STRONG SOLUTIONS FOR GENERALIZED NEWTONIAN FLUIDS

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ABSTRACT. We consider the motion of a generalized Newtonian fluid, where the extra stress tensor is induced by a potential with p–structure ($p = 2$ corresponds to the Newtonian case). We focus on the three dimensional case with periodic boundary conditions and extend the existence result for strong solutions for small times from $p > \frac{5}{3}$ (see [16]) to $p > \frac{7}{5}$. Moreover, for $\frac{7}{5} < p \leq 2$ we improve the regularity of the velocity field and show that $\mathbf{u} \in C([0,T], W^{1,6(p-1)-\varepsilon}_{\operatorname{div}}(\Omega))$ for all $\varepsilon > 0$. Within this class of regularity, we prove uniqueness for all $p > \frac{7}{5}$. We generalize these results to the case when p is space and time dependent and to the system governing the flow of electrorheological fluids as long as $\frac{7}{5} < \inf p(t,x) \leq \sup p(t,x) \leq 2$.

1. INTRODUCTION

In this paper we show existence of strong solutions to the following system describing the motion of generalized Newtonian fluids[1]:

$$
\begin{aligned}
\partial_t \mathbf{u} - \operatorname{div}(\mathbf{S}(\mathbf{D}\mathbf{u})) + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} + \boldsymbol{\nabla}\pi &= \mathbf{f}, && \text{on } I \times \Omega, \\
\operatorname{div} \mathbf{u} &= 0, && \text{on } I \times \Omega, \\
\mathbf{u}(0) &= \mathbf{u}_0, && \text{on } \Omega.
\end{aligned} \tag{1}
$$

We consider the three dimensional space periodic case, i.e. let Ω be the three dimensional torus and let $I = [0, T]$ with $T > 0$. The functions $\mathbf{u}$, π, and $\mathbf{f}$ denote the velocity, the pressure, and the external force. The function $\mathbf{u}_0$ is a given initial value. By $\mathbf{D}\mathbf{u}$ we denote the symmetric part of the gradient $\boldsymbol{\nabla}\mathbf{u}$, i.e. $\mathbf{D}\mathbf{u} = \frac{1}{2}(\boldsymbol{\nabla}\mathbf{u} + (\boldsymbol{\nabla}\mathbf{u})^\top)$. We assume that the extra stress $\mathbf{S}$ is induced by a p–potential F as defined below. Standard examples for $\mathbf{S}$ are

$$
\mathbf{S}(\mathbf{D}\mathbf{u}) \equiv (1 + |\mathbf{D}\mathbf{u}|^2)^{\frac{p-2}{2}} \, \mathbf{D}\mathbf{u}, \qquad \mathbf{S}(\mathbf{D}\mathbf{u}) \equiv (1 + |\mathbf{D}\mathbf{u}|)^{p-2} \, \mathbf{D}\mathbf{u},
$$

with $1 < p < \infty$. We compensate the missing boundary conditions by restricting the solutions to ones with mean value zero. This ensures that the Poincaré inequality remains valid.

Mathematical results for the system (1) can be found in [13, 15, 19, 2, 16, 18, 4].[2] We want to mention that the existence of global strong solutions to the problem

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[1]We refer to [16] for an extensive discussion on such fluids.

[2]In the monograph [16] the reader can find a detailed discussion of the problem (cf. [8, 23, 10, 11, 7, 17] for more recent results).

(1), i.e. $\mathbf{u} \in L^\infty(I, V_p) \cap L^2(I, W^{2,2}(\Omega)), \partial_t \mathbf{u} \in L^2(I, L^2(\Omega))$, satisfies for almost all $t \in I$,

$$\langle \partial_t \mathbf{u}, \boldsymbol{\varphi}\rangle + \langle \mathbf{S}(\mathbf{Du}), \mathbf{D}\boldsymbol{\varphi}\rangle + \langle \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}, \boldsymbol{\varphi}\rangle = \langle \mathbf{f}, \boldsymbol{\varphi}\rangle \qquad \forall \boldsymbol{\varphi} \in V_p, \tag{2}$$

is established for $p \geq \frac{3d+2}{d+2}$ (cf. [16, 19, 2]), where d is the dimension of Ω. The existence of a *local* in time strong solution for arbitrary data and the existence of a global strong solution for small data is proved in the case $p > \frac{3d-4}{d}$ (cf. [18, 16]). Moreover, in [16] it has been shown that for $p > \frac{5}{3}$ there exists strong solutions for short time and large data

$$\begin{aligned} \|\boldsymbol{\nabla}\mathbf{u}\|_{L^\infty(I,L^2(\Omega))} + \|\partial_t \mathbf{u}\|_{L^2(I,L^2(\Omega))} &\leq C, \\ \|\mathcal{I}_\Phi(\mathbf{u})\|_{L^1(I)} &\leq C, \end{aligned} \tag{3}$$

where $\mathcal{I}_\Phi(\mathbf{u}) \approx \int_\Omega (1+|\mathbf{Du}|)^{p-2}|\boldsymbol{\nabla}^2\mathbf{u}|^2\,dx$. It has been shown in [20] that this solution further satisfies

$$\|\partial_t \mathbf{u}\|_{L^\infty(I,L^2(\Omega))} + \|\mathcal{J}_\Phi(\mathbf{u})\|_{L^1(I)} \leq C, \tag{4}$$

where $\mathcal{J}_\Phi(\mathbf{u}) \approx \int_\Omega (1+|\mathbf{Du}|)^{p-2}|\boldsymbol{\nabla}\partial_t\mathbf{u}|^2\,dx$.

In this paper[3] we prove the short time existence of strong solutions for large data for all $\frac{7}{5} < p \leq 2$. Moreover, we show that the solution satisfies additionally to (3) and (4)

$$\|\mathcal{I}_\Phi(\mathbf{u})\|_{L^{\frac{5p-6}{2-p}}(I)} \leq C. \tag{5}$$

We refer to theorem 17 for the precise statement of this result. From (3) and (4) we deduce that $\mathbf{u} \in C([0,T], W^{1,6(p-1)-\varepsilon}_{\operatorname{div}}(\Omega))$ for all $\varepsilon > 0$, especially $\mathbf{u} \in C(I, W^{1,\frac{12}{5}}_{\operatorname{div}}(\Omega))$. We will show that every weak solution $\mathbf{v} \in C(I, W^{1,\frac{12}{5}}_{\operatorname{div}}(\Omega))$ of (1) satisfies $\mathbf{v} = \mathbf{u}$, especially $\mathbf{u}$ is unique within its class of regularity (see theorem 19).

System (1) is usually studied under the assumption that p is constant, with $1 < p < \infty$. Motivated by the model for the motion of electrorheological fluids in [21, 22] which has been further studied in [25], we are also interested in the case, where p is a function in space and time. Electrorheological fluids are a special type of smart fluids which change their material properties due to the application of an electric field. In the model in [22] p is not a constant but a function of the electric field $\mathbf{E}$, i.e. $p = p(|\mathbf{E}|^2)$. The electric field itself is a solution to the quasi–static Maxwell equations and is not influenced by the motion of the fluid. Therefore it is possible to consider (1) for a given function $p : \Omega \times I \to (1, \infty)$. In this case we speak of a time and space dependent potential. Due to the nature of the Maxwell equations it is reasonable to consider smooth p. We define $p^- := \inf_{\Omega\times I} p$ and $p^+ := \sup_{\Omega\times I} p$. In [24] it has been proven that there exists a strong solution to (1) for large time and data as long as $p \in W^{1,\infty}(I \times \Omega)$ and[4]

$$\tfrac{11}{5} < p^- \leq p^+ < p^- + \tfrac{4}{3}.$$

The extra condition $p^+ < p^- + \frac{4}{3}$ is due to the use of classical Sobolev spaces. The reason is that the energy $\int_\Omega |\mathbf{Du}|^p\,dx$ cannot be fully expressed in terms of classical Sobolev spaces if p is non–constant. In that case the generalized Sobolev spaces

[3] This paper is based on the PhD thesis [4] of L. Diening

[4] The case of Dirichlet boundary conditions is treated in [25].

$W^{k,p(\cdot)}$ provide the right setting. See [12, 5, 25] and the references therein for a definition of these spaces and applications to fluid mechanics.

In section 8 we generalize our results on short time existence to the case, where p is non–constant, i.e. we prove short time existence of strong solutions for large data as long as $\frac{7}{5} < p^- \leq p^+ \leq 2$. Moreover, we show that the solution satisfies (3) and (4), while (5) will be replaced by

$$\|\mathcal{I}_\Phi(\mathbf{u})\|_{L^{\frac{5p^- - 6}{2-p^-}}(I)} \leq C.$$

See theorem 21 and corollary 22 for the precise statement. As before we deduce that the solution is unique within its class of regularity (cf. theorem 23). Finally we also extend these results to the system governing the motion of electrorheological fluids.

2. The Potential and the Extra Stress

We assumte that the extra stress tensor $\mathbf{S}$ is induced by a p–potential. In this section we give a definition of a p–potential and derive basic properties of it. We will consider only the case p constant. The case p non–constant will be covered by section 8.

Since we are dealing with functions from $\Omega \times \mathbb{R}^{n\times n}$ to $\mathbb{R}$, we will distinguish the partial derivatives by ∂_i and ∂_{jk}, a single index means a partial derivative with respect to the i-th space coordinate, while a double index represents a partial derivative with respect to the (j,k)-component of the underlying space of $n \times n$-matrices. By $\boldsymbol{\nabla}$ we denote the space gradient, while $\boldsymbol{\nabla}_{n\times n}$ denotes the matrix consisting of the partial derivatives with respect to the space of matrices. In a few cases we use d_i instead of ∂_i to indicate a total derivative. Note that by $\mathbf{B}^{\mathrm{sym}}$ we denote the symmetric part of a matrix $\mathbf{B} \in \mathbb{R}^{n\times n}$, i.e. $\mathbf{B}^{\mathrm{sym}} = \frac{1}{2}(\mathbf{B} + \mathbf{B}^\top)$. Further let $\mathbb{R}^{n\times n}_{\mathrm{sym}}$ be the subspace of $\mathbb{R}^{n\times n}$ consisting of the symmetric matrices. Moreover we use C as a constant which is generic but does not depend on the ellipticity constants. For the notation of the function spaces see section 4. We will assume that $1 < p \leq 2$ throughout the whole paper.

Definition 1. *Let $1 < p \leq 2$ and let $F : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ be a convex function, which is C^2 on $\mathbb{R}^{\geq 0}$, such that $F(0) = 0$, $F'(0) = 0$. Assume that the induced function $\Phi : \mathbb{R}^{n\times n} \to \mathbb{R}^{\geq 0}$, defined through $\Phi(\mathbf{B}) = F(|\mathbf{B}^{\mathrm{sym}}|)$, satisfies*

$$\sum_{jklm} (\partial_{jk}\partial_{lm}\Phi)(\mathbf{B}) C_{jk} C_{lm} \geq \gamma_1 (1 + |\mathbf{B}^{\mathrm{sym}}|^2)^{\frac{p-2}{2}} |\mathbf{C}^{\mathrm{sym}}|^2, \tag{6}$$

$$\big|(\boldsymbol{\nabla}^2_{n\times n}\Phi)(\mathbf{B})\big| \leq \gamma_2 (1 + |\mathbf{B}^{\mathrm{sym}}|^2)^{\frac{p-2}{2}} \tag{7}$$

*for all $\mathbf{B}, \mathbf{C} \in \mathbb{R}^{n\times n}$ with constants $\gamma_1, \gamma_2 > 0$. Such a function F, resp. Φ, is called a p-**potential** and the corresponding constants γ_1, γ_2 are called the ellipticity constants of F, resp. Φ.*

We define the extra stress $\mathbf{S}$ induced by F, resp. Φ, by

$$\mathbf{S}(\mathbf{B}) := \boldsymbol{\nabla}_{n\times n}\Phi(\mathbf{B}) = F'(|\mathbf{B}^{\mathrm{sym}}|)\frac{\mathbf{B}^{\mathrm{sym}}}{|\mathbf{B}^{\mathrm{sym}}|}$$

for all $\mathbf{B} \in \mathbb{R}^{n\times n} \setminus \{\mathbf{0}\}$. We will see in remark 2 that $\mathbf{S}$ can be continuously extended by $\mathbf{S}(\mathbf{0}) = \mathbf{0}$. Note that $\mathbf{S}(\mathbf{B})$ does only depend on the symmetric part of $\mathbf{B}$.

Standard examples are

$$F_1(s) = \int_0^s (1+a^2)^{\frac{p-2}{2}} a\, da \qquad \text{and} \qquad F_2(s) = \int_0^s (1+a)^{p-2} a\, da.$$

Remark 2. *Observe that for all* $\mathbf{B} \in \mathbb{R}^{n\times n} \setminus \{\mathbf{0}\}$

$$\begin{aligned}(\partial_{jk}\Phi)(\mathbf{B}) &= F'(|\mathbf{B}^{\mathrm{sym}}|)\tfrac{B^{\mathrm{sym}}_{jk}}{|\mathbf{B}^{\mathrm{sym}}|},\\ (\partial_{jk}\partial_{lm}\Phi)(\mathbf{B}) &= F'(|\mathbf{B}^{\mathrm{sym}}|)\Big(\tfrac{\delta^{\mathrm{sym}}_{jk,lm}}{|\mathbf{B}^{\mathrm{sym}}|} - \tfrac{B^{\mathrm{sym}}_{jk}B^{\mathrm{sym}}_{lm}}{|\mathbf{B}^{\mathrm{sym}}|^3}\Big) + F''(|\mathbf{B}^{\mathrm{sym}}|)\tfrac{B^{\mathrm{sym}}_{jk}}{|\mathbf{B}^{\mathrm{sym}}|}\tfrac{B^{\mathrm{sym}}_{lm}}{|\mathbf{B}^{\mathrm{sym}}|},\end{aligned}$$

where $\delta^{\mathrm{sym}}_{jk,lm} := \frac{1}{2}(\delta_{jl}\delta_{km} + \delta_{jm}\delta_{kl})$. *Hence*

$$\sum_{jklm} (\partial_{jk}\partial_{lm}\Phi)(\mathbf{B})B_{jk}B_{lm} = F''(|\mathbf{B}^{\mathrm{sym}}|)|\mathbf{B}^{\mathrm{sym}}|^2.$$

So by (6) *and* (7) *we conclude that for all* $\mathbf{B} \in \mathbb{R}^{n\times n} \setminus \{\mathbf{0}\}$

$$\gamma_1(1+|\mathbf{B}^{\mathrm{sym}}|^2)^{\frac{p-2}{2}} \le F''(|\mathbf{B}^{\mathrm{sym}}|) \le \gamma_2(1+|\mathbf{B}^{\mathrm{sym}}|^2)^{\frac{p-2}{2}}. \tag{8}$$

Since $F'' \in C^2(\mathbb{R}^{\ge 0})$, *this estimate also holds for* $\mathbf{B} = \mathbf{0}$. *From the formula above for* $(\partial_{jk}\Phi)(\mathbf{B})$, *the continuity of* F' *at zero with* $F'(0) = 0$, *and the boundedness of* $B^{\mathrm{sym}}_{jk}/|\mathbf{B}^{\mathrm{sym}}|$ *in* $\mathbb{R}^{n\times n} \setminus \{\mathbf{0}\}$, *we deduce*

$$\lim_{|\mathbf{B}|\to 0} S_{jk}(\mathbf{B}) = \lim_{|\mathbf{B}|\to 0} (\partial_{jk}\Phi)(\mathbf{B}) = 0. \tag{9}$$

Remark 3. *Let* $\mathbf{B}, \mathbf{C} \in \mathbb{R}^{n\times n}$. *Due to* $\Phi(\mathbf{B}) = F(|\mathbf{B}^{\mathrm{sym}}|)$, *we have* $\Phi(\mathbf{B}) = \Phi(\mathbf{B}^{\mathrm{sym}})$, *thus the* $\partial_{jk}\partial_{lm}\Phi$ *are symmetric in* j,k *and* l,m *and* $(j,k),(l,m)$. *This implies that*

$$\sum_{jklm} (\partial_{jk}\partial_{lm}\Phi)(\mathbf{B})C_{jk}C_{lm} = \sum_{jklm} (\partial_{jk}\partial_{lm}\Phi)(\mathbf{B}^{\mathrm{sym}})C^{\mathrm{sym}}_{jk}C^{\mathrm{sym}}_{lm}, \tag{10}$$

$$(\boldsymbol{\nabla}_{n\times n}\Phi)(\mathbf{B}) = (\boldsymbol{\nabla}_{n\times n}\Phi)(\mathbf{B}^{\mathrm{sym}}), \tag{11}$$

$$(\boldsymbol{\nabla}^2_{n\times n}\Phi)(\mathbf{B}) = (\boldsymbol{\nabla}^2_{n\times n}\Phi)(\mathbf{B}^{\mathrm{sym}}). \tag{12}$$

Thus it suffices to verify (6), (7) *for all symmetric matrices. Since later we will mostly deal with symmetric matrices, we will in some cases leave out the symmetrization of the matrices, i.e. we will use* $\mathbf{B}$ *instead of* $\mathbf{B}^{\mathrm{sym}}$ *and restrict the admitted matrices to the symmetric ones.*

As in [16], one can deduce from (6) and (7) the following properties of $\mathbf{S}$.

Theorem 4. *There exist constants* $c_1, c_2 > 0$ *independent of* γ_1 *and* γ_2 *such that for all* $\mathbf{B}, \mathbf{C} \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ *there holds*

$$\mathbf{S}(\mathbf{0}) = \mathbf{0}, \tag{13}$$

$$\begin{aligned}\sum_{ij}(S_{ij}(\mathbf{B}) - S_{ij}(\mathbf{C}))(B_{ij} - C_{ij}) &\ge c_1\gamma_1(1+|\mathbf{B}|^2+|\mathbf{C}|^2)^{\frac{p-2}{2}}|\mathbf{B}-\mathbf{C}|^2,\\ \sum_{ij} S_{ij}(\mathbf{B})B_{ij} &\ge c_1\,\gamma_1(1+|\mathbf{B}|^2)^{\frac{p-2}{2}}|\mathbf{B}|^2,\end{aligned} \tag{14}$$

$$\begin{aligned}|\mathbf{S}(\mathbf{B}) - \mathbf{S}(\mathbf{C})| &\le c_2\,\gamma_2(1+|\mathbf{B}|^2+|\mathbf{C}|^2)^{\frac{p-2}{2}}\,|\mathbf{B}-\mathbf{C}|,\\ |\mathbf{S}(\mathbf{B})| &\le c_2\,\gamma_2(1+|\mathbf{B}|^2)^{\frac{p-2}{2}}\,|\mathbf{B}|.\end{aligned} \tag{15}$$

The same inequalities hold true for all $\mathbf{B}, \mathbf{C} \in \mathbb{R}^{n\times n}$ *if* $\mathbf{B}$ *and* $\mathbf{C}$ *is replaced on the right-hand side by* $\mathbf{B}^{\mathrm{sym}}$ *and* $\mathbf{C}^{\mathrm{sym}}$.

3. Special Energies

Later in our a priori estimates we will encounter the following two important expressions

$$\mathcal{I}_\Phi(t,\mathbf{u}) := \Big\langle \sum_r \sum_{jk\alpha\beta} (\partial_{\alpha\beta}\partial_{jk}\Phi)\big(\mathbf{Du}(t)\big)\partial_r D_{\alpha\beta}\mathbf{u}(t), \partial_r D_{jk}\mathbf{u}(t)\Big\rangle, \tag{16}$$

$$\mathcal{J}_\Phi(t,\mathbf{u}) := \Big\langle \sum_{jk\alpha\beta} (\partial_{\alpha\beta}\partial_{jk}\Phi)\big(\mathbf{Du}(t)\big)\partial_t D_{\alpha\beta}\mathbf{u}(t), \partial_t D_{jk}\mathbf{u}(t)\Big\rangle, \tag{17}$$

where Φ is a p–potential and $\mathbf{u}$ denotes a sufficiently smooth function over the space time cylinder. The brackets $\langle\cdot,\cdot\rangle$ denote integration over the space domain Ω. These two expressions will arise when we are going to test the equation of motion with $-\Delta\mathbf{u}$, resp. "$\partial_t^2\mathbf{u}$". Since $\mathcal{I}_\Phi$ and $\mathcal{J}_\Phi$ are very similar, it is useful to introduce another functor $\mathcal{G}_\Phi$ by

$$\mathcal{G}_\Phi(t,\mathbf{w},\mathbf{v}) := \Big\langle \sum_{jk\alpha\beta} (\partial_{\alpha\beta}\partial_{jk}\Phi)(\mathbf{Dw}(t)) D_{\alpha\beta}\mathbf{v}(t), D_{jk}\mathbf{v}(t)\Big\rangle, \tag{18}$$

where $\mathbf{w} : I\times\Omega \to \mathbb{R}^d$ and $\mathbf{v} : I\times\Omega\to\mathbb{R}^d$ (or $\mathbf{v} : I\times\Omega\to\mathbb{R}^{d\times d}$) are sufficiently smooth functions. Mostly we will simply write $\mathcal{I}_\Phi(\mathbf{u})$, $\mathcal{J}_\Phi(\mathbf{u})$, and $\mathcal{G}_\Phi(\mathbf{w},\mathbf{v})$ instead $\mathcal{I}_\Phi(t,\mathbf{u})$, $\mathcal{J}_\Phi(t,\mathbf{u})$, and $\mathcal{G}_\Phi(t,\mathbf{w},\mathbf{v})$. We have

$$\mathcal{I}_\Phi(\mathbf{u}) = \mathcal{G}_\Phi(\mathbf{u},\boldsymbol{\nabla}\mathbf{u}), \qquad \mathcal{J}_\Phi(\mathbf{u}) = \mathcal{G}_\Phi(\mathbf{u},\partial_t\mathbf{u}). \tag{19}$$

Due to the properties of Φ we estimate

$$\mathcal{G}_\Phi(\mathbf{w},\mathbf{v}) \geq \gamma_1 \int_\Omega (1+|\mathbf{Dw}|^2)^{\frac{p-2}{2}} |\mathbf{Dv}|^2\,dx. \tag{20}$$

The expression $(1+|\mathbf{Dw}|^2)^{\frac12}$ will appear quite often in all the chapters, so it is very useful to introduce the shortcut

$$\widetilde{D}\mathbf{w} := (1+|\mathbf{Dw}|^2)^{\frac12}. \tag{21}$$

As a consequence

$$\mathcal{I}_\Phi(\mathbf{u}) \geq C\,\gamma_1 \int_\Omega (\widetilde{D}\mathbf{u})^{p-2}|\boldsymbol{\nabla}\mathbf{Du}|^2\,dx, \tag{22}$$

$$\mathcal{J}_\Phi(\mathbf{u}) \geq C\,\gamma_1 \int_\Omega (\widetilde{D}\mathbf{u})^{p-2}|\partial_t\mathbf{Du}|^2\,dx. \tag{23}$$

Note that

$$\partial_j\partial_k u_m = \partial_j D_{km}\mathbf{u} + \partial_k D_{mj}\mathbf{u} - \partial_m D_{jk}\mathbf{u}. \tag{24}$$

which implies $|\boldsymbol{\nabla}^2\mathbf{u}| \leq 3|\boldsymbol{\nabla}\mathbf{Du}| \leq 6\,|\boldsymbol{\nabla}^2\mathbf{u}|$. Thus, $|\boldsymbol{\nabla}\mathbf{Du}|$ can always be replaced by $|\boldsymbol{\nabla}^2\mathbf{u}|$ (and vice versa) by increasing the multiplicative constant.

Closely connected to the quantities $\mathcal{I}_\Phi(\mathbf{u})$ and $\mathcal{J}_\Phi(\mathbf{u})$ is the function $(\widetilde{D}\mathbf{u})^{\frac{p}{2}}$, which will be important when examining the regularity of solutions.

Lemma 5. *Let Φ be a p–potential. Then there exists $C > 0$, such that for all (sufficiently smooth) $\mathbf{u}$ and almost all times $t \in I$ there holds*

$$\gamma_1 \Big\| \boldsymbol{\nabla}\Big(\big(\widetilde{D}\mathbf{u}(t)\big)^{\frac{p}{2}}\Big)\Big\|_2^2 \le C\, \mathcal{I}_\Phi(t,\mathbf{u}). \tag{25}$$

$$\gamma_1 \Big\| \partial_t\Big(\big(\widetilde{D}\mathbf{u}(t)\big)^{\frac{p}{2}}\Big)\Big\|_2^2 \le C\, \mathcal{J}_\Phi(t,\mathbf{u}). \tag{26}$$

Proof. Observe that

$$\boldsymbol{\nabla}\big((\widetilde{D}\mathbf{u})^{\frac{p}{2}}\big) = \sum_{jk} \tfrac{p}{2}(\widetilde{D}\mathbf{u})^{\frac{p-4}{2}}(D_{jk}\mathbf{u})\,(\boldsymbol{\nabla} D_{jk}\mathbf{u}). \tag{27}$$

Raising this to the power of two and integrating over Ω proves the first inequality. If we replace $\boldsymbol{\nabla}$ in the calculations above by ∂_t, we get the result for $\mathcal{J}_\Phi(\mathbf{u})$. □

In the following we will derive more useful estimates for $\mathcal{G}_\Phi(\mathbf{w},\mathbf{v})$, $\mathcal{I}_\Phi(\mathbf{u})$ and $\mathcal{J}_\Phi(\mathbf{u})$:

Lemma 6. *Let Φ be a p–potential. Then for all (sufficiently smooth) $\mathbf{u}$ and almost all times $t \in I$ there holds*

$$\gamma_1 \|\boldsymbol{\nabla}^2\mathbf{u}(t)\|_p^p \le C\, \mathcal{I}_\Phi(t,\mathbf{u}) + \gamma_1 \|\widetilde{D}\mathbf{u}(t)\|_p^p. \tag{28}$$

$$\gamma_1 \|\partial_t \mathbf{D}\mathbf{u}(t)\|_p^p \le C\, \mathcal{J}_\Phi(t,\mathbf{u}) + \gamma_1 \|\widetilde{D}\mathbf{u}(t)\|_p^p. \tag{29}$$

Proof. Note that for all $q \in [1,2]$, $a \ge 0$, $b \ge 1$ there holds

$$a^q \le a^2 b^{q-2} + b^q. \tag{30}$$

Indeed, there is nothing to prove if $q = 2$, so let $1 \le q < 2$. In this case $1 < \frac{2}{q} < \infty$, and Young's inequality gives

$$a^q = (a^2 b^{q-2})^{\frac{q}{2}} (b^{\frac{(2-q)q}{2}}) \overset{\text{Young}}{\le} a^2 b^{q-2} + b^q.$$

Now (30) implies

$$|\boldsymbol{\nabla}^2\mathbf{u}|^p \le (\widetilde{D}\mathbf{u})^{p-2} |\boldsymbol{\nabla}^2\mathbf{u}|^2 + (\widetilde{D}\mathbf{u})^p.$$

Since in general $|\boldsymbol{\nabla}^2\mathbf{u}| \le 3|\boldsymbol{\nabla}\mathbf{D}\mathbf{u}|$ (see (24)) we deduce

$$\begin{aligned}
\|\boldsymbol{\nabla}^2\mathbf{u}\|_p^p &\le \int_\Omega (\widetilde{D}\mathbf{u})^{p-2}|\boldsymbol{\nabla}^2\mathbf{u}|^2\,dx + \|\widetilde{D}\mathbf{u}\|_p^p \\
&\le 9 \int_\Omega (\widetilde{D}\mathbf{u})^{p-2}|\boldsymbol{\nabla}\mathbf{D}\mathbf{u}|^2\,dx + \|\widetilde{D}\mathbf{u}\|_p^p \\
&\overset{(22)}{\le} \frac{C}{\gamma_1}\,\mathcal{I}_\Phi(\mathbf{u}) + \|\widetilde{D}\mathbf{u}\|_p^p.
\end{aligned}$$

The estimate for $\partial_t\mathbf{D}\mathbf{u}$ follows analogously. □

Lemma 7. *Let Φ be a p–potential. Then for all (sufficiently smooth) $\mathbf{u}$ and $\mathbf{v}$, for all $1 \le q \le 2$, and almost every $t \in I$ there holds:*

$$\|\mathbf{D}\mathbf{v}(t)\|_q \le \tfrac{C}{\gamma_1}\,\big(\tfrac{1}{\gamma_1}\mathcal{G}_\Phi(t,\mathbf{w},\mathbf{v})\big)^{\frac{1}{2}} \|(\widetilde{D}\mathbf{w}(t))^{\frac{2-p}{2}}\|_{\frac{2q}{2-q}}, \tag{31}$$

where $\frac{2q}{2-q} = \infty$ for $q = 2$.

Proof. Observe that $1 \le \frac{2}{q} < \infty$ and $1 < (\frac{2}{q})' = \frac{2}{2-q} \le \infty$. Further for $1 \le q < 2$

$$\begin{aligned}\|\mathbf{D}\mathbf{v}\|_q^q &= \int_\Omega \Big((\widetilde{D}\mathbf{w})^{p-2}|\mathbf{D}\mathbf{v}|^2\Big)^{\frac{q}{2}} (\widetilde{D}\mathbf{w})^{\frac{(2-p)q}{2}} dx \\ &\le \left(\int_\Omega (\widetilde{D}\mathbf{w})^{p-2}|\mathbf{D}\mathbf{v}|^2 dx\right)^{\frac{q}{2}} \big\|(\widetilde{D}\mathbf{w})^{\frac{(2-p)q}{2}}\big\|_{\frac{2}{2-q}} \\ &= \left(\int_\Omega (\widetilde{D}\mathbf{w})^{p-2}|\mathbf{D}\mathbf{v}|^2 dx\right)^{\frac{q}{2}} \big\|(\widetilde{D}\mathbf{w})^{\frac{(2-p)}{2}}\big\|^q_{\frac{2q}{2-q}}.\end{aligned}$$

This and (20) prove the lemma for $q < 2$. The case $q = 2$ is similar. □

Note that this lemma is applicable to $\mathcal{I}_\Phi(\mathbf{u}) = \mathcal{G}_\Phi(\mathbf{u}, \boldsymbol{\nabla}\mathbf{u})$ and as well to $\mathcal{J}_\Phi(\mathbf{u}) = \mathcal{G}_\Phi(\mathbf{u}, \partial_t\mathbf{u})$. Analogously we have

Lemma 8. *Let Φ be a p–potential. Then for all (sufficiently smooth) $\mathbf{u}$ and $\mathbf{v}$ and for all $1 \le q \le 2$ there holds:*

$$\|\mathbf{D}(\mathbf{u}-\mathbf{v})\|_q \le C \big\langle \mathbf{S}(\mathbf{D}\mathbf{u}) - \mathbf{S}(\mathbf{D}\mathbf{v}), \mathbf{D}(\mathbf{u}-\mathbf{v})\big\rangle^{\frac{1}{2}} \|(\widetilde{D}\mathbf{u})^{\frac{2-p}{2}} + (\widetilde{D}\mathbf{v})^{\frac{2-p}{2}}\|_{\frac{2q}{2-q}},$$

where $\frac{2q}{2-q} = \infty$ for $q = 2$.

Proof. Analogously to the proof of lemma 7

$$\begin{aligned}\|\mathbf{D}(\mathbf{u}-\mathbf{v})\|_q^q &= \int_\Omega \Big((\widetilde{D}\mathbf{u} + \widetilde{D}\mathbf{v})^{p-2}|\mathbf{D}(\mathbf{u}-\mathbf{v})|^2\Big)^{\frac{q}{2}} (\widetilde{D}\mathbf{u} + \widetilde{D}\mathbf{v})^{\frac{(2-p)q}{2}} dx \\ &\le \left(\int_\Omega (\widetilde{D}\mathbf{u}+\widetilde{D}\mathbf{v})^{p-2}|\mathbf{D}(\mathbf{u}-\mathbf{v})|^2 dx\right)^{\frac{q}{2}} \big\|(\widetilde{D}\mathbf{u}+\widetilde{D}\mathbf{v})^{\frac{(2-p)q}{2}}\big\|_{\frac{2}{2-q}} \\ &\overset{(14)}{\le} \frac{C}{\gamma_1} \big\langle \mathbf{S}(\mathbf{D}\mathbf{u}) - \mathbf{S}(\mathbf{D}\mathbf{v}), \mathbf{D}(\mathbf{u}-\mathbf{v})\big\rangle^{\frac{q}{2}} \big\|(\widetilde{D}\mathbf{u})^{\frac{(2-p)}{2}} + (\widetilde{D}\mathbf{v})^{\frac{(2-p)}{2}}\big\|^q_{\frac{2q}{2-q}}.\end{aligned}$$

This proves the lemma for $q < 2$. The case $q = 2$ is similar. □

Note that the following estimates for $\mathcal{I}_\Phi$ and $\mathcal{J}_\Phi$ will depend on the dimension of the underlying space, which is in our case three dimensional.

Lemma 9. *For all (sufficiently smooth) $\mathbf{u}$ and almost every $t \in I$ there holds*

$$\gamma_1 \|\widetilde{D}\mathbf{u}(t)\|_{3p}^p \le C \Big(\mathcal{I}_\Phi(t, \mathbf{u}) + \|\widetilde{D}\mathbf{u}(t)\|_p^p\Big). \tag{32}$$

Proof. By lemma 5 and the embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$ there holds

$$\begin{aligned}\gamma_1 \|\widetilde{D}\mathbf{u}\|_{3p}^p &= \gamma_1 \|(\widetilde{D}\mathbf{u})^{\frac{p}{2}}\|_6^2 \le C\,\gamma_1 (\|\boldsymbol{\nabla}((\widetilde{D}\mathbf{u})^{\frac{p}{2}})\|_2^2 + \|(\widetilde{D}\mathbf{u})^{\frac{p}{2}}\|_2^2) \\ &\le C\,\big(\mathcal{I}_\Phi(\mathbf{u}) + \gamma_1 \|\widetilde{D}\mathbf{u}\|_p^p\big).\end{aligned}$$

This proves the lemma. □

Lemma 10. *For all (sufficiently smooth)* $\mathbf{u}$ *with* $\langle \mathbf{u}, 1\rangle = 0$ *and almost every* $t \in I$ *there holds*

$$\|\mathbf{u}(t)\|^p_{2,\frac{3p}{p+1}} \le C\,(\mathcal{I}_\Phi(t,\mathbf{u}) + 1), \tag{33}$$

$$\|\partial_t \mathbf{u}(t)\|^p_{1,\frac{3p}{p+1}} \le C\,\mathcal{J}_\Phi(t,\mathbf{u})^{\frac{p}{2}}\big(\mathcal{I}_\Phi(t,\mathbf{u}) + 1\big)^{\frac{2-p}{2}}, \tag{34}$$

$$\le C(\mathcal{J}_\Phi(t,\mathbf{u}) + \mathcal{I}_\Phi(t,\mathbf{u}) + 1). \tag{35}$$

Proof. From lemma 7 we deduce

$$\begin{aligned}
\|\boldsymbol{\nabla}\mathbf{D}\mathbf{u}\|_{\frac{3p}{p+1}} &\le C\,\mathcal{I}_\Phi(\mathbf{u})^{\frac12}\,\|(\widetilde{D}\mathbf{u})^{\frac{2-p}{2}}\|_{\frac{6p}{2-p}} \\
&\le C\,\mathcal{I}_\Phi(\mathbf{u})^{\frac12}\,\|\widetilde{D}\mathbf{u}\|_{3p}^{\frac{2-p}{2}} \\
&\le C\,\mathcal{I}_\Phi(\mathbf{u})^{\frac12}\big(1 + \|\mathbf{D}\mathbf{u}\|_{3p}\big)^{\frac{2-p}{2}} \\
&\le C\,\mathcal{I}_\Phi(\mathbf{u})^{\frac12}\big(1 + C\,\|\boldsymbol{\nabla}\mathbf{D}\mathbf{u}\|_{\frac{3p}{p+1}}\big)^{\frac{2-p}{2}}, \quad \text{since } \langle \mathbf{u}, 1\rangle = 0.
\end{aligned}$$

This implies

$$\|\boldsymbol{\nabla}\mathbf{D}\mathbf{u}\|^p_{\frac{3p}{p+1}} \le C\,(\mathcal{I}_\Phi(\mathbf{u}) + 1).$$

Since $|\boldsymbol{\nabla}^2\mathbf{u}| \le 3\,|\boldsymbol{\nabla}\mathbf{D}\mathbf{u}|$ (see (24)) and $\langle \mathbf{u}, 1\rangle = 0$, we get

$$\|\mathbf{u}\|^p_{2,\frac{3p}{p+1}} \le C\,(\mathcal{I}_\Phi(\mathbf{u}) + 1).$$

Analogously we can use lemma 7 to get

$$\begin{aligned}
\|\partial_t\mathbf{D}\mathbf{u}\|_{\frac{3p}{p+1}} &\le C\,\mathcal{J}_\Phi(\mathbf{u})^{\frac12}\,\|(\widetilde{D}\mathbf{u})^{\frac{2-p}{2}}\|_{\frac{6p}{2-p}} \\
&\le C\,\mathcal{J}_\Phi(\mathbf{u})^{\frac12}\big(1 + C\,\|\boldsymbol{\nabla}\mathbf{D}\mathbf{u}\|_{\frac{3p}{p+1}}\big)^{\frac{2-p}{2}}, \qquad \text{since } \langle \mathbf{u}, 1\rangle = 0 \\
&\overset{(33)}{\le} C\,\mathcal{J}_\Phi(\mathbf{u})^{\frac12}\big(1 + C\,(\mathcal{I}_\Phi(\mathbf{u}) + 1)^{\frac1p}\big)^{\frac{2-p}{2}} \\
&\le C\,\mathcal{J}_\Phi(\mathbf{u})^{\frac12}\big(1 + \mathcal{I}_\Phi(\mathbf{u})\big)^{\frac{2-p}{2p}}.
\end{aligned}$$

Now $\langle \mathbf{u}, 1\rangle = 0$ and Korn's inequality imply

$$\|\partial_t\mathbf{u}\|_{1,\frac{3p}{p+1}} \le C\,\|\partial_t\mathbf{D}\mathbf{u}\|_{\frac{3p}{p+1}} \le C\,\mathcal{J}_\Phi(\mathbf{u})^{\frac12}\big(1 + \mathcal{I}_\Phi(\mathbf{u})\big)^{\frac{2-p}{2p}},$$

which proves (34). The rest is an application of Young's inequality. □

4. Galerkin Approximation - The Case $\frac32 < p \le 2$

Let us introduce the spaces, which we will need later. As before let Ω be the three dimensional torus. By $(L^q(\Omega), \|\cdot\|_q)$, resp. $(W^{k,q}(\Omega), \|\cdot\|_{k,q})$, we denote the classical Lebesgue and Sobolev spaces. For a Banach space X we denote by $L^q([0,T], X)$ the Bochner space with q–integrability and values in X. We will also make use of the space $C(I, X)$ of continuous functions with values in X.

Due to the constraint $\operatorname{div}\mathbf{u} = 0$ of incompressibility we introduce spaces of divergence-free functions. For $1 < q < \infty$ and $k \in \mathbb{N}_0$ let

$$\begin{aligned}
\mathcal{V} &:= \{\varphi \in C^\infty_{\text{per}}(\Omega)\,:\, \operatorname{div}\varphi = 0,\ \langle \varphi, 1\rangle = 0\}, \\
L^q_{\text{div}}(\Omega) &:= \overline{(\mathcal{V}, \|\cdot\|_q)}, \\
W^{k,q}_{\text{div}}(\Omega) &:= \overline{(\mathcal{V}, \|\cdot\|_{k,q})},
\end{aligned}$$

where $\overline{(\mathcal{V}, \|\cdot\|)}$ is the closure of $\mathcal{V}$ with respect to the given norm and $\langle f, g\rangle \equiv \int_\Omega f\, g\, dx$ is the scalar product with respect to the space.

In order to prove theorem 17 we will use a Galerkin approximation and derive some a priori estimates for the approximative solutions $\mathbf{u}^N$. In section 6 we will pass to the limit $N \to \infty$ showing the existence of a desired solution $\mathbf{u}$ of system (1).

Let $\{\boldsymbol{\omega}^r\}$ denote the set consisting of the eigenvectors of the Stokes operator denoted by A. Let λ_r be the corresponding eigenvalues and $X_N = \operatorname{span}\{\boldsymbol{\omega}^1, \dots, \boldsymbol{\omega}^N\}$. Note that $\langle \boldsymbol{\omega}^r, 1\rangle = 0$. Define $P^N\mathbf{u} = \sum_{r=1}^N \langle \mathbf{u}, \boldsymbol{\omega}^r\rangle\, \boldsymbol{\omega}^r$. Then

$$\lambda_r \langle \boldsymbol{\omega}^r, \mathbf{u}^N\rangle = \langle A\, \boldsymbol{\omega}^r, \mathbf{u}^N\rangle = \langle \boldsymbol{\nabla}\boldsymbol{\omega}^r, \boldsymbol{\nabla}\mathbf{u}^N\rangle \tag{36}$$

and $P^N : W^{s,2} \to (X_N, \|\cdot\|_{s,2})$ are uniformly continuous for all $0 \le s \le 3$. (See [16, 24] for a proof.)

Let us define $\mathbf{u}^N(t,x) = \sum_{r=1}^N c_r^N(t)\boldsymbol{\omega}^r(x)$ and $\mathbf{f}^N = P^N\mathbf{f}$, where the coefficients $c_r^N(t)$ solve the Galerkin system (for all $1 \le r \le N$)

$$\begin{aligned} \big\langle \partial_t \mathbf{u}^N, \boldsymbol{\omega}^r\big\rangle + \big\langle \mathbf{S}(\mathbf{D}\mathbf{u}^N), \mathbf{D}\boldsymbol{\omega}^r\big\rangle + \big\langle (\mathbf{u}^N \cdot \boldsymbol{\nabla})\mathbf{u}^N, \boldsymbol{\omega}^r\big\rangle &= \langle \mathbf{f}^N, \boldsymbol{\omega}^r\rangle, \\ \mathbf{u}^N(0) &= P^N\mathbf{u}_0. \end{aligned} \tag{37}$$

We will show that this Galerkin approximation has a short time solution $\mathbf{u}^N$, which will converge to a solution of (1). Since the matrix $\langle \boldsymbol{\omega}_j, \boldsymbol{\omega}_k\rangle$ with $j, k = 1, \dots, N$ is positive definite, the Galerkin system (37) can be rewritten as a system of ordinary differential equations. This in turn fulfills the Carathéodory conditions and is therefore solvable locally in time, i.e. on a small time interval $I^* = [0, T^*)$. In theorem 17 we assume that $\mathbf{f} \in L^\infty(I, W^{1,2}(\Omega))$ and $\partial_t\mathbf{f} \in L^2(I, L^2(\Omega))$ and thus $\mathbf{f}^N = P^N\mathbf{f} \in L^\infty(I, W^{1,2}(\Omega))$ and $\partial_t\mathbf{f}^N = P^N(\partial_t\mathbf{f}) \in L^2(I, L^2(\Omega))$. This implies $c_r^N, \partial_t c_r^N, \partial_t^2 c_r^N \in L^2(I^*)$. Thus $\mathbf{u}^N, \partial_t\mathbf{u}^N, \partial_t^2\mathbf{u}^N \in L^2(I^*, X_N)$. (Note that the norms may depend on N). To ensure solvability for large times at least for this finite dimensional problem we have to establish a first a priori estimate.

Since $\mathbf{u}^N, \partial_t\mathbf{u}^N, \partial_t^2\mathbf{u}^N \in L^2(I^*, X_N)$, we can test the Galerkin system (37) with $\mathbf{u}^N$ and get

$$\tfrac{1}{2} d_t \|\mathbf{u}^N\|_2^2 + \langle \mathbf{S}(\mathbf{D}\mathbf{u}^N), \mathbf{D}\mathbf{u}^N\rangle = \langle \mathbf{f}^N, \mathbf{u}^N\rangle.$$

Note that $\langle (\mathbf{u}^N \cdot \boldsymbol{\nabla})\mathbf{u}^N, \mathbf{u}^N\rangle = 0$ due to $\operatorname{div}\mathbf{u}^N = 0$. The coercivity of $\mathbf{S}$ (see (14)), the continuity of P^N on $L^2(\Omega)$, and Gronwall's inequality imply

$$\tfrac{1}{2}\max_{[0,T^*]} \|\mathbf{u}^N\|_2^2 + \int_0^{T^*} \|\mathbf{D}\mathbf{u}^N\|_p^p\, dt \le C\int_0^T \|\mathbf{f}\|_2^2\, dt + \tfrac{1}{2}\|\mathbf{u}_0\|_2^2 \le C(T, \mathbf{f}, \mathbf{u}_0).$$

This implies

$$\|c_r^N\|_{L^\infty(I^*)} \le C(T, \mathbf{f}, \mathbf{u}_0), \qquad 1 \le r \le N.$$

As a consequence we can iterate Carathéodory's theorem to push the solvability of the Galerkin system (37) up to any fixed time interval $I = [0, T)$. Hence, independently of N

$$\|\mathbf{u}\|_{L^\infty(I, L^2(\Omega))} + \|\mathbf{u}\|_{L^p(I, W^{1,p}(\Omega))} \le C, \tag{38}$$

where we have used Korn's inequality.

We got the first a priori estimate by using $\mathbf{u}^N$ as a test function. To derive our second a priori estimate we want to use $A\mathbf{u}^N$ as a test function. The special choice of base functions $\boldsymbol{\omega}^r$ ensures that we do not leave X_N, the space of admissible test functions. More explicitly we multiply the r-th equation of the Galerkin system (37) by $\lambda_r c_r^N$ and use (36) to obtain

$$\langle \partial_t \mathbf{u}^N, A\mathbf{u}^N \rangle) - \big\langle \mathbf{S}(\mathbf{D}\mathbf{u}^N), \mathbf{D}A\mathbf{u}^N \big\rangle - \big\langle (\mathbf{u}^N \cdot \boldsymbol{\nabla})\mathbf{u}^N, A\mathbf{u}^N \big\rangle = \langle \boldsymbol{\nabla}\mathbf{f}^N, \boldsymbol{\nabla}\mathbf{u}^N \rangle.$$

Due to the periodicity we have $A = -\Delta$, so

$$\tfrac{1}{2} d_t \|\boldsymbol{\nabla}\mathbf{u}^N\|_2^2 - \langle \mathbf{S}(\mathbf{D}\mathbf{u}^N), \mathbf{D}\Delta\mathbf{u}^N \rangle - \big\langle (\mathbf{u}^N \cdot \boldsymbol{\nabla})\mathbf{u}^N, \Delta\mathbf{u}^N \big\rangle = \langle \boldsymbol{\nabla}\mathbf{f}^N, \boldsymbol{\nabla}\mathbf{u}^N \rangle.$$

Using partial integration and the properties of $\mathbf{S}$ we deduce (cf. [16]):

$$\big\langle (\mathbf{u}^N \cdot \boldsymbol{\nabla})\mathbf{u}^N, -\Delta\mathbf{u}^N \big\rangle = \sum_{ijk} \int_\Omega \partial_k u_i^N \partial_i u_j^N \partial_k u_j^N \, dx, \tag{39}$$

$$\begin{aligned} \langle \mathbf{S}(\mathbf{D}\mathbf{u}^N), -\mathbf{D}\Delta\mathbf{u}^N \rangle &= \sum_{ijklm} \int_\Omega (\partial_{ij}\partial_{kl}\Phi)(\mathbf{D}\mathbf{u}^N) \partial_m D_{ij}\mathbf{u}^N \, \partial_m D_{kl}\mathbf{u}^N \, dx \\ &\overset{(14)}{\geq} C\,\mathcal{I}_\Phi(\mathbf{u}^N). \end{aligned} \tag{40}$$

Thus we have

$$\tfrac{1}{2} d_t \|\boldsymbol{\nabla}\mathbf{u}^N\|_2^2 + c\,\mathcal{I}_\Phi(\mathbf{u}^N) \leq \|\boldsymbol{\nabla}\mathbf{u}^N\|_3^3 + \big|\langle \boldsymbol{\nabla}\mathbf{f}^N, \boldsymbol{\nabla}\mathbf{u}^N \rangle\big|, \tag{41}$$

If $p > \frac{11}{5}$ one can show that $\|\boldsymbol{\nabla}\mathbf{u}^N\|_3^3 \leq C_\varepsilon \|\boldsymbol{\nabla}\mathbf{u}^N\|_p^p \|\boldsymbol{\nabla}\mathbf{u}^N\|_2^2 + \varepsilon\mathcal{I}_\Phi(\mathbf{u}^N)$ (see [16]), which enables us to apply Gronwall's inequality after absorbing $\varepsilon\mathcal{I}_\Phi(\mathbf{u}^N)$ on the left hand side. This would gives us a global estimate. If $p > \frac{5}{3}$ we can show that $\|\boldsymbol{\nabla}\mathbf{u}^N\|_3^3 \leq C_\varepsilon \|\boldsymbol{\nabla}\mathbf{u}^N\|_p^p \|\boldsymbol{\nabla}\mathbf{u}^N\|_2^R + \varepsilon\,\mathcal{I}_\Phi(\mathbf{u}^N)$ for some constant $1 < R < \infty$ and thereafter absorb $\varepsilon\mathcal{I}_\Phi(\mathbf{u}^N)$ on the left hand side and apply a local version of Gronwall's inequality (see lemma 24). Instead of using Gronwall's inequality it is also possible to divide the inequality by $(1 + \|\boldsymbol{\nabla}\mathbf{u}^N\|_2)^R$ as was done in [16] and derive the same local estimates. This in turn implies enough regularity for $\mathbf{u}^N$ to justify all the later testing of the Galerkin system with "$\partial_t \mathbf{u}^N$" and "$\partial_t \mathbf{u}^N \partial_t$". Nevertheless we will not make use of these facts, since we are also interested in smaller values of p than $\frac{5}{3}$. What we do is, we test immediately with "$\partial_t \mathbf{u}^N \partial_t$" to get in addition to (41) another estimate. Then we will use the resulting two estimates at the same time to derive quite strong a priori estimates for $\mathbf{u}^N$ for values up to $p > \frac{3}{2}$ in this section and up to $p > \frac{7}{5}$ in the next section.

Let us take the time derivative of the Galerkin system (37):

$$\big\langle \partial_t^2 \mathbf{u}^N, \boldsymbol{\omega}^r \big\rangle + \big\langle \partial_t \mathbf{S}(\mathbf{D}\mathbf{u}^N), \mathbf{D}\boldsymbol{\omega}^r \big\rangle + \langle \partial_t((\mathbf{u}^N \cdot \boldsymbol{\nabla})\mathbf{u}^N), \boldsymbol{\omega}^r \rangle = \langle \partial_t \mathbf{f}^N, \boldsymbol{\omega}^r \rangle,$$

for $1 \leq r \leq N$. Since $\mathbf{u}^N \in W^{2,2}(I, X_n)$, this makes sense and we can even test with $\partial_t \mathbf{u}^N \in W^{1,2}(I, X_n)$:

$$\begin{aligned} \tfrac{1}{2} d_t \|\partial_t \mathbf{u}^N\|_2^2 &+ \big\langle \partial_t(\mathbf{S}(\mathbf{D}\mathbf{u}^N)), \partial_t \mathbf{D}\mathbf{u}^N \big\rangle \\ &+ \langle \partial_t((\mathbf{u}^N \cdot \boldsymbol{\nabla})\mathbf{u}^N), \partial_t \mathbf{u}^N \rangle = \langle \partial_t \mathbf{f}^N, \partial_t \mathbf{u}^N \rangle. \end{aligned}$$

Once again the second term on the left-hand side has a sign, namely

$$\begin{aligned} \big\langle \partial_t(\mathbf{S}(\mathbf{D}\mathbf{u}^N)), \partial_t \mathbf{D}\mathbf{u}^N \big\rangle &= \sum_{ikl} \big\langle (\partial_{ij}\partial_{kl}\Phi)(\mathbf{D}\mathbf{u}^N) \partial_t D_{ij}\mathbf{u}^N, \partial_t D_{kl}\mathbf{u}^N \big\rangle \\ &\overset{(14)}{\geq} c\,\mathcal{J}_\Phi(\mathbf{u}^N). \end{aligned}$$

This yields

$$d_t\|\partial_t \mathbf{u}^N\|_2^2 + c\,\mathcal{J}_\Phi(\mathbf{u}) \le \big|\langle \partial_t((\mathbf{u}^N\cdot\boldsymbol{\nabla})\mathbf{u}^N), \partial_t\mathbf{u}^N\rangle\big| + |\langle \partial_t \mathbf{f}^N, \partial_t\mathbf{u}^N\rangle|. \tag{42}$$

Recall that

$$\tfrac{1}{2}d_t\|\boldsymbol{\nabla}\mathbf{u}^N\|_2^2 + c\,\mathcal{I}_\Phi(\mathbf{u}^N) \le \|\boldsymbol{\nabla}\mathbf{u}^N\|_3^3 + \big|\langle \boldsymbol{\nabla}\mathbf{f}^N, \boldsymbol{\nabla}\mathbf{u}^N\rangle\big|. \tag{43}$$

By a first view we have gained nothing. We have to control one more bad term, namely $|\langle \partial_t((\mathbf{u}^N\cdot\boldsymbol{\nabla})\mathbf{u}^N), \partial_t\mathbf{u}^N\rangle|$, but we only got more information about the time derivative of $\mathbf{u}^N$. But the critical term $\|\boldsymbol{\nabla}\mathbf{u}^N\|_3^3$, which gave the lower bound for p has no time derivatives. The next lemma shows that $\mathcal{J}_\Phi(\mathbf{u}^N)$ reveals indeed more information.

Lemma 11. *Let $1 < q < \infty$, then for almost every $t \in I$*

$$\begin{aligned} d_t\big(\|\widetilde{D}\mathbf{u}(t)\|_q^q\big) &\le q\,C\,\mathcal{J}_\Phi(t,\mathbf{u})^{\frac12}\big(\|\widetilde{D}\mathbf{u}(t)\|_{2q-p}^{2q-p}\big)^{\frac12} \\ &\le \varepsilon\mathcal{J}_\Phi(t,\mathbf{u}) + C_\varepsilon\|\widetilde{D}\mathbf{u}(t)\|_{2q-p}^{2q-p}, \end{aligned} \tag{44}$$

where $\widetilde{D}\mathbf{u} \equiv (1+|\mathbf{D}\mathbf{u}|^2)^{\frac12}$.

Proof. Note that

$$\partial_t\big((\widetilde{D}\mathbf{u})^q\big) = q(\widetilde{D}\mathbf{u})^{q-2}(D_{jk}\mathbf{u})\,(\partial_t D_{jk}\mathbf{u}).$$

Hence

$$\begin{aligned} d_t(\|\widetilde{D}\mathbf{u}\|_q^q) &\le q\int_\Omega (\widetilde{D}\mathbf{u})^{q-1}|\partial_t\mathbf{D}\mathbf{u}|\,dx \\ &= q\int_\Omega (\widetilde{D}\mathbf{u})^{\frac{p-2}{2}}|\partial_t\mathbf{D}\mathbf{u}|\,(\widetilde{D}\mathbf{u})^{q-\frac{p}{2}}\,dx \\ &\le q\,C\,\mathcal{J}_\Phi(\mathbf{u})^{\frac12}\big(\|\widetilde{D}\mathbf{u}\|_{2q-p}^{2q-p}\big)^{\frac12}, \end{aligned}$$

where $\|\widetilde{D}\mathbf{u}\|_{2q-p}^{2q-p} \equiv \int_\Omega(\widetilde{D}\mathbf{u})^{2q-p}\,dx$ even if $2q-p<1$. The rest is an implication of Young's inequality. □

This lemma enables us to produce $d_t(\|\widetilde{D}\mathbf{u}^N\|_q^q)$ on the left-hand side of (42) if we add $C\,\|\widetilde{D}\mathbf{u}^N\|_{2q-p}^{2q-p}$ to the right-hand side. We have three critical terms to control:

$$\|\boldsymbol{\nabla}\mathbf{u}^N\|_3^3, \qquad |\langle \partial_t((\mathbf{u}^N\cdot\boldsymbol{\nabla})\mathbf{u}^N), \partial_t\mathbf{u}^N\rangle|, \qquad C\,\|\widetilde{D}\mathbf{u}^N\|_{2q-p}^{2q-p}.$$

The first and the second one will be easier to estimate for large q, but the third one for small q. The problem now is to find the optimal choice for q. We start by examining which values of q are needed for the first and the second term. In the view of lemma 24, we will be able to control arbitrary powers of $\|\widetilde{D}\mathbf{u}^N\|_q^q$ and $\|\partial_t\mathbf{u}^N\|_2^2$. Note that we will skip the index N of $\mathbf{u}^N$ to keep the notations simple.

Lemma 12. *Let $q > \frac{9-3p}{2}$, then there exists a constant $R_1 = R_1(p) > q$, such that*

$$\|\boldsymbol{\nabla}\mathbf{u}\|_3^3 \le C_\varepsilon\|\widetilde{D}\mathbf{u}\|_q^{R_1} + \varepsilon\mathcal{I}_\Phi(\mathbf{u}) + \varepsilon.$$

Proof. If $q \ge 3$, then there is nothing to prove, so assume $q < 3$. We can interpolate $L^3(\Omega) = [L^q(\Omega), L^{3p}(\Omega)]_\theta$ with

$$\frac13 = \frac{(1-\theta)}{q} + \frac{\theta}{3p} \quad\Leftrightarrow\quad \theta = \frac{(3-q)p}{3p-q}, \quad 1-\theta = \frac{q(p-1)}{3p-q}.$$

Therefore

$$\|\nabla\mathbf{u}\|_3^3 \le \|\nabla\mathbf{u}\|_q^{3(1-\theta)}\|\nabla\mathbf{u}\|_{3p}^{3\theta}.$$

If $3\theta < p$, there exists an $\delta > 1$ such that

$$\begin{aligned}\|\nabla\mathbf{u}\|_3^3 &\le C_\varepsilon\|\nabla\mathbf{u}\|_q^{3(1-\theta)\delta'} + \varepsilon\|\nabla\mathbf{u}\|_{3p}^p\\ &\le C_\varepsilon\|\nabla\mathbf{u}\|_q^{3(1-\theta)\delta'} + C\,\varepsilon\|\nabla\mathbf{u}\|_{1,\frac{3p}{p+1}}^p\\ &\le C_\varepsilon\|\nabla\mathbf{u}\|_q^{3(1-\theta)\delta'} + \varepsilon C\,(\mathcal{I}_\Phi(\mathbf{u})+1),\end{aligned}$$

where we have used lemma 10. So by Korn's inequality

$$\|\nabla\mathbf{u}\|_3^3 \le C_{\varepsilon_2}\|\widetilde{D}\mathbf{u}\|_q^{3(1-\theta)\delta'} + \varepsilon_2\mathcal{I}_\Phi(\mathbf{u}) + \varepsilon_2.$$

We still have to verify $3\theta < p$, but this is equivalent to

$$\frac{3(3-q)p}{3p-q} < p \quad\Leftrightarrow\quad \frac{9-3p}{2} < q,$$

which holds due to the assumptions on q. □

Lemma 13. *Let $q > \frac{9-3p}{2}$, then there exist constants $R_2 = R_2(p) > 2$ and $R_3 = R_3(p) > q$ such that*

$$|\langle(\partial_t\mathbf{u}\cdot\nabla)\mathbf{u},\partial_t\mathbf{u}\rangle| \le \varepsilon\mathcal{J}_\Phi(\mathbf{u}) + C_\varepsilon\big(\|\partial_t\mathbf{u}\|_2^{R_2} + \|\widetilde{D}\mathbf{u}\|_q^{R_3}+1\big).$$

Proof. Note that lemma 7 ($q \mapsto \frac{2q}{2-p+q}$) implies

$$\begin{aligned}\|\partial_t\mathbf{D}\mathbf{u}\|_{\frac{2q}{2-p+q}} &\le C\,\mathcal{J}_\Phi(\mathbf{u})^{\frac12}\|(\widetilde{D}\mathbf{u})^{\frac{2-p}{2}}\|_{\frac{2q}{2-p}}\\ &\le C\,\mathcal{J}_\Phi(\mathbf{u})^{\frac12}\|\widetilde{D}\mathbf{u}\|_q^{\frac{2-p}{2}}.\end{aligned}\tag{45}$$

Furthermore $W^{1,\frac{2q}{2-p+q}}(\Omega) \hookrightarrow L^{\frac{6q}{6-3p+q}}(\Omega)$. Since $\frac{9-3p}{2} < q$ is equivalent to $\frac{2q}{q-1} < \frac{6q}{6-3p+q}$ we can use the interpolation

$$L^{\frac{2q}{q-1}}(\Omega) = [L^2(\Omega), L^{\frac{6q}{6-3p+q}}(\Omega)]_\theta.$$

This and Korn's inequality implies

$$\begin{aligned}|\langle(\partial_t\mathbf{u}\cdot\nabla)\mathbf{u},\partial_t\mathbf{u}\rangle| &\le \|\partial_t\mathbf{u}\|_{\frac{2q}{q-1}}^2\|\nabla\mathbf{u}\|_q\\ &\le C\,\|\partial_t\mathbf{u}\|_2^{2(1-\theta)}\|\partial_t\mathbf{u}\|_{\frac{6q}{6-3p+q}}^{2\theta}\|\nabla\mathbf{u}\|_q\\ &\le C\,\|\partial_t\mathbf{u}\|_2^{2(1-\theta)}\|\partial_t\nabla\mathbf{u}\|_{\frac{2q}{2-p+q}}^{2\theta}\|\nabla\mathbf{u}\|_q\\ &\overset{(45)}{\le} C\,\|\partial_t\mathbf{u}\|_2^{2(1-\theta)}\Big(\mathcal{J}_\Phi(\mathbf{u})^{\frac12}\|\widetilde{D}\mathbf{u}\|_q^{\frac{2-p}{2}}\Big)^{2\theta}\|\nabla\mathbf{u}\|_q\\ &\le \varepsilon\mathcal{J}_\Phi(\mathbf{u}) + C_\varepsilon\big(\|\partial_t\mathbf{u}\|_2^{R_2} + \|\widetilde{D}\mathbf{u}\|_q^{R_3}+1\big).\end{aligned}$$

□

It is indeed interesting that both terms $|\langle\partial_t((\mathbf{u}^N\cdot\nabla)\mathbf{u}^N),\partial_t\mathbf{u}^N\rangle|$ and $\|\nabla\mathbf{u}^N\|_3^3$ require the same bound for q, which is $q > \frac{9-3p}{2}$. Now we have to find the upper bound for q, in order to control $\|\nabla\mathbf{u}^N\|_{2q-p}^{2q-p}$. Unfortunately this requires extensive calculations, so we will postpone this to the next section. Since the calculations for $p > \frac32$ are a lot simpler, we will finish this section by outlining how to proceed in this simpler case.

So let us assume for the rest of this section that $p > \frac{3}{2}$. Set $q := \frac{3+p}{2}$, then $2q - p \leq 3$, so

$$\|\nabla \mathbf{u}^N\|_{2q-p}^{2q-p} \leq C\,(\|\nabla \mathbf{u}^N\|_3^3 + 1).$$

That means that $\|\nabla \mathbf{u}^N\|_{2q-p}^{2q-p}$ can be controlled if $\|\nabla \mathbf{u}^N\|_3^3$ can be controlled. But the choice of q and $p > \frac{3}{2}$ ensures that $q > \frac{9-3p}{2}$. Hence by lemma 12, lemma 13, Korn's inequality, and the above calculations we get

$$\begin{aligned}
&d_t\big(\|\nabla \mathbf{u}^N\|_2^2\big) + c\,\mathcal{I}_\Phi(\mathbf{u}^N) \\
&\qquad \leq C\,\big(1 + \|\widetilde{D}\mathbf{u}^N\|_q^{R_1} + |\langle \nabla \mathbf{f}^N, \nabla \mathbf{u}^N\rangle|\big), \\
&d_t\big(\|\partial_t \mathbf{u}^N\|_2^2\big) + d_t\big(\|\widetilde{D}\mathbf{u}^N\|_q^q\big) + c\,\mathcal{J}_\Phi(\mathbf{u}^N) \\
&\qquad \leq C\,\big(1 + \|\partial_t \mathbf{u}^N\|_2^{R_2} + \|\widetilde{D}\mathbf{u}^N\|_q^{R_3} + |\langle \partial_t \mathbf{f}^N, \partial_t \mathbf{u}^N\rangle|\big).
\end{aligned}$$

The remaining terms involving $\mathbf{f}^N$ are easy to control:

$$\begin{aligned}
|\langle \nabla \mathbf{f}^N, \nabla \mathbf{u}^N\rangle| &\leq \|P^N \mathbf{f}\|_{1,2}\|\nabla \mathbf{u}^N\|_2 \leq C\,\|\mathbf{f}\|_{1,2}\|\nabla \mathbf{u}^N\|_2 \\
&\leq C\,\|\mathbf{f}\|_{1,2}^2 + C\,\|\widetilde{D}\mathbf{u}^N\|_q^2, \\
|\langle \partial_t \mathbf{f}^N, \partial_t \mathbf{u}^N\rangle| &\leq \|P^N(\partial_t \mathbf{f})\|_2\|\partial_t \mathbf{u}^N\|_2 \leq C\,\|\partial_t \mathbf{f}\|_2\|\partial_t \mathbf{u}^N\|_2 \\
&\leq C\,\|\partial_t \mathbf{f}\|_2^2 + C\,\|\partial_t \mathbf{u}^N\|_2^2.
\end{aligned}$$

Overall

$$\begin{aligned}
&d_t\big(\|\nabla \mathbf{u}^N\|_2^2\big) + d_t\big(\|\partial_t \mathbf{u}^N\|_2^2\big) + d_t\big(\|\widetilde{D}\mathbf{u}^N\|_q^q\big) + c\,\mathcal{I}_\Phi(\mathbf{u}^N) + c\,\mathcal{J}_\Phi(\mathbf{u}^N) \\
&\quad \leq C\,\big(1 + \|\widetilde{D}\mathbf{u}^N\|_q^{\max\{R_1,R_3,2\}} + \|\partial_t \mathbf{u}^N\|_2^{\max\{R_2,2\}} + \|\mathbf{f}\|_{1,2}^2 + \|\partial_t \mathbf{f}\|_2^2\big).
\end{aligned}$$

Now lemma 24 ensures that for small times, i.e. T' is small, we get boundedness of the following expressions (uniformly in N):

$$\begin{aligned}
&\|\partial_t \mathbf{u}^N\|_{L^\infty(I',L^2(\Omega))}^2, \quad \|\partial_t \mathbf{u}^N\|_{L^\infty(I',L^2(\Omega))}^2, \quad \|\nabla \mathbf{u}^N\|_{L^\infty(I',L^q(\Omega))}^q, \\
&\|\mathcal{I}_\Phi(\mathbf{u}^N)\|_{L^1(I')}, \qquad \|\mathcal{J}_\Phi(\mathbf{u}^N)\|_{L^1(I')},
\end{aligned}$$

where $I' = [0, T']$. Later in section 6 will see that these a priori estimates are sufficient to pass to the limit $N \to \infty$ to get a solution $\mathbf{u}$ of our original problem (1). But beforehand we will show in the next section how to derive similar a priori estimates in the more general case $\frac{7}{5} < p \leq 2$.

5. The case $p > \frac{7}{5}$

If p is smaller than $\frac{3}{2}$, we have to do more subtle calculations. We cannot just add (43) and (42) in order to get control of $\|\widetilde{D}\mathbf{u}^N\|_{2q-p}^{2q-p}$. Recall that we need $q > \frac{9-3p}{2}$ in order to control the terms $\|\nabla \mathbf{u}^N\|_3^3$ and $|\langle(\partial_t \mathbf{u} \cdot \nabla)\mathbf{u}, \partial_t \mathbf{u}\rangle|$. But this implies $2q - p > 3$. So $\|\widetilde{D}\mathbf{u}^N\|_{2q-p}^{2q-p}$ is worse than $\|\nabla \mathbf{u}^N\|_3^3$. Since $\|\widetilde{D}\mathbf{u}^N\|_{2q-p}^{2q-p}$ grows with respect to q a lot faster than $\|\widetilde{D}\mathbf{u}^N\|_q^q$, the term $\|\widetilde{D}\mathbf{u}^N\|_{2q-p}^{2q-p}$ requires a preferably small choice of q. But since we cannot control $\|\nabla \mathbf{u}^N\|_3^3$ for $q = \frac{9-3p}{2}$, we certainly cannot control the worse term $\|\widetilde{D}\mathbf{u}^N\|_{2q-p}^{2q-p}$ for $q = \frac{9-3p}{2}$ and thus for no $q \geq \frac{9-3p}{2}$. Hence we must proceed in a different way.

The central idea is that we have not made use of the term $d_t\|\nabla \mathbf{u}^N\|_2^2$. Since it contains less information than $d_t\|\widetilde{D}\mathbf{u}^N\|_q^q$, there is no need to extract information out of it. So we try to transfer $d_t\|\nabla \mathbf{u}^N\|_2^2$ in its original form $\langle \partial_t \mathbf{u}^N, -\Delta \mathbf{u}^N\rangle$ to the right-hand side of (43). This gives

$$\mathcal{I}_\Phi(\mathbf{u}^N) \le C\left(\|\nabla \mathbf{u}^N\|_3^3 + |\langle \nabla \mathbf{f}^N, \nabla \mathbf{u}^N\rangle| + |\langle \partial_t \mathbf{u}^N, -\Delta \mathbf{u}^N\rangle|\right), \tag{46}$$

The disadvantage is that we have to control one extra term, but the advantage is that we can raise this inequality to the r-th power. This gives, as long as we can control the right-hand side, information on $\mathcal{I}_\Phi(\mathbf{u}^N)^r$, which can be used to control $\|\widetilde{D}\mathbf{u}\|_{2q-p}^{2q-p}$ for higher values of q.

Before we calculate the maximal allowed r and the resulting q we will reduce (46) to a more suitable form: Lemma 12 implies that for $q > \frac{9-3p}{2}$ there holds

$$\mathcal{I}_\Phi(\mathbf{u}^N) \le C\left(1 + \|\widetilde{D}\mathbf{u}^N\|_q^{R_1} + |\langle \nabla \mathbf{f}^N, \nabla \mathbf{u}^N\rangle| + |\langle \partial_t \mathbf{u}^N, -\Delta \mathbf{u}^N\rangle|\right).$$

Since

$$\|\mathbf{f}^N\|_{L^\infty(I,W^{1,2}(\Omega))} = \|P^N \mathbf{f}\|_{L^\infty(I,W^{1,2}(\Omega))} \le C\,\|\mathbf{f}\|_{L^\infty(I,W^{1,2}(\Omega))} \le C,$$

this reduces to

$$\mathcal{I}_\Phi(\mathbf{u}^N) \le C\left(1 + \|\widetilde{D}\mathbf{u}^N\|_q^{R_1} + |\langle \partial_t \mathbf{u}^N, -\Delta \mathbf{u}^N\rangle|\right). \tag{47}$$

Since we can control arbitrary powers of $\|\widetilde{D}\mathbf{u}^N\|_q$ by the local Gronwall's lemma 24, we see that the convective and the force terms do not raise difficulties for $q > \frac{9-3p}{2}$, even if we raise the inequality to the r-th power. The following lemma gives control of the remaining term $|\langle \partial_t \mathbf{u}^N, -\Delta \mathbf{u}^N\rangle|$.

Lemma 14. *For $1 < p \le 2$ there holds*

$$|\langle \partial_t \mathbf{u}, \Delta \mathbf{u}\rangle| \le C\,\|\partial_t \mathbf{u}\|_2^{\frac{4(p-1)}{3p-2}}\,\mathcal{J}_\Phi(\mathbf{u})^{\frac{2-p}{2(3p-2)}}\,(\mathcal{I}_\Phi(\mathbf{u})+1)^{\frac{p+2}{2(3p-2)}}.$$

Proof. With the help of lemma 10 we conclude

$$\begin{aligned}
|\langle \partial_t \mathbf{u}, \Delta \mathbf{u}\rangle| &\le \|\partial_t \mathbf{u}\|_{\frac{3p}{2p-1}}\,\|\mathbf{u}\|_{2,\frac{3p}{p+1}}\\
&\le C\,\|\partial_t \mathbf{u}\|_{\frac{3p}{2p-1}}\,(\mathcal{I}_\Phi(\mathbf{u})+1)^{\frac{1}{p}}\\
&\le \|\partial_t \mathbf{u}\|_2^{1-\theta}\|\partial_t \mathbf{u}\|_{1,\frac{3p}{1+p}}^{\theta}\,(\mathcal{I}_\Phi(\mathbf{u})+1)^{\frac{1}{p}}\\
&\le \|\partial_t \mathbf{u}\|_2^{1-\theta}\big(\mathcal{J}_\Phi(\mathbf{u})^{\frac12}(\mathcal{I}_\Phi(\mathbf{u})+1)^{\frac{2-p}{2p}}\big)^{\theta}(\mathcal{I}_\Phi(\mathbf{u})+1)^{\frac{1}{p}}
\end{aligned}$$

with

$$\frac{2p-1}{3p} = \frac{1-\theta}{2} + \frac{\theta}{3p}.$$

Therefore $\theta = \frac{2-p}{3p-2}$ and $1-\theta = \frac{4p-4}{3p-2}$ and $\frac{2-p}{2p}\cdot\theta + \frac{1}{p} = \frac{p+2}{2(3p-2)}$. This proves the lemma. □

This lemma and (47) imply

$$\begin{aligned}
\mathcal{I}_\Phi(\mathbf{u}^N) \le C\big(1 &+ \|\widetilde{D}\mathbf{u}^N\|_q^{R_1}\\
&+ \|\partial_t \mathbf{u}^N\|_2^{\frac{4(p-1)}{3p-2}}\,\mathcal{J}_\Phi(\mathbf{u}^N)^{\frac{2-p}{2(3p-2)}}\,(\mathcal{I}_\Phi(\mathbf{u}^N)+1)^{\frac{p+2}{2(3p-2)}}\big).
\end{aligned}$$

Thus by Young's inequality for $p > \frac{6}{5}$

$$\mathcal{I}_\Phi(\mathbf{u}^N) \le C\left(1 + \|\widetilde{D}\mathbf{u}^N\|_q^{R_1} + \|\partial_t \mathbf{u}^N\|_2^{\frac{8(p-1)}{5p-6}}\,\mathcal{J}_\Phi(\mathbf{u}^N)^{\frac{2-p}{5p-6}}\right). \tag{48}$$

We are finally at the point where we can raise the inequality to the r-th power

$$\mathcal{I}_\Phi(\mathbf{u}^N)^r \le C(r)\left(1 + \|\widetilde{D}\mathbf{u}^N\|_q^{rR_1} + \|\partial_t\mathbf{u}^N\|_2^{r\frac{8(p-1)}{5p-6}}\mathcal{J}_\Phi(\mathbf{u}^N)^{r\frac{2-p}{5p-6}}\right).$$

As long as $r < \frac{5p-6}{2-p}$, the last term can be broken up into a large power of $\|\partial_t\mathbf{u}^N\|_2^2$ and $\mathcal{J}_\Phi(\mathbf{u}^N)$. We summarize

Lemma 15. *Let $\frac{4}{3} < p \le 2$, $q > \frac{9-3p}{2}$, and $1 \le r < \frac{5p-6}{2-p}$, then there exist constants $R_4 = R_4(p)$, $R_5 = R_5(p)$, such that*

$$\mathcal{I}_\Phi(\mathbf{u}^N)^r \le C_\varepsilon\left(1 + \|\widetilde{D}\mathbf{u}^N\|_q^{R_4} + \|\partial_t\mathbf{u}^N\|_2^{R_5}\right) + \varepsilon\mathcal{J}_\Phi(\mathbf{u}^N).$$

As in the case $p > \frac{3}{2}$ we calculate from (42) by using lemma 13 for $q > \frac{9-3p}{2}$

$$d_t\left(\|\partial_t\mathbf{u}^N\|_2^2\right) + c\,\mathcal{J}_\Phi(\mathbf{u}^N) \le C\left(1 + \|\partial_t\mathbf{u}^N\|_2^{R_2} + \|\widetilde{D}\mathbf{u}^N\|_q^{R_3} + \|\partial_t\mathbf{f}\|_2^2\right),$$

where we have used $\|\partial_t\mathbf{f}^N\|_2 = \|P^N(\partial_t\mathbf{f})\|_2 \le C\,\|\partial_t\mathbf{f}\|_2$. Hence by lemma 11 and lemma 15 for $q > \frac{9-3p}{2}$ and $r < \frac{5p-6}{2-p}$

$$\begin{aligned} &\mathcal{I}_\Phi(\mathbf{u}^N)^r \le C_\varepsilon\left(1 + \|\widetilde{D}\mathbf{u}^N\|_q^{R_4} + \|\partial_t\mathbf{u}^N\|_2^{R_5}\right) + \varepsilon\mathcal{J}_\Phi(\mathbf{u}^N), \\ &d_t\left(\|\partial_t\mathbf{u}^N\|_2^2\right) + d_t\left(\|\widetilde{D}\mathbf{u}^N\|_q^q\right) + c\,\mathcal{J}_\Phi(\mathbf{u}^N) \\ &\qquad \le C\left(1 + \|\partial_t\mathbf{u}^N\|_2^{R_2} + \|\widetilde{D}\mathbf{u}^N\|_q^{R_3} + \|\widetilde{D}\mathbf{u}^N\|_{2q-p}^{2q-p} + \|\partial_t\mathbf{f}\|_2^2\right). \end{aligned} \tag{49}$$

Different from the case $p > \frac{3}{2}$ we can use $\mathcal{I}_\Phi(\mathbf{u}^N)^r$ to control $\|\widetilde{D}\mathbf{u}^N\|_{2q-p}^{2q-p}$.

Lemma 16. *Let $1 < p \le 2$, $p < q < \min\{\frac{3p(r+1)}{3+r}, 2p\}$ and $r \ge 1$, then there exists a constant $R_6 > 1$, such that*

$$\|\widetilde{D}\mathbf{u}\|_{2q-p}^{2q-p} \le C_\varepsilon\|\widetilde{D}\mathbf{u}\|_q^{R_6} + \varepsilon\left(\mathcal{I}_\Phi(\mathbf{u})^r + 1\right).$$

Proof. From the assumptions we know that $p < q < 2p$, which implies $q < 2q - p < 3p$. Hence we can interpolate $L^{2q-p}(\Omega) = [L^q(\Omega), L^{3p}(\Omega)]_\theta$ with $\theta = \frac{3p(q-p)}{(2q-p)(3p-q)}$ and $1 - \theta = \frac{2q(2p-q)}{(2q-p)(3p-q)}$. Now we can estimate

$$\begin{aligned} \|\widetilde{D}\mathbf{u}\|_{2q-p}^{2q-p} &\le \|\widetilde{D}\mathbf{u}\|_q^{(1-\theta)(2q-p)}\|\widetilde{D}\mathbf{u}\|_{3p}^{\theta(2q-p)} \\ &\overset{(33)}{\le} \|\widetilde{D}\mathbf{u}\|_q^{(1-\theta)(2q-p)}\left(\mathcal{I}_\Phi(\mathbf{u}) + 1\right)^{\frac{\theta(2q-p)}{p}}. \end{aligned}$$

If $\frac{\theta(2q-p)}{p} < r$, then we can use Young's inequality with $\delta := \frac{pr}{\theta(2q-p)} > 1$ to obtain the desired result

$$\|\widetilde{D}\mathbf{u}\|_{2q-p}^{2q-p} \le C_\varepsilon\|\widetilde{D}\mathbf{u}\|_q^{(1-\theta)(2q-p)\delta'} + \varepsilon\left(\mathcal{I}_\Phi(\mathbf{u})^r + 1\right).$$

Still we have to verify the condition $\frac{\theta(2q-p)}{p} < r$. For this note that

$$\frac{\theta(2q-p)}{p} < r \quad \Leftrightarrow \quad \frac{3(q-p)}{3p-q} < r \quad \Leftrightarrow \quad q < \frac{3p(r+1)}{3+r},$$

which holds due to the assumptions on q. □

This lemma and (49) imply

$$\begin{aligned} &d_t\left(\|\partial_t\mathbf{u}^N\|_2^2\right) + d_t\left(\|\widetilde{D}\mathbf{u}^N\|_q^q\right) + c\,\mathcal{J}_\Phi(\mathbf{u}^N) + c\,\mathcal{I}_\Phi(\mathbf{u}^N)^r \\ &\qquad \le C\left(1 + \|\partial_t\mathbf{u}^N\|_2^{\max\{R_2,R_5\}} + \|\widetilde{D}\mathbf{u}^N\|_q^{\max\{R_3,R_4,R_6\}} + \|\partial_t\mathbf{f}\|_2^2\right) \end{aligned} \tag{50}$$

as long as

$$\max\left\{p, \tfrac{9-3p}{2}\right\} < q < \min\left\{\tfrac{3p(r+1)}{3+r}, 2p\right\} \quad \text{and} \quad r < \tfrac{5p-6}{2-p}. \tag{51}$$

This is the crucial estimate, which will on the one hand provide us with the desired a priori estimate and on the other hand restrict the value of admissible p. It is therefore of importance to determine the exact range of p for which we can find suitable q, r which fulfill (51). We will do so now: Since $\frac{3p(r+1)}{3+r}$ is increasing in r we can always find a suitable r if and only if p, q fulfill

$$\max\left\{p, \tfrac{9-3p}{2}\right\} < q < \min\left\{6(p-1), 2p\right\}.$$

The existence of a suitable q is in turn equivalent to $p > \frac{7}{5}$. Hence we have shown that for all $p > \frac{7}{5}$ we can find suitable r and q, such that (51) is valid, e.g $q = \frac{12}{5}$ and $r = \frac{5}{3}$. Before we can apply lemma 24 to ensure uniform a priori estimates for $\mathbf{u}^N$, we have to take a look at the initial data, namely $\|(\nabla \mathbf{u}^N)(0)\|_q$ and $\|(\partial_t \mathbf{u}^N)(0)\|_2$. The first one is easily bounded by

$$\begin{aligned} \|(\nabla \mathbf{u}^N)(0)\|_q &= \|\nabla P^N \mathbf{u}_0\|_q \le C\, \|P^N \mathbf{u}_0\|_{1,2p} \\ &\le C\, \|P^N \mathbf{u}_0\|_{2,2} \le C\, \|\mathbf{u}_0\|_{2,2} \le C. \end{aligned}$$

To bound $(\partial_t \mathbf{u}^N)(0)$ let $\boldsymbol{\varphi} \in L^2(\Omega)$ with $\|\boldsymbol{\varphi}\|_2 \le 1$, then

$$\begin{aligned} |\langle \partial_t \mathbf{u}^N, \boldsymbol{\varphi}\rangle| &= |\langle \partial_t \mathbf{u}_0^N, P^N \boldsymbol{\varphi}\rangle| \\ &= |\langle \operatorname{div} \mathbf{S}(\mathbf{D}\mathbf{u}_0^N)) + (\mathbf{u}_0^N \cdot \nabla)\mathbf{u}_0^N - \mathbf{f}^N(0), P^N \boldsymbol{\varphi}\rangle| \\ &\le \|\nabla \mathbf{S}(\mathbf{D}\mathbf{u}_0^N)\|_2 + C\, \|\mathbf{u}_0\|_{2,2}^2 + \|\mathbf{f}^N(0)\|_2 \\ &\le C\, \|(\widetilde{D}\mathbf{u}_0^N)^{p-2} \nabla \mathbf{D}\mathbf{u}_0^N\|_2 + C\, \|\mathbf{u}_0\|_{2,2}^2 + \|\mathbf{f}^N(0)\|_2 \\ &\le C\left(\|\mathbf{u}_0\|_{2,2} + \|\mathbf{u}_0\|_{2,2}^2 + \|\mathbf{f}(0)\|_2\right) \le C, \end{aligned}$$

since $p \le 2$. Here we have used that $\mathbf{f} \in L^\infty(I, W^{1,2}(\Omega))$ and that $\partial_t \mathbf{f} \in L^2(I, L^2(\Omega))$ implies $\mathbf{f} \in C(I, L^2(\Omega))$. Thus $\|(\partial_t \mathbf{u}^N)(0)\|_2 \le C$. So we can apply lemma 24 to (50) and get for small times $I' = [0, T']$

$$\|\partial_t \mathbf{u}^N\|_{L^\infty(I', L^2(\Omega))} + \|\nabla \mathbf{u}^N\|_{L^\infty(I', L^q(\Omega))} \tag{52}$$

$$+\|\mathcal{I}_\Phi(\mathbf{u}^N)\|_{L^r(I')} + \|\mathcal{J}_\Phi(\mathbf{u}^N)\|_{L^1(I')} \le C. \tag{53}$$

We use (48) to get rid of the r–dependence:

$$\|\mathcal{I}_\Phi(\mathbf{u}^N)\|_{L^{\frac{5p-6}{2-p}}(I')} \le C, \tag{54}$$

where $\frac{5p-6}{2-p} = \infty$ if $p = 2$.

In the next section we will show that these a priori estimates are by far enough to pass to the limit $N \to \infty$.

6. Passage to the Limit

Theorem 17. *Let $\frac{7}{5} < p \le 2$. Let Φ be a p–potential with induced extra stress $\mathbf{S}$, i.e. $\mathbf{S} = \nabla_{n\times n}\Phi$. Assume that*

$$\|\mathbf{f}\|_{L^\infty(I, W^{1,2}(\Omega))} + \|\partial_t \mathbf{f}\|_{L^2(I, L^2(\Omega))} + \|\mathbf{u}_0\|_{W^{2,2}_{\operatorname{div}}(\Omega)} \le K.$$

Then there exists a constant $T' = T'(K)$ with $0 < T' < T$, such that the system (1) *has a strong solution* $\mathbf{u}$ *on* $I' = [0, T']$ *satisfying*

$$\begin{aligned} \|\partial_t \mathbf{u}\|_{L^\infty(I', L^2(\Omega))} &+ \|\mathbf{u}\|_{L^\infty(I', W^{1,\frac{12}{5}}(\Omega))} \\ &+ \|\mathcal{J}_\Phi(\mathbf{u})\|_{L^1(I')} + \|\mathcal{I}_\Phi(\mathbf{u})\|_{L^{\frac{5p-6}{2-p}}(I')} \le C. \end{aligned} \tag{55}$$

Proof. In sections 4 and 5 we have proven the existence of approximative solutions $\mathbf{u}^N$, which solve (37) and satisfy

$$\begin{aligned} \|\partial_t \mathbf{u}^N\|_{L^\infty(I', L^2(\Omega))} &+ \|\nabla \mathbf{u}^N\|_{L^\infty(I', L^q(\Omega))} \\ &+ \|\mathcal{I}_\Phi(\mathbf{u}^N)\|_{L^{\frac{5p-6}{2-p}}(I')} + \|\mathcal{J}_\Phi(\mathbf{u}^N)\|_{L^1(I')} \le C. \end{aligned} \tag{56}$$

Since $q = \frac{12}{5}$ and $r = \frac{5}{3}$ was an admissible choice within the derivation of the a priori estimates, we can assume $q \ge \frac{12}{5}$. Estimate (56) especially implies $\|\mathcal{I}_\Phi(\mathbf{u}^N)\|_{L^1(I')} \le C$, so by lemma 6 we get $\|\nabla^2 \mathbf{u}^N\|_{p, I' \times \Omega}^p \le C$ and therefore $\|\mathbf{u}^N\|_{L^p(I', W^{2,p}(\Omega))} \le C$, since $\langle \mathbf{u}^N, 1 \rangle = 0$. Overall we can pick a subsequence (still denoted by $\mathbf{u}^N$) with

$$\mathbf{u}^N \rightharpoonup \mathbf{u} \qquad \text{in } L^p(I', W^{2,p}(\Omega)), \tag{57}$$

$$\mathbf{u}^N \overset{*}{\rightharpoonup} \mathbf{u} \qquad \text{in } L^\infty(I', W^{1,\frac{12}{5}}(\Omega)), \tag{58}$$

$$\partial_t \mathbf{u}^N \overset{*}{\rightharpoonup} \partial_t \mathbf{u} \qquad \text{in } L^\infty(I', L^2(\Omega)), \tag{59}$$

where we have used that the weak limit of distributions on $I \times \Omega$ is unique. Since $W^{2,p}(\Omega) \hookrightarrow\hookrightarrow W^{1,2}(\Omega)$ for $p > \frac{7}{5}$, the lemma of Aubin–Lions implies the existence of a subsequence, such that

$$\nabla \mathbf{u}^N \to \nabla \mathbf{u} \qquad \text{in } L^2(I' \times \Omega). \tag{60}$$

As a consequence we get convergence of the convective term

$$(\mathbf{u}^N \cdot \nabla)\mathbf{u}^N \to (\mathbf{u} \cdot \nabla)\mathbf{u} \qquad \text{in } L^{\frac{4}{3}}(I' \times \Omega). \tag{61}$$

Observe that

$$\begin{aligned} \|\mathbf{S}(\mathbf{D}\mathbf{u}^N)\|_{L^2(I' \times \Omega)} &\overset{(15)}{\le} C\, \|(\widetilde{D}\mathbf{u}^N)^{p-1}\|_{L^2(I' \times \Omega)} \\ &\le C\,(1 + \|\nabla \mathbf{u}^N\|_{L^2(I' \times \Omega)}) \le C. \end{aligned} \tag{62}$$

On the other hand by (60) $\mathbf{D}\mathbf{u}^N \to \mathbf{D}\mathbf{u}$ a.e. in $I' \times \Omega$, so

$$\mathbf{S}(\mathbf{D}\mathbf{u}^N) \to \mathbf{S}(\mathbf{D}\mathbf{u}) \qquad \text{a.e. in } I' \times \Omega \tag{63}$$

due to the continuity properties of $\mathbf{S}$. Now Vitali's theorem, (62), and (63) imply

$$\mathbf{S}(\mathbf{D}\mathbf{u}^N) \to \mathbf{S}(\mathbf{D}\mathbf{u}) \qquad \text{a.e. in } L^1(I' \times \Omega). \tag{64}$$

Choose $\boldsymbol{\omega}^r$ and $\varphi \in C_0^\infty(I')$, then we can conclude from (37), (59), (61), and (64) that

$$\int_{I'} \varphi\Big(\langle \partial_t \mathbf{u}, \boldsymbol{\omega}^r \rangle + \langle \mathbf{S}(\mathbf{D}\mathbf{u}), \mathbf{D}\boldsymbol{\omega}^r \rangle + \langle (\mathbf{u} \cdot \nabla)\mathbf{u}, \boldsymbol{\omega}^r \rangle\Big)\, dt = \int_{I'} \varphi \langle \mathbf{f}, \boldsymbol{\omega}^r \rangle dt.$$

Furthermore $\mathbf{u}$ fulfills

$$\|\partial_t \mathbf{u}\|_{L^2(I' \times \Omega)} + \|\mathbf{S}(\mathbf{D}\mathbf{u})\|_{L^1(I' \times \Omega)} + \|(\mathbf{u} \cdot \nabla)\mathbf{u}\|_{L^{\frac{4}{3}}(I' \times \Omega)} \le C.$$

Since $\{\boldsymbol{\omega}^1, \boldsymbol{\omega}^2, \ldots\}$ is dense in $W^{s,2}_{\operatorname{div}}(\Omega)$ and $W^{s,2}_{\operatorname{div}}(\Omega) \hookrightarrow W^{1,\infty}_{\operatorname{div}}(\Omega)$ for $s > \frac{5}{2}$, we deduce that

$$\int\limits_{I'} \varphi\Big(\big\langle \partial_t \mathbf{u}, \boldsymbol{\omega}\big\rangle + \langle \mathbf{S}(\mathbf{D}\mathbf{u}), \mathbf{D}\boldsymbol{\omega}\rangle + \langle (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}, \boldsymbol{\omega}\rangle\Big)\,dt = \int\limits_{I'} \varphi\langle \mathbf{f}, \boldsymbol{\omega}\rangle dt$$

is fulfilled for all $\boldsymbol{\omega} \in W^{s,2}_{\operatorname{div}}(\Omega)$, especially for all $\boldsymbol{\omega} \in \mathcal{V}$. Note that

$$\big\langle \partial_t \mathbf{u}, \boldsymbol{\omega}\big\rangle, \langle \mathbf{S}(\mathbf{D}\mathbf{u}), \mathbf{D}\boldsymbol{\omega}\rangle, \langle (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}, \boldsymbol{\omega}\rangle, \langle \mathbf{f}, \boldsymbol{\omega}\rangle \in L^1(I')$$

so

$$\big\langle \partial_t \mathbf{u}, \boldsymbol{\omega}\big\rangle + \langle \mathbf{S}(\mathbf{D}\mathbf{u}), \mathbf{D}\boldsymbol{\omega}\rangle + \langle (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}, \boldsymbol{\omega}\rangle = \langle \mathbf{f}, \boldsymbol{\omega}\rangle \tag{65}$$

for all $\boldsymbol{\omega} \in \mathcal{V}$ and a.e. $t \in I'$. It remains to show that $\mathbf{u}(0) = \mathbf{u}_0$. But this follows from the parabolic embedding

$$\begin{aligned}\|P^N \mathbf{u}_0 - \mathbf{u}(0)\|_2 &= \|\mathbf{u}^N(0) - \mathbf{u}(0)\|_2 \\ &\leq C \underbrace{\|\mathbf{u}^N - \mathbf{u}\|^{\frac{1}{2}}_{L^2(I',L^2(\Omega))}}_{\to 0} \underbrace{\|\partial_t \mathbf{u}^N - \partial_t \mathbf{u}\|^{\frac{1}{2}}_{L^2(I',L^2(\Omega))}}_{\leq C} \to 0.\end{aligned} \tag{66}$$

Since $P^N \mathbf{u}_0 \to \mathbf{u}_0$ in $L^2(\Omega)$ we get $\mathbf{u}(0) = \mathbf{u}_0$. Overall we have shown by (65) and (66) that $\mathbf{u}$ satisfies (1) in the weak sense. It remains to prove the norm estimates for $\mathbf{u}$, $\mathcal{I}_\Phi(\mathbf{u})$ and $\mathcal{J}_\Phi(\mathbf{u})$. First of all, from (58) and (59) there follows

$$\|\partial_t \mathbf{u}\|_{L^\infty(I',L^2(\Omega))} + \|\mathbf{u}\|_{L^\infty(I',W^{1,\frac{12}{5}}(\Omega))} \leq C.$$

Define $H : I \times \Omega \times \mathbb{R}^{d\times d} \times \mathbb{R}^{d\times d} \to \mathbb{R}$ by

$$H(t,x,\mathbf{y},\mathbf{z}) := \sum_{jk\alpha\beta} (\partial_{\alpha\beta}\partial_{jk}\Phi)(t,x,\mathbf{y})\, z_{\alpha\beta}\, z_{jk},$$

then

(a) $H \geq 0$,
(b) H is measurable in (t,x) for all $\mathbf{y}, \mathbf{z}$,
(c) H is continuous in $\mathbf{z}$ and $\mathbf{y}$ for almost every $(t,x) \in I' \times \Omega$,
(d) H is convex in $\mathbf{z}$ for all $\mathbf{y}$ and almost every $(t,x) \in I' \times \Omega$.

Furthermore

$$\|\mathcal{J}_\Phi{}^A(\mathbf{u}^N)\|_{L^1(I',L^1(\Omega))} = \Big\| H(\mathbf{D}\mathbf{u}^N, \partial_t \mathbf{D}\mathbf{u}^N)\Big\|_{L^1(I'\times\Omega)}, \tag{67}$$

$$\|\mathcal{I}_\Phi{}^A(\mathbf{u}^N)\|_{L^1(I',L^1(\Omega))} = \Big\| \sum_{k=1}^d H(\mathbf{D}\mathbf{u}^N, \partial_k \mathbf{D}\mathbf{u}^N)\Big\|_{L^1(I'\times\Omega)}. \tag{68}$$

Due to lemma 10 and $\mathcal{I}_\Phi(\mathbf{u}^N)_{L^1(I)} + \mathcal{J}_\Phi(\mathbf{u}^N)_{L^1(I)} \leq C$ we have

$$\|\partial_t \mathbf{u}^N\|_{L^p(I,W^{1,\frac{3p}{p+1}}(\Omega))} \leq C.$$

Thus we can pass to a subsequence (still denoted by $\mathbf{u}^N$) with

$$\partial_t \boldsymbol{\nabla}\mathbf{u}^N \rightharpoonup \partial_t \boldsymbol{\nabla}\mathbf{u} \qquad \text{in } L^p(I', L^{\frac{3p}{p+1}}(\Omega)). \tag{69}$$

Note that (69), (57), and (60) imply

$$\begin{aligned} \nabla^2 \mathbf{u}^N &\rightharpoonup \nabla^2 \mathbf{u} && \text{in } L^1(I' \times \Omega), \\ \partial_t \nabla \mathbf{u}^N &\rightharpoonup \partial_t \nabla \mathbf{u} && \text{in } L^1(I' \times \Omega), \\ \nabla \mathbf{u}^N &\to \nabla \mathbf{u} && \text{in } L^1(I' \times \Omega). \end{aligned}$$

Thus from the semicontinuity theorem of De Giorgi ([9], pg. 132), (56), and (67) follows

$$\|\mathcal{J}_\Phi{}^A(\mathbf{u})\|_{L^1(I')} \le C. \tag{70}$$

Furthermore H, $\mathbf{Du}^N$, $\partial_k \mathbf{Du}^N$ fulfill all the requirements of corollary 26. Thus we deduce from (56) and corollary 26

$$\|\mathcal{I}_\Phi(\mathbf{u})\|_{L^{\frac{5p-6}{2-p}}(I')} \le C. \tag{71}$$

This proves the theorem. □

The next corollary shows what regularity for $\mathbf{u}$ can be deduced from (55). This justifies that we call $\mathbf{u}$ a "strong" solution.

Corollary 18. *Let* $\mathbf{u}$ *be the solution of theorem 17, then*

$$\begin{aligned} \mathbf{u} &\in L^{\frac{p(5p-6)}{2-p}}(I', W^{2,\frac{3p}{p+1}}(\Omega)), \\ \partial_t^2 \mathbf{u} &\in L^2(I', (W^{1,2}_{\mathrm{div}}(\Omega))'), \\ (\widetilde{D}\mathbf{u})^{\frac{p}{2}} &\in C(I', L^{\frac{12(p-1)}{p}, \frac{4(p-1)}{2-p}}) \qquad \textit{(Lorentz space)}. \end{aligned}$$

For all $1 \le s < 6(p-1)$ *there holds*

$$\mathbf{u} \in C(I', W^{1,s}(\Omega)).$$

Furthermore there exists a pressure π *with*

$$\nabla \pi \in L^{\frac{2(5p-6)}{2-p}}(I', L^2(\Omega))$$

such that

$$\partial_t \mathbf{u} - \operatorname{div}(\mathbf{S}(\mathbf{Du})) + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla \pi = \mathbf{f} \tag{72}$$

a.e. in $I' \times \Omega$.

Proof. From (55) and lemma 5 we deduce that

$$\begin{aligned} (\widetilde{D}\mathbf{u})^{\frac{p}{2}} &\in L^{\frac{2(5p-6)}{2-p}}(I', W^{1,2}(\Omega)), \\ \partial_t\big((\widetilde{D}\mathbf{u})^{\frac{p}{2}}\big) &\in L^2(I', L^2(\Omega)). \end{aligned}$$

Thus by theorem 35 with $\theta = \frac{2-p}{4(p-1)}$ we get

$$\begin{aligned} (\widetilde{D}\mathbf{u})^{\frac{p}{2}} &\in C(I', [W^{1,2}(\Omega), L^2(\Omega)]_{\theta, \frac{1}{\theta}}) \\ &= C(I', B^{\frac{5p-6}{4(p-1)}, 2}_{\frac{4(p-1)}{2-p}}(\Omega)) && \text{Besov Space} \\ &\hookrightarrow C(I', L^{\frac{12(p-1)}{p}, \frac{4(p-1)}{2-p}}(\Omega)) && \text{Lorentz Space.} \end{aligned}$$

For more details regarding Besov spaces and Lorentz spaces see Bergh, Löfström [3] and Triebel [26]. Let $1 \le s < 6(p-1)$, then

$$(\widetilde{D}\mathbf{u})^{\frac{p}{2}} \in C(I', L^{\frac{12(p-1)}{p}, \frac{4(p-1)}{2-p}}(\Omega)) \hookrightarrow C(I', L^{\frac{2s}{p}}(\Omega)).$$

As a consequence $\|(\widetilde{D}\mathbf{u})^{\frac{p}{2}}\|_{\frac{2s}{p}} \in C(I')$, so $\mathbf{Du} \in C(I', L^s(\Omega))$. From Korn's inequality we deduce

$$\mathbf{u} \in C(I', W^{1,s}(\Omega)).$$

From $\|(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}\|_2 \le \|\mathbf{u}\|_{1,\frac{12}{5}}$ and the choice $s := \frac{12}{5} < 6(p-1)$ we deduce

$$(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} \in C(I', L^2(\Omega)).$$

From (55) and lemma 10 we deduce that

$$\|\mathbf{u}\|_{L^{\frac{p(5p-6)}{2-p}}(I', W^{2,\frac{3p}{p+1}}(\Omega))} \le C.$$

Further note that

$$|\boldsymbol{\nabla}(\mathbf{S}(\mathbf{Du}))| \le C\,(\widetilde{D}\mathbf{u})^{p-2}|\boldsymbol{\nabla}\mathbf{Du}|,$$

so

$$\|\boldsymbol{\nabla}(\mathbf{S}(\mathbf{Du}))\|_2 \le C\,\mathcal{I}_\Phi(\mathbf{u})^{\frac{1}{2}}.$$

Thus $\mathbf{S}(\mathbf{Du}) \in L^{\frac{2p(5p-6)}{2-p}}(I', W^{1,2}(\Omega))$. We have shown that all the terms $\partial_t\mathbf{u}$, $-\operatorname{div}(\mathbf{S}(\mathbf{Du}))$, and $(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}$ in (65) are in $L^{\frac{2(5p-6)}{2-p}}(I', L^2(\Omega))$. Thus De Rahm's theorem ensures the existence of a pressure π with $\boldsymbol{\nabla}\pi \in L^{\frac{2(5p-6)}{2-p}}(I', L^2(\Omega))$. From

$$|\partial_t(\mathbf{S}(\mathbf{Du}))| \le C\,(\widetilde{D}\mathbf{u})^{p-2}|\partial_t\mathbf{Du}|$$

and lemma 5 we deduce

$$\|\partial_t(\mathbf{S}(\mathbf{Du}))\|_2^2 \le C\,\mathcal{J}_\Phi(\mathbf{u}).$$

This and (72) prove $\partial_t^2\mathbf{u} \in L^2(I', (W^{1,2}_{\operatorname{div}}(\Omega))')$. This proves the corollary. □

7. Uniqueness

Theorem 19. *Let $\frac{7}{5} < p \le 2$ and let $\mathbf{u}$ and $\mathbf{v}$ be weak solutions of* (1) *with*

$$\mathbf{u}, \mathbf{v} \in C(I, W^{1,\frac{12}{5}}(\Omega)).$$

Then $\mathbf{u} = \mathbf{v}$.

Proof. Let $\mathbf{e} := \mathbf{u} - \mathbf{v}$. We take the difference of the equations of $\mathbf{u}$ and $\mathbf{v}$ and use $\mathbf{e}$ as a test function, then

$$\langle\partial_t\mathbf{e},\mathbf{e}\rangle + \langle\mathbf{S}(\mathbf{Du}) - \mathbf{S}(\mathbf{Dv}), \mathbf{Du} - \mathbf{Dv}\rangle + \langle(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} - (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}, \mathbf{e}\rangle = 0.$$

This reduces to

$$\tfrac{1}{2}d_t\|\mathbf{e}\|_2^2 + \langle\mathbf{S}(\mathbf{Du}) - \mathbf{S}(\mathbf{Dv}), \mathbf{Du} - \mathbf{Dv}\rangle \le \big|\langle(\mathbf{e}\cdot\boldsymbol{\nabla})\mathbf{u}, \mathbf{e}\rangle\big|. \tag{73}$$

Since $p > \frac{7}{5}$, there exists $q > \frac{8}{5}$ with

$$\frac{2-p}{2}\cdot\frac{2q}{2-q} < \frac{12}{5}.$$

Thus

$$\|(\widetilde{D}\mathbf{u})^{\frac{2-p}{2}} + (\widetilde{D}\mathbf{v})^{\frac{2-p}{2}}\|_{\frac{2q}{2-q}} \in L^\infty(I).$$

Lemma 8 implies

$$\|\mathbf{De}\|_q \le C\,\big\langle\mathbf{S}(\mathbf{Du}) - \mathbf{S}(\mathbf{Dv}), \mathbf{Du} - \mathbf{Dv}\big\rangle^{\frac{1}{2}}.$$

Korn's inequality implies $\|\mathbf{e}\|_{\frac{3q}{3-q}} \le C\,\|\mathbf{e}\|_{1,q} \le C\,\|\mathbf{De}\|_q$. So by (73)

$$\tfrac{1}{2}d_t\|\mathbf{e}\|_2^2 + c\,\|\mathbf{e}\|_{\frac{3q}{3-q}}^2 \le \big|\langle(\mathbf{e}\cdot\boldsymbol{\nabla})\mathbf{u},\mathbf{e}\rangle\big| \le \|\mathbf{e}\|_{\frac{24}{7}}^2\,\|\mathbf{u}\|_{1,\frac{12}{5}} \le C\,\|\mathbf{e}\|_{\frac{24}{7}}^2.$$

Since $q > \frac{8}{5}$ there holds $2 < \frac{3q}{3-q} < \frac{24}{7}$ and $L^{\frac{24}{7}}(\Omega) = [L^2(\Omega), L^{\frac{3q}{3-q}}(\Omega)]_\theta$ with $0 < \theta < 1$. Thus for $\delta > 0$

$$\tfrac{1}{2}d_t\|\mathbf{e}\|_2^2 + c\,\|\mathbf{e}\|_{\frac{3q}{3-q}}^2 \le C\,\|\mathbf{e}\|_2^{2(1-\theta)}\|\mathbf{e}\|_{\frac{3q}{3-q}}^{2\theta} \le C_\delta\|\mathbf{e}\|_2^2 + \delta\|\mathbf{e}\|_{\frac{3q}{3-q}}^2.$$

Gronwall's inequality implies $\mathbf{e} = 0$, i.e. $\mathbf{u} = \mathbf{v}$. □

Note that for $p > \frac{7}{5}$ we have derived for small times the existence of a strong solution $\mathbf{u}$ with $\mathbf{u} \in C(I', W^{1,s}(\Omega))$ for all $1 \le s < 6(p-1)$. Especially this solution satisfies $\mathbf{u} \in C(I', W^{1,\frac{12}{5}}(\Omega))$. Theorem 19 above ensures that this solution is unique within the class of strong solutions in $C(I', W^{1,\frac{12}{5}}(\Omega))$. It is interesting to observe that the uniqueness as proven above exactly holds up to the same bound $p > \frac{7}{5}$ for which we have derived the existence of such solutions.

8. Space and Time Dependent Potentials

In the previous sections we have assumed that $\mathbf{S}$ is induced by a p–potential, where p is constant with $\frac{7}{5} < p \le 2$. In the study of electrorheological fluids it is necessary to admit p, which may vary in space and time. We are especially interested in the model studied in [21, 22, 24, 25], i.e.

$$\begin{aligned} \operatorname{div}\mathbf{E} &= 0,\\ \operatorname{curl}\mathbf{E} &= \mathbf{0}, \end{aligned} \tag{74}$$

$$\begin{aligned} \rho_0\partial_t\mathbf{u} - \operatorname{div}\mathbf{S} + \rho_0(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} + \boldsymbol{\nabla}\pi &= \rho_0\mathbf{f} + \chi^E\,[\boldsymbol{\nabla}\mathbf{E}]\mathbf{E},\\ \operatorname{div}\mathbf{u} &= 0, \end{aligned} \tag{75}$$

where $\mathbf{E}$ is the electric field, $\mathbf{P}$ the polarization, ρ_0 the constant density, $\mathbf{u}$ the velocity, π the pressure, $\mathbf{f}$ the mechanical, χ^E the dielectric susceptibility and the extra stress $\mathbf{S}$ is given by

$$\begin{aligned} \mathbf{S} = \alpha_{21}\big((1+|\mathbf{D}|^2)^{\frac{p-1}{2}} - 1\big)\mathbf{E}\otimes\mathbf{E} + (\alpha_{31} + \alpha_{33}|\mathbf{E}|^2)(1+|\mathbf{D}|^2)^{\frac{p-2}{2}}\mathbf{D}\\ + \alpha_{51}(1+|\mathbf{D}|^2)^{\frac{p-2}{2}}(\mathbf{DE}\otimes\mathbf{E} + \mathbf{E}\otimes\mathbf{DE}). \end{aligned} \tag{76}$$

Moreover, p is function of the electric field, i.e. $p = p(|\mathbf{E}|^2)$. Note that (74) decouples from (75). Thus, while solving (75) we can assume that $\mathbf{E}$ and p are given functions. Due to the nature of the Maxwell equations (74) it is reasonable to consider smooth $\mathbf{E}$ and p. We define $p^- := \inf_{\Omega\times I} p$ and $p^+ := \sup_{\Omega\times I} p$. Here we study the simplified model

$$\begin{aligned} \partial_t\mathbf{u} - \operatorname{div}\big(\mathbf{S}(\mathbf{Du})\big) + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} + \boldsymbol{\nabla}\pi &= \mathbf{f},\\ \operatorname{div}\mathbf{u} &= 0, \end{aligned} \tag{77}$$

with

$$\mathbf{S}(\mathbf{Du}) = (1+|\mathbf{Du}|^2)^{\frac{p-2}{2}}\mathbf{Du},$$

where p is a given function with $p \in W^{1,\infty}(I\times\Omega)$. We will show that all the results of the previous sections may be transfered to this model. Especially, we will prove short time existence of strong solutions of (77) for large data as long as $\frac{7}{5} < p^- \le p^+ \le 2$. We will also show, that this solution is unique within its class of regularity. Instead of repeating all the steps as in the case where p is constant, we will indicate all necessary changes in the calculations.

Definition 20. *Let $p : I \times \Omega \to (1,2]$ be a $W^{1,\infty}(I \times \Omega)$ function with $1 < p^- := \inf p \le p^+ := \sup p \le 2$. Let $F : I \times \Omega \times \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ be such that for a.e. $(t,x) \in I \times \Omega$ the function $F(t,x,\cdot)$ is a $p(t,x)$–potential (see definition 1) and the ellipticity constants do not depend on (t,x), i.e. the function $\Phi : \mathbb{R}^{n\times n} \to \mathbb{R}^{\geq 0}$ defined through $\Phi(t,x,\mathbf{B}) = F(t,x,|\mathbf{B}^{\mathrm{sym}}|)$ satisfies*

$$\sum_{jklm} (\partial_{jk}\partial_{lm}\Phi)(t,x,\mathbf{B}) C_{jk} C_{lm} \ge \gamma_1 (1 + |\mathbf{B}^{\mathrm{sym}}|^2)^{\frac{p(t,x)-2}{2}} |\mathbf{C}^{\mathrm{sym}}|^2, \tag{78}$$

$$\big|(\boldsymbol{\nabla}^2_{n\times n}\Phi)(t,x,\mathbf{B})\big| \le \gamma_2 (1 + |\mathbf{B}^{\mathrm{sym}}|^2)^{\frac{p(t,x)-2}{2}} \tag{79}$$

for all $\mathbf{B}, \mathbf{C} \in \mathbb{R}^{n\times n}$ with constants $\gamma_1, \gamma_2 > 0$. Further we assume that F is continuously differentiable with respect to t and x and that $(\partial_t F)(t,x,\cdot) : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$, resp. $(\partial_j F)(t,x,\cdot) : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$, is a C^1–function on $\mathbb{R}^{\geq 0}$, resp. a C^2–function on $\mathbb{R}^{>0}$, for all $t \in I$, $x \in \Omega$. Moreover, assume that for $j = 1, \dots, d$

$$\begin{aligned}
&(\partial_t F)(t,x,0) = 0,\\
&(\partial_t F)(t,x,R) > 0 \qquad \text{for all } R > 0,\\
&(\partial_j F)(t,x,0) = 0,\\
&(\partial_j F)(t,x,R) > 0 \qquad \text{for all } R > 0,\\
&\big|(\partial_t \boldsymbol{\nabla}_{n\times n}\Phi)(t,x,\mathbf{B})\big| \le \gamma_3 (1 + |\mathbf{B}^{\mathrm{sym}}|^2)^{\frac{p(t,x)-1}{2}} \ln(1 + |\mathbf{B}^{\mathrm{sym}}|),\\
&\big|(\boldsymbol{\nabla}\boldsymbol{\nabla}_{n\times n}\Phi)(t,x,\mathbf{B})\big| \le \gamma_3 (1 + |\mathbf{B}^{\mathrm{sym}}|^2)^{\frac{p(t,x)-1}{2}} \ln(1 + |\mathbf{B}^{\mathrm{sym}}|),
\end{aligned} \tag{80}$$

with $\gamma_3 > 0$. Such a function F, resp. Φ, is called a **time and space dependent p-potential** *and the corresponding constants γ_1, γ_2 and γ_3 are called the ellipticity and growth constants of F, resp. Φ. The function p is called the* **exponent** *of the potential. As in definition 1 we define the extra stress $\mathbf{S}$ by $\mathbf{S} := \boldsymbol{\nabla}_{n\times n}\Phi$.*

The standard examples of such $\mathbf{S}$ are

$$\mathbf{S}(\mathbf{Du}) \equiv (1 + |\mathbf{Du}|^2)^{\frac{p-2}{2}}\,\mathbf{Du}, \qquad \mathbf{S}(\mathbf{Du}) \equiv (1 + |\mathbf{Du}|)^{p-2}\,\mathbf{Du},$$

where $p \in W^{1,\infty}(I \times \Omega)$, especially (77) is included.

Assume for the rest of this section that F, resp. Φ is a time and space dependent p–potential and $\mathbf{S} := \boldsymbol{\nabla}_{n\times n}\Phi$. Then remark 2, remark 3, and theorem 4 still hold true. We will now transfer the estimates of section 2 to the case of time and space dependent p–potentials. The involved generic constants C may depend on $\|p\|_{W^{1,\infty}(I\times\Omega)}$.

We define the functionals $\mathcal{I}_\Phi$, $\mathcal{J}_\Phi$, and $\mathcal{G}_\Phi$ as in (16), (17), and (18). Then (22) and (23) hold true without change, while (25) and (26) must be modified to

$$\gamma_1 \|\boldsymbol{\nabla}\big((\widetilde{D}\mathbf{u})^{\frac{p}{2}}\big)\|_2^2 \le C\,\Big(\mathcal{I}_\Phi(\mathbf{u}) + \int_\Omega (\widetilde{D}\mathbf{u})^p \ln^2(\widetilde{D}\mathbf{u})\,dx\Big), \tag{81}$$

$$\gamma_1 \|\partial_t\big((\widetilde{D}\mathbf{u})^{\frac{p}{2}}\big)\|_2^2 \le C\,\Big(\mathcal{J}_\Phi(\mathbf{u}) + \int_\Omega (\widetilde{D}\mathbf{u})^p \ln^2(\widetilde{D}\mathbf{u})\,dx\Big). \tag{82}$$

In order to explain why there appears the logarithmic term, we will deduce (81) explicitly: Note that

$$\big|\boldsymbol{\nabla}\big((\widetilde{D}\mathbf{u})^{\frac{p}{2}}\big)\big| \le C\,(\widetilde{D}\mathbf{u})^{\frac{p-2}{2}} |\boldsymbol{\nabla}^2\mathbf{u}| + |\boldsymbol{\nabla} p|\,(\widetilde{D}\mathbf{u})^{\frac{p}{2}}\,\ln(\widetilde{D}\mathbf{u}). \tag{83}$$

Now (83), $\|\boldsymbol{\nabla} p\|_{L^\infty(I\times\Omega)} \le C$, and (22) prove (81).

Furthermore, (28) and (29) hold true, if $\|\cdot\|_p^p$ is exchanged by $\rho_p(\cdot)$, where $\rho_p(\cdot, g) : I \to \mathbb{R}^{\ge 0}$ is defined by

$$\rho_p(t,g) := \int_\Omega |g(t,x)|^{p(t,x)}\, dx.$$

Especially

$$\gamma_1 \rho_p(t, \boldsymbol{\nabla}^2 \mathbf{u}) \le C\, \mathcal{I}_\Phi(t,\mathbf{u}) + \gamma_1 \rho_p(t, \widetilde{D}\mathbf{u}). \tag{84}$$

$$\gamma_1 \rho_p(t, \partial_t \mathbf{D}\mathbf{u}) \le C\, \mathcal{J}_\Phi(t,\mathbf{u}) + \gamma_1 \rho_p(t, \widetilde{D}\mathbf{u}). \tag{85}$$

Mostly we will simply write $\rho_p(g)$ instead of $\rho_p(t,g)$. Moreover, (31) and lemma 8 hold true without change. The estimates of lemma 9 and 10 will in our case be modified to

$$\gamma_1 \big(\rho_{3p}(\widetilde{D}\mathbf{u})\big)^{\frac{1}{3}} \le C\, \Big(\mathcal{I}_\Phi(\mathbf{u}) + \int_\Omega |\widetilde{D}\mathbf{u}|^p \ln^2(\widetilde{D}\mathbf{u})\, dx + \rho_p(\widetilde{D}\mathbf{u})\Big).$$

and

$$\|\mathbf{u}\|^{p^-}_{2,\frac{3p^-}{p^-+1}} \le C\,(\mathcal{I}_\Phi(\mathbf{u}) + 1), \tag{86}$$

$$\|\partial_t \mathbf{u}\|^{p^-}_{1,\frac{3p^-}{p^-+1}} \le C\, \mathcal{J}_\Phi(\mathbf{u})^{\frac{p^-}{2}} \big(\mathcal{I}_\Phi(\mathbf{u}) + 1\big)^{\frac{2-p^-}{2}} \le C(\mathcal{J}_\Phi(\mathbf{u}) + \mathcal{I}_\Phi(\mathbf{u}) + 1). \tag{87}$$

With all these estimates above it is possible to transfer all calculations of section 4 and 5 to the case of a time and space dependent p–potential. Indeed, the test function $\mathbf{u}$ can be applied without change. Especially we get

$$\|\mathbf{u}^N\|^2_{L^\infty(I,L^2(\Omega))} + \int_I \int_\Omega |\mathbf{D}\mathbf{u}^N|^{p(x,t)}\, dx\, dt \le C. \tag{88}$$

The estimates for the test functions $\boldsymbol{\Delta}\mathbf{u}$ and "$\partial_t^2 \mathbf{u}$" involve additional terms of lower order, which can always be estimated. As an example we will in analogy to (40) consider the nonlinear main part $\operatorname{div}(\mathbf{S}(\mathbf{D}\mathbf{u}))$ when tested with $-\boldsymbol{\Delta}\mathbf{u}$, i.e.

$$\begin{aligned}
\langle \mathbf{S}(\mathbf{D}\mathbf{u}^N), \mathbf{D}\boldsymbol{\Delta}\mathbf{u}^N\rangle &= \sum_{ijklm} \int_\Omega (\partial_{ij}\partial_{kl}\Phi)(\mathbf{D}\mathbf{u}^N) \partial_m D_{ij}\mathbf{u}^N\, \partial_m D_{kl}\mathbf{u}^N\, dx \\
&\quad + \sum_{klm} \int_\Omega (\partial_m \partial_{kl}\Phi)(\mathbf{D}\mathbf{u}^N)\, \partial_m D_{kl}\mathbf{u}^N\, dx \\
&\overset{(16),(80)}{\ge} \mathcal{I}_\Phi(\mathbf{u}^N) - C \int_\Omega (\widetilde{D}\mathbf{u}^N)^{\frac{p-1}{2}}\, \ln(1 + |\mathbf{D}\mathbf{u}^N|) |\boldsymbol{\nabla}^2 \mathbf{u}^N|\, dx \\
&\ge \mathcal{I}_\Phi(\mathbf{u}^N) - \varepsilon \int_\Omega (\mathbf{D}\mathbf{u}^N)^{p-2}\, |\boldsymbol{\nabla}^2 \mathbf{u}^N|^2\, dx \\
&\quad - C_\varepsilon \int_\Omega (\widetilde{D}\mathbf{u}^N)^p\, \ln^2(1 + |\mathbf{D}\mathbf{u}^N|)\, dx \\
&\overset{(22)}{\ge} \tfrac{c}{2}\, \mathcal{I}_\Phi(\mathbf{u}^N) - C_\varepsilon \int_\Omega (\widetilde{D}\mathbf{u}^N)^p \, \big(1 + \ln^2(\widetilde{D}\mathbf{u}^N)\big)\, dx.
\end{aligned}$$

Thus we can derive an analogy to (47) with an additional logarithmic term on the right hand side. Repeating the calculations of section 5 we get (compare with (47))

$$\mathcal{I}_\Phi(\mathbf{u}^N) \le C\Big(1+\|\widetilde{D}\mathbf{u}^N\|_q^{R_1}+|\langle\partial_t\mathbf{u}^N,\Delta\mathbf{u}^N\rangle|+\int_\Omega(\widetilde{D}\mathbf{u}^N)^p\big(1+\ln^2(\widetilde{D}\mathbf{u}^N)\big)dx\Big). \tag{89}$$

Note that for every $\delta>0$ there holds

$$(\widetilde{D}\mathbf{u}^N)^p\left(1+\ln^2(\widetilde{D}\mathbf{u}^N)\right) \le C_\delta(\widetilde{D}\mathbf{u}^N)^{p+\delta}.$$

Especially, if we choose $p+\delta\le q$, then we can hide the logarithmic term in (89) in $\|\widetilde{D}\mathbf{u}^N\|_q^q$, i.e.

$$\mathcal{I}_\Phi(\mathbf{u}^N) \le C\big(1+\|\widetilde{D}\mathbf{u}^N\|_q^{R_1}+|\langle\partial_t\mathbf{u}^N,\Delta\mathbf{u}^N\rangle|\big). \tag{90}$$

This proves that (47) holds even in the case of a space and time dependent potential. The rest of the calculactions in section 5, e.g. testing with "$\partial_t^2\mathbf{u}$", can be carried out in the same way, that is all logarithmic terms will be hiden in $\|\widetilde{D}\mathbf{u}^N\|_q^q$. Overall, this proves that the crucial estimates (50) and (51) remain valid, where p in (51) has to be replaced by p^-, the lower bound of p. Thus, we have the following result for short time existence for the system (77):

Theorem 21. *Let $p\in W^{1,\infty}(I\times\Omega)$ with $\frac{7}{5}<p^-\le p^+\le 2$. Let $\mathbf{S}$ be induced by a space and time dependent p–potential Φ, i.e. $\mathbf{S}=\boldsymbol{\nabla}_{n\times n}\Phi$. Let $\frac{7}{5}<p^-\le p^+\le 2$ and*

$$\|\mathbf{f}\|_{L^\infty(I,W^{1,2}(\Omega))}+\|\partial_t\mathbf{f}\|_{L^2(I,L^2(\Omega))}+\|\mathbf{u}_0\|_{W^{2,2}_{\operatorname{div}}(\Omega)}\le K.$$

Then there exists a constant $T'=T'(K)$ with $0<T'<T$, such that the system (77) has a strong solution $\mathbf{u}$ on $I'=[0,T']$. Further

$$\begin{aligned}&\|\partial_t\mathbf{u}\|_{L^\infty(I',L^2(\Omega))}+\|\mathbf{u}\|_{C(I',W^{1,\frac{12}{5}}(\Omega))}\\&+\|\mathcal{J}_\Phi(\mathbf{u})\|_{L^1(I')}+\|\mathcal{I}_\Phi(\mathbf{u})\|_{L^{\frac{5p^--6}{2-p^-}}(I')}\le C.\end{aligned} \tag{91}$$

Corollary 22. *Let $\mathbf{u}$ be the solution of theorem 21, then*

$$\begin{aligned}\mathbf{u}&\in L^{\frac{p^-(5p^--6)}{2-p^-}}(I',W^{2,\frac{3p^-}{p^-+1}}(\Omega)),\\ \partial_t^2\mathbf{u}&\in L^2(I',(W^{1,2}_{\operatorname{div}}(\Omega))'),\\ (\widetilde{D}\mathbf{u})^{\frac{p}{2}}&\in C(I',L^{\frac{12(p^--1)}{p^-},\frac{4(p^--1)}{2-p^-}})\qquad\text{(Lorentz space).}\end{aligned}$$

For all $1\le s<6(p^--1)$ there holds

$$\mathbf{u}\in C(I',W^{1,s}(\Omega)).$$

Furthermore there exists a pressure π with

$$\boldsymbol{\nabla}\pi\in L^{\frac{2(5p^--6)}{2-p^-}}(I',L^2(\Omega))$$

such that

$$\partial_t\mathbf{u}-\operatorname{div}(\mathbf{S}(\mathbf{Du}))+(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}+\boldsymbol{\nabla}\pi=\mathbf{f} \tag{92}$$

a.e. in $I'\times\Omega$.

Proof. Using (91) we proceed as in corollary 18. The logarithmic terms that appear due to the space and time dependency of p (see (81) and (82)) can easily be controlled due to $\|\mathbf{u}\|_{C(I',W^{1,\frac{12}{5}}(\Omega))} < \infty$. □

The following theorem shows that the solutions are unique within their class of regularity.

Theorem 23. *Let $p, \mathbf{S}$ be as in theorem 21. Let $\mathbf{u}$ and $\mathbf{v}$ be weak solutions of* (77) *with*

$$\mathbf{u}, \mathbf{v} \in C(I, W^{1,\frac{12}{5}}(\Omega)).$$

Then $\mathbf{u} = \mathbf{v}$.

We want to remark that the results of theorem 21, corollary, 22, and theorem 23 can easily be generalized to the case of electrorheological fluids: In the model (74) and (75) the equation of motion (75) is the crucial one. The equations for $\mathbf{E}$ namely (74) decouple, i.e. $\mathbf{E}$ is the solution of the quasi–static Maxwell equations. These equations are well studied and we assume therefore that a smooth unique solution $\mathbf{E}$ of (74) is given and we want to solve (75). We further assume that the dependence of $p = p(|\mathbf{E}|^2)$ on $\mathbf{E}$ is sufficiently smooth, such that $p \in W^{1,\infty}(\Omega)$. Since ρ_0 is a constant, we can simplify (75) to (77), which is exaclty the system we have studied above. The only difference is that the extra stress defined in (76) is not induced by a time and space dependend p–potential Φ as defined in defintion 20. Nevertheless under suitable conditions on α_{21}, α_{31}, α_{33}, and α_{51} (see [25]) the extra stress $\mathbf{S}$ still satisfies the monotonicity conditions of theorem 4. Moreover, if we generalize the definitions of $\mathcal{I}_\Phi$, $\mathcal{J}_\Phi$, and $\mathcal{G}_\Phi$ by

$$\begin{aligned}
\mathcal{I}_\Phi(t, \mathbf{u}) &:= \Big\langle \sum_r \sum_{jk\alpha\beta} (\partial_{\alpha\beta} S_{jk})\big(\mathbf{Du}(t)\big)\partial_r D_{\alpha\beta}\mathbf{u}(t), \partial_r D_{jk}\mathbf{u}(t)\Big\rangle, \\
\mathcal{J}_\Phi(t, \mathbf{u}) &:= \Big\langle \sum_{jk\alpha\beta} (\partial_{\alpha\beta} S_{jk})\big(\mathbf{Du}(t)\big)\partial_t D_{\alpha\beta}\mathbf{u}(t), \partial_t D_{jk}\mathbf{u}(t)\Big\rangle, \\
\mathcal{G}_\Phi(t, \mathbf{w}, \mathbf{v}) &:= \Big\langle \sum_{jk\alpha\beta} (\partial_{\alpha\beta} S_{jk})(\mathbf{Dw}(t)) D_{\alpha\beta}\mathbf{v}(t), D_{jk}\mathbf{v}(t)\Big\rangle,
\end{aligned}$$

then it can easily be shown that (81) till (87) still hold. These estimates allow to generalize theorem 21, corollary 22, and 23 to system (75) for electrorheological fluids.

9. Appendix

Lemma 24 (local version of Gronwall's lemma)**.** *Let $T, \alpha, c_0 > 0$ be given constants and let $h \in L^1(0,T)$ with $h \geq 0$ a.e. in $[0,T]$, $0 \leq f \in C^1([0,T])$ satisfy*

$$f'(t) \leq h(t) + c_0\, f(t)^{1+\alpha}\,. \tag{93}$$

Let $t_0 \in [0,T]$ be such that $\alpha\, c_0\, H(t_0)^\alpha\, t_0 < 1$, where

$$H(t) := f(0) + \int_0^t h(s)\, ds\,.$$

Then for all $t \in [0, t_0]$ there holds

$$f(t) \leq H(t) + H(t)\Big(\big(1 - \alpha\, c_0\, H(t)^\alpha\, t\big)^{-\frac{1}{\alpha}} - 1\Big)\,.$$

Proof. Let $\delta > 0$ be small such that $\alpha\, c_0\, (H(t_0)+\delta)^\alpha\, t_0 < 1$. Let $t_1 \in [0, t_0]$ be fixed. Then $H_\delta := H(t_1) + \delta$ satisfies $\alpha\, c_0\, (H(t_1)+\delta)^\alpha\, t_0 < 1$. Define $y : [0, t_1] \to \mathbb{R}^{\geq 0}$ by

$$y(t) := \int_0^t f(s)^{1+\alpha}\, ds\,.$$

Then $y \in C^1([0, t_1])$, $y(0) = 0$, $y \geq 0$, and for all $t \in [0, t_1]$

$$\begin{aligned} y'(t) = f(t)^{1+\alpha} = \Big(f(0) + \int_0^t f'(s)\, ds \Big)^{1+\alpha} &\overset{(93)}{\leq} \big(H(t) + c_0\, y(t)\big)^{1+\alpha} \\ &\leq \big(H_\delta + c_0\, y(t)\big)^{1+\alpha}\,. \end{aligned} \tag{94}$$

Further, let $g \in C^1([0, t_1])$ be given by

$$g(t) := -\big(H_\delta + c_0\, y(t)\big)^{-\alpha}\,.$$

Then

$$g'(t) = \alpha\, c_0\, \big(H_\delta + c_0\, y(t)\big)^{-1-\alpha}\, y'(t) \overset{(94)}{\leq} \alpha\, c_0\,.$$

Thus $g(t) \leq g(0) + \alpha\, c_0\, t$ for $t \in [0, t_1]$ and therefore

$$-\big(H_\delta + c_0\, y(t)\big)^{-\alpha} \leq -H_\delta^{-\alpha} + \alpha\, c_0\, t\,.$$

Since $\alpha\, c_0\, H_\delta^\alpha\, t_1 < 1$, we estimate

$$H_\delta + c_0\, y(t) \leq \big(H_\delta^{-\alpha} - \alpha\, c_0\, t\big)^{-\frac{1}{\alpha}}$$

and

$$\begin{aligned} y(t) &\leq c_0^{-1} \Big(\big(H_\delta^{-\alpha} - \alpha\, c_0\, t\big)^{-\frac{1}{\alpha}} - H_\delta \Big) \\ &= c_0^{-1}\, H_\delta \Big(\big(1 - H_\delta^\alpha\, \alpha\, c_0\, t\big)^{-\frac{1}{\alpha}} - 1 \Big)\,. \end{aligned}$$

The limit $\delta \to 0$ implies for the choice $t = t_1$

$$y(t_1) \leq c_0^{-1}\, H(t_1) \Big(\big(1 - H(t_1)^\alpha\, \alpha\, c_0\, t_1\big)^{-\frac{1}{\alpha}} - 1 \Big)\,.$$

From (93) we deduce $f(t_1) \leq H(t_1) + c_0\, y(t_1)$. Since $t_1 \in [0, t_0]$ was arbitrary, this proves the lemma. $\square$

Lemma 25. *Let Ω be a domain in $\mathbb{R}^d$ and $I := [0, T] \subset \mathbb{R}$, $T > 0$. Further let $F : I \times \Omega \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ with*

(a) *$F \geq 0$,*
(b) *F measurable in t, x for a.a. $\mathbf{y}, \mathbf{z}$,*
(c) *$F(t, x, \cdot, \cdot)$ continuous for a.a. t, x,*
(d) *$F(t, x, \mathbf{y}, \cdot)$ be convex for all $\mathbf{y}$ and a.a. t, x.*

If $\mathbf{w}^N \to \mathbf{w}$ in $L^1_{\mathrm{loc}}(I \times \Omega)$ and $\boldsymbol{\nabla} \mathbf{w}^N \rightharpoonup \boldsymbol{\nabla} \mathbf{w}$ in $L^1_{\mathrm{loc}}(I \times \Omega)$, then for all r, s with $1 \leq r < \infty$, $1 < s < \infty$ or $r = s = 1$ there holds

$$\Big\| F(t, x, \mathbf{w}, \boldsymbol{\nabla} \mathbf{w}) \Big\|_{L^r(I, L^s(\Omega))} \leq \liminf_N \Big\| F(t, x, \mathbf{w}^N, \boldsymbol{\nabla} \mathbf{w}^N) \Big\|_{L^r(I, L^s(\Omega))}.$$

Proof. If $r = s = 1$ the result follows immediately from theorem 1 (De Giorgi) in [9], pg. 132. So let us assume $1 \le r < \infty$, $1 < s < \infty$. Since $L^s(\Omega)$ is reflexive, the dual of $L^r(I, L^s(\Omega))$ is $L^{r'}(I, L^{s'}(\Omega))$ (see [6]) and for all $f \in L^r(I, L^s(\Omega))$

$$\|f\|_{L^r(I,L^s(\Omega))} = \sup_{\|\varphi\|_{L^{r'}(I,L^{s'}(\Omega))} \le 1} |\langle f, \varphi \rangle|.$$

For $\varphi \in L^{r'}(I, L^{s'}(\Omega))$ with $\varphi \ge 0$ and $\|\varphi\|_{L^{r'}(I,L^{s'}(\Omega)} \le 1$ let $F_\varphi : I \times \Omega \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ be defined by

$$F_\varphi(t, x, \mathbf{y}, \mathbf{z}) := F(t, x, \mathbf{y}, \mathbf{z})\varphi(t, x).$$

Then F_φ fulfills

(a) $F_\varphi \ge 0$,
(b) F_φ measurable in t, x for a.a. $\mathbf{y}, \mathbf{z}$,
(c) $F_\varphi(t, x, \cdot, \cdot)$ continuous for a.a. t, x,
(d) $F_\varphi(t, x, \mathbf{y}, \cdot)$ be convex for all $\mathbf{y}$ and a.a. t, x.

Hence by theorem 1 (De Giorgi) from [9], pg. 132

$$\begin{aligned} &\iint_{I\,\Omega} F(t, x, \mathbf{w}, \nabla \mathbf{w})\varphi(t, x)\, dx\, dt \\ &\qquad \le \liminf_N \iint_{I\,\Omega} F(t, x, \mathbf{w}^N, \nabla \mathbf{w}^N)\varphi(t, x)\, dx\, dt \\ &\qquad \le \liminf_N \left\| F(t, x, \mathbf{w}^N, \nabla \mathbf{w}^N) \right\|_{L^r(I,L^s(\Omega))}. \end{aligned} \tag{95}$$

Since $F \ge 0$, the norm formula for F reduces to

$$\begin{aligned} &\left\| F(t, x, \mathbf{w}, \nabla \mathbf{w}) \right\|_{L^r(I,L^s(\Omega))} \\ &\qquad = \sup_{\substack{\|\varphi\|_{L^{r'}(I,L^{s'}(\Omega))} \le 1 \\ \varphi \ge 0}} \iint_{I\,\Omega} F(t, x, \mathbf{w}, \nabla \mathbf{w})\varphi(t, x)\, dx\, dt. \end{aligned}$$

This and (95) proves the lemma. □

Corollary 26. *Let Ω be a domain in $\mathbb{R}^n$ and $I := [0, T] \subset \mathbb{R}$, $T > 0$. Let $F : I \times \Omega \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{G} : I \times \Omega \times \mathbb{R}^m \to \mathbb{R}^{n \times n}$ with*

$$F(t, x, \mathbf{y}, \mathbf{z}) = \sum_{\alpha,\beta=1}^{n} G_{\alpha\beta}(t, x, y) z_\alpha z_\beta.$$

Moreover, assume G satisfies

(a) $\mathbf{G} \ge 0$, i.e. is positive semidefinite,
(b) $\mathbf{G}$ measurable in t, x for a.a. $\mathbf{y}$,
(c) $\mathbf{G}(t, x, \cdot)$ continuous for a.a. t, x.

If $\mathbf{w}^N \to \mathbf{w}$ in $L^1_{\mathrm{loc}}(I \times \Omega)$ and $\nabla \mathbf{w}^N \rightharpoonup \nabla \mathbf{w}$ in $L^1_{\mathrm{loc}}(I \times \Omega)$, then for all r, s with $1 \le r < \infty$, $1 \le s < \infty$ there holds

$$\left\| F(t, x, \mathbf{w}, \nabla \mathbf{w}) \right\|_{L^r(I,L^s(\Omega))} \le \liminf_N \left\| F(t, x, \mathbf{w}^N, \nabla \mathbf{w}^N) \right\|_{L^r(I,L^s(\Omega))}.$$

Proof. For $\frac{1}{2} < q < 1$ define F_q by

$$F_q(t,x,\mathbf{y},\mathbf{z}) := (F(t,x,\mathbf{y},\mathbf{z}))^q$$

then F_q fulfills the requirements of lemma 25. Thus

$$\begin{aligned}\Big\|F(t,x,\mathbf{w},\boldsymbol{\nabla}\mathbf{w})\Big\|_{L^r(I,L^s(\Omega))} &= \Big\|F_q(t,x,\mathbf{w},\boldsymbol{\nabla}\mathbf{w})\Big\|^{\frac{1}{q}}_{L^{\frac{r}{q}}(I,L^{\frac{s}{q}}(\Omega))} \\ &\le \liminf_N \Big\|F_q(t,x,\mathbf{w}^N,\boldsymbol{\nabla}\mathbf{w}^N)\Big\|^{\frac{1}{q}}_{L^{\frac{r}{q}}(I,L^{\frac{s}{q}}(\Omega))}. \\ &= \liminf_N \Big\|F(t,x,\mathbf{w}^N,\boldsymbol{\nabla}\mathbf{w}^N)\Big\|_{L^r(I,L^s(\Omega))}.\end{aligned}$$

This proves the corollary. □

We will now prove an interpolation theorem, which is very useful in deriving estimates for solutions of parabolic problems pointwise in time. Suppose that u is a solution to a parabolic problem, such that u, resp. $\partial_t u$, are in the Bochner spaces $L^{p_0}(I,A_0)$, resp. $L^{p_1}(I,A_1)$, where A_0 and A_1 are Banach spaces. Then we will see that u is with respect to the time a continuous function with values in the real interpolation space $\overline{A}_{\theta,q} := [A_0,A_1]_{\theta,q}$, where $\theta = \theta(p_0,p_1)$ and $q = q(p_0,p_1)$. The proof will be quite standard and we will mainly compile results which can be found in [14], [3] and [1]. Nevertheless to the knowledge of the authors there exists no exact statement of this result in literature. Since this interpolation result plays a fundamental role in this paper it is indispensable to prove it in some detail.

Definition 27. *Let $A_0 \subset A_1$ be two Banach spaces. Let $1 \le p_j < \infty$, $\alpha_j \in \mathbb{R}$ and $\eta_j = \alpha_j + \frac{1}{p_j}$ for $j = 0,1$. Let X denote the space*

$$X(A_0,A_1) = \{u : u \in L^1_{\mathrm{loc}}(\mathbb{R}^{\ge 0},A_0), u' \in L^1_{\mathrm{loc}}(\mathbb{R}^{\ge 0},A_1)\}.$$

We shall work with the Banach spaces $V = V(p_0,\alpha_0,A_0;p_1,\alpha_1,A_1)$ with

$$\begin{aligned}V &= \{u : u \in X(A_0,A_1), \|u\|_V < \infty\},\\ \|u\|_V &= \max\big\{\|t^{\alpha_0}u\|_{L^{p_0}(\mathbb{R}^{\ge 0},A_0)}, \|t^{\alpha_1}u'\|_{L^{p_1}(\mathbb{R}^{\ge 0},A_1)}\big\}.\end{aligned}$$

These spaces have been introduced by J. L. Lions and J. Peetre [14], but J. Bergh and J. Löfström have defined similar Banach spaces $\widetilde{V}(\overline{A},\overline{p},\overline{\eta})$ (see [3], corollary 3.12.3). The interconnection of these spaces is that both $\widetilde{V}(\overline{A},\overline{p},\overline{\eta})$ and $V(p_0,\alpha_0,A_0;p_1\alpha_1,A_1)$ are norm–equivalent, if we choose $\eta_j = \alpha_j + \frac{1}{p_j}$ with $j = 0,1$ (see [3], remark 3.14.12).

Lemma 28. *Let $u \in X(A_0,A_1)$ then there exists $b \in A_0 + A_1 = A_1$ such that*

$$u(t) = b + \int_1^t u'(\tau)d\tau, \quad \text{a.e. in } \mathbb{R}^{>0}.$$

Especially u is continuous on $(0,\infty)$ with values in A_1.

Proof. The proof of this lemma is standard. We refer to Adams [1], lemma 7.12, for a similar result. □

Hence every function $u \in X$ has a limit $u(0^+)$ in $A_0 + A_1 = A_1$. The lemma still holds true if $\mathbb{R}^{>0}$ is replaced by an intervall $I = [0,T]$.

Definition 29. *By $T = T(p_0, \alpha_0, A_0; p_1, \alpha_1, A_1)$ we denote the space of traces u of functions $v \in V = V(p_0, \alpha_0, A_0; p_1, \alpha_1, A_1)$ equipped with the quotient norm*

$$\|u\|_T = \inf_{v(0^+)=u} \|v\|_V.$$

Then $T(p_0, \alpha_0, A_0; p_1, \alpha_1, A_1)$ is a Banach space.

Theorem 30. *Let $1 \le p_j < \infty$, $\alpha_j \in \mathbb{R}$, and $\eta_j = \alpha_j + \frac{1}{p_j}$ with $j = 0, 1$. Further let θ and p be given by*

$$\theta = \frac{\eta_0}{\eta_0 + 1 - \eta_1}, \qquad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

If $\eta_0 > 0$ and $\eta_1 < 1$, then

$$T(p_0, \alpha_0, A_0; p_1, \alpha_1, A_1) = \overline{A}_{\theta,p}.$$

Proof. J. Bergh and J. Löfström have shown that their trace space $T(\overline{A}, \overline{p}, \theta)$ is under the stated conditions $\eta_0 > 0$ and $\eta_1 < 1$ equivalent to the trace space $T(p_0, \alpha_0, A_0; p_1, \alpha_1, A_1)$ of J. L. Lions and J. Peetre (see [3], remark 3.14.12). Further J. Bergh and J. Löfström showed that $T(\overline{A}, \overline{p}, \theta) = \overline{A}_{\theta,p}$ (see [3], pg. 72-75). □

Remark 31. *Let $1 \le p_0 < \infty$, $1 < p_1 < \infty$ and $\alpha_0 = \alpha_1 = 0$, then $\eta_0 = \frac{1}{p_0} > 0$ and $\eta_1 = \frac{1}{p_1} < 1$. Hence the requirements of theorem 30 are automatically fulfilled for $T(p_0, 0, A_0; p_1, 0, A_1)$. This case will later be of great importance. Hence we define:*

Definition 32. *For $1 \le p_j < \infty$ with $j = 0, 1$, let*

$$V(p_0, A_0; p_1, A_1) := V(p_0, 0, A_0; p_1, 0, A_1),$$
$$T(p_0, A_0; p_1, A_1) := T(p_0, 0, A_0; p_1, 0, A_1).$$

Theorem 33. *Let $1 \le p_j < \infty$ with $j = 0, 1$ and*

$$\theta = \frac{p_1}{p_1 + p_1 p_0 - p_0}, \tag{96}$$

then

$$T(p_0, A_0; p_1, A_1) = \overline{A}_{\theta, \frac{1}{\theta}} \tag{97}$$

and

$$V(p_0, A_0; p_1, A_1) \hookrightarrow L^\infty(\mathbb{R}^{\ge 0}, \overline{A}_{\theta,\frac{1}{\theta}}) \cap C(\mathbb{R}^{\ge 0}, \overline{A}_{\theta,\frac{1}{\theta}}).$$

Furthermore

$$\|u\|_{L^\infty(\mathbb{R}^{\ge 0}, \overline{A}_{\theta,\frac{1}{\theta}})} \le \|u\|_V, \tag{98}$$

$$\|u\|_{L^\infty(\mathbb{R}^{\ge 0}, \overline{A}_{\theta,\frac{1}{\theta}})} \le C\, \|u\|^{1-\theta}_{L^{p_0}(\mathbb{R}^{\ge 0}, A_0)} \|u'\|^{\theta}_{L^{p_1}(\mathbb{R}^{\ge 0}, A_1)}. \tag{99}$$

Proof. In the notation of theorem 30 we have $\eta_0 = \frac{1}{p_0} > 0$ and $\eta_1 = \frac{1}{p_1} < 1$ and

$$\theta = \frac{\eta_0}{\eta_0 + 1 - \eta_1} = \frac{\frac{1}{p_0}}{\frac{1}{p_0} + 1 - \frac{1}{p_1}} = \frac{p_1}{p_1 + p_1 p_0 - p_0}.$$

Furthermore

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} = \frac{(p_1-1)+1}{p_1+p_1p_0-p_0} = \theta.$$

This proves (97).

Due to lemma 28 we can assume $u \in C(\mathbb{R}^{\geq 0}, A_0 + A_1 = A_1)$. Let $u \in V = V(p_0, 0, A_0; p_1, 0, A_1)$. For $h > 0$ we define $\tau_h u$ by $(\tau_h u)(t) = u(t+h)$, then $\tau_h u \in V \cap C(\mathbb{R}^{\geq 0}, A_0 + A_1)$. By definition of T and theorem 30 there holds

$$\|u(h)\|_{\overline{A}_{\theta,\frac{1}{\theta}}} = \|(\tau_h u)(0)\|_{\overline{A}_{\theta,\frac{1}{\theta}}} \leq \|\tau_h u\|_V \leq \|u\|_V.$$

This proves the embedding $V \hookrightarrow L^\infty(\mathbb{R}^{\geq 0}, \overline{A}_{\theta,\frac{1}{\theta}})$ and estimate (98). In order to show $V \hookrightarrow C(\mathbb{R}^{\geq 0}, \overline{A}_{\theta,\frac{1}{\theta}})$ let $t \geq 0$ and $h \to 0$ (for $t = 0$ let $h \downarrow 0$), then

$$\begin{aligned}
&\|u(t+h) - u(t)\|_{\overline{A}_{\theta,\frac{1}{\theta}}} = \|(\tau_{t+h}u - \tau_t u)(0)\|_{\overline{A}_{\theta,\frac{1}{\theta}}} \\
&\quad \leq \|(\tau_{t+h}u - \tau_t u)\|_V \\
&\quad \leq \max\big\{\underbrace{\|\tau_{t+h}u - \tau_t u\|_{L^{p_0}(\mathbb{R}^{\geq 0}, A_0)}}_{\xrightarrow{h} 0}, \underbrace{\|\tau_{t+h}u' - \tau_t u'\|_{L^{p_1}(\mathbb{R}^{\geq 0}, A_1)}}_{\xrightarrow{h} 0}\big\} \\
&\quad \xrightarrow{h} 0.
\end{aligned}$$

Still we have to prove the logarithmic convex inequality for the norms of u. This will be done by a scaling argument. For $\lambda > 0$ let $\pi_\lambda u$ be defined by $(\pi_\lambda u)(t) = u(\lambda t)$. Then we have

$$\begin{aligned}
\|u(0)\|_{\overline{A}_{\theta,\frac{1}{\theta}}} &= \|(\pi_\lambda u)(0)\|_{\overline{A}_{\theta,\frac{1}{\theta}}} \\
&\leq C\,\|\pi_\lambda u\|_V \\
&\leq C\,\max\big\{\|\pi_\lambda u\|_{L^{p_0}(\mathbb{R}^{\geq 0}, A_0)}, \|(\pi_\lambda)u'\|_{L^{p_1}(\mathbb{R}^{\geq 0}, A_1)}\big\} \\
&= C\,\max\big\{\lambda^{-\frac{1}{p_0}}\|u\|_{L^{p_0}(\mathbb{R}^{\geq 0}, A_0)}, \lambda^{1-\frac{1}{p_1}}\|u'\|_{L^{p_1}(\mathbb{R}^{\geq 0}, A_1)}\big\}.
\end{aligned}$$

The minimum of the right-hand side over all $\lambda > 0$ is attained at λ_0 with

$$\lambda_0^{-\frac{1}{p_0}}\|u\|_{L^{p_0}(\mathbb{R}^{\geq 0}, A_0)} = \lambda_0^{1-\frac{1}{p_1}}\|u'\|_{L^{p_1}(\mathbb{R}^{\geq 0}, A_1)}.$$

Hence

$$\begin{aligned}
\|u(0)\|_{\overline{A}_{\theta,\frac{1}{\theta}}} &\leq C\,\|u\|_{L^{p_0}(\mathbb{R}^{\geq 0}, A_0)}^{\frac{1-\frac{1}{p_1}}{\frac{1}{p_0}+1-\frac{1}{p_1}}}\|u'\|_{L^{p_1}(\mathbb{R}^{\geq 0}, A_1)}^{\frac{\frac{1}{p_0}}{\frac{1}{p_0}+1-\frac{1}{p_1}}} \\
&= C\,\|u\|_{L^{p_0}(\mathbb{R}^{\geq 0}, A_0)}^{1-\theta}\|u'\|_{L^{p_1}(\mathbb{R}^{\geq 0}, A_1)}^{\theta}.
\end{aligned}$$

This implies the desired inequality for $u(0)$. For $h > 0$ consider

$$\begin{aligned}
\|u(h)\|_{\overline{A}_{\theta,\frac{1}{\theta}}} &= \|(\tau_h)u(0)\|_{\overline{A}_{\theta,\frac{1}{\theta}}} \\
&\leq C\,\|\tau_h u\|_{L^{p_0}(\mathbb{R}^{\geq 0}, A_0)}^{1-\theta}\|\tau_h'\|_{L^{p_1}(\mathbb{R}^{\geq 0}, A_1)}^{\theta} \\
&\leq C\,\|u\|_{L^{p_0}(\mathbb{R}^{\geq 0}, A_0)}^{1-\theta}\|u'\|_{L^{p_1}(\mathbb{R}^{\geq 0}, A_1)}^{\theta}.
\end{aligned}$$

□

If u is the solution of a parabolic differential equation on an interval $I = [0, T]$, than $u(t)$ is not defined for all $t > 0$.

Definition 34. *Let $A_0 \subset A_1$ be two Banach spaces and $I = [0,T]$ with $0<T\leq\infty$. Let $1 \leq p_j < \infty$ with $j = 0,1$. Let X_I denote the space*

$$X_I(A_0,A_1) = \{u \,:\, u \in L^1_{\text{loc}}(I,A_0), u' \in L^1_{\text{loc}}(I,A_1)\}.$$

We shall work with the Banach spaces $V_I = V(I;p_0,\alpha_0,A_0;p_1,\alpha_1,A_1)$ with

$$V_I = \{u \,:\, u \in X_I(A_0,A_1), \|u\|_{V_I} < \infty\},$$
$$\|u\|_{V_I} = \max\big\{\|t^{\alpha_0}u\|_{L^{p_0}(I,A_0)}, \|t^{\alpha_1}u'\|_{L^{p_1}(I,A_1)}\big\}.$$

Note that $V(p_0,\alpha_0,A_0;p_1,\alpha_1,A_1) = V_{\mathbb{R}^{\geq 0}}(p_0,\alpha_0,A_0;p_1,\alpha_1,A_1)$.

Theorem 35. *Let $0 < T \leq \infty$, $I = [0,T]$, $1 \leq p_j < \infty$ with $j = 0,1$ and*

$$\theta = \frac{p_1}{p_1 + p_1 p_0 - p_0},$$

then

$$V_I(p_0,A_0;p_1,A_1) \hookrightarrow C(I,\overline{A}_{\theta,\frac{1}{\theta}}).$$

Furthermore

$$\|u\|_{L^\infty(I,\overline{A}_{\theta,\frac{1}{\theta}})} \leq C \max\big\{\|u\|_{L^{p_0}(I,A_0)},\ \|u\|^{1-\theta}_{L^{p_0}(I,A_0)}\|u'\|^{\theta}_{L^{p_1}(I,A_1)}\big\}.$$

Proof. Let $u \in V_I := V_I(p_0,A_0;p_1,A_1)$. Let $\varphi \in C_0^\infty(\mathbb{R})$ with $0 \leq \varphi \leq 1$, $\varphi|_{[0,\frac{2}{3}T]} = 1$, and $\varphi|_{[\frac{4}{5}T,\infty)} = 0$. Let $w = u\cdot\varphi$, then $w \in V := V(p_0,A_0;p_1,A_1)$. By theorem 33 we conclude $w \in C(\mathbb{R}^{\geq 0},\overline{A}_{\theta,\frac{1}{\theta}})$, hence $u \in C([0,\frac{2}{3}T],\overline{A}_{\theta,\frac{1}{\theta}})$. If we substitute u by $\overline{u}$ with $\overline{u}(t) := u(T-t)$ and w by $\overline{w}$ with $\overline{w} = \overline{u}\cdot\varphi$, we get $u \in C([\frac{1}{3}T,T],\overline{A}_{\theta,\frac{1}{\theta}})$. Altogether we have proven $u \in C(I,\overline{A}_{\theta,\frac{1}{\theta}})$, which implies $V_I \to C(I,\overline{A}_{\theta,\frac{1}{\theta}})$ (as a linear mapping). Furthermore we know that

$$\begin{aligned}\|u\|_{L^\infty(I,\overline{A}_{\theta,\frac{1}{\theta}})} &\leq \|w\|_{L^\infty([0,\frac{T}{2}],\overline{A}_{\theta,\frac{1}{\theta}})} + \|\overline{w}\|_{L^\infty([\frac{T}{2},T],\overline{A}_{\theta,\frac{1}{\theta}})}\\ &\leq \|w\|_V + \|\overline{w}\|_V.\end{aligned}$$

We will now show that the right-hand side is bounded independently of u. For this we will only examine $\|w\|_V$, for $\|\overline{w}\|_V$ can be handled by exchanging w by $\overline{w}$ in the following calculations. Recall that $\|u\|_{V_I} \leq 1$, so by lemma 28 there holds $\|u'\|_{L^\infty(I,A_1)} \leq C\|u\|_{V_I}$.

$$\|w\|_{L^{p_0}(\mathbb{R}^{\geq 0},A_0)} = \|u\cdot\varphi\|_{L^{p_0}(\mathbb{R}^{\geq 0},A_0)} \leq \|\varphi\|_\infty\|u\|_{L^{p_0}(I,A_0)} \leq C_\varphi\,\|u\|_{L^{p_0}(I,A_0)}.$$

$$\begin{aligned}\|w'\|_{L^{p_1}(\mathbb{R}^{\geq 0},A_1)} &= \|(u\cdot\varphi)'\|_{L^{p_1}(\mathbb{R}^{\geq 0},A_1)}\\ &\leq \|u'\cdot\varphi\|_{L^{p_1}(\mathbb{R}^{\geq 0},A_1)} + \|\,u\cdot\varphi'\|_{L^{p_1}(\mathbb{R}^{\geq 0},A_1)}\\ &= \|u'\cdot\varphi\|_{L^{p_1}(\mathbb{R}^{\geq 0},A_1)} + \|\,u\cdot\varphi'\|_{L^{p_1}(I,A_1)}\\ &\leq \|\varphi\|_\infty \underbrace{\|u'\|_{L^{p_1}(I,A_1)}}_{\leq\|u\|_{V_I}} + \|\varphi\|_{1,\infty}\underbrace{\|u\|_{L^{p_1}(I,A_1)}}_{\leq C\,\|u\|_{V_I}}\\ &\leq \big(1 + 2\,C\,\|\varphi\|_{1,\infty}\big)\,\|u\|_{V_I}\\ &\leq C_\varphi\,\|u\|_{V_I}.\end{aligned}$$

Hence

$$\begin{aligned}\|w\|_V &\le \|w\|^{1-\theta}_{L^{p_0}(\mathbb{R}^{\ge 0},A_0)}\|w'\|^{\theta}_{L^{p_1}(\mathbb{R}^{\ge 0},A_1)}\\ &\le C_\varphi \|u\|^{1-\theta}_{L^{p_0}(I,A_0)}\|u\|^{\theta}_{V_I}\\ &\le C_\varphi\|u\|_{V_I}.\end{aligned}$$

This shows on the one hand that $V_I \to C(I,\overline{A}_{\theta,\frac{1}{\theta}})$ and on the other hand that

$$\begin{aligned}\|u\|_{L^\infty(I,\overline{A}_{\theta,\frac{1}{\theta}})} &\le \|w\|_{L^\infty([0,\frac{T}{2}],\overline{A}_{\theta,\frac{1}{\theta}})} + \|\overline{w}\|_{L^\infty([\frac{T}{2},T],\overline{A}_{\theta,\frac{1}{\theta}})}\\ &\le C_\varphi\|u\|^{1-\theta}_{L^{p_0}(I,A_0)}\|u\|^{\theta}_{V_I}\\ &= C_\varphi \max\big\{\|u\|_{L^{p_0}(I,A_0)}, \|u\|^{1-\theta}_{L^{p_0}(I,A_0)}\|u'\|^{\theta}_{L^{p_1}(I,A_1)}\big\}.\end{aligned}$$

This proves the theorem. □

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