqrt{\frac{n}{n-1}}(\nabla_{Y_j}Z, X_j) + C\sqrt{\frac{n}{n-1}}(\nabla_Z Y_j, X_j) \\
&\quad - i([H_{x_j-x_n}, Y_j], X_j) - C\sqrt{\frac{n}{n-1}}([Z, Y_j], X_j) \\
&= -\frac{C^2}{4}\frac{n}{n-1} + \frac{2-C^2}{2}\frac{n}{n-1} + 2 - \frac{n}{n-1} \\
&= -\frac{3C^2}{4}\frac{n}{n-1} + 2.
\end{aligned}
$$

For $K(X_j, X_k)$, we have

$$
\begin{aligned}
K(X_j, X_k) &= -(ad_{[X_j,X_k]_{\mathfrak{h}}}X_k, X_j) \\
&= \frac{1}{2}([E_{x_j-x_k} + E_{-x_j+x_k}, X_k], X_j) \\
&= \frac{1}{2}(X_j, X_j) \\
&= \frac{1}{2}.
\end{aligned}
$$

□

Corollary 3.1.6. *The scalar curvature is*

$$s = -2\,n\,C^2 + n^2 + n - 1.$$

Proof. The scalar curvature is

$$
\begin{aligned}
s &= 2\sum_{j=1}^{n-1} K(Z, X_j) + 2\sum_{j=1}^{n-1} K(Z, Y_j) + 2\sum_{j=1}^{n-1} K(X_j, Y_j) \\
&\quad + \sum_{j\neq k}(K(X_j, X_k) + K(Y_j, Y_k)) + \sum_{j\neq k}(K(X_j, Y_k)) \\
&= -2\,n\,C^2 + 4(n-1) + 2\binom{n-1}{2} \\
&= -2\,n\,C^2 + n^2 + n - 1.
\end{aligned}
$$

□

Corollary 3.1.7. *There is one $\widetilde{U(n)}$-invariant metric with constant sectional curvature $\frac{1}{2}$, and it corresponds to the case $C^2 = 2\frac{n-1}{n}$. In this case, the scalar curvature is*

$$s = (n-1)(n-2).$$

3.2 *Spin*-structures

We choose a maximal torus in $SO(\mathfrak{p})$ such that it extends the maximal torus of $\widetilde{U(n-1)}$.

Lemma 3.2.1. *The weights of* $\mathfrak{p}$ *as an* $SO(\mathfrak{p})$ *representation are* $\{0, \lambda_1, \dots, \lambda_{n-1}\}$, *where* 0 *corresponds to* $\mathfrak{t}_{\mathfrak{p}}$ *and* λ_j *correspond to* $\mathfrak{g}_{x_j - x_n} - \mathfrak{g}_{-x_j + x_n}$ *in the decomposition* (3.1). *Moreover, when restricted to* H, *the weight* λ_j *is* x_j.

Proposition 2.3.1 applied to the odd-dimensional sphere $S^{2n-1} = \widetilde{U(n)}/\widetilde{U(n-1)}$ gives:

Lemma 3.2.2. *The Spin-representation* S *has the weights*

$$\frac{1}{2}(\pm\lambda_1 \pm \dots \pm \lambda_{n-1}).$$

There is a decomposition of the Spin-representation as

$$S = \underbrace{\mathbb{C}^2 \otimes \dots \otimes \mathbb{C}^2}_{n-1 \text{ times}},$$

such that the k*th copy of* $\mathbb{C}^2$ *corresponds to the weight* λ_k. *The orthonormal basis* $\{Z, X_1, Y_1, \dots, X_{n-1}, Y_{n-1}\}$ *acts on* S *in the following way:* $\gamma(X_k)$ *and* $\gamma(Y_k)$ *act trivially on all the copies of* $\mathbb{C}^2$ *but the* k*th one, where the action is given by*

$$\gamma(X_k) \quad \text{acts via} \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and

$$\gamma(Y_k) \quad \text{acts via} \quad \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

Therefore

$$\gamma(X_j)\gamma(Y_j) \quad \text{acts via} \quad \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$

The action of $\gamma(Z)$ *is*

$$\gamma(Z)\Phi = (-1)^n i \epsilon_1 \epsilon_2 \dots \epsilon_{n-1} \Phi,$$

for Φ *in the weight space* $\frac{1}{2}(\epsilon_1\lambda_1 + \epsilon_2\lambda_2 + \dots + \epsilon_{n-1}\lambda_{n-1})$.

As a consequence of Lemma 2.3.5 we see that for $H_\alpha \in \mathfrak{t}_{n-1}$ we have

$$\begin{aligned}
\sigma \circ \widetilde{ad}(H_\alpha) &= \sum_{j=1}^{n-1} \frac{([H_\alpha, X_j], Y_j)}{4} \gamma(X_j)\gamma(Y_j) \\
&\quad + \sum_{j=1}^{n-1} \frac{([H_\alpha, X_j], Y_j)}{4} \gamma(Y_j)\gamma(X_j) \\
&= -\sum_{j=1}^{n-1} \frac{i}{4}(x_j - x_n)(H_\alpha)\gamma(X_j)\gamma(Y_j) \\
&\quad + \sum_{j=1}^{n-1} \frac{i}{4}(x_j - x_n)(H_\alpha)\gamma(Y_j)\gamma(X_j) \\
&= \sum_{j=1}^{n-1} (x_j - x_n)(H_\alpha)\gamma(X_j)\gamma(Y_j).
\end{aligned}$$

Lemma 3.2.3. *Via $\sigma \circ \widetilde{ad}$, S becomes a representation of H, with weights*

$$\frac{1}{2}(\pm x_1 \pm x_2 \pm \ldots \pm x_{n-1}).$$

It splits into $n+1$ irreducible representations of H, which we denote by S_r, according to the number of minuses in the expression of the weight. The highest weight of S_r is

$$\frac{1}{2}(x_1 + \ldots + x_{n-r} \underbrace{-x_{n-r+1} - \ldots - x_{n-1}}_{r}).$$

Since $(S, \sigma \circ \widetilde{ad})$ is a representation of $\widetilde{U(n-1)}$, the spinor bundle $\mathbb{S}$ on $S^{2n-1} = \widetilde{U(n)}/\widetilde{U(n-1)}$ becomes a homogeneous vector bundle, and the Dirac operator is a $\widetilde{U(n)}$-equivariant operator. According to Frobenius Reciprocity, the space of spinors, $\Omega^0(S^{2n-1};\mathbb{S})$ decomposes under the action of $\widetilde{U(n)}$ into

$$\Omega^0(S^{2n-1};\mathbb{S}) = \sum V \otimes (V^* \otimes S)^{\widetilde{U(n-1)}}, \tag{3.2}$$

where the sum is after all the irreducible representations of $\widetilde{U(n)}$.

Lemma 3.2.4. *The Dirac operator acts preserving the decomposition 3.2, and restricted to each piece, it has the form*

$$D(w \otimes v^* \otimes \Phi) = \sum_{a=1}^{2n-1} (w \otimes v^*) \otimes \gamma(e_a) \nabla^{Spin}_{e_a} \Phi + \sum_{a=1}^{2n-1} \pi(e_a)(w \otimes v^*) \otimes \gamma(e_a) \Phi \tag{3.3}$$

Here, ∇^{Spin} denotes the *Spin*-connection induced on the spinor bundle by the Riemannian connection. From Lemma 3.1.4 we obtain the explicit description of the *Spin*-connection:

Lemma 3.2.5. *The Spin-connection, $\nabla^{Spin}_{e_a}$ acts on S via $-\frac{1}{4}\sum_{b=1}^{2n-1} \gamma(\nabla_{e_a} e_b)\gamma(e_b)$. More explicitly*

$$\begin{aligned}
\nabla^{Spin}_{Z} &= -\frac{2-C^2}{4C}\sqrt{\frac{n}{n-1}}\sum_{j=1}^{n}\gamma(X_j)\gamma(Y_j), \\
\nabla^{Spin}_{X_j} &= \frac{C}{4}\sqrt{\frac{n}{n-1}}\,\gamma(Z)\gamma(Y_j), \\
\nabla^{Spin}_{Y_j} &= -\frac{C}{4}\sqrt{\frac{n}{n-1}}\,\gamma(Z)\gamma(X_j).
\end{aligned}$$

Remark. The term involving the *Spin*-connection in the expression of the Dirac operator is

$$\begin{aligned}\sum_{a=1}^{2n-1}\gamma(e_a)\nabla^{Spin}_{e_a} = &-\frac{2-C^2}{4\,C}\sum_{j=1}^{n-1}\gamma(Z)\gamma(X_j)\gamma(Y_j)\\ &+\frac{C}{4}\sqrt{\frac{n}{n-1}}\sum_{j=1}^{n-1}\gamma(X_j)\gamma(Z)\gamma(Y_j)\\ &-\frac{C}{4}\sqrt{\frac{n}{n-1}}\sum_{j=1}^{n-1}\gamma(Y_j)\gamma(Z)\gamma(X_j)\\ =&-\frac{2+C^2}{4\,C}\sqrt{\frac{n}{n-1}}\,\gamma(Z)\sum_{j=1}^{n-1}\gamma(X_j)\gamma(Y_j).\end{aligned} \tag{3.4}$$

We denote by D_1 the first term of the Dirac operator, and by D_2 the second term. Explicitly, when applied to a spinor of the form $\Psi = w\otimes v^*\otimes\Phi$ we have

$$\begin{aligned} D_1(w\otimes v^*\otimes\Phi) &= -\frac{2+C^2}{4\,C}\sqrt{\frac{n}{n-1}}\left(w\otimes v^*\otimes\left(\gamma(Z)\sum_{j=1}^{n-1}\gamma(X_j)\gamma(Y_j)\right)\Phi\right),\\ D_2(w\otimes v^*\otimes\Phi) &= \sum_{a=1}^{2n-1}\pi(e_a)(w\otimes v^*)\otimes\gamma(e_a)\Phi\\ &= -\sum_{a=1}^{2n-1} w\otimes(v^*\circ\pi(e_a))\otimes\gamma(e_a)\Phi \end{aligned}$$

3.3 Gelfand-Cetlin rule

In order to proceed to compute the spectrum we need another piece of information, which is the decomposition of an irreducible representation of $\widetilde{U(n)}$ into irreducible representations of $\widetilde{U(n-1)}$. This is known in literature as the *Gelfand-Cetlin rule*, and our exposure is adapted from [21].

From the Theorem of the Highest Weight, we know that irreducible representations of $\widetilde{U(n)}$ are in one-to-one correspondence to dominant analytically integral weights. Once we choose the fundamental Weyl chamber of $\widetilde{U(n)}$ to be $x_1 > x_2 > \ldots > x_{n-1} > x_n$, the dominant weights are weights of the form

$$\mu_1x_1 + \mu_2x_2 + \ldots + \mu_{n-1}x_{n-1} + \mu_nx_n,$$

with

$$\mu_1 \geq \mu_2 \geq \ldots \geq \mu_{n-1} \geq \mu_n.$$

Lemma 3.3.1. *The dominant analytically integral weights are of two types:*

1. *All $\mu_i \in \mathbb{Z}$ with*

$$\mu_1 \geq \mu_2 \geq \ldots \geq \mu_{n-1} \geq \mu_n.$$

These are dominant weights which are analytically integral for $U(n)$ also. They give representations of $\widetilde{U(n)}$ which descend to $U(n)$.

2. *All $\mu_i \in \mathbb{Z} + \frac{1}{2}$ with*
$$\mu_1 \geq \mu_2 \geq \ldots \geq \mu_{n-1} \geq \mu_n.$$
These weights are not analytically integral with respect $U(n)$, and $-1 \in \widetilde{U(n)}$ acts as $-Id$.

Proposition 3.3.2. *Let V_μ be an irreducible representation of $\widetilde{U(n)}$, with highest weight $\mu_1 x_1 + \mu_2 x_2 + \ldots + \mu_{n-1} x_{n-1} + \mu_n x_n$, and $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_{n-1} \geq \mu_n$. Then, under the action of $\widetilde{U(n-1)}$ it splits into irreducible representations, indexed by all the $(n-1)$-tuples $\nu_1 \geq \nu_2 \geq \ldots \geq \nu_{n-1}$, satisfying*

$$\mu_1 \geq \nu_1 \geq \mu_2 \geq \nu_2 \geq \ldots \geq \mu_{n-1} \geq \nu_{n-1} \geq \mu_n.$$

Each such representation V_ν appears only once into the decomposition.

Proof. According to the Poincaré-Birkhoff- Witt Theorem, all the possible weights appearing in the representation V_μ are obtained by applying the following type of operator to a highest vector v
$$E^{k_1}_{-x_1+x_2} E^{k_2}_{-x_2+x_3} \ldots E^{k_{n-1}}_{-x_{n-1}+x_n} v,$$
with $k_1 \leq \mu_1 - \mu_2$, $k_2 - k_1 \leq \mu_2 - \mu_3$, and so on. (The reason for this constrain is that if w is a vector in that space, then $E^{\mu_1-\mu_2+1}_{x_1-x_2} w = 0$, and it follows that $k_1 \leq \mu_1 - \mu_2$). Therefore, the possible weights are of the form

$$(\mu_1 - k_1)x_1 + (\mu_2 + k_1 - k_2)x_2 + \ldots + (\mu_{n-1} + k_{n-2} - k_{n-1})x_{n-1} + (\mu_n + k_{n-1})x_n.$$

When restricted to $\widetilde{U(n-1)}$ the possible weights are going to be of the form

$$(\mu_1 - k_1)x_1 + (\mu_2 + k_1 - k_2)x_2 + \ldots + (\mu_{n-1} + k_n - k_{n-1})x_{n-1}.$$

We have to decide which ones can be highest weights for irreducible representations of $\widetilde{U(n-1)}$. This means that we look at the fundamental Weyl chamber of $\widetilde{U(n-1)}$ which is of the form $x_1 \geq x_2 \geq \ldots \geq x_{n-1}$, and ask to have the following condition satisfied:

$$\mu_1 - k_1 \geq \mu_2 + k_1 - k_2 \geq \ldots \geq \mu_{n-1} + k_{n-2} - k_{n-1}.$$

Putting everything together, we have

$$\mu_1 \geq \mu_1 - k_1 \geq \mu_2 \geq \mu_2 + k_1 - k_2 \geq \ldots \geq \mu_{n-2} \geq \mu_{n-2} + k_{n-3} - k_{n-2} \geq \mu_{n-1} \geq \mu_{n-1} + k_{n-2} - k_{n-1} \geq \mu_n.$$

□

Corollary 3.3.3. *On the highest weight space of V_ν, H_λ acts via multiplication by $n \sum_{j=1}^{n-1} \nu_j - (n-1) \sum_{j=1}^{n} \mu_j$.*

Proof. From the relations

$$\begin{cases} \nu_1 = \mu_1 - k_1 \\ \nu_2 = \mu_2 + k_1 - k_2 \\ . \\ . \\ . \\ \nu_{n-2} = \mu_{n-2} + k_{n-3} - k_{n-2} \\ \nu_{n-1} = \mu_{n-1} + k_{n-2} - k_{n-1} \end{cases},$$

it turns out that

$$k_{n-1} = \sum_{j=1}^{n-1} \mu_j - \sum_{j=1}^{n-1} \nu_j.$$

Therefore the highest weight space in V_ν corresponds to the weight space

$$\nu_1 x_1 + \nu_2 x_2 + \ldots + \nu_{n-1} x_{n-1} + (\sum_{j=1}^{n} \mu_j - \sum_{j=1}^{n-1} \nu_j) x_n,$$

in V_μ and H_λ acts on this space via

$$\nu_1 + \nu_2 + \ldots + \nu_{n-1} - (n-1)(\sum_{j=1}^{n} \mu_j - \sum_{j=1}^{n-1} \nu_j) = n \sum_{j=1}^{n-1} \nu_j - (n-1) \sum_{j=1}^{n} \mu_j.$$

□

Corollary 3.3.4. *The irreducible representations of $\widetilde{U(n)}$, V_μ, for which $(V_\mu^* \otimes S)^{\widetilde{U(n-1)}}$ is non-trivial are*

1. $\mu = (a + \frac{1}{2} \geq \frac{1}{2} \geq \ldots \geq \frac{1}{2})$, *with a a positive integer. V_μ contains a copy of S_0, and the dimension of the space $(V_\mu^* \otimes S)^{\widetilde{U(n-1)}}$ is* 1.
2. $\mu = (-\frac{1}{2} \geq -\frac{1}{2} \geq \ldots \geq -b - \frac{1}{2})$, *with b a positive integer integer. V_μ contains a copy of S_{n-1}, and the dimension of the space $(V_\mu^* \otimes S)^{\widetilde{U(n-1)}}$ is* 1.
3. $\mu = (a + \frac{1}{2} \geq \ldots \geq \frac{1}{2} \geq \underbrace{-\frac{1}{2} \geq \ldots \geq -\frac{1}{2}}_{r} \geq -b - \frac{1}{2})$, *with a and b positive integers. Here r takes values from* 0 *to $n-1$. V_μ contains a copy of S_r and a copy of S_{r+1} in the decomposition into irreducible under $\widetilde{U(n-1)}$. The dimension of $(V_\mu^* \otimes S)^{\widetilde{U(n-1)}}$ is* 2.

Proof. It follows from the Schur Lemma and the Gelfand-Cetlin rule, together with the decomposition of S into irreducible representations of $\widetilde{U(n-1)}$. □

Also, the Gelfand-Cetlin rule gives us an explicit basis for the representation V_μ, which agrees with the decomposition under $\widetilde{U(n-1)}$, and then under $\widetilde{U(n-2)}$ and so on. This is called *the Gelfand-Cetlin basis* and its best described in terms of tableau of integers (or half integers) of the form

$$\begin{pmatrix} M_{1,n} & & M_{2,n} & \dots & M_{n-1,n} & & M_{n,n} \\ & M_{1,n-1} & \dots & \dots & \dots & M_{n-1,n-1} & \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ & & M_{1,2} & & M_{2,2} & & \\ & & & M_{1,1} & & & \end{pmatrix} \tag{3.5}$$

where each row is the highest weight of the irreducible representation of $\widetilde{U(k)}$ whose space contains the vector in question. There is a constraint on the numbers in the tableau coming from the Gelfand-Cetlin rule: each number $M_{i,k}$ has to satisfy $M_{i+1,k} \geq M_{i,k} \geq M_{i+1,k-1}$. We will denote such a tableau by $\mathcal{M}$, and the corresponding vector by $v_{\mathcal{M}}$. There are explicit description of the weights of the vectors $v_{\mathcal{M}}$, as well as how the raising a lowering operators act on them (Želobenko [21], Theorem 3 page 188 and Theorem 7 page 205).

3.4 The Spectrum of the Dirac operator

Theorem 3.4.1. *The Dirac operator on S^{2n-1} with the $\widetilde{U(n-1)}$-invariant metric of parameter C has the following spectrum*

1. *The eigenspace $V_{a\geq\frac{1}{2}\geq\dots\geq\frac{1}{2}}$ with eigenvalue*

$$(-1)^n \, \frac{(2+C^2)\,n+4a}{4\,C} \, \sqrt{\frac{n-1}{n}}. \tag{3.6}$$

2. *The eigenspace $V_{-\frac{1}{2}\geq\dots\geq-\frac{1}{2}\geq-b-\frac{1}{2}}$, with eigenvalue*

$$\frac{(2+C^2)\,n+4b}{4\,C} \, \sqrt{\frac{n-1}{n}}. \tag{3.7}$$

3. *The eigenspace $V_{a+\frac{1}{2}\geq\dots\geq\frac{1}{2}\geq\underbrace{-\frac{1}{2}\geq\dots\geq-\frac{1}{2}}_{r}\geq-b-\frac{1}{2}}$, with eigenvalues*

$$\frac{1}{2\sqrt{2}}\left[(-1)^{n+r}\;\tfrac{2+C^2-2C}{2C}\sqrt{\tfrac{n}{n-1}} \pm\sqrt{\begin{matrix}\tfrac{n}{n-1}\left(-\tfrac{(2+C^2)\,(n-2r-1)}{4C}+\tfrac{n-r-1-(n-1)(a-b)}{n}\right)\left(\tfrac{(2+C^2)\,(n-2r-3)}{4C}+\tfrac{r+1+(n-1)(a-b)}{n+1}\right)\\ -\,4(a+n-r-1)(b+r+1)\end{matrix}}\;\right]. \tag{3.8}$$

Proof. Since the Dirac operator, D, and the operators D_1 and D_2 which give $D = D_1 + D_2$ are all $\widetilde{U(n)}$-equivariant operators. According to the decomposition given by the Frobenius reciprocity, they split into

$$D|_V : V\otimes(V^*\otimes S)^{\widetilde{U(n-1)}} \to V\otimes(V^*\otimes S)^{\widetilde{U(n-1)}},$$

with V an irreducible representation of $\widetilde{U(n)}$.

On each of these subspaces the operator has the form

$$D(w\otimes v^*\otimes\Phi) = \sum_{a=1}^{2n-1} \pi(e_a)(w\otimes v^*)\otimes\gamma(e_a)\Phi + \sum_{a=1}^{2n-1} (w\otimes v^*)\otimes\gamma(e_a)\nabla^{Spin}_{e_a}\Phi$$

From (3.4), it follows that the first term in the expression of the Dirac operator is

$$D_1(\Psi) = -\frac{2+C^2}{4\,C}\sqrt{\frac{n}{n-1}}\left(1\otimes\sum_{j=1}^{n-1}\gamma(Z)\gamma(X_j)\gamma(Y_j)\right)\Psi.$$

Since $(V^*\otimes S)^H = \oplus_{r=0}^{n-1}(V^*\otimes S_r)^H$, for $\Psi_r \in (V^*\otimes S_r)^H$, we have

$$D_1\Psi_r = (-1)^{n+r}\,\frac{2+C^2}{4\,C}\,(n-2r-1)\sqrt{\frac{n}{n-1}}\,\Psi_r.$$

The second term in the expression of the Dirac operator is

$$D_2\Psi = \pi(Z)\otimes\gamma(Z)\Psi + \sum_{j=1}^{n-1}\pi(X_j)\otimes\gamma(X_j)\Psi + \sum_{j=1}^{n-1}\pi(Y_j)\otimes\gamma(Y_j)\Psi \tag{3.9}$$

We can say more about it, by applying it to $\Psi_r \in (V^*\otimes S_r)^H$, of the form $\Psi_r = v_r^*\otimes\Phi_r$, with v_r in the irreducible component of V corresponding to S_r, and $\Phi_r \in S_r$.

$$D_2(v^*\otimes\Phi_r) = -(v^*\circ\pi(Z))\otimes\gamma(Z)\Phi_r - \sum_{j=1}^{n-1}(v^*\circ\pi(X_j))\otimes\gamma(X_j)\Phi_r - \sum_{j=1}^{n-1}(v^*\circ\pi(Y_j))\otimes\gamma(Y_j)\Phi_r \tag{3.10}$$

Now we proceed to a case by case analysis of the spectrum of the Dirac operator:
Case 1. For $\mu = \{a+\frac{1}{2} \geq \frac{1}{2} \geq \ldots \geq \frac{1}{2}\}$.
In this case $(V_\mu^*\otimes S)^{\widetilde{U(n-1)}} = (V_\mu^*\otimes S_0)^{\widetilde{U(n-1)}}$, and it is one dimensional. Therefore, D restricted to this space is multiplication by a constant and this is the eigenvalue.
Let $\Psi_0 = v_0^*\otimes\Phi_0$ denote the generator. Then the first part of the Dirac operator is

$$D_1\Psi_0 = (-1)^n\,\frac{2+C^2}{4C}\,(n-1)\sqrt{\frac{n}{n-1}}\Psi_0$$

The only contribution from D_2 is from the term $\pi(Z)\otimes\gamma(Z)$. All the others are going to be zero.

$$\begin{aligned} D_2\Psi_0 &= \pi(Z)\otimes\gamma(Z)(v^*\otimes\Phi) \\ &= -\frac{i}{\sqrt{(n-1)n}\,C}(-v^*\circ\pi(H_\lambda))\otimes(-1)^n i\Phi_0 \\ &= (-1)^{n-1}\frac{1}{\sqrt{(n-1)n}\,C}(-(n-1)a) \\ &= (-1)^n\frac{a}{C}\sqrt{\frac{n-1}{n}}. \end{aligned}$$

Here, $v^*\circ\pi(H_\lambda) = -(n-1)\,a$ since v belongs to the weights space with weight

$(\frac{1}{2}, \ldots, \frac{1}{2}, a + \frac{1}{2})$. Putting the two terms together, it follows that

$$D\Psi_0 = (-1)^n \frac{(2 + C^2)\, n + 4\, a}{4C} \sqrt{\frac{n-1}{n}}.$$

Case 2. For $\mu = \{-\frac{1}{2} \geq \ldots \geq -\frac{1}{2} \geq -b - \frac{1}{2}\}$.
In this case $(V_\mu^* \otimes S)^{\widetilde{U(n-1)}} = (V_\mu^* \otimes S_{n-1})^{\widetilde{U(n-1)}}$, and it is one dimensional, and let Ψ_{n-1} denote the generator. The first part of the Dirac operator is

$$D_1 \Psi_{n-1} = \frac{2 + C^2}{4\, C}\, (n-1) \sqrt{\frac{n}{n-1}} \Psi_{n-1}$$

For the D_2-part, the only contribution is also from the term $\pi(Z) \otimes \gamma(Z)$.

$$\begin{aligned} D_2 \Psi_{n-1} &= -\pi(Z) \otimes \gamma(Z) \Psi_{n-1} \\ &= -\frac{i}{\sqrt{(n-1)n}\, C} \pi(H_\lambda)(v_{n-1}^*) \otimes \gamma(Z) \Phi_n \\ &= -\frac{1}{\sqrt{(n-1)n}\, C} \, (-v_{n-1}^* \circ \pi(H_\lambda)) \otimes (-1)^{2n-1} i \Psi_{n-1} \\ &= \frac{(n-1)\, b}{\sqrt{(n-1)n}\, C} \Psi_{n-1} \\ &= \frac{b}{C} \sqrt{\frac{n-1}{n}} \Psi_{n-1}. \end{aligned}$$

Putting the two terms together, it follows that

$$D\Psi_{n-1} = \frac{(2 + C^2)n + 4b}{4C} \sqrt{\frac{n-1}{n}} \Psi_{n-1}.$$

Case 3. For $\mu = \{a + \frac{1}{2} \geq \ldots \geq \frac{1}{2} \geq \underbrace{-\frac{1}{2} \geq \ldots \geq -\frac{1}{2}}_{r} \geq -b - \frac{1}{2}\}$.

We have

$$(V_\mu^* \otimes S)^{\widetilde{U(n-1)}} = (V_\mu^* \otimes S_r)^{\widetilde{U(n-1)}} \oplus (V_\mu^* \otimes S_{r+1})^{\widetilde{U(n-1)}},$$

which is 2-dimensional. We denote by Ψ_r, and Ψ_{r+1}, a generator for $(V_\mu^* \otimes S_r)^{\widetilde{U(n-1)}}$, respectively $(V_\mu^* \otimes S_{r+1})^{\widetilde{U(n-1)}}$. For the first summand in the Dirac operator we have

$$D_1 \Psi_r = (-1)^{n+r+1} \frac{2 + C^2}{4\, C} \, (n - 2r - 1) \sqrt{\frac{n}{n-1}} \Psi_r$$

and

$$D_1 \Psi_{r+1} = (-1)^{n+r-1} \frac{2 + C^2}{4\, C} \, (n - 2r - 3) \sqrt{\frac{n}{n-1}} \Psi_{r+1}.$$

When we apply $\pi(Z)\otimes\gamma(Z)$ to Ψ_r, we obtain

$$\begin{aligned}
\pi(Z)\otimes\gamma(Z)\,\Psi_r &= -\frac{i}{\sqrt{(n-1)n}\,C}\,\pi(H_\lambda)(v_r^*)\otimes\gamma(Z)\Phi_r \\
&= -\frac{i}{\sqrt{(n-1)n}\,C}\,\left(-v_r^*\circ\pi(H_\lambda)\right)\otimes(-1)^{n+r}i\Phi_r \\
&= (-1)^{n+r}\frac{1}{\sqrt{(n-1)n}\,C}\left(-\left((n-r-1)\frac{1}{2}-r\frac{1}{2}-(n-1)(a-b-\frac{1}{2})\right)\right)v_r^*\otimes\Phi_r \\
&= (-1)^{n+r-1}\frac{1}{\sqrt{(n-1)n}\,C}\,(\frac{1}{2}(n-r-1)-\frac{1}{2}r-(n-1)\,(a-b-\frac{1}{2}))v_r^*\otimes\Phi_r \\
&= (-1)^{n+r-1}\frac{1}{\sqrt{(n-1)n}\,C}\,(n-r-1-(n-1)\,(a-b))\,\Psi_r.
\end{aligned}$$

In the same spirit,

$$\begin{aligned}
\pi(Z)\otimes\gamma(Z)\,\Psi_{r+1} &= -\frac{i}{\sqrt{(n-1)\,n}\,C}\,\pi(H_\lambda)(v_r^*)\otimes\gamma(Z)\Phi_{r+1} \\
&= (-1)^{n+r-1}\frac{1}{\sqrt{(n-1)n}\,C}\,(r+1+(n-1)(a-b))\,\Psi_{r+1}.
\end{aligned}$$

For the remaining terms in the expression of the Dirac operator on $(V_\mu^*\otimes S)^{\widetilde{U(n-1)}}$, we have

$$\begin{aligned}
\textstyle\sum_{j=1}^{n-1}\pi(X_j)\otimes\gamma(X_j)\Psi_r &= (a+n-r-1)(b+r+1)\Psi_{r+1}, \\
\textstyle\sum_{j=1}^{n-1}\pi(Y_j)\otimes\gamma(X_j)\Psi_r &= (a+n-r-1)(b+r+1))\Psi_{r+1}, \\
\textstyle\sum_{j=1}^{n-1}\pi(X_j)\otimes\gamma(X_j)\Psi_{r+1} &= \Psi_r, \\
\textstyle\sum_{j=1}^{n-1}\pi(Y_j)\otimes\gamma(Y_j)\Psi_{r+1} &= \Psi_r.
\end{aligned}$$

In order to get this formulas we used the Gelfand-Cetlin basis and the description of the raising and lowering operators on this basis.
The matrix for the Dirac operator is

$$\begin{bmatrix}
(-1)^{n+r}\left(\frac{(2+C^2)\,(n-2r-1)}{4\,C})\sqrt{\frac{n}{n-1}}-\frac{(n-1)(1-a+b)-r}{\sqrt{(n-1)n}}\right) & \frac{2(a+n-r-1)(b+r+1)}{\sqrt{2}} \\
\frac{2}{\sqrt{2}} & (-1)^{n+r-1}\left(\frac{(2+C^2)\,(n-2r-3)}{4\,C}\sqrt{\frac{n}{n-1}}+\frac{r+1+(n-1)\,(a-b)}{\sqrt{(n-1)\,n}}\right)
\end{bmatrix}.$$

The eigenvalues of this matrix are

$$\frac{1}{2\sqrt{2}}\left[(-1)^{n+r}\frac{2+C^2-2C}{2C}\sqrt{\frac{n}{n-1}}\pm\right.$$
$$\left.\pm\sqrt{\frac{n}{n-1}\left(-\frac{(2+C^2)(n-2r-1)}{4C}+\frac{(n-1)(1-a+b)-r}{n}\right)\left(\frac{(2+C^2)(n-2r-3)}{4C}+\frac{r+1+(n-1)(a-b)}{n}\right)-4(a+n-r-1)(b+r+1)}\right].$$

□

When we consider the case of the metric with constant sectional curvature, i.e. the round metric on S^{2n+1}, which we obtain setting $C^2 = 2\,\frac{n-1}{n}$, the spectrum of the Dirac operator becomes really friendly.

Corollary 3.4.2. *For the round metric on S^{2n-1}, the spectrum of the Dirac operator is*

$$
\begin{aligned}
&V_{a+\frac{1}{2}\geq\frac{1}{2}\geq\ldots\geq\frac{1}{2}} && \textit{with eigenvalue} \quad (-1)^n\,\tfrac{2n-1+2\,a}{2\sqrt{2}}\\
&V_{-\frac{1}{2}\geq\ldots\geq-\frac{1}{2}\geq-b-\frac{1}{2}} && \textit{with eigenvalue} \quad \tfrac{2n-1+2\,b}{2\sqrt{2}}\\
&V_{a+\frac{1}{2}\geq\ldots\geq\frac{1}{2}\geq\underbrace{-\frac{1}{2}\geq\ldots\geq-\frac{1}{2}}_{r}\geq-b-\frac{1}{2}} && \textit{with eigenvalues} \quad \tfrac{(-1)^{n+r}+2(n+a+b)}{2\sqrt{2}}\\
& && \qquad\qquad\qquad\quad\ \tfrac{(-1)^{n+r}-2(n+a+b)}{2\sqrt{2}}.
\end{aligned}
\tag{3.11}
$$

Remark. From the explicit description of the spectrum for the round metric we see that the space of harmonic spinors (the kernel of the Dirac operator) is trivial. This can be seen directly as a consequence of the Weitzenböck formula, since the scalar curvature is strictly positive. A consequence of our brute computation of the spectrum is that in the general case of a $\widetilde{U(n)}$-invariant metric the space of harmonic spinors is always trivial for $n \geq 3$. The case $n = 2$ is special, and it corresponds to S^3 with the Berger metric on it. Hitchin, [10], noticed that by changing the round metric into any other invariant metric the space of harmonic spinors on S^3 can be made as big as we want.

Chapter 4

The eta-invariant for the odd-dimensional sphere

Definition 4.0.3. For a selfadjoint elliptic operator D on a compact Riemannian manifold M, the η-function, $\eta(s)$, is defined as

$$\eta(s) = \sum_{\lambda \in \Sigma'(D)} \operatorname{sgn}(\lambda)\, |\lambda|^{-s}, \tag{4.1}$$

for $\mathrm{R}e(s) > \dim M$. Here $\Sigma(D)$ is the spectrum of D and $\Sigma'(D) = \Sigma(D) \setminus 0$. Moreover, if a compact group G acts, and D is equivariant with respect to this group action, then the eigenspaces E_λ are finite-dimensional G-modules, and we set

$$\eta(g, s) = \sum_{\lambda \in \Sigma'(D)} \operatorname{Trace}(g|_{E_\lambda})\, \operatorname{sgn}(\lambda)\, |\lambda|^{-s}. \tag{4.2}$$

These functions were introduced and studied in [1, 2], where it is shown that for a Dirac type of operator, $\eta(g, s)$ has a meromorphic extension to the whole $\mathbb{C}$; one, moreover, which is holomorphic at zero.

4.1 The untwisted Dirac operator on the sphere

We continue with our set-up from Chapter 3. We consider the odd-dimensional sphere S^{2n-1} endowed with the round metric. We view it as the homogeneous space $\widetilde{U(n)}/\widetilde{U(n-1)}$, and the round metric has the property that it is a $\widetilde{U(n)}$-invariant metric. S^{2n-1} admits a $\widetilde{U(n)}$-invariant *Spin*-structure. The spectrum of the Dirac operator is given by Corollary 3.4.2. The eigenspaces are representations of $\widetilde{U(n)}$, and the goal of this section is to compute $\eta(g, s)$ for g in $\widetilde{U(n)}$. Since $\widetilde{U(n)}$ is a compact Lie group, any element g is conjugated to an element in the maximal torus T_n.

Theorem 4.1.1. *For $g \in T_n$, such that 1 is not an eigenvalue, we have*

$$\eta(g) = (-1)^n (\det g)^{\frac{1}{2}} \frac{2}{\det(I_n - g)}. \tag{4.3}$$

If 1 *is an eigenvalue of* g*, then*

$$\eta(g) = 0. \tag{4.4}$$

Remark. Since the function $\eta(g,s)$ is invariant under the conjugation, the above proposition gives us the entire function $\eta(g,0)$ on $\widetilde{U(n)}$.

Proof. With the explicit formula for the spectrum, we have to compute 3 sums for the η-function

$$\eta_1(g,s) = (-1)^n \sum_{a\geq 0} \left(\frac{2n-1+2a}{2\sqrt{2}}\right)^{-s} \text{Trace}\,(g|_{V_{a+\frac{1}{2}\geq\frac{1}{2}\geq\ldots\geq\frac{1}{2}}}). \tag{4.5}$$

$$\eta_2(g,s) = \sum_{b\geq 0} \left(\frac{2n-1+2b}{2\sqrt{2}}\right)^{-s} \text{Trace}\,(g|_{V_{-\frac{1}{2}\geq\ldots\geq-\frac{1}{2}\geq-b-\frac{1}{2}}}). \tag{4.6}$$

$$\begin{aligned}\eta_3(g,s) = &\sum_{\substack{a,b\geq 0\\ 0\leq r\leq n-1}} \left(\frac{(-1)^{n+r}+2n+2(a+b)}{2\sqrt{2}}\right)^{-s} \text{Trace}\,(g|_{V_{a,r,b}})\\ &- \sum_{\substack{a,b\geq 0,\\ 0\leq r\leq n-1}} \left(\frac{(-1)^{n+r-1}+2(n-2r-2)+2(a+b)}{2\sqrt{2}}\right)^{-s} \text{Trace}\,(g|_{V_{a,r,b}}),\end{aligned} \tag{4.7}$$

where $V_{a,r,b}$ is a shortcut for the irreducible representation of $\widetilde{U(n)}$ with highest weight $\mu = (a+\frac{1}{2} \geq \frac{1}{2} \geq \ldots \geq \frac{1}{2} \geq \underbrace{-\frac{1}{2} \geq \ldots \geq -\frac{1}{2}}_{r} \geq -b-\frac{1}{2}$, We do a case by case analysis of all these partial η-functions.

Case 1. The representation with highest weight $a+\frac{1}{2} \geq \frac{1}{2} \geq \ldots \geq \frac{1}{2}$ can be viewed as the tensor product of two irreducible representations with highest weights $a \geq 0 \geq \ldots \geq 0$ and $\frac{1}{2} \geq \ldots \geq \frac{1}{2}$. The first one, $V_{a\geq 0\geq\ldots\geq 0}$, descends to a representation of $U(n)$. Its character is H^n_a, the *ath symmetric polynomial* in n variables, i.e. the sum of all distinct monomials of degree a, [9]. The second representation is half of the determinant representation and its character is the square root of the determinant. With this set up the function $\eta_1(g,s)$ becomes:

$$\eta_1(g,s) = (-1)^n \sum_{a\geq 0} \left(\frac{2n-1+2a}{2\sqrt{2}}\right)^{-s} \det\,(g)^{\frac{1}{2}}\, H^n_a(g).$$

Evaluating this expression at $s=0$, we obtain

$$\begin{aligned}\eta_1(g,0) &= (-1)^n \det\,(g)^{\frac{1}{2}} \sum_{a\geq 0} H^n_a(g)\\ &= (-1)^n \det\,(g)^{\frac{1}{2}} \frac{1}{\det\,(I_n - g)}\end{aligned}$$

where for the last equality we used the following identity for symmetric polynomials, [9]:

$$\prod_{i=1}^{n} \frac{1}{1-x_i t} = \sum_{a\geq 0} H^n_a\, t^a. \tag{4.8}$$

Case 2. The computations for $\eta_2(g,0)$ follow the same recipe as for the case of $\eta_1(g,0)$. The half determinant representation is replaced by the minus half determinant representation and the character of the representation $V_{0\geq\ldots\geq 0\geq -b}$ is $H_b(g^{-1})$. And $\eta_2(g,0)$ becomes

$$\eta_2(g,0) = \det{(g)}^{-\frac{1}{2}} \frac{1}{\det{(I_n - g^{-1})}}.$$

Case 3. In the equation (4.7) we see that when we make the $s=0$ the entire expression is convergent. And, in fact it goes to zero. This is because the coefficients in front of the traces are all going to be one, and all the terms with the same trace appear twice in the sum: once with a positive sign and once with minus. Therefore

$$\eta_3(g,0) = 0. \tag{4.9}$$

Adding up the results of this case-by-case analysis , we obtain

$$\begin{aligned} \eta(g) &= (-1)^n \det{(g)}^{\frac{1}{2}} \frac{1}{\det{(I_n - g)}} + \det{(g^{-1})}^{-\frac{1}{2}} \frac{1}{\det{(I_n - g^{-1})}} \\ &= (-1)^n \frac{2}{\det{(I_n - g)}} (\det g)^{\frac{1}{2}}. \end{aligned} \tag{4.10}$$

It remains to study the case when g has 1 as an eigenvalue. It is sufficient to study the situation when 1 is an eigenvalue with multiplicity one. In order to proceed, we need the following result

Fact. Consider the Zeta-function

$$\zeta(s,\alpha) = \sum_{k=0}^{\infty} (\alpha + k)^{-s},$$

where α is a constant. A consequence of Hermite's formula, [23], gives the value of the Zeta function at $s=0$

$$\zeta(0,\alpha) = \frac{1}{2} - \alpha.$$

With this in mind, we have to compute just the analytical continuations at 0 of η_1 and η_2. This is because, the argument we used to prove that $\eta_3(g,0)$ vanishes in the general case, still holds for the case we analyze now.

We assume that the eigenvalues of g are $x_1,\ldots,x_{n-1},x_n$. Moreover because g has 1 as an eigenvalue, we can assume that the multiplicity of this eigenvalue is 1, and that $x_n = 1$ (the cases where the multiplicity is greater than one, are treated in a similar way). With this,

$$\begin{aligned}
\eta_1(g,s) &= (-1)^n (\det g)^{\frac{1}{2}} \sum_{a\geq 0} \left(\frac{2n-1+2a}{2\sqrt{2}}\right)^{-s} H_a^n(x_1,\dots,x_{n-1},x_n) \\
&= (-1)^n (\det g)^{\frac{1}{2}} \sum_{a\geq 0} \left(\frac{2n-1+2a}{2\sqrt{2}}\right)^{-s} H_a^n(x_1,\dots,x_{n-1},1) \\
&= (-1)^n (\det g)^{\frac{1}{2}} \sum_{a\geq 0} \left(\frac{2n-1+2a}{2\sqrt{2}}\right)^{-s} \sum_{j=0}^{a} H_j^{n-1}(x_1,\dots,x_{n-1}) \\
&= (-1)^n (\det g)^{\frac{1}{2}} (\sqrt{2})^s \Bigg(\sum_{a\geq 0} \left(\frac{2n-1}{2}+a\right)^{-s} H_0^{n-1}(x_1,\dots,x_{n-1}) \\
&\qquad + \sum_{a\geq 1} \left(\frac{2n-1}{2}+a\right)^{-s} H_1^{n-1}(x_1,\dots,x_{n-1}) + \dots + \\
&\qquad + \sum_{a\geq k} \left(\frac{2n-1}{2}+a\right)^{-s} H_k^{n-1}(x_1,\dots,x_{n-1}) + \dots \Bigg).
\end{aligned}$$

Using the analytical continuation for the Zeta-function, we obtain

$$\begin{aligned}
\eta_1(g,0) &= (-1)^n (\det g)^{\frac{1}{2}} \sum_{k\geq 0} \left(\frac{1}{2} - \frac{2n-1}{2} + k\right) H_k^{n-1}(x_1,\dots x_{n-1}) \\
&= (-1)^n (\det g)^{\frac{1}{2}} \left(\left(\frac{1}{2} - \frac{2n-1}{2}\right) \sum_{k\geq 0} H_k^{n-1}(x_1,\dots,x_{n-1}) + \sum_{k\geq 0} k H_k^{n-1}(x_1,\dots,x_{n-1}) \right) \\
&= (-1)^n (\det g)^{\frac{1}{2}} \left(\left(\frac{1}{2} - \frac{2n-1}{2}\right) \frac{1}{\det(I_{n-1}-g')} + \frac{1}{\det(I_{n-1}-g')} \sum_{j=0}^{n-1} \frac{x_j}{(1-x_j)^2} \right) \\
&= (-1)^n \frac{(\det g')^{\frac{1}{2}}}{\det(I_{n-1}-g')} \left(-(n-1) + \sum_{j=0}^{n-1} \frac{x_j}{(1-x_j)^2} \right).
\end{aligned}$$

Here g' represents the $(n-1)\times(n-1)$ matrix with eigenvalues $x_1,\dots,x_{n-1}$, and I_{n-1} represents the identity $(n-1)\times(n-1)$ matrix. In the same way,

$$\begin{aligned}
\eta_2(g,0) &= \frac{\det(g')^{-\frac{1}{2}}}{\det(I_n - g'^{-1})} \left(-(n-1) + \sum_{j=0}^{n-1} \frac{x_j^{-1}}{(1-x_j^{-1})^2} \right) \\
&= (-1)^{n-1} \frac{\det(g')^{\frac{1}{2}}}{\det(I_{n-1}-g')} \left(-(n-1) + \sum_{j=0}^{n-1} \frac{x_j}{(1-x_j)^2} \right).
\end{aligned}$$

Since $\eta(g,0) = \eta_1(g,0) + \eta_2(g,0)$, from these two formulas it follows that

$$\eta(g,0) = 0,$$

when 1 is an eigenvalue of g. □

4.2 Orbifold η-invariant

Let Γ be a finite subgroup of $SU(n)$. Since $\widetilde{U(n)}$ is a double cover of $U(n)$, which contains $SU(n)$, and therefore Γ, it follows that the structures we defined on S^{2n-1} (Riemannian metric, *Spin*-structure, Dirac operator) are Γ-invariant. We consider the quotient S^{2n-1}/Γ. For almost all the finite groups of $SU(n-1)$ (except the case $n = 2$, which corresponds to the lens spaces S^3/Γ), the quotient has singularities. Nevertheless, thinking of it as an orbifold, it still makes sense to talk about *Spin*-structure on it and Dirac operator.

Definition 4.2.1. The *orbifold η-invariant* is defined to be

$$\eta = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \eta(\gamma). \tag{4.11}$$

With the explicit formula of the η-character, (4.3), and using the fact that for an element $\gamma \in \Gamma$ we have $\det(\gamma) = 1$, it follows

Corollary 4.2.2. *The orbifold η-invariant for S^{2n-1}/Γ is*

$$\eta = (-1)^n \sum_{\substack{\gamma \in \Gamma \\ 1 \text{ not an eigenvalue for } \gamma}} \frac{2}{\det(I_n - \gamma)} \tag{4.12}$$

4.3 S^1-equivariant η-invariant

The center of $\widetilde{U(n)}$, Z, is S^1. Since the action of this S^1 on S^{2n-1}, commutes with the action of Γ, we want to consider a S^1-equivariant version of the η-invariant.

Definition 4.3.1. The *S^1-equivariant η-invariant* on S^{2n-1} is

$$\eta^{S^1}(t) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \eta(t\gamma), \tag{4.13}$$

for $t \in S^1$, and we view $t\,\gamma$ as an element in $\widetilde{U(n)}$.

Corollary 4.3.2. *The S^1-equivariant η-invariant is*

$$\eta^{S^1}(t) = (-1)^n t^{\frac{n}{2}} \sum_{\gamma \in \Gamma} \frac{1}{\det(I_n - t\gamma)}. \tag{4.14}$$

Remark. This formula makes sense, because we view the S^1-equivariant η-invariant as an element in the ring of Laurent polynomials $\mathbb{C}[[t]]$.

Proof. It follows from the general formula for $\eta(g, 0)$, (4.3), applied to $g = t\gamma$. $\square$

Note. The computation of the S^1-equivariant η-invariant is the main reason for which we are dealing with the sphere as the homogeneous space $\widetilde{U(n)}/\widetilde{U(n-1)}$, instead of the straightforward choice $SU(n)/SU(n-1)$. The motivation is that, when we look at the orbifold $\mathbb{C}^n/\Gamma$, there is an action of $\mathbb{C}^*$ by dilations on $\mathbb{C}^n$, which commutes with the action of Γ. Restricted to the boundary, we have the action of S^1, the center of $\widetilde{U(n)}$. This is the most symmetry we can get in the general case of a finite subgroup of $SU(n)$. In the special case of a finite abelian subgroup Γ of $SU(n)$, there is extra-symmetry coming from the action of $(\mathbb{C}^*)^n$ the n-dimensional torus, on $\mathbb{C}^n/\Gamma$. In this way, the orbifold $\mathbb{C}^n/\Gamma$ becomes a toric variety, and this property gives us special techniques to study the geometry of the singularity and its resolutions of singularities.

4.4 Twisted Dirac operator

All we discussed about in the previous sections can be adapted to the case of a twisted Dirac operator, by a homogeneous vector bundle $\mathbb{E}$. We know from Chapter 2 that such a vector bundle if given by a representation, E, of $\widetilde{U(n)}$. From Frobenius Reciprocity it follows that we are going to have the same eigenvalues for the Dirac operator, and the corresponding eigenspaces are going to be the initial vector spaces tensored with E. With this in mind, all the statements and the proofs from the previous section go through without any modification, provided that we endowed E with a $\widetilde{U(n)}$-invariant connection. We can condense everything in the following result.

Proposition 4.4.1. *For the Dirac operator associated to the homogeneous vector bundle* $\mathbb{E}$*,*

$$D_E : \Omega(S^{2n-1}, \mathbb{S}\otimes\mathbb{E}) \to \Omega(S^{2n-1}, \mathbb{S}\otimes\mathbb{E}),$$

the twisted η-character *is*

$$\eta_E(g) = \begin{cases} (-1)^{n+1}(\det g)^{\frac{1}{2}} \frac{2\,\chi_E(g)}{\det(I-g)} & \textit{if } 1 \textit{ is not an eigenvalue of } g, \\ 0 & \textit{if } 1 \textit{ is an eigenvalue of } g. \end{cases} \tag{4.15}$$

Moreover, the twisted orbifold η-invariant *on* S^{2n-1}/Γ *is*

$$\eta_E = (-1)^n \frac{1}{|\Gamma|} \sum_{\substack{\gamma\in\Gamma \\ 1 \text{ not an eigenvalue of } \gamma}} \frac{2\,\chi_E(\gamma)}{\det(I_n-\gamma)} \tag{4.16}$$

and its S^1*-equivariant version is*

$$\eta_E^{S^1}(t) = (-1)^n t^{\frac{n}{2}} \frac{\dim_{\mathbb{C}} E}{|\Gamma|} \sum_{\gamma\in\Gamma} \frac{2\chi_E(\gamma)}{\det(I-t\gamma)}. \tag{4.17}$$

Chapter 5

Molien series

5.1 Algebraic Geometry Set Up

Let Γ be a finite subgroup of $SL(n, \mathbb{C})$. Consider the set of irreducible representations $\{R_0, R_1, \ldots, R_p\}$, where R_0 stands for the trivial one dimensional representation. There is a natural action of Γ on $\mathbb{C}^n$, and we consider the quotient singularity $\mathbb{C}^n/\Gamma$. This quotient has the structure of a complex $n+1$-dimensional orbifold.

Its ring of regular functions $\mathbb{A}[\mathbb{C}^n/\Gamma]$ is given by the subring

$$\mathbb{C}[X_1, X_2, \ldots, X_n]^\Gamma \subset \mathbb{C}[X_1, X_2, \ldots, X_n]$$

of regular functions on $\mathbb{C}^n$ which are invariant under the action of Γ. This ring is called *the ring of invariants*. The variety $\mathbb{C}^n/\Gamma$ inherits $p+1$ distinct orbifold sheaves $\{\mathcal{R}_j\}_{j=0}^p$ consisting of those functions, $f(X_1, \ldots, X_n)$ on $\mathbb{C}^n$ such that

$$f \in \mathbb{C}[X_1, \ldots, X_n]^\Gamma_{R_j},$$

where $\mathbb{C}[X_1, \ldots, X_n]^\Gamma_{R_j}$ is the isotopic component corresponding to R_j in the decomposition of the ring of polynomials $\mathbb{C}[X_1, \ldots, X_n]$ under the action of Γ, i.e.

$$\mathbb{C}[X_1, \ldots, X_n] = \bigoplus_{j=0}^{p} \mathbb{C}[X_1, \ldots, X_n]^\Gamma_{R_j}. \tag{5.1}$$

Note that $\mathbb{C}[X_1, \ldots X_n]^\Gamma_{R_0} = \mathbb{C}[X_1, \ldots, X_n]^\Gamma$ is the space of sections of the orbifold sheaf $\mathcal{R}_0$, which is the structure sheaf of the variety $\mathbb{C}^n/\Gamma$, $\mathcal{O}_{\mathbb{C}^n/\Gamma}$. It follows from the Schur's lemma that each $\mathbb{C}[X_1, \ldots, X_n]^\Gamma_{R_j}$ is a module over the ring of invariants $\mathbb{C}[X_1, \ldots, \mathbb{C}_n]^\Gamma$, and we call it the *module of R_j-relative invariants*. Note that this is the same as saying that the space of sections of the orbifold sheaf $\mathcal{R}_j$, $\Gamma(\mathbb{C}^n/G, \mathcal{R}_j)$, is a module over the ring of regular functions on $\mathbb{C}^n/\Gamma$.

To simplify the notations we are going to denote by A the ring of polynomials in n-variables, and with this notation the decomposition (5.1) becomes

$$A = A^\Gamma \ \oplus \ A^\Gamma_{R_1} \ \oplus \ \ldots \ \oplus \ A^\Gamma_{R_p}. \tag{5.2}$$

On the polynomial algebra we have a grading given by the degree.

$$A = \bigoplus_{k\geq 0} A_n, \quad A_k A_l \subset A_{k+l}, \quad A_0 = \mathbb{C}. \tag{5.3}$$

Since the action of the group Γ preserves this grading, it follows that A^Γ has a structure of a graded algebra,

$$A^\Gamma = \bigoplus_{k\geq 0}(A^\Gamma \cap A_k) = \bigoplus_{k\geq 0} A^\Gamma_k,$$

and all the other isotopic components, A^Γ_R, have the structure of a graded A^Γ-module,

$$A^\Gamma_R = \bigoplus_{k\geq 0} A^\Gamma_{R,k}, \quad A^\Gamma_k \cdot A^\Gamma_{R,l} \subset A^\Gamma_{R,k+l}.$$

We would like to know the dimension of the homogeneous component A^Γ_k, consisting of Γ-invariant polynomials of degree k, or more generally the dimension of the homogeneous piece of degree k in A^Γ_R. This kind of information is encoded in the *Hilbert series* (sometimes called the *Poincaré series*)

$$p(A^\Gamma_R, t) = \sum_{k=0}^{\infty} \dim_{\mathbb{C}}(A^\Gamma_{R,k})\, t^k.$$

5.2 Molien series

In this paper we are going to use a modified version of the Hilbert series which is called the Molien series. For a reference, see Stanley's survey article, [20], and the references therein.

Definition 5.2.1. The *Molien series* for A^Γ_R is defined to be

$$\Phi_R(t) = \frac{1}{\dim R} \sum_{k\geq 0} \dim_{\mathbb{C}}(A^\Gamma_{R,k})\, t^k \tag{5.4}$$

Corollary 5.2.2. *We have*

$$\sum_R \dim_{\mathbb{C}}(R)\ \Phi_R(t) = (1-t)^{-n}. \tag{5.5}$$

Remark. A consequence of the Hilbert-Serre Theorem is that the Molien series is a rational function of t.

The following theorem gives an interpretation of the rational function $\Phi_R(t)$ in terms of the finite group.

Theorem 5.2.3. (Molien) *Let Γ be a finite subgroup of $SL(n, \mathbb{C})$, and let R be an irreducible representation of Γ. Then the Molien series, $\Phi(A^\Gamma_R, t)$, of the module of R-relative invariants is given by*

$$\Phi_R(t) = \frac{1}{|\Gamma|} \sum_{\gamma\in\Gamma} \frac{R^*(\gamma)}{\det(I_n - t\gamma)}, \tag{5.6}$$

where I_n is the identity matrix and R^* is the dual character to R.

Note. This theorem is valid in the more general case of a finite group of $Gl(n, \mathbb{C})$.

Remark. In the particular case of the ring of invariants, we have

$$\Phi(t) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \frac{1}{|\gamma|}. \tag{5.7}$$

Corollary 5.2.4. *If R is an irreducible representation of Γ,*

$$\Phi_R(\frac{1}{t}) = (-1)^n \, t^n \, \Phi_{R^*}(t).$$

In the particular case R is the trivial representation, this formula becomes

$$\Phi(\frac{1}{t}) = (-1)^n \, t^n \, \Phi(t).$$

The following Corollary tells us that all the isotopic components appearing in the decomposition (5.1) are non-trivial.

Corollary 5.2.5. *For every irreducible representation R of Γ, the module of R-relative invariants, A_R^Γ, is non-trivial.*

Proof. In the expression (5.6) for $\Phi_R(t)$, exactly one term, corresponding to $\gamma = I_n$ has a pole of order n at $t = 1$, while the other terms have poles of order strictly less than n. Therefore $\Psi_R(t) \neq 0$, so A_R^Γ must be non-trivial. □

In the following Proposition we gather together a few results about the structure of the ring of invariants, A^Γ.

Proposition 5.2.6.

(i) *The ring of invariants A^Γ has n but not $n+1$ algebraically independent invariants over $\mathbb{C}$. In particular, the Krull dimension of A^Γ,* dim.Krull(A^Γ)*, is n. Equivalently,* dim.Krull(A^Γ) *is the order to which $t = 1$ is a pole of the rational function $\Phi(t)$.*

(ii) *A^Γ is a Cohen-Macaulay algebra.*

(iii) *We can always find a* homogeneous system of parameters $\{\theta_1, \ldots, \theta_n\}$, *with* deg$\theta_i = d_i$, *and a set of homogeneous invariants $\{\zeta_1, \ldots, \zeta_p\}$,* deg$\zeta_i = e_i$, *such that*

$$A^\Gamma = \bigoplus_{j=1}^{p} \mathbb{C}[\theta_1, \ldots, \theta_n] \, \zeta_j.$$

(iv) *The Molien series of the ring of invariants is*

$$\Phi(t) = \frac{t^{e_1} + \ldots + t^{e_p}}{(1 - t^{d_1}) \ldots (1 - t^{d_n})}. \tag{5.8}$$

(v) If we assume $0 = e_1 \le e_2 \le \ldots \le e_p$, *then*

$$e_p = \sum_{j=1}^{n} (d_j - 1) \; - 1,$$

$$e_i + e_{i-p+1} = e_p,$$

$$p\,|G| = d_1 \cdot d_2 \cdot \ldots \cdot d_n.$$

Proof. We refer to [20] for a proof. We just sketch the proof of (v) since it uses the Molien series. From Corollary 5.2.4 and (iv) of the Proposition, we have

$$\Phi(\frac{1}{t}) = (-1)^n \; \frac{\sum_{i=1}^{p} t^{d_1+\ldots+d_n-e_i}}{\prod_{j=1}^{n}(1-t^{d_j})}$$

$$= (-1)^n \; t^n \; \Phi(t),$$

This implies that

$$\sum_{i=1}^{p} t^{d_1+\ldots+d_n-e_i} = \sum_{i=1}^{p} t^{m+e_i}.$$

Because of the order we have on the e_i's, it follows that

$$d_1 + \ldots + d_n - e_i = m + e_{p-i+1},$$

for all $1 \le i \le p$. In particular, for $i = 1$ we obtain

$$e_p = \sum_{j=1}^{n} d_j - n,$$

and therefore the first relation. Plugging this into the equation for general i, we obtain the second relation.
For the last relation, we notice that since

$$\Phi(t) = \frac{t^{e_1} + \ldots + t^{e_p}}{(1-t^{d_1})\ldots(1-t^{d_n})},$$

then the coefficient of the pole at $t = 1$ of the Molien series is

$$\frac{p}{d_1 \cdot \ldots \cdot d_n}.$$

On the other hand, from Molien's theorem, it follows that the coefficient of the pole at $t = 1$ is coming from the contribution of the identity, and it is $\frac{1}{|G|}$. Therefore the third relation. $\square$

We have similar results for the module of R-relative invariants. First of all we need to define the notion of Krull dimension for the module of R-relative invariants. Proposition 5.2.6 tells us how to define it.

Definition 5.2.7. We define the *Krull dimension* of the module of R-relative invariants, dim.KrullA_R^Γ, to be the order to which $t = 1$ is a pole of $\Phi_R(t)$.

Proposition 5.2.8. *Let R be an irreducible representation of Γ, and consider the module of R-relative invariants A_R^Γ.*

(i) The Krull dimension of A_R^Γ is n.

(ii) A_R^Γ is a Cohen-Macaulay module.

(iii) A_R^Γ is a finitely generated A^Γ-module. Moreover, A_R^Γ is generated by homogeneous polynomials of degree not exceeding $|G|$, $\{\rho_1, \dots \rho_r\}$

$$A_R^\Gamma = \bigoplus_{i=1}^{r} \mathbb{C}[\theta_1, \dots, \theta_n]\, \rho_i$$

5.3 The relation between the η-invariant and the Molien series

The set up is the following: Γ finite subgroup of $SU(n)$, and let R be an irreducible representation of Γ. There is a natural action of Γ on $\mathbb{C}^n$. To the algebraic variety $\mathbb{C}^n/\Gamma$ we associate the Molien series of R-relative invariants $\Phi_R(t)$. To the boundary, S^{2n-1}/Γ, of the geometric orbifold IC^n/Γ we associate the η-invariant $\eta_R(t)$. All we had built up until now, allows us to state the following nice result which relates the algebraic and geometrical side of the quotient singularity $\mathbb{C}^n/\Gamma$.

Theorem 5.3.1. *The η-invariant is related to the Molien series by the following relation*

$$\eta_R(t) = \frac{(-1)^n}{2} \operatorname{Res}_1 \frac{\Phi_{R^*}(t)}{1-t}, \tag{5.9}$$

where R is the dual representation to R.

Remark. If we consider the S^1-equivariant η-invariant, $\eta_R^{S^1}(t)$, then this is related to the Molien series via

$$\eta_R^{S^1}(t) = 2\,(-1)^n\, t^{\frac{n}{2}} \dim_{\mathbb{C}}(R)\; \Phi_{R^*}(t). \tag{5.10}$$

We can further describe the Molien series in terms of η-invariants associated to the singularity. Consider the Laurent expansion about $t = 1$ of the Molien series $\Phi_R(t)$:

$$\Psi_R(t) = \frac{B_n^R}{(1-t)^n} + \frac{B_{n-1}^R}{(1-t)^{n-1}} + \dots + \frac{B_1^R}{1-t} + \theta^R(t). \tag{5.11}$$

From Molien's Theorem it is easy to notice that

$$B_n^R = \frac{1}{|G|},$$

for any R irreducible representation of Γ. Also, Theorem 5.3.1 implies that

$$\theta^R(1) = \frac{(-1)^n}{2} \operatorname{Res}_1 \frac{\Phi_R(t)}{(1-t)}.$$

Now, the coefficient B^R_{n-1} is given by the following result:

Proposition 5.3.2. *Let Γ be a finite subgroup of $Sl(n,\mathbb{C})$. The coefficient B^R_{n-1} in the Laurent expansion of $\Phi_R(t)$ at $t=1$ is 0 for all irreducible representations R of Γ.*

Proof. We consider the Molien series $\Phi_R(t)$ and we subtract the contribution from the identity. The remaining expression has a pole of order at most n at $t=1$. An element $\gamma \in \Gamma$ contributes to this pole if and only if it fixes a subspace of $\mathbb{C}^n$ of dimension $n-1$. This is because such an element which contributes to the pole of order $n-1$ must have the eigenvalue 1 with multiplicity $n-1$. But since we are dealing with a finite subgroup of $Sl(n,\mathbb{C})$, there are no elements with such a property. □

Remark. If we drop the assumption that the finite group sits in $SL(n,\mathbb{C})$, and we consider it as a finite group of $GL(n,\mathbb{C})$, then this coefficient is not going to be zero. An element $1 \neq \gamma$ with the property that it fixes a subspace of $\mathbb{C}^n$ of dimension $n-1$ it is called **pseudoreflection**. Let λ be the eigenvalue which is not zero. Then

$$\frac{1}{\det(I-gt)} = \frac{1}{(1-t)^{n-1}(1-\lambda t)},$$

and after multiplying by $(1-t)^{n-1}$, the value at $t=1$ is $\frac{1}{1-\lambda}$. We observe that

$$\frac{1}{1-\lambda} + \frac{1}{1-\lambda^{-1}} = 1.$$

So if we pair together each element with its inverse, we see that the total contribution from all the pseudoreflections is equal to one half of the number of them. Self-inverse elements have $\lambda = -1$, so their contribution is also one half. Therefore, in the case of the ring of invariants, the Laurent series expansion of the Molien series is

$$\Phi(t) = \frac{1}{|\Gamma|}\frac{1}{(1-t)^n} + \frac{1}{|\Gamma|}\frac{r/2}{(1-t)^{n-1}} + \dots,$$

where r is the number of pseudoreflections.

In order to attempt to describe the other coefficients in the Laurent expansion (5.11) we need to understand better the geometry of the orbifold $\mathbb{C}^n/\Gamma$. For each subgroup of Γ, A, we consider the *fixed point set*

$$V_A = \{x \in \mathbb{C}^n |\, ax = x \text{ for all } a \in A\}.$$

Note that $V_\Gamma = \{0\}$, the origin in $\mathbb{C}^n$, and that for the trivial subgroup we have $V_{\{I_n\}} = \mathbb{C}^n$. Let x be in $\mathbb{C}^n$. Then the corresponding point in $\mathbb{C}^n/\Gamma$, $[x]$, is a singular point if and only if the subgroup of Γ fixing x is nontrivial. This is equivalent to saying that $x \in V_A$ for A a nontrivial subgroup of Γ. Therefore the singular set of $\mathbb{C}^n/\Gamma$ is

$$S = \bigcup_{A \text{ non-trivial subgroup of } \Gamma} V_A/\Gamma.$$

Let W_A be the orthogonal complements to V_A in $\mathbb{C}^n$, so that $V_A \oplus W_A = \mathbb{C}^n$. There is an induced action of A on W_A, so that $\mathbb{C}^n/A = V_A \times W_A/A$. For a generic point $x \in V_A$, the subgroup of Γ fixing x is A, and the singularity of $\mathbb{C}^n/\Gamma$ at $[x]$ is locally modeled on the product $V_A \times W_A/A$.

Proposition 5.3.3. *For the Molien series of the ring of invariants, $\Phi(t)$, the coefficient B_k in the Laurent expansion at $t = 1$ is*

$$B_k = \frac{(-1)^{n-k}}{2} \sum_{\substack{A \text{ non-trivial subgroup of } \Gamma \\ \dim V_A = k}} \frac{|A|}{|\Gamma|} \, \eta(W_A/A), \tag{5.12}$$

where $\eta(W_A/A)$ is the eta-invariant for the boundary at infinity of the quotient singularity W_A/A.

Proof. From Molien's Theorem we see that we obtain the contribution to B_k from elements $\gamma \in \Gamma$ which have 1 as an eigenvalue with multiplicity k. Let A be a subgroup generated by such an element. Then $\dim V_A = k$, and the elements in this subgroup contribute to B_k with

$$\frac{1}{|\Gamma|} \sum_{\substack{a \in A \\ a \text{ has 1 as an eigenvalue} \\ \text{with multiplicity } k}} \frac{1}{\det\left(I_{n-k} - a\right)},$$

which is

$$\frac{(-1)^{n-k}}{2} \, \frac{|A|}{|\Gamma|} \, \eta(W_A/A).$$

Adding up after all the subgroups A of Γ with the property that $\dim V_A = k$, we obtain the desired formula. □

If R is an irreducible representation of Γ, then as a representation of A it decomposes into

$$R = Q_1 \oplus \ldots \oplus Q_{r_A},$$

where $\{Q_1, \ldots, Q_{r_A}\}$ are irreducible representations of A possibly repeating themselves. With this set-up and the same proof as Proposition 5.3.3, we have

Proposition 5.3.4. *The coefficient B_k^R in the Laurent expansion at $t = 1$ of the Molien series for the module of R-relative invariants is*

$$B_k^R = (-1)^{n-k} \sum_{\substack{A \text{ non-trivial subgroup of } \Gamma \\ \dim V_A = k}} \frac{|A|}{|\Gamma|} \sum_{j=1}^{r_A} \eta_{Q_j}(W_A/A). \tag{5.13}$$

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