Advances in Differential Equations Volume 6, Number 3, March 2001, pp. 257–302

ON WEAK SOLUTIONS TO A CLASS OF NON-NEWTONIAN INCOMPRESSIBLE FLUIDS IN BOUNDED THREE-DIMENSIONAL DOMAINS: THE CASE $p \geq 2$

J. Málek and J. Nečas

Mathematical Institute of Charles University
Sokolovská 83, 18675 Prague 8, Czech Republic

M. Růžička

Institute of Applied Mathematics, University of Bonn
Beringstraße 4–6, 53115 Bonn, Germany

(Submitted by: Emmanuele DiBenedetto)

Abstract. Existence and regularity properties of solutions for the evolutionary system describing unsteady flows of incompressible fluids with shear dependent viscosity are studied. The problem is considered in a bounded, smooth domain of $\mathbb{R}^3$ with Dirichlet boundary conditions. The nonlinear elliptic operator, which is related to the stress tensor, has p structure. The paper deals with the case $p \geq 2$, for which the existence of weak solutions is proved. If $p \geq \frac{9}{4}$ then a weak solution is strong and unique among all weak solutions.

1. Introduction and setting of the problem. This paper deals with unsteady flows of an incompressible fluid in a bounded domain $\Omega \subset \mathbb{R}^d$, $d > 1$ described by the system of equations

$$\begin{aligned} \operatorname{div} \mathbf{v} &= 0\,, \\ \rho \frac{\partial \mathbf{v}}{\partial t} - \operatorname{div} \mathbf{T}^E + \rho v_k \frac{\partial \mathbf{v}}{\partial x_k} &= -\nabla \pi + \rho \mathbf{f}\,, \end{aligned} \tag{1.1}$$

where $\mathbf{v} = (v_1, v_2, \ldots, v_d)$ is the velocity, π represents the pressure, ρ is a positive constant determining the density of a material, $\mathbf{f} = (f_1, f_2, \ldots, f_d)$ stands for the given external body forces and $\mathbf{T}^E$ denotes the extra stress tensor.

Accepted for publication in revised form December 1999.
AMS Subject Classifications: 35K35, 35K55, 76D99, 76D05.

Let a finite $T > 0$ be given. Then all functions are evaluated at (t, x), where $t \in [0, T]$ and $x \in \Omega$. We also use the usual summation convention throughout the whole text. Let us further denote by $\mathbf{D}$ the symmetric part of the velocity gradient; i.e., $\mathbf{D} = \mathbf{D}(\mathbf{v}) \equiv \frac{1}{2}\left[(\nabla \mathbf{v}) + (\nabla \mathbf{v})^T\right]$. We consider a constitutive relation for $\mathbf{T}^E$ of the form

$$\mathbf{T}^E = \mathbf{T}(\mathbf{D})\,. \tag{1.2}$$

Before specifying precise assumptions for $\mathbf{T}$, let us first think of the following four examples: for $p > 1$ and $\nu_0 > 0$ we set

$$\begin{aligned} &\text{(a)} \qquad \mathbf{T}^E = 2\nu_0 |\mathbf{D}|^{p-2}\mathbf{D}\,,\\ &\text{(b)} \qquad \mathbf{T}^E = 2\nu_0 (1 + |\mathbf{D}|^2)^{\frac{p-2}{2}}\mathbf{D}\,,\\ &\text{(c)} \qquad \mathbf{T}^E = 2\nu_0 (1 + |\mathbf{D}|)^{p-2}\mathbf{D}\,,\\ &\text{(d)} \qquad \mathbf{T}^E = 2\nu_0 (1 + |\mathbf{D}|^{p-2})\mathbf{D}\,, \end{aligned} \tag{1.3}$$

where $|\mathbf{D}|$ denotes the usual Euclidean matrix-norm.

On the one hand, the models (a)–(d) have some joint properties. Firstly, for $p = 2$ all formulas reduce to the Stokes law, i.e., $\mathbf{T}^E = 2\nu_0\mathbf{D}$, and (1.1) turns into the Navier-Stokes system, which is a model for Newtonian fluids.

Secondly, we can easily construct scalar potentials to $\mathbf{T}^E$ in (1.3) (a)–(d). In fact, all examples for $\mathbf{T}^E$ can be written in the form

$$\mathbf{T}^E = 2\nu_0 \mu(|\mathbf{D}|^2)\mathbf{D}\,, \tag{1.4}$$

where $\mu : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is the generalized viscosity function. The corresponding potential $\Phi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ to $\mathbf{T}^E$ given in (1.4) is defined by

$$\Phi(|\mathbf{D}|^2) \equiv \nu_0 \int_0^{|\mathbf{D}|^2} \mu(s)\,ds\,,$$

and we have for $r, s = 1, 2, \dots, d$

$$T_{rs}(\cdot) = \partial_{rs}\Phi(\cdot) \equiv \frac{\partial\Phi(\cdot)}{\partial D_{rs}}\,, \qquad \Phi(0) = \partial_{rs}\Phi(0) = 0\,.$$

Considering a simple shear flow, i.e., $\mathbf{v} = (v_1(x_2), 0, 0)$, the quantity $\kappa \equiv |v_1'(x_2)|$ ($= 2|\mathbf{D}|$ in the considered motion) is called shear rate. This is why the fluids constituted by (1.3) are sometimes named *fluids with shear-dependent viscosity*. Models belonging to this class of non-Newtonian fluid mechanics are frequently used in engineering practice, as discussed in Málek, Rajagopal, Růžička [13], for example.

Finally, it is worth remarking that all models (a)–(d) in (1.3) satisfy the p-coercivity condition, i.e.,

$$\mathbf{T}^E \cdot \mathbf{D} \geq 2\nu_0 |\mathbf{D}|^p \tag{1.5}$$

and they have $(p-1)$ growth, which means $|\mathbf{T}^E| \leq c(1+|\mathbf{D}|)^{p-1}$, $c>0$. On the other hand, despite their similar structure, the graphs of $\mu(|\mathbf{D}|^2)$ differ dramatically from each other (cf. Figures 1.1 and 1.2 in [11]). The different asymptotic behaviour of $\mu(s)$ as $s \to 0^+$ or $s \to \infty$, makes the class of investigated models robust and therefore very useful.

We complement the equations (1.1) by an initial condition

$$\mathbf{v}(0,\cdot) = \mathbf{v}_0(\cdot) \qquad \text{in } \Omega\,, \tag{1.6}$$

and by Dirichlet boundary conditions

$$\mathbf{v}(t,x) = \mathbf{0} \qquad \text{for all } (t,x) \in [0,T) \times \partial\Omega\,. \tag{1.7}$$

Simpler than (1.7) are space-periodic boundary conditions. In that case, Ω is a d-dimensional cube with sides of finite length $L>0$ and

$$\mathbf{v}, \pi \text{ are periodic with period } L \text{ in each variable } x_i\,, i=1,2,\dots,d\,. \tag{1.8}$$

By the *Problem (NS-Dir)$_p$*, we will mean the problem (1.1), (1.2), (1.6), (1.7), while we will call the problem (1.1), (1.2), (1.6), (1.8) the *Problem (NS-Per)$_p$*. Since no assumptions for $\mathbf{T}$ in (1.2) have been specified yet, one should have in mind the examples (1.3) (a)–(d).

Our long-lasting aim is to study the global-in-time existence of (weak) solutions to both the *Problem (NS-Dir)$_p$* and the *Problem (NS-Per)$_p$*, and to investigate their further qualitative properties[1] in dependence on the parameter p.

Regarding the *Problem (NS-Dir)$_p$*, the first mathematical investigations go back to Ladyzhenskaya's lecture at the International Mathematical Congress in 1966, where she proposed, among others, to study the system (1.1),

[1] Here, however, we discuss the questions of uniqueness and regularity of weak solutions only.

(1.6), (1.7) and (1.3)(d) with $p = 4$. Later on these first results were extended, and presented in further contributions of Ladyzhenskaya; cf. [5], [6] and [7]. Combining monotone operator theory and compactness arguments, she proved the existence of weak solutions to all models (a)–(d) in (1.3) if $p \geq 1 + \frac{2d}{d+2}$ and their uniqueness if $p \geq \frac{d+2}{2}$. See also Lions [8] for a comparable proof of the same results. Recently, Amann [1] showed the existence of regular (classical) solutions to the *Problem (NS-Dir)$_p$* provided that the data $\mathbf{f}$ and $\mathbf{v}_0$ are small and assuming that the tensor function $\mathbf{T}$ in (1.2) satisfies

$$\partial_{rs}\mathbf{T}(\mathbf{0}) = const. > 0\,.$$

Thus, in particular, these results are related to the models (b), (c) in (1.3) if $p \geq 1$ and to (d) if $p \geq 2$.

More results are known about the *Problem (NS-Per)$_p$* due to a series of papers Bellout, Bloom, Nečas [3] (see also Bellout, Bloom, Nečas [2]), Málek, Nečas, Růžička [9], Málek, Rajagopal, Růžička [13] and finally the monograph Málek, Nečas, Rokyta, Růžička [11] which will be sometimes used as a reference for detailed explanations and proofs of some assertions.

Excluding the example (a) in (1.3) for $p > 2$, the following has been proved in these publications:

the existence of a weak solution	for $p \in (\frac{3d}{d+2}, \frac{2d}{d-2})$	if $d = 3\,,4\,;$
the existence of a strong solution	for $p \geq 1 + \frac{2d}{d+2}$	if $d \geq 3\,;$
	for $p > 1$	if $d = 2\,;$
uniqueness of the weak solution	for $p \geq 1 + \frac{2d}{d+2}$	if $d \geq 2\,.$

It is natural to ask whether the same results are valid for the *Problem (NS-Dir)$_p$*, too. Since the superquadratic case ($p > 2$) and the subquadratic case ($p < 2$) require different approaches, we investigate the former in this paper, and we devote to the latter a forthcoming paper. We also concentrate, mainly for methodological reasons, on three-dimensional situations, i.e., $\Omega \subset \mathbb{R}^3$. If $d = 2$ and $p \geq 2$, the existence and uniqueness of weak solutions follows already from Ladyzhenskaya [5], [6] and [7] and Lions [8], while the regularity (i.e., the existence of the unique strong solution) can be obtained following the lines of the present paper. Finally, we also restrict ourselves to the very interesting[2] and most complicated case $p \in [2, 3)$. The case $p \geq 3$ will require a slightly different technique which we intend to investigate later.

[2]The interest in studying the case $p \in [2, 3)$ is based on the fact that up to the present

It might be worth remarking that if the nonlinearity $\mathbf{T}$ had a potential Φ depending on the modulus of the full velocity gradient $|\nabla \mathbf{v}|$ and not only on the modulus of the symmetric velocity gradient $|\mathbf{D}(\mathbf{v})|$, then the results in the preprint [10] hold not only for a larger set of parameters p ($p \in [2,6)$), but also provide stronger information on regularity of $\mathbf{v}$. We want to emphasize, however, that although the preprint [10] is formulated for models where the potential Φ depends on $|\mathbf{D}|$, it holds only for cases when the potential Φ of the nonlinearity $\mathbf{T}$ depends on $|\nabla \mathbf{v}|$, since Lemma 7.18 in [10] is correct only in this case and wrong if Φ depends on $|\mathbf{D}|$.

Before formulating the main result, we will define some useful function spaces and notions. Let $\big(X(\Omega), \|\cdot\|_{X(\Omega)}\big)$ be a Banach space of scalar functions defined in Ω. Then $X(\Omega)^3$ (respectively $X(\Omega)^{3\times 3}$) represents the space of vector-valued (respectively tensor-valued) functions whose components belong to $X(\Omega)$. Let further p, $q > 1$ and $k > 0$. Then $\big(L^p(\Omega), \|\cdot\|_p\big)$ denotes the usual Lebesgue spaces and $\big(W^{k,p}(\Omega), \|\cdot\|_{k,p}\big)$ is used for standard Sobolev spaces. Finally, by

$$\Big(L^q((0,T); X(\Omega)), \Big(\int_0^T \|\cdot\|_{X(\Omega)}^q dt\Big)^{1/q}\Big)$$

we denote Bochner spaces. Sometimes we write I instead of $(0,T)$ and Q_T instead of $I \times \Omega$. Also as usual, $\mathcal{D}(\Omega)$ denotes the space of smooth functions with compact support in Ω. We further define

$$\begin{aligned} \mathcal{V} &\equiv \big\{\boldsymbol{\psi} \in \mathcal{D}(\Omega)^3 : \operatorname{div} \boldsymbol{\psi} = 0\big\}\,, \\ H &\equiv \text{the closure of } \mathcal{V} \text{ with respect to the } \|\cdot\|_2\text{–norm}\,, \\ V_p &\equiv \text{the closure of } \mathcal{V} \text{ with respect to the } \|\nabla\cdot\|_p\text{–norm}\,. \end{aligned}$$

In order to give the definition of weak and strong solutions to the *Problem (NS-Dir)$_p$*, we will specify assumptions on the tensor function $\mathbf{T}$ from (1.2).

results, the existence of weak solutions to the *Problem (NS-Dir)$_p$* was known only for $p \geq 11/5$ and for the special linear case when $p = 2$ (this means for the Navier-Stokes equations). The results presented here cover this gap, providing the existence of weak solutions to the *Problem (NS-Dir)$_p$* for $p \geq 2$. In addition, we also give information about the integrability of second derivatives if $p \geq 9/4$, which is important for the investigation of large-time behavior in the range $p \in [9/4, 5/2)$, since in this range this information is essential while if $p \in [5/2, \infty)$ the global existence of a finite-dimensional attractor can be proved using $L^\infty(0,\infty; W^{1,p})$ regularity only, which is known if $p \geq 5/2$ for a weak solution (see [12] for more details).

Let $\mathbb{R}^{3\times3}_{sym} \equiv \{\mathbf{D} \in \mathbb{R}^{3\times3} : D_{ij} = D_{ji}, \quad i,j = 1,2,3\}$. We assume the existence of a potential $\Phi : \mathbb{R}^+_0 \to \mathbb{R}^+_0$ and constants $C_1, C_2 > 0$ such that for some $p > 1$ and for all $r,s,m,n = 1,2,3$, $\mathbf{B},\mathbf{D} \in \mathbb{R}^{3\times3}_{sym}$

$$T_{rs}(\mathbf{D}) = \partial_{rs}\Phi(|\mathbf{D}|^2)\,, \quad \Phi(0) = \partial_{rs}\Phi(0) = 0\,, \tag{1.9}$$

$$\partial_{ij}\partial_{kl}\Phi(|\mathbf{D}|^2)B_{ij}B_{kl} \geq C_1(1+|\mathbf{D}|)^{p-2}|\mathbf{B}|^2\,, \tag{1.10}$$

$$\big|\partial_{rs}\partial_{mn}\Phi(|\mathbf{D}|^2)\big| \leq C_2(1+|\mathbf{D}|)^{p-2}\,. \tag{1.11}$$

Remark 1.12. It is easy to check that examples (b)–(d) in (1.3) satisfy (1.9)–(1.11), while the analogy of the condition (1.10) for the example (a) reads

$$\partial_{ij}\partial_{kl}\Phi(|\mathbf{D}|^2)B_{ij}B_{kl} \geq C_1|\mathbf{D}|^{p-2}|\mathbf{B}|^2\,.$$

We do not consider this case in this paper.

Definition 1.13. A function $\mathbf{v}$ is said to be a weak solution to the *Problem (NS-Dir)$_p$* if and only if $\mathbf{v} \in L^\infty(I;H) \cap L^p(I;V_p)$ and the weak formulation

$$\int_{Q_T}\big[-\mathbf{v}\cdot\frac{\partial\boldsymbol{\varphi}}{\partial t} + T_{ij}(\mathbf{D}(\mathbf{v}))D_{ij}(\boldsymbol{\varphi}) - v_k\mathbf{v}\cdot\frac{\partial\boldsymbol{\varphi}}{\partial x_k}\big]\,dx\,dt = \int_{Q_T}\mathbf{f}\cdot\boldsymbol{\varphi}\,dx\,dt + \int_\Omega \mathbf{v}_0\cdot\boldsymbol{\varphi}\,dx \tag{1.14}$$

is fulfilled for all $\boldsymbol{\varphi} \in \mathcal{D}(-\infty,T;\mathcal{V})$.

Definition 1.15. A function $\mathbf{v}$ is said to be a strong solution to the *Problem (NS-Dir)$_p$* if and only if

$$\mathbf{v} \in C(I;H) \cap L^\infty(I;V_p) \cap L^{\frac{2}{p-1}}(I;W^{2,\frac{6}{p+1}}(\Omega)^3)\,, \quad \frac{\partial\mathbf{v}}{\partial t} \in L^2(I;L^2(\Omega)^3)\,,$$

and for all $\boldsymbol{\varphi} \in V_p$ and almost every $t \in I$

$$\big(\frac{\partial\mathbf{v}(t)}{\partial t},\boldsymbol{\varphi}\big) + \int_\Omega T_{ij}(\mathbf{D}(\mathbf{v}(t)))D_{ij}(\boldsymbol{\varphi})\,dx + \big(v_k(t)\frac{\partial\mathbf{v}(t)}{\partial x_k},\boldsymbol{\varphi}\big) = \big(\mathbf{f}(t),\boldsymbol{\varphi}\big)\,. \tag{1.16}$$

The brackets $(\mathbf{h},\mathbf{g})$ stand for $\int_\Omega \mathbf{h}\cdot\mathbf{g}\,dx$, where $\mathbf{h}\cdot\mathbf{g} \in L^1(\Omega)$.

Now we formulate the main result of this paper.

Theorem 1.17. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain, $\partial\Omega \in \mathcal{C}^3$. Let $p \in [2,3)$, $\mathbf{v}_0 \in V_p$ and $\mathbf{f} \in L^2(I;L^2(\Omega)^3)$. Assume that $\mathbf{T}$ in* (1.2) *satisfies* (1.9)–(1.11). *Then there exists a weak solution $\mathbf{v}$ to the Problem (NS-Dir)$_p$. Moreover, if*

$$p \geq \frac{9}{4} = \frac{11}{5} + 0.05 \tag{1.18}$$

then the solution is strong and unique, and for all $\Omega_0 \subset\subset \Omega$ *and* $0 \le t_1 < t_2 \le T$

$$\int_{t_1}^{t_2} \int_{\Omega_0} (1 + |\mathbf{D}(\mathbf{v})|)^{p-2} |\mathbf{D}(\nabla \mathbf{v})|^2 dx\, dt \ < \infty \,. \tag{1.19}$$

In particular, $\mathbf{v} \in L^2(I; W^{2,2}_{\mathrm{loc}}(\Omega)^3)$*. The tangential derivatives* $\frac{\partial \mathbf{v}}{\partial \tau^r}$*,* $r = 1, 2$*, satisfy an estimate analogous to* (1.19)*. Thus,* $\nabla \frac{\partial \mathbf{v}}{\partial \tau^r} \in L^2(I; L^2(\Omega)^{3\times 3})$*.*

Remark 1.20. By the results in [7] and [8], the existence of weak solutions is obtained for $p \ge \frac{11}{5}$. Thus Theorem 1.17 fills the gap between 2 and $\frac{11}{5}$. Further, for p satisfying (1.18) we obtain regularity properties for $\mathbf{v}$. However, these results are slightly worse than those for the *Problem (NS-Per)*$_p$, where strong solutions exists for all $p \ge \frac{11}{5}$ and belong to the space $L^2(I; W^{2,2}(\Omega)^3)$. It might be useful to emphasize that we are able to strengthen our results significantly if the nonlinearity $\mathbf{T}$ comes from a potential depending on the modulus of the full velocity gradient of $\mathbf{v}$. More precisely, if $\mathbf{T} = \partial\Phi(|\nabla \mathbf{v}|)$, then the weak solutions exist for $p \in [2, 6)$, and the existence of strong solutions is guaranteed for $p \in (\frac{20}{9}, 6)$. In addition, the strong solutions belong to $L^{p'}(I; W^{2,p'}(\Omega)^3)$, $p' = p/(p-1)$; see [10]. We want to warn the reader before possible confusions: the preprint [10] is formulated for $\mathbf{T} = \partial\Phi(|\mathbf{D}|)$; however, due to the wrong proof of Lemma 7.18 it is valid only for $\mathbf{T}$ of the type $\mathbf{T} = \partial\Phi(|\nabla \mathbf{v}|)$.

The proof of Theorem 1.17 is split into several sections. In the next section we clarify, using some formal arguments, the main difficulty of the problem connected with low regularity of the pressure. This difficulty is overcome by constructing an appropriate twofold approximation of the original problem based on both the mollification of the convective term $v_{\varepsilon k} \frac{\partial \mathbf{v}}{\partial x_k}$ and on a quadratic approximation of the potential Φ, denoted by Φ_A. Section 3 deals with the existence and regularity of the weak solution to the approximate problem. In Section 4 we carry out the limiting process from Φ_A to Φ, while in Section 5 we let $\varepsilon \to 0$ and we finally obtain the results stated in Theorem 1.17. The appendix contains some helpful assertions.

2. Definition of the approximate problem. We start with a lemma collecting some consequences of the assumption (1.9)–(1.11). The proof can be found for example in Málek, Nečas, Rokyta, Růžička [11], Chapter 5, Lemma 1.19 and Lemma 1.35.

Lemma 2.1. *Let* $\mathbf{T}$ *and* Φ *satisfy* (1.9)–(1.11)*. Assume that* $p \ge 2$*. Then there exist* $C_i, i = 3, \dots, 7$*, such that for all* $\mathbf{B} \in \mathbb{R}^{3\times 3}_{\mathrm{sym}}$

$$\mathbf{T}(\mathbf{B}) \cdot \mathbf{B} \ge C_3 (1 + |\mathbf{B}|^{p-2}) |\mathbf{B}|^2 \,, \tag{2.2}$$

$$\big|\mathbf{T}(\mathbf{B})\big| \le C_4(1+|\mathbf{B}|)^{p-1}\,, \tag{2.3}$$

$$C_5(1+|\mathbf{B}|^{p-2})|\mathbf{B}|^2 \le C_3(\tfrac{1}{2}+\tfrac{1}{p}|\mathbf{B}|^{p-2})|\mathbf{B}|^2 \le \Phi(|\mathbf{B}|^2) \le C_6(1+|\mathbf{B}|)^p \tag{2.4}$$

and for all $\mathbf{B}$, $\mathbf{D} \in \mathbb{R}^{3\times 3}_{\mathrm{sym}}$

$$\big(\mathbf{T}(\mathbf{B})-\mathbf{T}(\mathbf{D})\big)\cdot\big(\mathbf{B}-\mathbf{D}\big) \ge C_7|\mathbf{B}-\mathbf{D}|^2\,. \tag{2.5}$$

Learning from the methods used for the *Problem (NS-Per)*$_p$, we start with a mollification of the convective term. Let $\omega \in \mathcal{D}(\mathbb{R}^3)$ be a usual mollification kernel with support in $B_1(0) \equiv \{x \in \mathbb{R}^3 : |x| < 1\}$ and $\int_{\mathbb{R}^3}\omega\,dx = 1$. For every $\varepsilon > 0$ we define $\omega_\varepsilon = \frac{1}{\varepsilon^3}\omega(\frac{x}{\varepsilon})$ and we put $\mathbf{v}_\varepsilon \equiv \mathbf{v} * \omega_\varepsilon$; i.e.,

$$\mathbf{v}_\varepsilon(x) = \frac{1}{\varepsilon^3}\int_{\mathbb{R}^3}\omega\big(\frac{x-y}{\varepsilon}\big)\mathbf{v}(y)\,dy\,.$$

It is easy to observe that

$$\|\mathbf{v}_\varepsilon\|_{2,2} \le \frac{1}{\varepsilon^2}\|\mathbf{v}\|_2\,, \tag{2.6}$$

and for $\mathbf{v} \in L^p(\Omega)$, always $\|\mathbf{v}_\varepsilon\|_p \le \|\mathbf{v}\|_p$.

Now, let us consider[3] the *Problem (NS-Dir)*$^\varepsilon_p$: to find $(\mathbf{v},\pi) = (\mathbf{v}^\varepsilon,\pi^\varepsilon)$ such that

$$\begin{gathered}\operatorname{div}\mathbf{v} = 0 \quad \text{in } Q_T\,,\\ \frac{\partial\mathbf{v}}{\partial t} - \operatorname{div}\mathbf{T}(\mathbf{D}(\mathbf{v})) + v_{\varepsilon k}\frac{\partial\mathbf{v}}{\partial x_k} = -\nabla\pi + \mathbf{f} \quad \text{in } Q_T\,,\\ \mathbf{v}(0,\cdot) = \mathbf{v}_0(\cdot) \quad \text{in } \Omega \qquad \text{and} \qquad \mathbf{v} = \mathbf{0} \ \text{ at } I\times\partial\Omega\,,\end{gathered} \tag{2.7}$$

where $\mathbf{T}$ satisfies (1.9)–(1.11).

Now we formally derive a priori estimates. First, we multiply equation $(2.7)_2$ by $\mathbf{v}$. Integrating over Q_T, using (2.2), partial integration and a p-version of Korn's inequality (see (6.2) in the appendix), we obtain

$$\|\mathbf{v}(t)\|_2^2 + C_3K_2^2\int_0^t\|\nabla\mathbf{v}\|_2^2\,d\tau + 2C_3\,K_p^p\int_0^t\|\nabla\mathbf{v}\|_p^p\,d\tau \le c(\mathbf{f},\mathbf{v}_0)\,, \tag{2.8}$$

[3]We put for simplicity $\rho \equiv 1$ in the sequel, because the size of the constant density plays no role in the whole analysis.

where $c(\mathbf{f}, \mathbf{v}_0)$ denotes a generic constant depending only on the data; in case of (2.8) $c(\mathbf{f}, \mathbf{v}_0) = \|\mathbf{v}_0\|_2^2 + c\|\mathbf{f}\|_2^2$. The estimate (2.8) is independent of ε, and we have

$$\mathbf{v} \in L^\infty(I; H) \cap L^p(I; V_p)\,. \tag{2.9}$$

Multiplying $(2.7)_2$ by $\frac{\partial \mathbf{v}}{\partial t}$ (which is also divergence-free and vanishing at $\partial\Omega$), integrating over Q_t and using the fact that

$$\int_\Omega T_{ij}(\mathbf{D}(\mathbf{v})) D_{ij}(\frac{\partial \mathbf{v}}{\partial t})\, dx = \frac{d}{dt} \int_\Omega \Phi(|\mathbf{D}(\mathbf{v})|^2)\, dx\,,$$

along with (2.4) we get

$$\int_0^t \|\frac{\partial \mathbf{v}}{\partial t}\|_2^2\, d\tau + K_p^p \|\nabla \mathbf{v}(t)\|_p^p \le c(\mathbf{f}, \mathbf{v}_0) + \int_{Q_T} |\mathbf{v}_\varepsilon|^2 |\nabla \mathbf{v}|^2\, dx\, dt \equiv J\,. \tag{2.10}$$

Now, since $\mathbf{v}$ satisfies (2.9) it is possible to prove that

$$J \le c(\mathbf{f}, \mathbf{v}_0) \qquad \text{if } \ p \ge \frac{12}{5}\,, \tag{2.11}$$

and

$$J \le c\,(\frac{1}{\varepsilon}, \mathbf{f}, \mathbf{v}_0) \qquad \text{if } \ p \in [2, \frac{12}{5})\,. \tag{2.12}$$

Indeed, using for $p \in [\frac{12}{5}, 3)$ the interpolation inequality

$$\|\mathbf{v}\|_{\frac{2p}{2-p}} \le \|\mathbf{v}\|_2^{\frac{5p-12}{5p-6}} \|\mathbf{v}\|_{\frac{3p}{3-p}}^{\frac{6}{5p-6}}$$

we have

$$\begin{aligned} \int_\Omega |\mathbf{v}_\varepsilon|^2 |\nabla \mathbf{v}|^2 dx &\le \|\nabla \mathbf{v}\|_p^2 \|\mathbf{v}_\varepsilon\|_{\frac{2p}{p-2}}^2 \le c\|\nabla \mathbf{v}\|_p^{p + \frac{p(16-5p)}{5p-6}} \|\mathbf{v}\|_2^{2\frac{5p-12}{5p-6}} \\ &\le c\|\mathbf{v}\|_2^{2\frac{5p-12}{5p-6}} \|\nabla \mathbf{v}\|_p^{p\frac{16-5p}{5p-6}} \|\nabla \mathbf{v}\|_p^p =: h\, \|\nabla \mathbf{v}\|_p^p\,. \end{aligned} \tag{2.13}$$

Observe that $\frac{16-5p}{5p-6} \le 1$ if $p \ge \frac{11}{5}$. Thus due to (2.9) and $h \in L^1(0,T)$ we can derive (2.11) for $p \in [\frac{12}{5}, 3)$ from (2.10) and Gronwall's lemma. The estimate (2.12) is a consequence of (2.6) and the imbedding $W^{2,2}(\Omega) \hookrightarrow L^\infty(\Omega)$. So we have from (2.10)

$$\int_0^t \|\frac{\partial \mathbf{v}}{\partial t}\|_2^2\, d\tau + \|\nabla \mathbf{v}(t)\|_p^p \le \begin{cases} c\,(\frac{1}{\varepsilon}, \mathbf{f}, \mathbf{v}_0) & \text{if } p \in [2, \frac{12}{5})\,, \\ c(\mathbf{f}, \mathbf{v}_0) & \text{if } p \ge \frac{12}{5}\,. \end{cases} \tag{2.14}$$

In order to pass to the limit in the nonlinear term given by $\mathbf{T}$ (at least for $p \in [2, \frac{11}{5}))$ and in order to obtain higher regularity properties (for $p \geq \frac{11}{5}$) we need more information about $\nabla \mathbf{v}$. For the *Problem (NS-Per)$_p$* we used $-\Delta \mathbf{v}$ as a test function which is divergence-free, but in the case of the *Problem (NS-Dir)$_p$* the term $-\Delta \mathbf{v}$ does not vanish at $\partial\Omega$.

Let $\Omega' \subset \bar{\Omega}' \subset \Omega$ and let $\xi \in \mathcal{D}(\Omega)$ be a cut-off function such that $\xi \in [0,1]$ in Ω'. Now, we multiply $(2.7)_2$ by $-\Delta \mathbf{v} \xi^2$, a test function vanishing at the boundary, but not being divergence-free. Thus the pressure π enters the picture. Using the divergence-free constraint we have $-\int_\Omega \nabla \pi \Delta \mathbf{v} \xi^2 \, dx = \int_\Omega \pi \Delta \mathbf{v} \nabla \xi^2 \, dx$.

In order to clarify a key point that leads to the final approximation, we will consider only the main terms, "forgetting" cut-off functions. This yields

$$\begin{aligned} &\tfrac{1}{2}\|\nabla \mathbf{v}(t)\|_2^2 + \int_0^t \int_\Omega \partial_{ij}\partial_{kl}\Phi(|\mathbf{D}(\mathbf{v})|^2) D_{ij}(\nabla \mathbf{v}) \cdot D_{kl}(\nabla \mathbf{v}) \, dx \, d\tau \\ &\quad \leq \int_0^t \int_\Omega |\mathbf{f} \cdot \Delta \mathbf{v}| + |\mathbf{v}_\varepsilon| \, |\nabla \mathbf{v}||\Delta \mathbf{v}| \, dx \, d\tau + \int_0^t \int_\Omega |\pi||\Delta \mathbf{v}| \, dx \, d\tau \, . \end{aligned} \tag{2.15}$$

By (1.10) and Young's inequality, we get

$$\begin{aligned} \tfrac{1}{2}\|\nabla \mathbf{v}(t)\|_2^2 + \int_0^t \|\nabla^{(2)} \mathbf{v}\|_2^2 \, d\tau &\leq c(\delta) \int_0^T \|\mathbf{f}\|_2^2 \, dt + c(\delta) \int_0^t \int_\Omega |\mathbf{v}_\varepsilon|^2 |\nabla \mathbf{v}|^2 dx \, d\tau \\ &\quad + \delta \int_0^t \|\Delta \mathbf{v}\|_2^2 \, d\tau + \int_0^t \int_\Omega |\pi| \, |\Delta \mathbf{v}| \, dx \, d\tau \\ &\underset{(2.11)}{\overset{(2.12)}{\leq}} c\,(\frac{1}{\varepsilon}, \mathbf{f}, \mathbf{v}_0) + \int_0^t \int_\Omega |\pi| \, |\Delta \mathbf{v}| \, dx \, d\tau + \delta \int_0^t \|\Delta \mathbf{v}\|_2^2 \, d\tau \, . \end{aligned} \tag{2.16}$$

Since $\operatorname{div} \mathbf{T}(\mathbf{D}(\cdot))$ maps $L^p(I; V_p) \to L^{p'}(I; V_p^*)$, $p' = \frac{p}{p-1}$, we see that $\pi \in L^{p'}(Q_T)$, where $p' \leq 2$. This fact causes the main difficulty, because we have no information about $\Delta \mathbf{v}$ in $L^p(Q_T)$. If we estimate

$$\int_0^t \int_\Omega |\pi| \, |\Delta \mathbf{v}| \, dx \, d\tau \leq c(\delta) \int_0^t \|\pi\|_2^2 \, d\tau + \delta \int_0^t \|\nabla^{(2)} \mathbf{v}\|_2^2 \, d\tau \, ,$$

then we need that $\pi \in L^2(Q_T)$ at least on the level of approximations. In order to obtain this, we approximate the p-potential Φ by quadratic potentials Φ_A in the following way: for $A > 1$ and $\mathbf{B} \in \mathbb{R}^{3\times3}_{\text{sym}}$ we define

$$\Phi_A(|\mathbf{B}|^2) = \begin{cases} \Phi(|\mathbf{B}|^2) & \text{if } |\mathbf{B}| \leq A \, , \\ \alpha_2 |\mathbf{B}|^2 + \alpha_1 |\mathbf{B}| + \alpha_0 & \text{if } |\mathbf{B}| > A \, , \end{cases} \tag{2.17}$$

where α_0, α_1, α_2 are such that $\Phi_A \in C^2(\mathbb{R}_0^+)$. This gives

$$\begin{aligned}\alpha_0 &= \Phi(A^2) + 2\Phi''(A^2)A^4 - \Phi'(A^2)A^2,\\ \alpha_1 &= -4\Phi''(A^2)A^3, \qquad \alpha_2 = 2\Phi''(A^2)A^2 + \Phi'(A^2)\,.\end{aligned} \tag{2.18}$$

Defining $\mathbf{T}^A$ by

$$\mathbf{T}^A(\mathbf{B}) \equiv \partial\Phi_A(|\mathbf{B}|^2)\,, \tag{2.19}$$

we see (compare with $(2.23)_1$ below) that $\mathbf{T}^A$ has linear growth and $\operatorname{div}\mathbf{T}^A(\mathbf{D}(\cdot))$ maps $L^2(I;V_2)$ into $L^2(I;V_2^*)$. Thus we finally come to the definition of the approximate problem.

Definition 2.20. (the *Problem (NS-Dir)*$_p^{\varepsilon,A}$) For given $\varepsilon > 0$ and $A > 1$, we look for $\mathbf{v} = \mathbf{v}^{\varepsilon,A}$ and $\pi = \pi^{\varepsilon,A}$ solving the system

$$\begin{gathered}\operatorname{div}\mathbf{v} = 0 \quad \text{in } Q_T\,,\\ \frac{\partial\mathbf{v}}{\partial t} - \operatorname{div}\mathbf{T}^A(\mathbf{D}(\mathbf{v})) + v_{\varepsilon k}\frac{\partial\mathbf{v}}{\partial x_k} = -\nabla\pi + \mathbf{f} \quad \text{in } Q_T\,,\\ \mathbf{v}(0,\cdot) = \mathbf{v}_0(\cdot) \text{ in } \Omega \quad \text{ and } \quad \mathbf{v} = \mathbf{0} \quad \text{at } I\times\partial\Omega\,.\end{gathered} \tag{2.21}$$

We will need several inequalities connected with the potential Φ_A, which we summarize in the following lemma.

Lemma 2.22. *Assume that* (1.9)–(1.11) *hold. Then* Φ_A, $A > 1$, *given by* (2.17), (2.18) *and* $\mathbf{T}_A$ *given by* (2.19) *satisfy* (*for* i, j, k, l, r, $s = 1,2,3$ *and all* $\mathbf{B},\mathbf{D} \in \mathbb{R}^{3\times3}_{\mathrm{sym}}$)

$$\begin{aligned}&|\partial\Phi_A(|\mathbf{D}|^2)| \le \tilde{C}_1(A)(1+|\mathbf{D}|)\,, \quad |\partial^2\Phi_A(|\mathbf{D}|^2)| \le \tilde{C}_2(A)\,,\\ &\partial_{ij}\Phi_A(|\mathbf{D}|^2)\,D_{ij} \ge C_3|\mathbf{D}|^2\,, \quad C_8|\mathbf{D}|^2 \le \Phi_A(|\mathbf{D}|^2) \le C_9(1+|\mathbf{D}|^2)^{\frac{p}{2}}\,,\end{aligned} \tag{2.23}$$

$$\partial_{ij}\partial_{kl}\Phi_A(|\mathbf{D}|^2)B_{ij}B_{kl} \ge C_{10}\begin{cases}(1+|\mathbf{D}|)^{p-2}|\mathbf{B}|^2 & \text{if } |\mathbf{D}| \le A\,,\\ (1+A)^{p-2}|\mathbf{B}|^2 & \text{if } |\mathbf{D}| > A\,,\end{cases} \tag{2.24}$$

$$|\partial_{rs}\Phi_A| \le C_{11}(1+|\Phi_A|)^{\frac{p-1}{p}}\,, \tag{2.25}$$

$$|\partial_{kl}\partial_{rs}\Phi_A(|\mathbf{D}|^2)| \le C_{12}\begin{cases}(1+|\mathbf{D}|)^{p-2} & \text{if } |\mathbf{D}| \le A\,,\\ (1+A)^{p-2} & \text{if } |\mathbf{D}| > A\,,\end{cases} \tag{2.26}$$

$$1+|\partial\Phi_A(|\mathbf{D}|^2)| \ge C_{13}\begin{cases}(1+|\mathbf{D}|)^{p-1} & \text{if } |\mathbf{D}| \le A\,,\\ (1+A)^{p-1} & \text{if } |\mathbf{D}| > A\,.\end{cases} \tag{2.27}$$

Proof. All assertions are evident if $|\mathbf{D}| \le A$. Therefore we assume $|\mathbf{D}| > A$ in what follows. We first observe that (1.10) yields (for all $\mathbf{B},\mathbf{D} \in \mathbb{R}^{3\times3}_{\mathrm{sym}}$)

$$4\Phi''(|\mathbf{D}|^2)(\mathbf{D}\cdot\mathbf{B})^2 + 2\Phi'(|\mathbf{D}|^2)|\mathbf{B}|^2 \ge C_1(1+|\mathbf{D}|)^{p-2}|\mathbf{B}|^2\,.$$

In particular, for $\mathbf{B} = \frac{1}{\sqrt{3}}\mathbf{I}$ and $\mathbf{D} = \frac{A}{\sqrt{3}}\mathbf{I}$ we obtain

$$4\Phi''(A^2)A^2 + 2\Phi'(A^2) \ge C_1(1+A)^{p-2}. \tag{2.28}$$

Similarly, from (2.2), we have

$$2\Phi'(|\mathbf{D}|^2) \ge C_3(1 + |\mathbf{D}|^{p-2})\,,$$

and C_3 is given by $C_3 = \frac{C_1}{2(p-1)}$ (cf. [11], Chapter 5, Lemma 1.19). Choosing $\mathbf{D} = \frac{A}{\sqrt{3}}\mathbf{I}$ gives

$$2\Phi'(A^2) \ge C_3(1 + A^{p-2})\,. \tag{2.29}$$

On the other hand, (1.11) and (2.3) imply

$$4\Phi''(|\mathbf{D}|^2)|\mathbf{D}|^2 + 2\Phi'(|\mathbf{D}|^2) \le \tilde{C}_2(1 + |\mathbf{D}|)^{p-2}. \tag{2.30}$$

$$2\Phi'(|\mathbf{D}|^2)|\mathbf{D}| \le C_4(1 + |\mathbf{D}|^{p-1}). \tag{2.31}$$

As

$$\begin{aligned} \partial_{ij}\Phi_A(|\mathbf{D}|^2) &= 2\alpha_2 D_{ij} + \alpha_1 \frac{D_{ij}}{|\mathbf{D}|}\,, \\ \partial_{ij}\partial_{rs}\Phi_A(|\mathbf{D}|^2) &= 2\alpha_2 \delta_{ir}\delta_{js} + \alpha_1 \frac{\delta_{ir}\delta_{js}}{|\mathbf{D}|} - \alpha_1 \frac{D_{ij}D_{rs}}{|\mathbf{D}|^3}\,, \end{aligned} \tag{2.32}$$

we see that $(2.23)_{1-3}$, (2.24) and (2.26) are consequences of (2.28)–(2.32). Let us verify (2.24), for example. We have ($|\mathbf{D}| > A$)

$$\begin{aligned} \partial_{ij}\partial_{rs}\Phi_A(|\mathbf{D}|^2)B_{ij}B_{kl} &= 2\alpha_2|\mathbf{B}|^2 + \alpha_1\frac{|\mathbf{B}|^2}{|\mathbf{D}|} - \alpha_1\frac{(\mathbf{B}\cdot\mathbf{D})^2}{|\mathbf{D}|^3} \qquad (2.33) \\ &= \big(4\Phi''(A^2)A^2 + 2\Phi'(A^2)\big)|\mathbf{B}|^2 - 4\Phi''(A^2)A^3\frac{|\mathbf{B}|^2}{|\mathbf{D}|} + 4\Phi''(A^2)A^3\frac{(\mathbf{B}\cdot\mathbf{D})^2}{|\mathbf{D}|^3}\,. \end{aligned}$$

Then either $\Phi''(A^2) \ge 0$ or $\Phi''(A^2) < 0$. If $\Phi''(A^2) \ge 0$, we rewrite (2.33) as

$$\begin{aligned} &2\Phi'(A^2)|\mathbf{B}|^2 + 4\Phi''(A^2)\frac{A^3}{|\mathbf{D}|^3}\big(|\mathbf{B}|^2|\mathbf{D}|^2(|\mathbf{D}| - A)^2 + A(\mathbf{D}\cdot\mathbf{B})^2\big) \\ &\ge 2\Phi'(A^2)|\mathbf{B}|^2 \overset{(2.29)}{\ge} C_3(1 + A^{p-2})|\mathbf{B}|^2\,. \end{aligned} \tag{2.34}$$

If $\Phi''(A^2)$ is negative then (2.33) can be written as

$$\begin{aligned}&\big(4\Phi''(A^2)A^2+2\Phi'(A^2)\big)|\mathbf{B}|^2-4\Phi''(A^2)\frac{A^3}{|\mathbf{D}|^3}\big(|\mathbf{B}|^2|\mathbf{D}|^2-(\mathbf{B}\cdot\mathbf{D})^2\big)\\&\overset{(2.28)}{\geq} C_1(1+A)^{p-2}|\mathbf{B}|^2\geq\frac{C_1}{2}(1+A^{p-2})|\mathbf{B}|^2\geq C_3(1+A^{p-2})|\mathbf{B}|^2,\end{aligned}\tag{2.35}$$

where we used the relation between C_1 and C_3 and the simple inequality $(1+A)^{p-2}\geq\frac{1}{2}(1+A^{p-2})$ valid for $p\geq 2$. Thus, we conclude that (2.24) holds independently of the sign of $\Phi''(A^2)$. Moreover, (2.33)–(2.35) with $\mathbf{B}=\mathbf{D}$ leads to

$$4\Phi''(A^2)A^2+2\Phi'(A^2)\geq C_3(1+A^{p-2})\,.\tag{2.36}$$

Next, (2.4) yields $(2.23)_4$. Indeed, we have

$$\Phi_A(|\mathbf{D}|^2)=2\Phi''(A^2)A^2(|\mathbf{D}|^2-2A|\mathbf{D}|+A^2)+\Phi'(A^2)(|\mathbf{D}|^2-A^2)+\Phi(A^2).$$

If $\Phi''(A^2)<0$ we rewrite and estimate the right-hand side as

$$\begin{aligned}&\big(2\Phi''(A^2)A^2+\Phi'(A^2)\big)\big(|\mathbf{D}|^2-A^2\big)+4\Phi''(A^2)A^3(A-|\mathbf{D}|)+\Phi(A^2)\\&\underset{(2.4)}{\overset{(2.36)}{\geq}}\frac{C_3}{2}(1+A^{p-2})(|\mathbf{D}|^2-A^2)+C_3(\frac{1}{2}+\frac{A^{p-2}}{p})A^2\\&=\frac{C_3}{2}|\mathbf{D}|^2+\frac{C_3}{2}A^{p-2}(|\mathbf{D}|^2-A^2)+\frac{C_3}{p}A^p\geq\frac{C_3}{2}|\mathbf{D}|^2\,,\end{aligned}$$

while if $\Phi''(A^2)\geq 0$ we have

$$\begin{aligned}\Phi_A(|\mathbf{D}|^2)&=2\Phi''(A^2)A^2(|\mathbf{D}|-A)^2+\Phi'(A^2)(|\mathbf{D}|^2-A^2)+\Phi(A^2)\\&\underset{(2.4)}{\overset{(2.29)}{\geq}}\frac{C_3}{2}(1+A^{p-2})(|\mathbf{D}|^2-A^2)+C_3(\frac{1}{2}+\frac{A^{p-2}}{p})A^2\,,\end{aligned}$$

which gives the same conclusion as above; i.e., the lower bound in $(2.23)_4$ is proved. The upper bound is clear. We also have

$$\partial_{ij}\Phi_A(|\mathbf{D}|^2)=\Big[\big(4\Phi''(A^2)A^2+2\Phi'(A^2)\big)\big(1-\frac{A}{|\mathbf{D}|}\big)+2\Phi'(A^2)\frac{A}{|\mathbf{D}|}\Big]D_{ij}\,.$$

Since all terms in front of D_{ij} are positive we obtain ($A > 1$)

$$|\partial\Phi_A(|\mathbf{D}|^2)| = 2\Phi'(A^2)A + \big(4\Phi''(A^2)A^2 + 2\Phi'(A^2)\big)(|\mathbf{D}| - A) \overset{(2.29)}{\underset{(2.36)}{\geq}} C_3(1 + A^{p-2})A \geq C_{13}(1 + A)^{p-1} .$$

This proves (2.27). Finally, to prove (2.25) for $|\mathbf{D}| > A$, it is enough ($\Phi_A \in C^2(\mathbb{R}_0^+)$) to show that for $F(|\mathbf{D}|) \equiv \big|\partial\Phi_A(|\mathbf{D}|^2)\big|^p \big(1+\Phi_A(|\mathbf{D}|^2)\big)^{1-p}$ the limit $\lim_{|\mathbf{D}|\to\infty} F(|\mathbf{D}|)$ is finite. However,

$$F(|\mathbf{D}|) = \frac{\big|2\alpha_2\mathbf{D} + \alpha_1\frac{\mathbf{D}}{|\mathbf{D}|}\big|^p}{\big(1 + \alpha_0 + \alpha_1|\mathbf{D}| + \alpha_2|\mathbf{D}|^2\big)^{p-1}} \leq \frac{(2\alpha_2)^p|\mathbf{D}|^{2-p}\big(1 + \frac{|\alpha_1|}{2\alpha_2|\mathbf{D}|}\big)^p}{\alpha_2^{p-1}\big(\frac{1+\alpha_0}{\alpha_2|\mathbf{D}|^2} + \frac{\alpha_1}{\alpha_2|\mathbf{D}|} + 1\big)^{p-1}}.$$

Therefore,

$$\lim_{|\mathbf{D}|\to\infty} F(|\mathbf{D}|) \leq 2^p\alpha_2\frac{1}{A^{p-2}} \overset{(2.30)}{\leq} 2\tilde{C}_2 .$$

3. On strong solutions to the ***Problem (NS-Dir)$_p^{\varepsilon,A}$*****.** The goal of this section is to prove the existence of a strong solution to the *Problem (NS-Dir)$_p^{\varepsilon,A}$* provided that $\varepsilon > 0$ and $A > 1$ are fixed. We say that a couple $(\mathbf{v}, \pi) \equiv (\mathbf{v}^{\varepsilon,A}, \pi^{\varepsilon,A})$ is a strong solution to the *Problem (NS-Dir)$_p^{\varepsilon,A}$* if

$$\begin{aligned} &\mathbf{v} \in L^\infty(I; V_2) \cap L^2(I; W^{2,2}(\Omega)^3) , \\ &\frac{\partial\mathbf{v}}{\partial t} \in L^2(I; L^2(\Omega)^3) , \quad \pi \in L^2(I; L^2(\Omega)) , \end{aligned} \tag{3.1}$$

and the weak formulation

$$\begin{aligned} (\frac{\partial\mathbf{v}(t)}{\partial t}, \boldsymbol{\varphi}) + \int_\Omega T_{ij}^A(\mathbf{D}(\mathbf{v}(t)))D_{ij}(\boldsymbol{\varphi})\, dx + (v_{\varepsilon k}(t)\frac{\partial\mathbf{v}(t)}{\partial x_k}, \boldsymbol{\varphi}) \\ = (\pi(t), \operatorname{div}\boldsymbol{\varphi}) + (\mathbf{f}(t), \boldsymbol{\varphi}) \end{aligned} \tag{3.2}$$

is fulfilled for all $\boldsymbol{\varphi} \in W_0^{1,2}(\Omega)^3$ and almost all $t \in I$. In fact, due to the linear growth of $\mathbf{T}^A$ and (3.1), a strong solution $(\mathbf{v}, \pi)$ satisfies the equation $(2.20)_2$ almost everywhere in Q_T.

A couple $(\mathbf{v}, \pi) = (\mathbf{v}^{\varepsilon,A}, \pi^{\varepsilon,A})$ will be called a weak solution to the *Problem (NS-Dir)$_p^{\varepsilon,A}$* if $\mathbf{v} \in L^\infty(I; V_2)$ and $(3.1)_{2-3}$ and (3.2) are valid.

Theorem 3.3. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain, $\partial\Omega \in \mathcal{C}^3$. Let $\mathbf{v}_0 \in V_p$ and $\mathbf{f} \in L^2(I; L^2(\Omega)^3)$ and $p \geq 2$. Then, for all $\varepsilon > 0$ and $A > 1$ there exists a strong solution $(\mathbf{v}, \pi) = (\mathbf{v}^{\varepsilon,A}, \pi^{\varepsilon,A})$ to the Problem (NS-Dir)$_p^{\varepsilon,A}$ such that $\mathbf{v}$ is unique in the class of weak solutions of the Problem (NS-Dir)$_p^{\varepsilon,A}$ and it holds that*

$$\|\mathbf{v}\|_{L^\infty(I;H)} + \|\mathbf{v}\|_{L^2(I;V_2)} \leq c(\mathbf{f}, \mathbf{v}_0) \tag{3.4}$$

$$\begin{aligned} \|\frac{\partial \mathbf{v}}{\partial t}\|^2_{L^2(I;L^2(\Omega)^3)} &+ \|\Phi_A(|\mathbf{D}(\mathbf{v})|^2)\|_{L^\infty(I;L^1(\Omega))} \\ &\leq c(\mathbf{f}, \mathbf{v}_0) + \int_{Q_T} |\mathbf{v}_\varepsilon|^2 |\nabla \mathbf{v}|^2 dx\, dt \leq c\,(\frac{1}{\varepsilon}, \mathbf{f}, \mathbf{v}_0) \end{aligned} \tag{3.5}$$

$$\begin{aligned} \|\pi\|^2_{L^2(I;L^2(\Omega))} &\leq \int_0^T \|\partial \Phi_A(|\mathbf{D}(\mathbf{v})|^2)\|_2^2\, dt + \int_{Q_T} |\mathbf{v}_\varepsilon|^2 |\nabla \mathbf{v}|^2 dx\, dt + c(\mathbf{f}) \\ &\leq c(\frac{1}{\varepsilon}, A, \mathbf{f}, \mathbf{v}_0)\,. \end{aligned} \tag{3.6}$$

Moreover,

$$\|\nabla^{(2)} \mathbf{v}\|^2_{L^2(I;L^2(\Omega)^{3\times 3})} \leq c(\frac{1}{\varepsilon}, A, \mathbf{f}, \mathbf{v}_0), \tag{3.7}$$

and $(2.20)_2$ *holds almost everywhere in* Q_T*.*

Proof. The proof consists of two parts. While the first one, including the existence of a unique weak solution, is standard (and we only sketch the proof), the second one, proving the regularity result (3.7), uses methods that are not so common, and we will give a detailed proof.

Using the Galerkin approach we can justify the a priori estimates (3.4) and (3.5), which are derived in the same way as (2.8) and (2.12). By (2.6) and (3.4) $v_{\varepsilon k} \frac{\partial \mathbf{v}}{\partial x_k} \in L^2(I; L^2(\Omega)^3)$. Due to (2.17), the operator

$$-\operatorname{div} \mathbf{T}^A(\mathbf{D}(\cdot)) : L^2(I; W_0^{1,2}(\Omega)^3) \to L^2(I; (W_0^{1,2}(\Omega)^3)^*)$$

is a strongly monotone operator. Thus combining the Galerkin method with the monotone operator theory we can prove the existence of a weak solution $\mathbf{v} = \mathbf{v}^{\varepsilon,A}$ satisfying (3.4) and (3.5). See Lions [8] for details, for example.

Defining $\mathbf{F} \in L^2(I; (W_0^{1,2}(\Omega)^3)^*)$ by

$$\begin{aligned} \langle \mathbf{F}(t), \boldsymbol{\varphi} \rangle \equiv (\frac{\partial \mathbf{v}(t)}{\partial t}, \boldsymbol{\varphi}) &+ \int_\Omega T^A_{ij}(\mathbf{D}(\mathbf{v}(t))) D_{ij}(\boldsymbol{\varphi})\, dx \\ &+ (v_{\varepsilon k}(t) \frac{\partial \mathbf{v}(t)}{\partial x_k}, \boldsymbol{\varphi}) - (\mathbf{f}(t), \boldsymbol{\varphi}) \end{aligned} \tag{3.8}$$

we see that for almost all $t \in I$ and all $\boldsymbol{\varphi} \in V_2$, $\langle \mathbf{F}(t), \boldsymbol{\varphi} \rangle = 0$. By De Rham's theorem and Theorem 6.7 there exists $\pi \in L^2(I; L^2(\Omega))$, $\int_\Omega \pi \, dx = 0$ such that

$$\langle \mathbf{F}(t), \boldsymbol{\varphi} \rangle = \langle \nabla \pi(t), \boldsymbol{\varphi} \rangle \qquad \forall \boldsymbol{\varphi} \in W_0^{1,2}(\Omega)^3 \, .$$

From (3.8) and known a priori estimates, we obtain (3.6), where we also used $(2.23)_1$. Thus the first part of the proof is finished.

In the rest of this section we will focus on (3.7), having at our disposal (3.4)–(3.6) and the weak formulation (3.2). The proof of (3.7) consists, as usual, of interior regularity and regularity near the boundary. In the latter case we will not flatten the boundary. We prefer to use a curvilinear system in order to derive estimate (3.7) in tangential and normal directions. Because of the missing boundary conditions for the pressure we obtain the estimates in the normal direction by applying the (pressure-eliminating) curl operator to the system $(2.21)_2$.

Let $T : \Omega_0 \subset \Omega \to \Omega$ be a diffeomorphism. Using (3.2) we get the identity

$$\begin{aligned} 0 = {} & \big(\frac{\partial \mathbf{v}}{\partial t}(Tx) - \frac{\partial \mathbf{v}}{\partial t}(x), \boldsymbol{\varphi}(x) \big) \\ & + \int_\Omega \Big[\partial_{ij} \Phi_A (|\mathbf{D}(\mathbf{v}(Tx))|^2) - \partial_{ij} \Phi_A (|\mathbf{D}(\mathbf{v}(x))|^2) \Big] D_{ij}(\boldsymbol{\varphi}(x)) \, dx \\ & + \Big((v_{\varepsilon k} \frac{\partial \mathbf{v}}{\partial x_k})(Tx) - (v_{\varepsilon k} \frac{\partial \mathbf{v}}{\partial x_k})(x), \boldsymbol{\varphi}(x) \Big) \\ & + \big(\nabla \pi(Tx) - \nabla \pi(x), \boldsymbol{\varphi}(x) \big) - \big(\mathbf{f}(Tx) - \mathbf{f}(x), \boldsymbol{\varphi}(x) \big) \\ & \equiv I_1 + I_2 + I_3 + I_4 + I_5 \end{aligned} \tag{3.9}$$

valid for "correct" $\boldsymbol{\varphi}$. In (3.9) we suppressed the dependence of $\mathbf{v}$, π, $\mathbf{f}$ and $\boldsymbol{\varphi}$ on t. If $\mathbf{g}$ denotes one of these quantities, then

$$\big(\mathbf{g}(Tx) - \mathbf{g}(x), \boldsymbol{\varphi}(x) \big) \equiv \int_\Omega \big(\mathbf{g}(t, Tx) - \mathbf{g}(t, x) \big) \cdot \boldsymbol{\varphi}(t, x) \, dx \, .$$

This convention is used in the sequel. In case of interior regularity, let $V' \subset\subset \Omega_0 \subset\subset \Omega$ be such that $\operatorname{dist}(\partial\Omega_0, \partial\Omega) = h_0 > 0$. Let $\mathbf{e}^r$, $r = 1, 2, 3$, be a basis of a coordinate system in $\mathbb{R}^3$. Setting for $r = 1, 2, 3$ and $h \in (0, h_0)$

$$T = T_{r,h} : x \mapsto x + h\mathbf{e}^r \, , \tag{3.10}$$

we then get $T : \Omega_0 \to \Omega$. Let us consider a cut-off function $\xi \in \mathcal{D}(\Omega_0)$ such that $\xi \in [0, 1]$ in Ω and $\xi \equiv 1$ in V'.

In the case of regularity (near the boundary) in tangential directions, let us consider one of the maps a_k, $k = 1, 2, \dots, N$, that locally describe $\partial\Omega$ (cf. [4], page 305). We know that for a certain $\alpha > 0$, $\partial\Omega$ is covered by sets $V^k \equiv \{x = (x', x_3) \in \mathbb{R}^3 : \; x' = (x_1, x_2); |x'| \le \alpha \text{ and } a_k(x') - \alpha < x_3 < a_k(x') + \alpha\}$, where $a_k \in C^3(B_\alpha(0'))$ and

$$\frac{\partial a_k(0')}{\partial x_s} = 0, \qquad (s = 1, 2). \tag{3.11}$$

Let us also define $V_+^k \equiv \{x \in \mathbb{R}^3 : |x'| \le \alpha \text{ and } a_k(x') < x_3 < a_k(x') + \alpha\}$, $V_-^k \equiv \{x \in \mathbb{R}^3 : |x'| \le \alpha \text{ and } a_k(x') - \alpha < x_3 < a_k(x')\}$; then $V^k = V_+^k \cup V_-^k \cup \{x \in \partial\Omega : |x'| \le \alpha \text{ and } x_3 = a_k(x')\}$. We finally choose sets Ω_0^k covering $\partial\Omega$ such that $\Omega_0^k \subset V^k$, $\text{dist}(\partial\Omega_0^k, \partial V^k) \ge h_0 > 0$.

Let us fix k and drop for simplicity the index k. Setting $\hat{\mathbf{e}}^1 \equiv (1, 0)$ and $\hat{\mathbf{e}}^2 \equiv (0, 1)$, we can define for $s = 1, 2$ and $h \in (0, h_0)$ the mapping $T = T_{s,h} : \Omega_0 \to V$ by

$$x \mapsto (x' + h\hat{\mathbf{e}}^s, x_3 + a(x' + h\hat{\mathbf{e}}^s) - a(x')) \equiv y\,. \tag{3.12}$$

Then the inverse mapping T^{-1} is given by $(x = T^{-1}(y))$

$$y \mapsto (y' - h\hat{\mathbf{e}}^s, x_3 + a(y' - h\hat{\mathbf{e}}^s) - a(y'))\,. \tag{3.13}$$

Put

$$\Delta^{\pm} a(x') = a(x' \pm h\hat{\mathbf{e}}^s) - a(x')\,. \tag{3.14}$$

Then

$$\Big(\frac{\partial T_i}{\partial x_j}(x)\Big)_{i,j=1,2,3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\partial \Delta^+ a}{\partial x_1}(x') & \frac{\partial \Delta^+ a}{\partial x_2}(x') & 1 \end{pmatrix} \tag{3.15}$$

and

$$\Big(\frac{\partial T_i^{-1}}{\partial y_j}(y)\Big)_{i,j=1,2,3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\partial \Delta^- a}{\partial y_1}(y') & \frac{\partial \Delta^- a}{\partial y_2}(y') & 1 \end{pmatrix}. \tag{3.16}$$

Both matrices in (3.15) and (3.16) have determinant equal to 1 (the same is true for T defined in (3.10)). The s-th tangential derivative ($s = 1, 2$) of any (scalar or vector) function g, denoted $\frac{\partial g}{\partial \tau^s}$, is defined by

$$\frac{\partial g}{\partial \tau^s}(x) \equiv \lim_{h \to 0} \frac{g(Tx) - g(x)}{h},$$

and it holds that

$$\frac{\partial g}{\partial \tau^s}(x) = \frac{\partial g}{\partial x_s}(x) + \frac{\partial g}{\partial x_3}(x)\frac{\partial a}{\partial x_s}(x')\,. \tag{3.17}$$

For the reader's convenience let us note that if $g \in W_0^{1,p}(\Omega)$, $p > 1$, then for all $h \in (0, h_0)$

$$\int_{\Omega_0} \Big|\frac{g(Tx) - g(x)}{h}\Big|^p \, dx \le c(a)\|\nabla g\|_p^p\,. \tag{3.18}$$

Indeed, setting $T_\lambda(x) = \big(x' + \lambda\hat{\mathbf{e}}^s, x_3 + a(x' + \lambda\hat{\mathbf{e}}^s) - a(x')\big)$, we can write

$$\begin{aligned}
&\int_{\Omega_0} \Big|\frac{g(Tx) - g(x)}{h}\Big|^p \, dx = \int_{\Omega_0} \Big|\int_0^1 \frac{\partial g(T_\lambda(x))}{\partial \tau^s}\, d\lambda\Big|^p \, dx \\
&\le \int_{\Omega_0}\int_0^1 \Big|\frac{\partial g(T_\lambda(x))}{\partial \tau^s}\Big|^p \, d\lambda\, dx \le \int_0^1 \int_\Omega \Big|\frac{\partial g(y)}{\partial \tau^s}\Big|^p \, dy\, d\lambda \overset{(3.17)}{\le} c(a)\|\nabla g\|_p^p\,.
\end{aligned}$$

On the other hand, if $g \in L^p(\Omega)$ and if for all $h \in (0, h_0)$

$$\int_{\Omega} \Big|\frac{g(Tx) - g(x)}{h}\Big|^p \, dx \le c_0 < \infty\,, \tag{3.19}$$

then $\dfrac{\partial g}{\partial \tau^s}$ exists (for $s = 1, 2$) in the sense of distributions and

$$\int_{\Omega} \Big|\frac{\partial g}{\partial \tau^s}\Big|^p \, dx \le c_0\,. \tag{3.20}$$

If further $V' \subset\subset \Omega_0$, we consider $\xi \in [0,1]$, $\xi \in \mathcal{D}(\Omega_0)$, $\xi \equiv 1$ in V'. Setting

$$\boldsymbol{\varphi} = \frac{1}{h^2}\big(\mathbf{v}(Tx) - \mathbf{v}(x)\big)\xi^2(x) \tag{3.21}$$

we see that interior regularity and regularity in tangential directions can be treated analogously. Since the mapping T is more complicated in the latter case, we will present a detailed proof for it. In order to shorten formulas let us denote $\mathbf{w}(x) \equiv \mathbf{v}(Tx) - \mathbf{v}(x)$. Let us take (3.21) as a test function in (3.9), and let us calculate the terms I_k, $k = 1, \dots, 5$, separately. We get

$$I_1 = \frac{1}{2}\frac{d}{dt}\int_{\Omega_0} \Big|\frac{\mathbf{w}}{h}\Big|^2 \xi^2 \, dx \,, \tag{3.22}$$

$$\begin{aligned} I_2 &= \frac{1}{h^2}\int_{\Omega_0} \big[\partial_{ij}\Phi_A(|\mathbf{D}(\mathbf{v}(Tx))|^2) - \partial_{ij}\Phi_A(|\mathbf{D}(\mathbf{v}(x))|^2)\big] D_{ij}(\mathbf{w}(x))\xi^2(x)\, dx \\ &\quad + \frac{2}{h^2}\int_{\Omega_0} \big[\partial_{ij}\Phi_A(|\mathbf{D}(\mathbf{v}(Tx))|^2) - \partial_{ij}\Phi_A(|\mathbf{D}(\mathbf{v}(x))|^2)\big] w_i(x)\xi(x)\frac{\partial \xi(x)}{\partial x_j}\, dx \\ &= J_1 + J_2 \,. \end{aligned} \tag{3.23}$$

By (2.24) and Lemma 6.5 from the appendix we have

$$\begin{aligned} J_1 &= \frac{1}{h^2}\int_{\Omega_0}\int_0^1 \partial_{ij}\partial_{kl}\Phi_A(|\mathbf{D}(\mathbf{v}) + \lambda \mathbf{D}(\mathbf{w})|^2) D_{ij}(\mathbf{w}) D_{kl}(\mathbf{w})\xi^2 \, d\lambda \, dx \\ &\ge C_{10}\int_{\Omega_0} \Big|\frac{\mathbf{D}(\mathbf{w})}{h}\Big|^2 \xi^2\, dx \ge C_{14}\int_{\Omega_0}\Big|\frac{\nabla \mathbf{w}}{h}\Big|^2\xi^2\, dx - c\int_{\Omega_0}\Big|\frac{\mathbf{w}}{h}\Big|^2 \big|\nabla \xi\big|^2\, dx \,, \end{aligned} \tag{3.24}$$

while

$$\begin{aligned} |J_2| &= \frac{2}{h^2}\Big|\int_{\Omega_0}\int_0^1 \partial_{ij}\partial_{kl}\Phi_A(|\mathbf{D}(\mathbf{v}) + \lambda\mathbf{D}(\mathbf{w})|^2) D_{kl}(\mathbf{w})\xi w_i \frac{\partial \xi}{\partial x_j}\, d\lambda\, dx\Big| \\ &\overset{(2.23)_2}{\le} \frac{2\tilde{C}_2(A)}{h^2}\int_{\Omega_0} \big|\nabla \mathbf{w}\big|\xi\, |\mathbf{w}||\nabla\xi|\, dx \\ &\le \frac{C_{14}}{8}\int_{\Omega_0}\Big|\frac{\nabla \mathbf{w}}{h}\Big|^2\xi^2\, dx + c\int_{\Omega_0}\Big|\frac{\mathbf{w}}{h}\Big|^2\big|\nabla\xi\big|^2\, dx \,. \end{aligned} \tag{3.25}$$

Note that in this section all generic constants c can depend on A. Further, the convective term gives

$$\begin{aligned} h^2 I_3 &= \int_{\Omega_0} w_{\varepsilon k}(x)\frac{\partial v_i(Tx)}{\partial x_k} w_i(x)\xi^2(x)\, dx + \int_{\Omega_0} v_{\varepsilon k}(x)\frac{\partial w_i(x)}{\partial x_k} w_i(x)\xi^2(x)\, dx \\ &= -\int_{\Omega_0}\frac{\partial v_{\varepsilon k}(Tx)}{\partial x_k} v_i(Tx) w_i(x)\xi^2(x)\, dx - \int_{\Omega_0} w_{\varepsilon k}(x) v_i(Tx)\frac{\partial w_i(x)}{\partial x_k}\xi^2(x)\, dx \\ &\quad - 2\int_{\Omega_0} w_{\varepsilon k}(x) v_i(Tx) w_i(x)\xi(x)\frac{\partial \xi(x)}{\partial x_k}\, dx - \int_{\Omega_0} v_{\varepsilon k}(x)|\mathbf{w}(x)|^2\xi(x)\frac{\partial\xi(x)}{\partial x_k}\, dx \\ &= J_3 + J_4 + J_5 + J_6 \,. \end{aligned} \tag{3.26}$$

Since $\operatorname{div} \mathbf{v}_\varepsilon = 0$, by (3.15)

$$J_3 = -\int_{\Omega_0} \frac{\partial v_{\varepsilon s}(Tx)}{\partial y_3} \frac{\partial \Delta^+ a(x')}{\partial x_s} v_i(Tx) w_i(x) \xi^2 \, dx \,, \tag{3.27}$$

where we have started to use the convention that whenever the index s appears twice in an expression we sum over s from 1 to 2. Using the regularity of the mollified function, the regularity of the boundary and (3.4), we get from (3.26) and (3.27)

$$|I_3| \leq \frac{C_{14}}{8} \int_{\Omega_0} \left|\frac{\nabla \mathbf{w}}{h}\right|^2 \xi^2 \, dx + c \int_{\Omega_0} \left|\frac{\mathbf{w}}{h}\right|^2 \left(|\nabla \xi|^2 + |\xi|^2\right) dx \,. \tag{3.28}$$

The pressure term I_4 requires more calculations. It is useful to start with an equivalent form

$$I_4 = \int_{\Omega_0} \frac{\partial \pi(x)}{\partial x_i} \left[\boldsymbol{\varphi}_i(Tx) - \boldsymbol{\varphi}_i(x)\right] dx = \int_{\Omega_0} \pi(x) \operatorname{div} \left(\boldsymbol{\varphi}(x) - \boldsymbol{\varphi}(T^{-1}x)\right) dx \,.$$

With $\boldsymbol{\varphi}$ as in (3.21) we obtain

$$\begin{aligned}
h^2 I_4 = & \int_{\Omega_0} \pi(x) \left[\operatorname{div} \mathbf{w}(x) \, \xi^2(x) - \operatorname{div} \mathbf{w}(T^{-1}x) \, \xi^2(T^{-1}x)\right] dx \\
& + 2 \int_{\Omega_0} \pi(x) \left[w_i(x) \xi(x) \frac{\partial \xi(x)}{\partial x_i} - w_i(T^{-1}x) \xi(T^{-1}x) \frac{\partial \xi(T^{-1}x)}{\partial x_i}\right] dx \\
= & \int_{\Omega_0} \pi(x) \operatorname{div} \left(\mathbf{v}(Tx) - 2\mathbf{v}(x) + \mathbf{v}(T^{-1}x)\right) \xi^2(x) \, dx \\
& + \int_{\Omega_0} \pi(x) \operatorname{div} \mathbf{w}(T^{-1}x) \left(\xi^2(x) - \xi^2(T^{-1}x)\right) dx \\
& + 2 \int_{\Omega_0} \pi(x) \left[v_i(Tx) - 2v_i(x) + v_i(T^{-1}x)\right] \xi(x) \frac{\partial \xi(x)}{\partial x_i} \, dx \\
& + 2 \int_{\Omega_0} \pi(x) w_i(T^{-1}x) \left(\xi(x) \frac{\partial \xi(x)}{\partial x_i} - \xi(T^{-1}x) \frac{\partial \xi(T^{-1}x)}{\partial x_i} \right) dx \\
= & J_7 + J_8 + J_9 + J_{10} \,.
\end{aligned} \tag{3.29}$$

Using further (3.15), (3.16) and the fact that $\operatorname{div} \mathbf{v} = 0$ in Q_T, we rewrite J_7

as follows:

$$\begin{aligned} J_7 &= \int_{\Omega_0} \pi(x)\left(\frac{\partial v_s(Tx)}{\partial y_3}\frac{\partial \Delta^+ a(x')}{\partial x_s} + \frac{\partial v_s(T^{-1}x)}{\partial y_3}\frac{\partial \Delta^- a(x')}{\partial x_s}\right)\xi^2(x)\,dx \\ &= \int_{\Omega_0} \pi(x)\left(\frac{\partial v_s(Tx)}{\partial y_3} - \frac{\partial v_s(x)}{\partial y_3}\right)\frac{\partial \Delta^+ a(x')}{\partial x_s}\xi^2(x)\,dx \\ &\quad + \int_{\Omega_0} \pi(x)\left(\frac{\partial v_s(x)}{\partial y_3} - \frac{\partial v_s(T^{-1}x)}{\partial y_3}\right)\frac{\partial \Delta^- a(x')}{\partial x_s}\xi^2(x)\,dx \\ &\quad + \int_{\Omega_0} \pi(x)\frac{\partial v_s(x)}{\partial y_3}\frac{\partial}{\partial x_s}\Big(a(x'+h\hat{\mathbf{e}}^r) - 2a(x') + a(x'-h\hat{\mathbf{e}}^r)\Big)\xi^2(x)\,dx \\ &= J_{11} + J_{12} + J_{13}\,. \end{aligned} \tag{3.30}$$

The last term in (3.9) is treated similarly; hence,

$$\begin{aligned} h^2 I_5 &= \int_{\Omega_0} f_i(x)\big(v_i(Tx) - 2v_i(x) + v_i(T^{-1}x)\big)\xi^2(x)\,dx \qquad (3.31) \\ &\quad + \int_{\Omega_0} f_i(x)\big(v_i(x) - v_i(T^{-1}x)\big)\big(\xi^2(x) - \xi^2(T^{-1}x)\big)\,dx = J_{14} + J_{15}\,. \end{aligned}$$

Now, since $\operatorname{div}\mathbf{w}(Tx) = \dfrac{\partial v_s(Tx)}{\partial y_3}\dfrac{\partial \Delta^+ a(x)}{\partial x_s}$, we see that

$$|J_8| \le c\int_{\Omega_0} \big|\pi(Tx)\big|\big|\nabla\mathbf{v}(Tx)\big|\big|\nabla\Delta^+ a(x)\big|\big|\xi(Tx) - \xi(x)\big|\,dx\,. \tag{3.32}$$

Hence

$$\frac{1}{h^2}\big(|J_8| + |J_{10}| + |J_{13}| + |J_{15}|\big) \le c\big(\|\pi\|_2^2 + \|\mathbf{f}\|_2^2 + \|\nabla\mathbf{v}\|_2^2\big) \tag{3.33}$$

and

$$\frac{1}{h^2}\big(|J_{11}| + |J_{12}|\big) \le c\|\pi\|_2^2 + \frac{C_{14}}{8}\int_{\Omega_0}\Big|\frac{\nabla\mathbf{w}}{h}\Big|^2\xi^2\,dx\,. \tag{3.34}$$

It remains to estimate J_9 and J_{14}. Denoting $\mathbf{g}(x) \equiv \frac{\mathbf{w}(x)}{h} = \frac{\mathbf{v}(Tx)-\mathbf{v}(x)}{h}$, we can rewrite J_9 as follows:

$$\begin{aligned} \frac{J_9}{h^2} &= \int_{\Omega_0} \pi(x)\frac{g_i(x) - g_i(T^{-1}x)}{h}\xi(x)\frac{\partial\xi(x)}{\partial x_i}\,dx \\ &= \int_{\Omega_0} \pi(x)\frac{1}{h}\big[g_i(x)\xi(x) - g_i(T^{-1}x)\xi(T^{-1}x)\big]\frac{\partial\xi(x)}{\partial x_i}\,dx \qquad (3.35) \\ &\quad + \int_{\Omega_0} \pi(x)\frac{1}{h}\big[\xi(T^{-1}x) - \xi(x)\big]g_i(T^{-1}x)\frac{\partial\xi(x)}{\partial x_i}\,dx\,. \end{aligned}$$

Since $\mathbf{g}\xi \in W_0^{1,2}(\Omega_0)$ for fixed h, we can use (3.18) to conclude that

$$\frac{|J_9|}{h^2} \le c(a)\|\pi\|_2\|\nabla\xi\|_\infty\Big(\int_{\Omega_0}\big|\frac{\nabla\mathbf{w}}{h}\big|^2\xi^2\,dx\Big)^{1/2} + \|\pi\|_2\|\nabla\xi\|_\infty\|\nabla\mathbf{v}\|_2\,. \tag{3.36}$$

Also

$$\frac{|J_{14}|}{h^2} \le \|\mathbf{f}\|_2\Big(\int_{\Omega_0}\big|\frac{\nabla\mathbf{w}}{h}\big|^2\xi^2\,dx\Big)^{1/2} + \|\nabla\mathbf{v}\|_2\|\mathbf{f}\|_2\|\nabla\xi\|_\infty\,. \tag{3.37}$$

Putting all calculations between (3.22) and (3.37) together, using Young's inequality and integrating over $(0,T)$ we finally obtain

$$\int_0^T\int_{\Omega_0}\big|\frac{\nabla\mathbf{w}}{h}\big|^2\xi^2\,dx\,dt \le c\int_0^T \|\nabla\mathbf{v}\|_2^2 + \|\pi\|_2^2 + \|\mathbf{f}\|_2^2\;dt + c(\mathbf{v}_0)\,. \tag{3.38}$$

Hence, (see (3.19) and (3.20))

$$\sum_{s=1}^{2}\sum_{i=1}^{3}\int_0^T\int_\Omega\big|\frac{\partial^2\mathbf{v}}{\partial x_i\partial\tau^s}\big|^2\xi^2\,dx\,dt \le c\,. \tag{3.39}$$

In case of interior regularity, we proceed in an analogous way with some simplifications due to the simpler structure of the mapping T; cf. (3.10). Thus (3.7) holds for all $\Omega' \subset\subset \Omega$. This implies that the equation $(2.21)_2$ holds almost everywhere.

In order to get (3.7) globally, we need an estimate of type (3.39) in the normal direction (which is locally x_3). We avoid the missing information about the pressure by taking the curl of $(2.21)_2$. Recall that for $\mathbf{h} = (h_1, h_2, h_3)$

$$\operatorname{curl}\mathbf{h} = \big(\frac{\partial h_3}{\partial x_2} - \frac{\partial h_2}{\partial x_3}, \frac{\partial h_1}{\partial x_3} - \frac{\partial h_3}{\partial x_1}, \frac{\partial h_2}{\partial x_1} - \frac{\partial h_1}{\partial x_2}\big)\,.$$

Applying the curl operator to $(2.21)_2$ we obtain three equations in $W^{-1,2}(\Omega)$; however, only the first two are useful for us. The first equation reads

$$\begin{aligned}\frac{\partial}{\partial x_2}\frac{\partial v_3}{\partial t} - \frac{\partial}{\partial x_3}\frac{\partial v_2}{\partial t} - \frac{\partial f_3}{\partial x_2} + \frac{\partial f_2}{\partial x_3} + \frac{\partial}{\partial x_2}\big[v_{\varepsilon k}\frac{\partial v_3}{\partial x_k}\big] - \frac{\partial}{\partial x_3}\big[v_{\varepsilon k}\frac{\partial v_2}{\partial x_k}\big]\\ - \frac{\partial^2}{\partial x_1\partial x_2}\partial_{31}\Phi_A - \frac{\partial^2}{\partial x_2\partial x_2}\partial_{32}\Phi_A - \frac{\partial^2}{\partial x_2\partial x_3}\partial_{33}\Phi_A\\ + \frac{\partial^2}{\partial x_3\partial x_1}\partial_{21}\Phi_A + \frac{\partial^2}{\partial x_3\partial x_2}\partial_{22}\Phi_A + \frac{\partial^2}{\partial x_3\partial x_3}\partial_{23}\Phi_A = 0\,,\end{aligned} \tag{3.40}$$

while the second equation has the form

$$\frac{\partial}{\partial x_3}\frac{\partial v_1}{\partial t} - \frac{\partial}{\partial x_1}\frac{\partial v_3}{\partial t} - \frac{\partial f_1}{\partial x_3} + \frac{\partial f_3}{\partial x_1} + \frac{\partial}{\partial x_3}\big[v_{\varepsilon k}\frac{\partial v_1}{\partial x_k}\big] - \frac{\partial}{\partial x_1}\big[v_{\varepsilon k}\frac{\partial v_3}{\partial x_k}\big]$$
$$+ \frac{\partial^2}{\partial x_1 \partial x_1}\partial_{31}\Phi_A + \frac{\partial^2}{\partial x_1 \partial x_2}\partial_{32}\Phi_A + \frac{\partial^2}{\partial x_1 \partial x_3}\partial_{33}\Phi_A \tag{3.41}$$
$$- \frac{\partial^2}{\partial x_1 \partial x_3}\partial_{11}\Phi_A - \frac{\partial^2}{\partial x_2 \partial x_3}\partial_{12}\Phi_A - \frac{\partial^2}{\partial x_3 \partial x_3}\partial_{13}\Phi_A = 0\,.$$

We will get the desired estimate from the last terms in equations (3.40) and (3.41) and from the equation

$$\frac{\partial^2 v_3}{\partial x_3^2} = -\frac{\partial^2 v_1}{\partial x_1 \partial x_3} - \frac{\partial^2 v_2}{\partial x_2 \partial x_3}\,, \tag{3.42}$$

which follows from the divergence-free constraint, taking the derivative with respect to x_3. Let us denote

$$G_1 \equiv \frac{\partial}{\partial x_3}\frac{\partial \Phi_A(|\mathbf{D}(\mathbf{v})|^2)}{\partial D_{13}} \quad \text{and} \quad G_2 \equiv \frac{\partial}{\partial x_3}\frac{\partial \Phi_A(|\mathbf{D}(\mathbf{v})|^2)}{\partial D_{23}}\,. \tag{3.43}$$

Clearly, for $i = 1, 2$,

$$\int_0^T \|\xi G_i\|_{-1,2}^2\, dt \le c \tag{3.44}$$

due to (3.4) and (2.23). From (3.40) and (3.41), we can also observe, using (3.39) and (3.17), that

$$\int_0^T \|\frac{\partial G_i}{\partial x_3}\xi\|_{-1,2}^2\, dt \le c + c\int_0^T \int_\Omega \sum_{i=1}^3 \sum_{s=1}^2 \big|\frac{\partial^2 \mathbf{v}}{\partial x_i \partial x_s}\big|^2\, dx\, dt\,. \tag{3.45}$$

Hence, by Theorem 6.7 on negative norms

$$\int_0^T \|\xi G_i\|_2^2\, dt \le c + c\int_0^T \int_\Omega \sum_{i=1}^3 \sum_{s=1}^2 \big|\frac{\partial^2 \mathbf{v}}{\partial x_i \partial x_s}\big|^2\, dx\, dt\,. \tag{3.46}$$

Directly from (3.43) we obtain the system

$$\partial_{23}\partial_{13}\Phi_A\frac{\partial D_{13}}{\partial x_3} + \partial_{23}\partial_{23}\Phi_A\frac{\partial D_{23}}{\partial x_3} \tag{3.47}$$
$$= \frac{G_2}{2} - \frac{1}{2}\sum_{r,s=1}^2 \partial_{23}\partial_{rs}\Phi_A\frac{\partial D_{rs}}{\partial x_3} - \frac{1}{2}\partial_{23}\partial_{33}\Phi_A\frac{\partial^2 v_3}{\partial x_3^2}$$

$$\partial_{13}\partial_{13}\Phi_A\frac{\partial D_{13}}{\partial x_3}+\partial_{13}\partial_{23}\Phi_A\frac{\partial D_{23}}{\partial x_3}$$
$$=\frac{G_1}{2}-\frac{1}{2}\sum_{r,s=1}^{2}\partial_{13}\partial_{rs}\Phi_A\frac{\partial D_{rs}}{\partial x_3}-\frac{1}{2}\partial_{13}\partial_{33}\Phi_A\frac{\partial^2 v_3}{\partial x_3^2}\,.$$

Since the matrix

$$\mathbb{A}\equiv\begin{pmatrix}\partial_{23}\partial_{13}\Phi_A & \partial_{23}\partial_{23}\Phi_A\\ \partial_{13}\partial_{13}\Phi_A & \partial_{13}\partial_{23}\Phi_A\end{pmatrix}\tag{3.48}$$

is positive definite thanks to (2.24), we can compute $\frac{\partial}{\partial x_3}D_{s3}$ $(s=1,2)$ from (3.47). Moreover, the coefficients of the matrix (3.48) are bounded due to $(2.23)_2$, so we get

$$\frac{\partial D_{s3}(\mathbf{v})}{\partial x_3}\approx\text{right-hand side of (3.47)}\,.\tag{3.49}$$

Because of

$$\frac{\partial D_{s3}(\mathbf{v})}{\partial x_3}=\frac{\partial^2 v_s}{\partial x_3^2}+\frac{\partial^2 v_3}{\partial x_s\partial x_3}\,,$$

we conclude from (3.49), (3.42) and (3.17) that

$$\sum_{i=1}^{3}\Big\|\frac{\partial^2 v_i}{\partial x_3^2}\xi\Big\|_2^2\le c+\sum_{i,j=1}^{3}\sum_{s=1}^{2}\Big\|\frac{\partial^2 v_i}{\partial x_j\partial\tau^s}\xi\Big\|_2^2+c\sup_{\Omega_0}\sum_{s=1}^{2}\Big|\frac{\partial a}{\partial x_s}\Big|\sum_{i=1}^{3}\Big\|\frac{\partial^2 v_i}{\partial x_3^2}\xi\Big\|_2^2.\tag{3.50}$$

Now, if we take Ω_0 small enough, we can arrange due to (3.11) that

$$c\sup_{\Omega_0}\sum_{s=1}^{2}\Big|\frac{\partial a}{\partial x_s}\Big|\le\frac{1}{2},$$

and the last term in (3.50) can be moved to the left-hand side. The proof of (3.7) is finished. Theorem 3.3 is proved. □

We will finish this section by proving a variant of inequality (3.5) which will be useful in Sections 4 and 5. For $t\in I$ we denote

$$E_A(t)\equiv 1+\|\Phi_A(|\mathbf{D}(\mathbf{v}^{\varepsilon,A}(t))|^2)\|_1\,.$$

Lemma 3.51. *Set* $\gamma_\mu(s) \equiv \frac{1}{1-\mu} s^{1-\mu}$ *for* $1 \neq \mu \geq 0$ *and* $\gamma_1(s) \equiv ln\ (s)$ *(the derivative of* γ_μ *is denoted by* γ'_μ*). Let* $p \in [2,6)$*. Then it holds for all* $t \in I$ *that*

$$\begin{aligned}\int_0^t \|\frac{\partial \mathbf{v}^{\varepsilon,A}}{\partial t}\|_2^2\, \gamma'_\mu(E_A(\tau))\, d\tau + \sup_{\tau\in[0,t]} \gamma_\mu(E_A(\tau)) \\ \leq c(\mathbf{f}, \mathbf{v}_0) + \int_0^t \gamma'_\mu(E_A(\tau)) \int_\Omega |\mathbf{v}_\varepsilon^{\varepsilon,A}|^2 |\nabla \mathbf{v}^{\varepsilon,A}|^2\, dx\, d\tau\,.\end{aligned} \tag{3.52}$$

Proof. Let again $\mathbf{v} = \mathbf{v}^{\varepsilon,A}$ and $\gamma(s) = \gamma_\mu(s)$. Multiplying equation $(2.21)_2$ by $\frac{\partial \mathbf{v}}{\partial t}\, \gamma'(E_A(\tau))$ and integrating over Q_t yields

$$\begin{aligned}\int_0^t \|\frac{\partial \mathbf{v}}{\partial t}\|_2^2\, \gamma'(E_A(\tau))\, d\tau - \int_0^t \gamma'(E_A(\tau)) \int_\Omega \frac{\partial}{\partial x_j} \partial_{ij} \Phi_A(|\mathbf{D}(\mathbf{v})|^2) \frac{\partial \mathbf{v}}{\partial \tau}\, dx\, d\tau \\ \leq c(\mathbf{f}) + \int_0^t \gamma'(E_A(\tau)) \int_\Omega |\mathbf{v}_\varepsilon|^2 |\nabla \mathbf{v}|^2\, dx\, d\tau\,.\end{aligned} \tag{3.53}$$

Denoting the second term in (3.53) by J we need to show that $J = \gamma(E_A(t)) - \gamma(E_A(0))$ in order to obtain (3.52). Since $\mathbf{v} \in C(I; V_2) \cap L^2(I; W^{2,2}(\Omega)^3)$ and $\frac{\partial \mathbf{v}}{\partial t} \in L^2(I; L^2(\Omega)^3)$ there exists a sequence $\{\mathbf{v}^n\} \subset C^\infty(I; \mathcal{V})$ such that

$$\begin{aligned}\mathbf{v}^n &\to \mathbf{v} && \text{strongly in } L^2(I; W^{2,2}(\Omega)^3)\,, \\ \mathbf{v}^n(t) &\to \mathbf{v}(t) && \text{strongly in } V_2 \text{ for all } t \in I\,, \\ \frac{\partial \mathbf{v}^n}{\partial t} &\to \frac{\partial \mathbf{v}}{\partial t} && \text{strongly in } L^2(I; L^2(\Omega)^3)\,.\end{aligned} \tag{3.54}$$

Then we have for all $t \in I$

$$\lim_{n\to\infty} \int_\Omega \Phi_A(|\mathbf{D}(\mathbf{v}^n(t))|^2)\, dx = \int_\Omega \Phi_A(|\mathbf{D}(\mathbf{v}(t))|^2)\, dx\,. \tag{3.55}$$

To see (3.55), we use the inequality

$$|\Phi_A(|\mathbf{B}|^2) - \Phi_A(|\mathbf{A}|^2)| \leq c\big(1 + |\mathbf{A}| + |\mathbf{B}|\big)|\mathbf{A} - \mathbf{B}|\,,$$

which follows from $(2.23)_1$, and $(3.54)_2$.

Observing that $E_A(\mathbf{v}^n(t)) \geq 1$ we have $\|\gamma'(E_A(\mathbf{v}^n(t)))\|_\infty \leq 1$. Thus we obtain $\gamma'(E_A(\mathbf{v}^n(t))) \overset{*}{\rightharpoonup} \chi(t)$ in $L^\infty(I)$ at least for a subsequence. Since

$\gamma'(E_A(\mathbf{v}^n(t))) \to \gamma'(E_A(\mathbf{v}(t)))$ for all $t \in I$ due to (3.55), it is easy to conclude that

$$\gamma'(E_A(\mathbf{v}^n(t))) \overset{*}{\rightharpoonup} \gamma'(E_A(\mathbf{v}(t))) \qquad \text{weakly-}^* \text{ in } L^\infty(I)\,. \tag{3.56}$$

Now, using (3.54) together with (3.56) gives

$$J = -\lim_{n\to\infty} \int_0^t \gamma'(E_A(\mathbf{v}^n(\tau))) \int_\Omega \frac{\partial}{\partial x_j} \partial_{ij} \Phi_A(|\mathbf{D}(\mathbf{v}^n)|^2) \frac{\partial \mathbf{v}^n}{\partial \tau}\, dx\, d\tau$$

from which we deduce

$$\begin{aligned} J &= \lim_{n\to\infty} \int_0^t \frac{d}{d\tau} \|\Phi_A(|\mathbf{D}(\mathbf{v}^n)|^2)\|_1\, \gamma'(E_A(\mathbf{v}^n(\tau)))\, d\tau \\ &= \lim_{n\to\infty} \int_0^t \frac{d}{d\tau} (\gamma(E_A(\mathbf{v}^n(\tau))))\, d\tau = \lim_{n\to\infty} \{\gamma(E_A(\mathbf{v}^n(t))) - \gamma(E_A(\mathbf{v}^n(0)))\}. \end{aligned}$$

Passing to the limit, which is allowed due to (3.55), the assertion follows.

4. Limiting process $A \to \infty$. The goal of this section is to pass to the limit as $A \to \infty$. This means to return from the quadratic approximations Φ_A to the original potential Φ. Since the convective term will still be mollified, we will come from the *Problem (NS-Dir)*$_p^{\varepsilon,A}$ to the *Problem (NS-Dir)*$_p^{\varepsilon}$ defined in (2.7). Nevertheless, in preparation for the limiting process $\varepsilon \to 0^+$, we will often indicate the dependence of the estimates on ε. We denote

$$K_{\varepsilon,A} \equiv \int_\Omega |\mathbf{v}_\varepsilon^{\varepsilon,A}|^2 |\nabla \mathbf{v}^{\varepsilon,A}|^2\, dx.$$

We already know from (3.4)–(3.6) that strong solutions $(\mathbf{v}^{\varepsilon,A}, \pi^{\varepsilon,A})$ of the *Problem (NS-Dir)*$_p^{\varepsilon,A}$ satisfy the following estimates:

$$\|\mathbf{v}^{\varepsilon,A}\|^2_{L^\infty(I;H)} + \|\mathbf{v}^{\varepsilon,A}\|^2_{L^2(I;V_2)} \le c(\mathbf{f}, \mathbf{v}_0), \tag{4.1}$$

$$\begin{aligned} \Big\|\frac{\partial \mathbf{v}^{\varepsilon,A}}{\partial t}\Big\|^2_{L^2(Q_T)} + \big\|\Phi_A(|\mathbf{D}(\mathbf{v}^{\varepsilon,A})|^2)\big\|_{L^\infty(I;L^1(\Omega))} \\ \le c(\mathbf{f}, \mathbf{v}_0)\Big(1 + \int_0^T K_{\varepsilon,A}\, dt\Big)\,, \end{aligned} \tag{4.2}$$

$$\|\pi^{\varepsilon,A}\|^2_{L^2(I;L^2(\Omega))} \le c(\mathbf{f},\mathbf{v}_0)\Big(1+\int_0^T K_{\varepsilon,A}\,dt\Big) + \int_0^T \big\|\partial\Phi_A(|\mathbf{D}(\mathbf{v}^{\varepsilon,A})|^2)\big\|_2^2\,dt\,. \tag{4.3}$$

Note that due to (4.1) and (2.6)

$$\int_0^T K_{\varepsilon,A}(t)\,dt \le c\,(\frac{1}{\varepsilon},\mathbf{f},\mathbf{v}_0)\,. \tag{4.4}$$

From Lemma 4.15 below with $\mu=0$ and (4.4) it follows that

$$\int_0^T \big\|\nabla\partial\Phi_A(|\mathbf{D}(\mathbf{v}^{\varepsilon,A})|^2)\big\|_{\frac{6}{p+1}}^{\frac{2}{p-1}}\,dt \le c\,(\frac{1}{\varepsilon},\mathbf{f},\mathbf{v}_0)\Big(1+\int_0^T \big\|\partial\Phi_A(|\mathbf{D}(\mathbf{v}^{\varepsilon,A})|^2)\big\|_2^2\,dt\Big)\,. \tag{4.5}$$

Due to (2.25) and (4.2) we also see

$$\big\|\partial\Phi_A(|\mathbf{D}(\mathbf{v}^{\varepsilon,A})|^2)\big\|^{p'}_{L^\infty(I;L^{p'}(\Omega))} \le c\,(\frac{1}{\varepsilon},\mathbf{f},\mathbf{v}_0)\,. \tag{4.6}$$

Recalling an interpolation inequality (valid for all $p\in[2,4]$)

$$\|g\|_2^2 \le c\|g\|_{p'}^{2(1-\lambda)}\|g\|_{1,\frac{6}{p+1}}^{2\lambda} \qquad \text{with } \lambda=\frac{3(p-2)}{(6-p)(p-1)}$$

we obtain

$$\int_0^T \big\|\partial\Phi_A\big\|_2^2\,dt \le T\big\|\partial\Phi_A\|^2_{L^\infty(I;L^{p'}(\Omega))} + c\,\big\|\partial\Phi_A\big\|^{2(1-\lambda)}_{L^\infty(I;L^{p'}(\Omega))}\int_0^T \big\|\nabla\partial\Phi_A\big\|^{2\lambda}_{\frac{6}{p+1}}\,dt\,. \tag{4.7}$$

Hence from Hölder's inequality and (4.5)–(4.7) we obtain

$$\int_0^T \big\|\nabla\partial\Phi_A(|\mathbf{D}(\mathbf{v}^{\varepsilon,A})|^2)\big\|_{\frac{6}{p+1}}^{\frac{2}{p-1}}\,dt \le c\,(\frac{1}{\varepsilon},\mathbf{f},\mathbf{v}_0)\Big(1+\int_0^T \big\|\nabla\partial\Phi_A(|\mathbf{D}(\mathbf{v}^{\varepsilon,A})|^2)\big\|_{\frac{6}{p+1}}^{\frac{2}{p-1}}\,dt\Big)^{(p-1)\lambda}, \tag{4.8}$$

which implies for $p \in [2,3)$

$$\int_0^T \left\|\nabla\partial\Phi_A(|\mathbf{D}(\mathbf{v}^{\varepsilon,A})|^2)\right\|_{\frac{6}{p+1}}^{\frac{2}{p-1}}\,dt \le c\,(\frac{1}{\varepsilon},\mathbf{f},\mathbf{v}_0)\,. \tag{4.9}$$

Inequality (4.9) gives (see the appendix, Lemma 6.12, inequality (6.14))

$$\int_0^T \left\|\mathbf{D}(\nabla\mathbf{v}^{\varepsilon,A})\right\|_{\frac{6}{p+1}}^{\frac{2}{p-1}}\,dt \le c\,(\frac{1}{\varepsilon},\mathbf{f},\mathbf{v}_0)\,, \tag{4.10}$$

and by inequality (6.4) we finally obtain

$$\int_0^T \|\nabla^{(2)}\mathbf{v}^{\varepsilon,A}\|_{\frac{6}{p+1}}^{\frac{2}{p-1}}\,dt \le c\,(\frac{1}{\varepsilon},\mathbf{f},\mathbf{v}_0)\,. \tag{4.11}$$

Considering also (4.1)–(4.4), $(2.23)_4$, (4.6), (4.7), (4.9) and the Aubin-Lions lemma (see Lions [8]) we get the existence of $\mathbf{v}^\varepsilon, \pi^\varepsilon$ such that for $A \to \infty$ (or at least for subsequences)

$$\begin{aligned}
\mathbf{v}^{\varepsilon,A} &\rightharpoonup \mathbf{v}^\varepsilon && \text{*-weakly in } L^\infty(I;V_2)\cap L^{\frac{2}{p-1}}(I;W^{2,\frac{6}{p+1}}(\Omega)^3)\,,\\
\frac{\partial\mathbf{v}^{\varepsilon,A}}{\partial t} &\rightharpoonup \frac{\partial\mathbf{v}^\varepsilon}{\partial t} && \text{weakly in } L^2(I;L^2(\Omega)^3)\,,\\
\pi^{\varepsilon,A} &\rightharpoonup \pi^\varepsilon && \text{weakly in } L^2(I;L^2(\Omega))\,,\\
\nabla\mathbf{v}^{\varepsilon,A} &\to \nabla\mathbf{v}^\varepsilon && \text{strongly in } L^{\frac{2}{p-1}}(I;L^{\tilde q}(\Omega)^{3\times3})\,,\\
\nabla\mathbf{v}^{\varepsilon,A} &\to \nabla\mathbf{v}^\varepsilon && \text{strongly in } L^s(I;L^s(\Omega)^{3\times3})\,,
\end{aligned} \tag{4.12}$$

where $\tilde q < \frac{6}{p-1}$ and $s < \frac{2(2p+1)}{3(p-1)}$.[4] Because the Φ_A's approximate Φ locally uniformly, it is not difficult to conclude from (4.12) that $\mathbf{v}^\varepsilon$ solves the *Problem (NS-Dir)*$_p^\varepsilon$.

Remark 4.13. Since $\frac{2(2p+1)}{3(p-1)} > 2$ for $p \in [2,4)$ we see from (4.12), (2.6) and Vitali's convergence lemma that for almost all $t \in I$

$$\lim_{A\to\infty}\int_0^t K_{\varepsilon,A}(\tau)\,d\tau = \int_0^t K_\varepsilon(\tau)\,d\tau\,, \tag{4.14}$$

[4] In $(4.12)_5$ we used the following parabolic imbedding: for $r = \frac{2}{p-1}$ and $q = \frac{6}{p+1}$ the space $L^\infty(I;L^2(\Omega))\cap L^r(I;W^{1,q}(\Omega))$ is continuously imbedded into $L^\tau(Q_T)$, where $\tau = \frac{2(2p+1)}{3(p-1)}$, which follows from the interpolation inequality $\|\mathbf{u}\|_s \le \|\mathbf{u}\|_2^{1-\alpha}\|\mathbf{u}\|_{\frac{3q}{3-q}}^\alpha$ with $\alpha = \frac{3(s-2)}{s(4-p)}$.

where $K_\varepsilon = \int_\Omega |\mathbf{v}_\varepsilon^\varepsilon|^2 |\nabla \mathbf{v}^\varepsilon|^2 \, dx$. Thus the estimates (4.2), (4.3) and (4.11) for $\frac{\partial \mathbf{v}^\varepsilon}{\partial t}$, π^ε and $\nabla^{(2)} \mathbf{v}^\varepsilon$ remain valid due to the weak lower semicontinuity of norms and (4.14).

Now, we formulate and prove Lemma 4.15, which implies (4.5), by setting $\mu = 0$, $q = \frac{6}{p+1}$ and $r = \frac{2}{p-1}$ in (4.16).

Before doing so, we shall relabel the interior cut-off function by ξ_0 and the cut-off function localized near the boundary by ξ_k, $k = 1, 2, \dots, N$. Let us also use the convention that if ξ_k occurs in any expression then we sum up over k from 1 to N.

Recall that $\gamma_\mu(s) = \frac{1}{1-\mu} s^{1-\mu}$ for $1 \neq \mu \geq 0$ and $\gamma_1(s) = \ln s$. By γ_μ' we mean the derivative of γ_μ. Finally, we set $\theta_A(s) \equiv \max(1+s, 1+A)$ and $E_A(t) \equiv 1 + \|\Phi_A(|\mathbf{D}(\mathbf{v}^{\varepsilon,A}(t))|^2)\|_1$.

Lemma 4.15. *For all $\varepsilon > 0$ and all $A > 1$ let $(\mathbf{v}, \pi) = (\mathbf{v}^{\varepsilon,A}, \pi^{\varepsilon,A})$ be a solution of the Problem (NS-Dir)$_p^{\varepsilon,A}$. For $q \in [1, \frac{6}{p+1}]$, $r \in (0, \frac{2}{p-1}]$ and $\mu \geq 0$ we have*

$$\int_0^T \left\|\nabla \partial \Phi_A(|\mathbf{D}(\mathbf{v})|^2)\right\|_q^r \gamma_\mu'(E_A)\, dt + \int_0^T Y \, dt \tag{4.16}$$
$$\leq c(\mathbf{f}, \mathbf{v}_0)\Big(1 + \int_0^T \big(K_{\varepsilon,A} + \|\partial \Phi_A\|_2^2\big)\gamma_\mu'(E_A)\, dt\Big),$$

where

$$Y \equiv \gamma_\mu'(E_A)\Big(\int_\Omega \theta_A(|\mathbf{D}(\mathbf{v})|)^{p-2}\big(|\mathbf{D}(\nabla \mathbf{v})|^2 \xi_0^2 + \sum_{r=1}^{2} |\mathbf{D}(\frac{\partial \mathbf{v}}{\partial \tau^r})|^2 \xi_k^2\big)\, dx\Big)^{\frac{r(p-1)}{2}}.$$

Proof. We split the proof into four steps. Step 1 deals with estimates of the second derivatives in the interior of the domain and tangential derivatives near the boundary. Step 2 gives the estimates of the full second velocity gradient near the boundary by means of the gradient of the tangential derivatives. Inequalities for the full second velocity gradient are derived in Step 3. Step 4 provides the derivation of (4.16).

Step 1: *Second derivative estimates in the interior and in tangential directions near the boundary.*

Since $(2.21)_2$ is valid almost everywhere in Q_T, we can directly multiply by $-\Delta \mathbf{v}^{\varepsilon,A} \xi_0^2$. Let us drop the indices ε, A, so that $\mathbf{v} = \mathbf{v}^{\varepsilon,A}$, $\pi = \pi^{\varepsilon,A}$ and let $I_1, \dots, I_5$ denote the terms coming from $(2.21)_2$ after multiplying by

$-\Delta \mathbf{v}\xi_0^2$ and integrating over Ω (precise definitions of I_k are given below). We have

$$|I_1| \equiv |\int_\Omega \frac{\partial \mathbf{v}}{\partial t} \cdot \Delta \mathbf{v}\xi_0^2\,dx| \le \frac{C_{14}}{8}\big\|\nabla^{(2)}\mathbf{v}\xi_0\big\|_2^2 + c\big\|\frac{\partial \mathbf{v}}{\partial t}\big\|_2^2, \tag{4.17}$$

$$\begin{aligned} I_2 &\equiv \int_{\Omega_0} \frac{\partial}{\partial x_j}\frac{\partial \Phi_A(|\mathbf{D}(\mathbf{v})|^2)}{\partial D_{ij}}\Delta v_i \xi_0^2\,dx \\ &= -\int_{\Omega_0} \frac{\partial \Phi_A(|\mathbf{D}(\mathbf{v})|^2)}{\partial D_{ij}} D_{ij}(\Delta \mathbf{v})\xi_0^2 dx - 2\int_{\Omega_0} \frac{\partial \Phi_A(|\mathbf{D}(\mathbf{v})|^2)}{\partial D_{ij}}\Delta v_i \xi_0 \frac{\partial \xi_0}{\partial x_j} dx \\ &= \int_{\Omega_0} \frac{\partial^2 \Phi_A(|\mathbf{D}(\mathbf{v})|^2)}{\partial D_{ij}\partial D_{kl}} D_{ij}(\nabla \mathbf{v}) D_{kl}(\nabla \mathbf{v})\xi_0^2\,dx \qquad (4.18) \\ &\quad + 2\int_{\Omega_0} \frac{\partial \Phi_A(|\mathbf{D}(\mathbf{v})|^2)}{\partial D_{ij}} D_{ij}(\frac{\partial \mathbf{v}}{\partial x_k})\xi_0 \frac{\partial \xi_0}{\partial x_k}\,dx \\ &\quad - 2\int_{\Omega_0} \frac{\partial \Phi_A(|\mathbf{D}(\mathbf{v})|^2)}{\partial D_{ij}}\Delta v_i \xi_0 \frac{\partial \xi_0}{\partial x_j}\,dx = J_1 + J_2 + J_3\,. \end{aligned}$$

Again we obtain

$$\begin{aligned} J_1 &\overset{(2.24)}{\ge} \frac{1}{2}J_1 + \frac{C_{10}}{2}\int_{\Omega_0} |\mathbf{D}(\nabla \mathbf{v})|^2 \xi_0^2\,dx \qquad (4.19) \\ &\overset{(6.6)}{\ge} \frac{1}{2}J_1 + C_{14}\sum_{i,j=1}^{3}\int_{\Omega_0} \big|\frac{\partial^2 \mathbf{v}}{\partial x_i \partial x_j}\big|^2 \xi_0^2\,dx - c\int_{\Omega_0} |\nabla \mathbf{v}|^2 |\nabla \xi_0|^2\,dx\,, \end{aligned}$$

and

$$|J_2 + J_3| \le \frac{C_{14}}{8}\big\|\nabla^{(2)}\mathbf{v}\xi_0\big\|_2^2 + \|\partial \Phi_A\|_2^2\,. \tag{4.20}$$

The convective term, the pressure and the body force are estimated easily. We have

$$|I_3| \equiv \big| -\int_{\Omega_0} \mathbf{v}_{\varepsilon k}\frac{\partial \mathbf{v}}{\partial x_k}\Delta \mathbf{v}\xi_0^2\,dx\big| \le \frac{C_{14}}{8}\big\|\nabla^{(2)}\mathbf{v}\xi_0\big\|_2^2 + c\,K_{\varepsilon,A} \tag{4.21}$$

and

$$I_4 \equiv \int_{\Omega_0} \frac{\partial \pi}{\partial x_i}\Delta v_i \xi_0^2\,dx = -2\int_{\Omega_0} \pi \Delta v_i \xi_0 \frac{\partial \xi_0}{\partial x_i}\,dx\,.$$

Hence

$$|I_4| \le c\|\pi\|_2\big\|\nabla^{(2)}\mathbf{v}\xi_0\big\|_2 \le c\|\pi\|_2^2 + \frac{C_{14}}{8}\big\|\nabla^{(2)}\mathbf{v}\xi_0\big\|_2^2\,. \tag{4.22}$$

Finally,

$$|I_5| \equiv |\int_\Omega \mathbf{f} \cdot \Delta \mathbf{v} \xi_0^2 \, dx| \le c\|\mathbf{f}\|_2^2 + \frac{C_{14}}{8} \big\|\nabla^{(2)} \mathbf{v} \xi_0\big\|_2^2 . \tag{4.23}$$

Collecting (4.17)–(4.23) we obtain (interior estimates)

$$\begin{aligned}\int_{\Omega_0} & \frac{\partial^2 \Phi_A(|\mathbf{D}(\mathbf{v})|^2)}{\partial D_{ij} \partial D_{kl}} D_{ij}(\nabla \mathbf{v}) D_{kl}(\nabla \mathbf{v}) \xi_0^2 \, dx + \int_{\Omega_0} \sum_{i,j,k=1}^{3} \big|\frac{\partial^2 v_i}{\partial x_j \partial x_k}\big|^2 \xi_0^2 \, dx \\ & \le c\Big(\|\mathbf{f}\|_2^2 + K_{\varepsilon,A} + \big\|\partial \Phi_A\big\|_2^2 + \big\|\frac{\partial \mathbf{v}}{\partial t}\big\|_2^2 + \|\pi\|_2^2 \Big) \\ & \le c\Big(\|\mathbf{f}\|_2^2 + K_{\varepsilon,A} + \big\|\partial \Phi_A\big\|_2^2 + \big\|\frac{\partial \mathbf{v}}{\partial t}\big\|_2^2 \Big) ,\end{aligned} \tag{4.24}$$

where we used the inequality (cf. (4.3) and (3.8))

$$\|\pi(t)\|_2^2 \le c\Big(\|\mathbf{f}(t)\|_2^2 + K_{\varepsilon,A}(t) + \big\|\partial \Phi_A(t)\big\|_2^2 + \big\|\frac{\partial \mathbf{v}(t)}{\partial t}\big\|_2^2 \Big) ,$$

valid for almost all $t \in I$. The same procedure works if we estimate the tangential derivatives. We obtain

$$\begin{aligned}\int_{\Omega_0} & \frac{\partial^2 \Phi_A(|\mathbf{D}(\mathbf{v})|^2)}{\partial D_{ij} \partial D_{kl}} D_{ij}(\frac{\partial \mathbf{v}}{\partial \tau^s}) D_{kl}(\frac{\partial \mathbf{v}}{\partial \tau^s}) \xi_k^2 \, dx + \sum_{i,j=1}^{3} \sum_{s=1}^{2} \int_{\Omega_0} \big|\frac{\partial^2 v_i}{\partial x_j \partial \tau^s}\big|^2 \xi_k^2 \, dx \\ & \le c\Big(\|\mathbf{f}\|_2^2 + K_{\varepsilon,A} + \big\|\partial \Phi_A\big\|_2^2 + \big\|\frac{\partial \mathbf{v}}{\partial t}\big\|_2^2 \Big) .\end{aligned} \tag{4.25}$$

Because of the careful estimates of the tangential derivatives presented in the previous section, we think it is not necessary to repeat more or less the same steps once more.

Step 2: *Evaluation of all second derivatives of the velocity near the boundary in terms of tangential derivatives.*

The aim of this step is to show that for $q \in [1, 2]$

$$\begin{aligned}\|\theta_A(|\mathbf{D}(\mathbf{v})|)^{p-2} \nabla^{(2)} \mathbf{v} \xi_k\|_q & \le c\Big(\|\mathbf{f}\|_2 + K_{\varepsilon,A}^{1/2} + \big\|\partial \Phi_A\big\|_2 + \big\|\frac{\partial \mathbf{v}}{\partial t}\big\|_2 \Big) \\ & \quad + c \sum_{r=1}^{2} \|\theta_A(|\mathbf{D}|)^{p-2} \nabla \frac{\partial \mathbf{v}}{\partial \tau^r} \xi_k\|_q .\end{aligned} \tag{4.26}$$

Set $\boldsymbol{\zeta} \equiv (D_{13}(\frac{\partial \mathbf{v}}{\partial x_3}), D_{23}(\frac{\partial \mathbf{v}}{\partial x_3}))$ and write θ_A instead of $\theta_A(|\mathbf{D}(\mathbf{v})|)$. Since

$$\sum_{i,j=1}^{3} \frac{\partial^2 \mathbf{v}}{\partial x_i \partial x_j} = \sum_{i=1}^{3} \sum_{r=1}^{2} \frac{\partial^2 \mathbf{v}}{\partial x_i \partial x_r} + \frac{\partial^2 \mathbf{v}}{\partial x_3^2} \tag{4.27}$$

and

$$\begin{aligned} \frac{\partial^2 v_s}{\partial x_3^2} &= 2D_{s3}(\frac{\partial \mathbf{v}}{\partial x_3}) - \frac{\partial^2 v_3}{\partial x_3 \partial x_s} \qquad s = 1, 2\,, \\ \frac{\partial^2 v_3}{\partial x_3^2} &= -\frac{\partial^2 v_1}{\partial x_1 \partial x_3} - \frac{\partial^2 v_2}{\partial x_2 \partial x_3}\,, \end{aligned}$$

we see that

$$|\nabla^{(2)} \mathbf{v}| \le c \Big| \sum_{i=1}^{3} \sum_{r=1}^{2} \frac{\partial^2 \mathbf{v}}{\partial x_i \partial x_r} \Big| + c\, |\boldsymbol{\zeta}|\,, \tag{4.28}$$

which leads (multiplying the last inequality by $\xi_k \theta_A^{p-2}$) to

$$\theta_A^{p-2} |\nabla^{(2)} \mathbf{v}| \xi_k \le c \sum_{i=1}^{3} \sum_{r=1}^{2} \theta_A^{p-2} \Big| \frac{\partial^2 \mathbf{v}}{\partial x_i \partial x_r} \Big| \xi_k + c\, \theta_A^{p-2} |\boldsymbol{\zeta}| \xi_k\,. \tag{4.29}$$

In order to estimate the last term in (4.29) we use (3.47), which can be re-written as (see (3.43) and (3.48) for the definition of $\mathbf{G} = (G_2, G_1)$ and $\mathbb{A}$)

$$\mathbb{A}\, \boldsymbol{\zeta} = \tfrac{1}{2} \mathbf{G} - \mathbb{H}\,, \tag{4.30}$$

where $\mathbb{H}$ includes the remaining terms in (3.47). Thanks to (3.42) and (2.26), $\mathbb{H}$ can be estimated by

$$|\mathbb{H}| \le c \sum_{r,s=1}^{2} \theta_A^{p-2} \Big| D_{rs}(\frac{\partial \mathbf{v}}{\partial x_3}) \Big| \le \tilde{c} \sum_{r=1}^{2} \theta_A^{p-2} \Big| \nabla \frac{\partial \mathbf{v}}{\partial x_r} \Big|\,.$$

As $(\mathbb{A}\,\boldsymbol{\eta}, \boldsymbol{\eta}) \ge C_0 \theta_A^{p-2} |\boldsymbol{\eta}|^2$, we see that (4.30) together with the last inequality yields

$$C_0 \theta_A^{p-2} |\boldsymbol{\zeta}| \le \tfrac{1}{2} |\mathbf{G}| + \tilde{c} \sum_{r=1}^{2} \theta_A^{p-2} \Big| \nabla \frac{\partial \mathbf{v}}{\partial x_r} \Big|\,.$$

Plugging this into (4.29) results in

$$\theta_A^{p-2}|\nabla^{(2)}\mathbf{v}|\xi_k \le c|\mathbf{G}|\xi_k + c\sum_{i=1}^{3}\sum_{r=1}^{2}\theta_A^{p-2}\Big|\frac{\partial^2\mathbf{v}}{\partial x_i \partial x_r}\Big|\xi_k\,. \tag{4.31}$$

With the help of the definition for $\frac{\partial}{\partial\tau_r}$, $r=1,2$ (cf. (3.17)), we deduce from (4.31) that

$$\theta_A^{p-2}|\nabla^{(2)}\mathbf{v}|\xi_k \le c|\mathbf{G}|\xi_k + c\sum_{i=1}^{3}\sum_{r=1}^{2}\theta_A^{p-2}\Big|\frac{\partial^2\mathbf{v}}{\partial x_i \partial \tau^r}\Big|\xi_k$$
$$+ c\sum_{r=1}^{2}\sup_{\Omega_k}\Big|\frac{\partial a}{\partial x_r}\Big|\,\theta_A^{p-2}\Big|\nabla^{(2)}\mathbf{v}\Big|\xi_k\,.$$

Due to (3.11) we can arrange the covering of the boundary in such a way that $c\sup_{\Omega_k}|\frac{\partial a}{\partial x_r}| < \frac{1}{2}$, which leads to

$$\theta_A^{p-2}|\nabla^{(2)}\mathbf{v}|\xi_k \le c|\mathbf{G}|\xi_k + c\sum_{r=1}^{2}\theta_A^{p-2}\Big|\nabla\frac{\partial\mathbf{v}}{\partial \tau^r}\Big|\xi_k\,. \tag{4.32}$$

The inequality (4.26) will follow from (4.32) by taking the L^q-norm of all terms provided that we can estimate $\|\mathbf{G}\xi_k\|_q$ in a suitable way. For this purpose we proceed similarly to Section 3 using the curl-operator.

By the negative norm theorem (cf. the appendix, Theorem 6.7) and equations (3.40) and (3.41) we have for $s=1,2$ (cf. (3.43))

$$\begin{aligned}\big\|G_s\xi_k\big\|_q &\le c\Big\|\frac{\partial}{\partial x_3}\frac{\partial\Phi_A}{\partial D_{s3}}\xi_k\Big\|_{-1,q} + \sum_{r=1}^{2}\sum_{i,k,l=1}^{3}c\Big\|\frac{\partial^2}{\partial x_i\partial x_r}\frac{\partial\Phi_A}{\partial D_{kl}}\xi_k\Big\|_{-1,q}\\ &\quad + c\|\operatorname{curl}\Big(\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}_\varepsilon\cdot\nabla\mathbf{v} + \mathbf{f}\Big)\xi_k\|_{-1,q} + c\Big\|\frac{\partial}{\partial x_3}\frac{\partial\Phi_A}{\partial D_{s3}}\nabla\xi_k\Big\|_{-1,q}\\ &= I_6 + \dots + I_9\,.\end{aligned} \tag{4.33}$$

We easily see that

$$I_6 + I_9 \le c\|\partial\Phi_A\|_2 \tag{4.34}$$

and

$$I_8 \le c(\mathbf{f},\mathbf{v}_0)\big(1 + K_{\varepsilon,A}^{1/2} + \|\frac{\partial\mathbf{v}}{\partial t}\|_2\big)\,. \tag{4.35}$$

Further, we get using (2.26) ($q' = q/(q-1)$)

$$I_7 \le \sup_{\substack{\phi \in W_0^{1,q'}(\Omega) \\ \|\phi\|_{1,q'} \le 1}} \int_{\Omega_0} \Big| \frac{\partial}{\partial x_r} \frac{\partial \Phi_A}{\partial D_{kl}} \xi_k \frac{\partial \phi}{\partial x_i} \Big| + \Big| \frac{\partial \Phi_A}{\partial D_{kl}} \Big(\frac{\partial \xi_k}{\partial x_i} \frac{\partial \phi}{\partial x_r} + \frac{\partial^2 \xi_k}{\partial x_i \partial x_r} \phi \Big) \Big| \, dx$$

$$\le c \Big(\int_{\Omega_0} \theta_A^{(p-2)q} |\nabla \frac{\partial \mathbf{v}}{\partial x_r}|^q \xi_k^q \, dx \Big)^{1/q} + c \big\| \partial \Phi_A \big\|_2 . \tag{4.36}$$

Now, using (4.33)–(4.36), (4.32) integrated over Ω and again the definition of $\frac{\partial}{\partial \tau^r}$, we finally come to (4.26).

Step 3: *Inequalities for the full second velocity gradient.*

We are going to estimate the second gradient of the velocity putting (4.24), (4.25) and (4.26) together. We first observe that by the Hölder's inequality we have

$$\begin{aligned} \|\theta_A^{p-2} \nabla^{(2)} \mathbf{v} \xi_0\|_q &+ \|\theta_A^{p-2} \nabla \frac{\partial \mathbf{v}}{\partial \tau^r} \xi_k\|_q \\ &\le c \Big(\|\nabla^{(2)} \mathbf{v} \xi_0\|_2 + \|\nabla \frac{\partial \mathbf{v}}{\partial \tau^r} \xi_k\|_2 \Big) \|\theta_A\|^{p-2}_{\frac{(p-2)2q}{2-q}} \\ &\overset{(2.27)}{\le} c \Big(\|\nabla^{(2)} \mathbf{v} \xi_0\|_2 + \|\nabla \frac{\partial \mathbf{v}}{\partial \tau^r} \xi_k\|_2 \Big) \|1 + \partial \Phi_A\|^{\frac{p-2}{p-1}}_{\frac{(p-2)2q}{(p-1)(2-q)}} . \end{aligned} \tag{4.37}$$

Requiring $W^{1,q}(\Omega) \hookrightarrow L^{\frac{(p-2)2q}{(p-1)(2-q)}}(\Omega)$ we obtain the condition $q \in [1, \frac{6}{p+1}]$ and

$$\|1 + \partial \Phi_A\|^{\frac{p-2}{p-1}}_{\frac{(p-2)2q}{(p-1)(2-q)}} \le c \Big(\|1 + \partial \Phi_A\|_1^{\frac{p-2}{p-1}} + \|\nabla \partial \Phi_A\|_q^{\frac{p-2}{p-1}} \Big) .$$

Using this, (4.37), Lemma 6.3, (4.24)–(4.26) and Lemma 6.12 we come to the conclusion that

$$\begin{aligned} \|\nabla \partial \Phi_A\|_q + \|\nabla^{(2)} \mathbf{v}\|_q &\le c \Big(\|\mathbf{f}\|_2 + K_{\varepsilon,A}^{1/2} + \big\|\partial \Phi_A\big\|_2 + \big\| \frac{\partial \mathbf{v}}{\partial t} \big\|_2 \Big) \\ + c \Big(\|\nabla \partial \Phi_A\|_q^{\frac{p-2}{p-1}} &+ \|1 + \partial \Phi_A\|_1^{\frac{p-2}{p-1}} \Big) \Big(\|\mathbf{f}\|_2 + K_{\varepsilon,A}^{1/2} + \big\|\partial \Phi_A\big\|_2 + \big\| \frac{\partial \mathbf{v}}{\partial t} \big\|_2 \Big), \end{aligned}$$

which yields with the help of Young's inequality

$$\begin{aligned} \|\nabla \partial \Phi_A\|_q + \|\nabla^{(2)} \mathbf{v}\|_q &\le c \Big(1 + \|\mathbf{f}\|_2 + K_{\varepsilon,A}^{1/2} + \big\|\partial \Phi_A\big\|_2 + \big\| \frac{\partial \mathbf{v}}{\partial t} \big\|_2 \Big) \\ &+ c \Big(1 + \|\mathbf{f}\|_2 + K_{\varepsilon,A}^{1/2} + \big\|\partial \Phi_A\big\|_2 + \big\| \frac{\partial \mathbf{v}}{\partial t} \big\|_2 \Big)^{p-1} \\ &+ c \Big(1 + \Big(\|\mathbf{f}\|_2 + K_{\varepsilon,A}^{1/2} + \big\| \frac{\partial \mathbf{v}}{\partial t} \big\|_2 \Big) \|\partial \Phi_A\|_2^{\frac{p-2}{p-1}} + \|\partial \Phi_A\|_2^{\frac{2p-3}{p-1}} \Big) . \end{aligned} \tag{4.38}$$

Since $\frac{2p-3}{p-1} \le p-1$ for $p \ge 2$ and $\frac{p-1}{p-2}$ is the dual exponent to $p-1$ we finally obtain by considering the r-th power of the inequality ($r > 0$)

$$\|\nabla\partial\Phi_A\|_q^r + \|\nabla^{(2)}\mathbf{v}\|_q^r \le c\Big(1 + \|\mathbf{f}\|_2^2 + K_{\varepsilon,A} + \big\|\partial\Phi_A\big\|_2^2 + \Big\|\frac{\partial\mathbf{v}}{\partial t}\Big\|_2^2\Big)^{\frac{r(p-1)}{2}}, \quad (4.39)$$

which we wanted to prove.

Step 4: *Derivation of* (4.16).

Taking the $\frac{r(p-1)}{2}$-th power of (4.24) and (4.25) and adding them to (4.39) we obtain the inequality with the right-hand side bounded by the right-hand side of (4.39) (up to a multiplicative constant). Then we multiply the so-obtained inequality by $\gamma'_\mu(E_A)$ and integrate it with respect to time over $(0, T)$. Inequality (4.16) then easily follows by requiring $\frac{r(p-1)}{2} \le 1$ and also taking (3.52) into account. □

We want to finish this section with four lemmas, which are consequences of (3.52) and (4.16) if we pass to the limit as $A \to \infty$.

Lemma 4.40. *Let $p \in [2, 6)$. Then we have for almost all $t \in I$*

$$\|\nabla\mathbf{v}^\varepsilon(t)\|_2^2 \le c(\mathbf{f}, \mathbf{v}_0) + \int_0^t K_\varepsilon(\tau)\, d\tau. \quad (4.41)$$

Proof. From (3.52) for $\mu = 0$ and $(2.23)_4$ we obtain

$$\|\nabla\mathbf{v}^{\varepsilon,A}(t)\|_2^2 \le c(\mathbf{f}, \mathbf{v}_0) + \int_0^t K_{\varepsilon,A}(\tau)\, d\tau\,. \quad (4.42)$$

From (4.14) and the lower semicontinuity of norms we immediately get (4.41). □

Lemma 4.43. *Let $p \in [2, 3)$ and let γ_μ be defined as in Lemma 3.51, $\mu \ge 0$. Then we have for almost all $t \in I$*

$$\operatorname*{ess\,sup}_{\tau\in(0,t)} \gamma_\mu(E(\tau)) \le c(\mathbf{f}, \mathbf{v}_0) + \int_0^t K_\varepsilon(\tau)\ \gamma'_\mu(E(\tau))\, d\tau\,, \quad (4.44)$$

where $E(t) \equiv 1 + \|\Phi(|\mathbf{D}(\mathbf{v}^\varepsilon(t))|^2)\|_1$.

Proof. First we will show that for almost all $t \in I$

$$\lim_{A\to\infty} \int_\Omega \Phi_A(|\mathbf{D}(\mathbf{v}^{\varepsilon,A}(t))|^2)\, dx = \int_\Omega \Phi(|\mathbf{D}(\mathbf{v}^\varepsilon(t))|^2)\, dx\,. \quad (4.45)$$

We have

$$\begin{aligned} &\Big|\int_\Omega \Phi_A(|\mathbf{D}(\mathbf{v}^{\varepsilon,A}(t))|^2)\,dx - \int_\Omega \Phi(|\mathbf{D}(\mathbf{v}^{\varepsilon}(t))|^2)\,dx\Big| \\ &\le \Big|\int_\Omega \Phi_A(|\mathbf{D}(\mathbf{v}^{\varepsilon,A}(t))|^2)\,dx - \int_\Omega \Phi_A(|\mathbf{D}(\mathbf{v}^{\varepsilon}(t))|^2)\,dx\Big| \\ &\quad + \Big|\int_\Omega \Phi_A(|\mathbf{D}(\mathbf{v}^{\varepsilon}(t))|^2)\,dx - \int_\Omega \Phi(|\mathbf{D}(\mathbf{v}^{\varepsilon}(t))|^2)\,dx\Big| \equiv I_1 + I_2\,. \end{aligned} \tag{4.46}$$

From (2.27) and $(2.23)_4$ follows

$$I_1 \le C\big(1 + \|\nabla\mathbf{v}^{\varepsilon,A}(t)\|_p^{p-1} + \|\nabla\mathbf{v}^{\varepsilon}(t)\|_p^{p-1}\big)\|\nabla\mathbf{v}^{\varepsilon,A}(t) - \nabla\mathbf{v}^{\varepsilon}(t)\|_p\,,$$

which together with $(4.12)_4$ gives for almost all $t \in I$

$$\lim_{A\to\infty} I_1 = 0\,, \tag{4.47}$$

since $\frac{6}{p-1} > p$ for $p \in [2,3)$. Further, from the definition of Φ_A (cf. (2.17)) we see that

$$I_2 = \Big|\int_{\{x:|\mathbf{D}(\mathbf{v}^\varepsilon(t,x))|\ge A\}} \Phi_A(|\mathbf{D}(\mathbf{v}^{\varepsilon}(t))|^2)\,dx - \Phi(|\mathbf{D}(\mathbf{v}^{\varepsilon}(t))|^2)\,dx\Big|\,. \tag{4.48}$$

But from $(4.12)_5$ we get for almost all $t \in I$

$$\lim_{A\to\infty} |\{x : |\mathbf{D}(\mathbf{v}^\varepsilon(t,x))| \ge A\}| = 0\,. \tag{4.49}$$

This and the bound $|\Phi_A(|\mathbf{D}(\mathbf{v}^\varepsilon)|^2) - \Phi(|\mathbf{D}(\mathbf{v}^\varepsilon)|^2)| \le c(1 + |\mathbf{D}(\mathbf{v}^\varepsilon)|^p)$ gives $\lim\limits_{A\to\infty} I_2 = 0$, which together with (4.47) proves (4.45). This in turn implies that

$$\begin{aligned} &\gamma_\mu(E_A(t)) \to \gamma_\mu(E(t)) &&\text{for almost every } t \in I\,, \\ &\gamma'_\mu(E_A(t)) \overset{*}{\rightharpoonup} \gamma'_\mu(E(t)) &&\text{weakly-}^* \text{ in } L^\infty(I)\,. \end{aligned} \tag{4.50}$$

Furthermore, we have (cf. (4.14))

$$\int_\Omega |\mathbf{v}_\varepsilon^{\varepsilon,A}|^2 |\nabla\mathbf{v}^{\varepsilon,A}|^2\,dx \to \int_\Omega |\mathbf{v}_\varepsilon^{\varepsilon}|^2 |\nabla\mathbf{v}^{\varepsilon}|^2\,dx \qquad \text{in } L^1(I)\,. \tag{4.51}$$

Since the first term in (3.52) is nonnegative we can pass to the limit in (3.52) as $A \to \infty$ due to (4.50) and (4.51). This gives (4.44). □

Lemma 4.52. *Let* $p \in [2,3)$ *and let* γ_μ *be as in Lemma* 3.51, $\mu \geq 0$. *Then we have for all* $q \in [1, \frac{6}{p+1}]$ *and all* $r \in [1, \frac{2}{p-1}]$

$$\int_0^T \|\nabla\partial\Phi(|\mathbf{D}(\mathbf{v}^\varepsilon(t))|^2)\|_q^r \,\gamma'_\mu(E(t))\,dt + \int_0^T Z(\mathbf{v}^\varepsilon(t))\gamma'_\mu(E(t))\,dt \tag{4.53}$$
$$\leq c(\mathbf{f},\mathbf{v}_0)\Big(1+\int_0^T K_\varepsilon(t)\,\gamma'_\mu(E(t))\,dt + \int_0^T \|\partial\Phi(|\mathbf{D}(\mathbf{v}^\varepsilon(t))|^2)\|_2^2\,\gamma'_\mu(E(t))\,dt\Big),$$

where

$$Z(\mathbf{v}^\varepsilon(t)) = \Big(\|\nabla^{(2)}\mathbf{v}^\varepsilon(t)\xi_0\|_2^2 + \sum_{r=1}^{2}\|\mathbf{D}(\frac{\partial \mathbf{v}^\varepsilon(t)}{\partial\tau^r})\,\xi_k\|_2^2\Big)^{\frac{r(p-1)}{2}}.$$

Proof. Let us for simplicity denote $\gamma(s) = \gamma_\mu(s)$. In Lemma 4.15 we proved for $r \leq \frac{2}{p-1}$

$$\int_0^T \|\nabla\partial\Phi_A(|\mathbf{D}(\mathbf{v}^{\varepsilon,A})|^2)\|_q^r \,\gamma'(E_A(t))\,dt + \int_0^T Z(\mathbf{v}^{\varepsilon,A}(t))\gamma'(E_A(t))\,dt \tag{4.54}$$
$$\leq c(\mathbf{f},\mathbf{v}_0)\Big(1+\int_0^T K_{\varepsilon,A}\,\gamma'(E_A(t))\,dt + \int_0^T \|\partial\Phi_A(|\mathbf{D}(\mathbf{v}^{\varepsilon,A})|^2)\|_2^2\,\gamma'(E_A(t))\,dt\Big).$$

Since Φ_A approximates Φ locally uniformly we get from $(4.12)_5$ that

$$\partial\Phi_A(|\mathbf{D}(\mathbf{v}^{\varepsilon,A})|^2) \to \partial\Phi(|\mathbf{D}(\mathbf{v}^\varepsilon)|^2) \qquad \text{almost everywhere in } Q_T\,. \tag{4.55}$$

From (4.6) and (4.9) we obtain[5]

$$\partial\Phi_A(|\mathbf{D}(\mathbf{v}^{\varepsilon,A})|^2) \quad \text{is bounded in } L^{\frac{2(p+3)}{3(p-1)}}(Q_T)\,, \tag{4.56}$$

which together with (cf. (2.25), $(2.23)_4$)

$$|\partial\Phi_A(|\mathbf{A}|^2)| \leq c(1+|\mathbf{A}|)^{p-1}$$

[5] Here we use the following parabolic imbedding: the space $L^\infty(I; L^{p'}(\Omega)) \cap L^r(I; W^{1,q}(\Omega))$ is continuously imbedded into $L^\tau(Q_T)$, where $\tau = \frac{2(3+p)}{3(p-1)}$, which follows from the interpolation inequality $\|\mathbf{u}\|_s \leq \|\mathbf{u}\|_{p'}^{1-\alpha}\|\mathbf{u}\|_{\frac{3q}{3-q}}^{\alpha}$ with $\alpha = \frac{6(r(p-1)-p)}{r(p-1)(6-p)}$.

and Vitali's convergence lemma gives (for $p \in [2,3)$)

$$\partial\Phi_A(|\mathbf{D}(\mathbf{v}^{\varepsilon,A})|^2) \to \partial\Phi(|\mathbf{D}(\mathbf{v}^{\varepsilon})|^2) \qquad \text{strongly in } L^2(Q_T)\,. \tag{4.57}$$

Thus, (4.57) and $(4.50)_2$ enable the limiting process in the second term on the right-hand side of (4.54). Similarly, using (4.14) and $(4.50)_2$, we can justify the limiting process in the first term on the right-hand side in (4.54). It remains to pass to the limit in the left-hand side of (4.54). For all $q < \infty$ we have

$$\gamma'(E_A(t)) \to \gamma'(E(t)) \qquad \text{strongly in } L^q(I)\,.$$

Therefore, by the Egorov theorem we know that there is a set I_δ, $|I \setminus I_\delta| \le \delta$, and a subsequence $A_\delta \to \infty$ such that

$$\gamma'(E_{A_\delta}(t)) \rightrightarrows \gamma'(E(t)) \qquad \text{uniformly in } I_\delta\,.$$

From this and boundedness of $\|\nabla\partial\Phi_A(|\mathbf{D}(\mathbf{v}^{\varepsilon,A})|^2)\|_q^r$ in $L^1(I)$ we conclude that

$$\liminf_{A_\delta\to\infty} \int_{I_\delta} \|\nabla\partial\Phi_{A_\delta}(|\mathbf{D}(\mathbf{v}^{\varepsilon,A_\delta})|^2)\|_q^r \Big(\gamma'(E_{A_\delta}(t)) - \gamma'(E(t))\Big)\,dt = 0\,.$$

Thus we have

$$\begin{aligned}\liminf_{A_\delta\to\infty} &\int_{I_\delta} \|\nabla\partial\Phi_{A_\delta}(|\mathbf{D}(\mathbf{v}^{\varepsilon,A_\delta})|^2)\|_q^r\,\gamma'(E_{A_\delta}(t))\,dt \\ &= \liminf_{A_\delta\to\infty} \int_{I_\delta} \|\nabla\partial\Phi_{A_\delta}(|\mathbf{D}(\mathbf{v}^{\varepsilon,A_\delta})|^2)\|_q^r\,\gamma'(E(t))\,dt\,.\end{aligned} \tag{4.58}$$

Since

$$0 < \gamma'(E(t)) \le 1 \tag{4.59}$$

we can define by $\nu \equiv \gamma'(E(t))\,dt$ a new measure, which is absolutely continuous with respect to dt. From (4.57) and (4.9) we can identify the weak limit of $\nabla\partial\Phi_A$ in $L^r(I;L^q(\Omega))$ as $\nabla\partial\Phi$. From this, (4.59), the boundedness of the right-hand side of (4.54) and the uniqueness of the weak limit we obtain that $\nabla\partial\Phi_A(|\mathbf{D}\mathbf{v}^{\varepsilon,A}|^2) \rightharpoonup \nabla\partial\Phi(|\mathbf{D}\mathbf{v}^{\varepsilon}|^2)$ also in $L^r(I,\nu;L^q(\Omega))$, and therefore

$$\begin{aligned}\liminf_{A_\delta\to\infty} &\int_{I_\delta} \|\nabla\partial\Phi_{A_\delta}(|\mathbf{D}(\mathbf{v}^{\varepsilon,A_\delta})|^2)\|_q^r\,\gamma'(E(t))\,dt \\ &\ge \int_{I_\delta} \|\nabla\partial\Phi(|\mathbf{D}(\mathbf{v}^{\varepsilon})|^2)\|_q^r\,\gamma'(E(t))\,dt\,.\end{aligned}$$

Since $\delta > 0$ was arbitrary and the terms on the right-hand side of (4.54) are bounded independent of δ we can conclude the proof for the first quantity on the left-hand side in (4.53) by a diagonal argument. The proof for the second term on the left-hand side of (4.53) follows along the same lines. $\square$

Lemma 4.60. *For $p \in [2,3)$, we have*

$$\int_0^T \int_\Omega (1+|\mathbf{D}(\mathbf{v}^\varepsilon)|)^{p-2}\Big(\|\nabla^{(2)}\mathbf{v}^\varepsilon \xi_0\|_2^2 + \sum_{r=1}^2 \|\mathbf{D}(\frac{\partial \mathbf{v}^\varepsilon}{\partial \tau^r})\,\xi_k\|_2^2\Big)\,dx\,dt$$
$$\leq c(\mathbf{f},\mathbf{v}_0)\Big(1+\int_0^T K_\varepsilon(t)\,dt + \int_0^T \|\partial\Phi(|\mathbf{D}(\mathbf{v}^\varepsilon(t))|^2)\|_2^2\,dt\Big). \tag{4.61}$$

Proof. In Lemma 4.15 we proved

$$\int_0^T \int_\Omega \theta_A(|\mathbf{D}(\mathbf{v}^{\varepsilon,A})|)^{p-2}\Big(\|\nabla^{(2)}\mathbf{v}^{\varepsilon,A}\xi_0\|_2^2 + \sum_{r=1}^2 \|\mathbf{D}(\frac{\partial \mathbf{v}^{\varepsilon,A}}{\partial \tau^r})\,\xi_k\|_2^2\Big)\,dx\,dt$$
$$\leq c(\mathbf{f},\mathbf{v}_0)\bigg(1+\int_0^T K_{\varepsilon,A}\,dt + \int_0^T \|\partial\Phi_A(|\mathbf{D}(\mathbf{v}^{\varepsilon,A})|^2)\|_2^2\,dt\bigg). \tag{4.62}$$

We have already shown in the proof of the previous lemma that we can pass to the limit as $A \to \infty$ in the right-hand side of (4.62). For the left-hand side we proceed similarly and use (4.12) and the Egorov theorem to conclude that for all $\delta > 0$ there is a set Q_δ, $|Q \setminus Q_\delta| \leq \delta$, and a subsequence A_δ such that

$$\theta_{A_\delta}(|\mathbf{D}(\mathbf{v}^{\varepsilon,A_\delta})|) \rightrightarrows 1+|\mathbf{D}(\mathbf{v}^\varepsilon)| \qquad \text{uniformly in } Q_\delta\,.$$

Now we finish the proof along the lines of Lemma 4.52.

5. Limiting process $\varepsilon \to 0$. In the previous section we obtained solutions $(\mathbf{v},\pi) = (\mathbf{v}^\varepsilon,\pi^\varepsilon)$ of the *Problem (NS-Dir)*$_p^\varepsilon$, that were based on estimates independent of A. These estimates, however, were dependent on ε through $K_\varepsilon = \int_\Omega |\mathbf{v}_\varepsilon^\varepsilon|^2 |\nabla \mathbf{v}^\varepsilon|^2\,dx$, or precisely through $\int_0^T K_\varepsilon\,dt$.

From the energy inequality (2.8) for the *Problem (NS-Dir)*$_p^\varepsilon$, Lemma 4.40 and Remark 4.13 we get for all $p \in [2,3)$ and almost all $t \in I$

$$\operatorname*{ess\,sup}_{t\in I} \|\mathbf{v}^\varepsilon(t)\|_2^2 + \int_0^T \|\nabla \mathbf{v}^\varepsilon\|_2^2\,dt + \int_0^T \|\nabla \mathbf{v}^\varepsilon\|_p^p\,dt \leq c(\mathbf{f},\mathbf{v}_0)\,, \tag{5.1}$$

$$\|\nabla \mathbf{v}^\varepsilon(t)\|_2^2 \leq c(\mathbf{f},\mathbf{v}_0) + \int_0^t K_\varepsilon(\tau)\,d\tau\,, \tag{5.2}$$

$$\int_0^T \|\frac{\partial \mathbf{v}^\varepsilon}{\partial t}\|_2^2\,dt \leq c(\mathbf{f},\mathbf{v}_0)\Big(1+\int_0^T K_\varepsilon(t)\,dt\Big). \tag{5.3}$$

It is worth noticing that in addition to $\int_0^T \|\nabla \mathbf{v}^\varepsilon\|_2^2\,dt$, which has already occurred in (4.1) or (3.4), we now also obtain the estimate of $\int_0^T \|\nabla \mathbf{v}^\varepsilon\|_p^p\,dt$

in (5.1), as follows from (2.2). Moreover, we proved in Lemma 4.43 and Lemma 4.52 that for $\mu \geq 0$

$$\operatorname*{ess\,sup}_{\tau\in(0,t)} \gamma_\mu(E(\tau)) \leq c(\mathbf{f},\mathbf{v}_0) + \int_0^t K_\varepsilon(\tau)\, \gamma'_\mu(E(\tau))\, d\tau\,, \tag{5.4}$$

$$\begin{aligned}\int_0^T \|\nabla\partial\Phi(|\mathbf{D}(\mathbf{v}^\varepsilon(t))|^2)\|^{\frac{2}{p-1}}_{\frac{6}{p+1}}\, \gamma'_\mu(E(t))\, dt + \int_0^T Z(\mathbf{v}^\varepsilon(t))\, \gamma'_\mu(E(t))\, dt \\ \leq c(\mathbf{f},\mathbf{v}_0)\Big(1 + \int_0^T K_\varepsilon(t)\, \gamma'_\mu(E(t))\, dt + \int_0^T \|\partial\Phi\|_2^2\, \gamma'_\mu(E(t))\, dt\Big)\,,\end{aligned} \tag{5.5}$$

where $\gamma_\mu(s) = \frac{1}{1-\mu} s^{1-\mu}$, for $\mu \geq 0$ and $\mu \neq 1$, and $\gamma_1(s) = \ln(s)$ and $Z(\mathbf{v}^\varepsilon(t))$ is defined in Lemma 4.52. Moreover, in the case $\mu = 0$ we also have from Lemma 4.60

$$\begin{aligned}\int_0^T \int_\Omega (1 + |\mathbf{D}(\mathbf{v}^\varepsilon)|)^{p-2}\Big(\|\nabla^{(2)}\mathbf{v}^\varepsilon \xi_0\|_2^2 + \sum_{r=1}^{2} \|\mathbf{D}(\frac{\partial \mathbf{v}^\varepsilon}{\partial \tau^r})\, \xi_k\|_2^2\Big)\, dx\, dt \\ \leq c(\mathbf{f},\mathbf{v}_0)\Big(1 + \int_0^T K_\varepsilon(t)\, dt + \int_0^T \|\partial\Phi(|\mathbf{D}(\mathbf{v}^\varepsilon(t))|^2)\|_2^2\, dt\Big)\,.\end{aligned} \tag{5.6}$$

Let us make two observations. Firstly, if $\mu \in [0,1]$ in (5.4) and the right-hand side is bounded, then $\{\mathbf{v}^\varepsilon\}$ lies in a ball of $L^\infty(I;V_p)$. If $\mu > 1$, then inequality (5.4) gives us no information. Secondly, denoting

$$I_{p,\frac{6}{p+1}}(t) \equiv \int_\Omega (1 + |\mathbf{D}(\mathbf{v}(t))|)^{\frac{6(p-2)}{p+1}} |\mathbf{D}(\nabla\mathbf{v}(t))|^{\frac{6}{p+1}}\, dx \tag{5.7}$$

and using Lemma 6.9, we obtain from (5.5) and (5.4) that

$$\begin{aligned}\gamma_\mu(E(t)) + \int_0^T \gamma'_\mu(E(t)) I^{\frac{p+1}{3(p-1)}}_{p,\frac{6}{p+1}}(t)\, dt + \int_0^T Z(\mathbf{v}(t))\, \gamma'_\mu(E(t))\, dt \\ \leq c(\mathbf{f},\mathbf{v}_0)\bigg(1 + \int_0^T \gamma'_\mu(E(t)) K_\varepsilon(t)\, dt + \int_0^T \gamma'_\mu(E(t)) \|\nabla\mathbf{v}\|^{2(p-1)}_{2(p-1)}\, dt\bigg)\,,\end{aligned} \tag{5.8}$$

in which the first term is omitted if $\mu > 1$.

Our last task is to estimate (uniformly with respect to ε) the right-hand sides of (5.2)–(5.4), (5.6) and (5.8).

(*i*) The case $p \in [\frac{12}{5}, 3)$. Let us first recall that by (2.4) the norm $\|\Phi(|\mathbf{D}(\mathbf{v})|^2)\|_1$ is equivalent to $\|\mathbf{D}(\mathbf{v})\|_p^p$. Thus, we can estimate K_ε in (5.4), with $\mu = 0$, in the same way as in (2.13), and we obtain

$$\|\mathbf{v}(t)\|_p^p \le c(\mathbf{f}, \mathbf{v}_0) + c \int_0^t \|\mathbf{v}\|_2^{2\frac{5p-12}{5p-6}} \|\nabla \mathbf{v}\|_p^{p\frac{16-5p}{5p-6}} \|\nabla \mathbf{v}\|_p^p \, d\tau \,,$$

which together with (5.1) and Gronwall's lemma gives the fact that

$$\mathbf{v}^\varepsilon \text{ is bounded in } L^\infty(I; V_p) \tag{5.9}$$

uniformly with respect to ε. Thus $\int_0^T K_\varepsilon \, dt$ is bounded independent of ε and it remains to bound the second term on the right-hand sides of (5.6) and (5.8) for $\mu = 0$. From (6.11) we deduce that

$$\|\nabla \mathbf{v}\|_6^{\frac{6(p-1)}{p+1}} \le C_{18}\big(1 + \|\mathbf{v}\|_p^{\frac{6(p-1)}{p+1}} + I_{p,\frac{6}{p+1}}(t)\big) \,, \tag{5.10}$$

which together with the interpolation inequality

$$\|\nabla \mathbf{v}\|_{2(p-1)} \le \|\nabla \mathbf{v}\|_6^{\frac{3(p-2)}{(6-p)(p-1)}} \|\nabla \mathbf{v}\|_p^{\frac{p(4-p)}{(6-p)(p-1)}}$$

implies that

$$\begin{aligned} \|\nabla \mathbf{v}\|_{2(p-1)}^{2(p-1)} &\overset{(5.10)}{\le} c\|\nabla \mathbf{v}\|_p^{\frac{2p(4-p)}{6-p}} \Big(1 + \|\nabla \mathbf{v}\|_p^{\frac{6(p-2)}{6-p}} + I_{p,\frac{6}{p+1}}^{\frac{(p-2)(p+1)}{(6-p)(p-1)}}\Big) \\ &\overset{(2.4)}{\le} c\|\Phi(|\mathbf{D}(\mathbf{v})|^2)\|_1^{\frac{2(4-p)}{6-p}} \Big(1 + \|\Phi(|\mathbf{D}(\mathbf{v})|^2)\|_1^{\frac{6(p-2)}{p(6-p)}} + I_{p,\frac{6}{p+1}}^{\frac{(p-2)(p+1)}{(6-p)(p-1)}}\Big) \,. \end{aligned} \tag{5.11}$$

We get from (5.8) ($\mu = 0$), (5.9) and the first inequality in (5.11)

$$\int_0^T I_{p,\frac{6}{p+1}}^{\frac{p+1}{3(p-1)}}(t)\, dt \le c(\mathbf{f}, \mathbf{v}_0)\Big(1 + \int_0^T I_{p,\frac{6}{p+1}}^{\frac{(p-2)(p+1)}{(6-p)(p-1)}}(t)\, dt\Big) \,. \tag{5.12}$$

However for $p \in [2, 3)$ we see that the exponent on the right-hand side of (5.12) is strictly less than the exponent on the left-hand side of (5.12). Thus the right-hand sides of (5.6) and (5.8) with $\mu = 0$ are bounded independent of ε, and we proved that

$$\int_0^T \|(1+|\mathbf{D}(\mathbf{v}^\varepsilon)|)^{\frac{p-2}{2}}\mathbf{D}(\nabla\mathbf{v}^\varepsilon)\|_{2,\mathrm{loc}}^2\,dt + \int_0^T \|(1+|\mathbf{D}(\mathbf{v}^\varepsilon)|)^{\frac{p-2}{2}}\mathbf{D}(\frac{\partial\mathbf{v}^\varepsilon}{\partial\tau^r})\|_2^2\,dt$$
$$\int_0^T \|\frac{\partial\mathbf{v}^\varepsilon}{\partial t}\|_2^2\,dt + \operatorname*{ess\,sup}_{t\in I}\|\nabla\mathbf{v}^\varepsilon(t)\|_p^p + \int_0^T \|\nabla^{(2)}\mathbf{v}^\varepsilon\|_{\frac{6}{p+1}}^{\frac{2}{p-1}}\,dt \le c(\mathbf{f},\mathbf{v}_0)\,. \tag{5.13}$$

From (5.13) and the Aubin-Lions lemma we obtain that for $\tilde{q} \in [1, \frac{6}{p-1})$

$$\nabla\mathbf{v}^\varepsilon \to \nabla\mathbf{v} \qquad \text{in } L^{\frac{2}{p-1}}(I; L^{\tilde{q}}(\Omega)^{3\times3})\,,\ \text{almost everywhere in } Q_T\,. \tag{5.14}$$

Using (5.1), (5.13), (5.14), (2.3) and Vitali's convergence lemma we pass to the limit as $\varepsilon \to 0$ in the weak formulation of $(2.7)_2$. Concerning the uniqueness we argue as in [11], Theorem 5.4.37.

(*ii*) The case $p \in [2, 12/5)$. We are going to estimate the terms on the right-hand side of (5.8). For the convective term we have

$$\begin{aligned} K_\varepsilon &\le c\|\nabla\mathbf{v}\|_p^2\|\nabla\mathbf{v}\|_{\frac{6p}{5p-6}}^2 \le c\|\nabla\mathbf{v}\|_p^{2(2-\lambda)}\|\nabla\mathbf{v}\|_6^{2\lambda} \\ &\overset{(2.4)}{\le} c\|\Phi(|\mathbf{D}(\mathbf{v})|^2)\|_1^{\frac{2(2-\lambda)}{p}}\big(1 + \|\Phi(|\mathbf{D}(\mathbf{v})|^2)\|_1^{\frac{2\lambda}{p}} + I_{p,\frac{6}{p+1}}^{\frac{2\lambda(p+1)}{6(p-1)}}\big)\,, \end{aligned}$$

where we have used the interpolation inequality $\|\mathbf{z}\|_{\frac{6p}{5p-6}} \le \|\mathbf{z}\|_p^{1-\lambda}\|\mathbf{z}\|_6^\lambda$ with $\lambda = \frac{12-5p}{6-p}$ and $1-\lambda = \frac{2(2p-3)}{6-p}$ and (5.10). From this, (5.11) and (5.8) we obtain that (recall $\gamma_\mu'(s) = s^{-\mu}$)

$$\begin{aligned} &\gamma_\mu(E(t)) + \int_0^T E(t)^{-\mu} I_{p,\frac{6}{p+1}}^{\frac{p+1}{3(p-1)}}(t)\,dt \\ &\le c(\mathbf{f},\mathbf{v}_0)\bigg(1 + \int_0^T E(t)^{\frac{4}{p}-\mu}\,dt + \int_0^T E(t)^{\frac{2(p-1)}{p}-\mu}\,dt \\ &\quad + \int_0^T \big\{E(t)^{-\mu} I_{p,\frac{6}{p+1}}^{\frac{p+1}{3(p-1)}}(t)\big\}^{\frac{3(p-2)}{6-p}} E(t)^{\frac{p+2}{6-p}+\mu(\frac{3(p-2)}{6-p}-1)}\,dt \\ &\quad + \int_0^T \big\{E(t)^{-\mu} I_{p,\frac{6}{p+1}}^{\frac{p+1}{3(p-1)}}(t)\big\}^{\lambda} E(t)^{\lambda+\frac{2(2-\lambda)}{p}+\mu(\lambda-1)}\,dt\bigg). \end{aligned} \tag{5.15}$$

Requiring now for the first two integrals on the right-hand side that the exponent is less than 1 yields

$$\mu_1 = \frac{4-p}{p}\,, \qquad \mu_2 = \frac{p-2}{p}\,. \tag{5.16}$$

Note that for $p \in [2,3)$ $\mu_2 \le \mu_1$. Furthermore, we require for the third integral on the right-hand side of (5.15)

$$\frac{3(p-2)}{6-p}\gamma = 1\,, \qquad \Big[\frac{p+2}{6-p} + \mu\big(\frac{3(p-2)}{6-p} - 1\big)\Big]\gamma' \le 1\,,$$

where $\frac{1}{\gamma} + \frac{1}{\gamma'} = 1$, which implies

$$\mu_3 = \frac{p-2}{2(3-p)}\,, \tag{5.17}$$

and for the fourth integral on the right-hand side

$$\lambda\delta = 1\,, \qquad \Big[\lambda + \frac{2(2-\lambda)}{p} + \mu(\lambda - 1)\Big]\delta' \le 1\,,$$

where $\frac{1}{\delta} + \frac{1}{\delta'} = 1$, which gives

$$\mu_4 = 2\frac{3-p}{2p-3}\,. \tag{5.18}$$

Note that for $p \in [2, \frac{12}{5})$, we have $\mu_3 \le \mu_4$ and $\mu_1 \le \mu_4$, and therefore we choose $\mu = \mu_4$, since $E(t)^{-1} \le 1$. By Young's inequality and (5.1) we obtain

$$\gamma_\mu(E(t)) + \int_0^T E(t)^{-\mu} I_{p,\frac{6}{p+1}}^{\frac{p+1}{3(p-1)}}(t)\,dt \le c(\mathbf{f}, \mathbf{v}_0)\,, \tag{5.19}$$

where the first term is omitted if $\mu > 1$. Note that

$$\mu \le 1 \quad \text{ if and only if } \quad p \ge \frac{9}{4}\,,$$

and therefore we obtain (5.9) for these p's. Now we go again into (5.15) and set $\mu = 0$. Using (5.9) and the fact that the exponents of the terms in the squiggly brackets in (5.15) are strictly less than 1 we easily obtain that the right-hand side of (5.15) is finite independent of ε. This together with (5.6) yields (5.12), and we can conclude the proof for these p's as in case (i). Due to the definition of $I_{p,\frac{6}{p+1}}$ we get from (5.19) for all $p \in [2, 9/4)$

$$\int_0^T \big(1 + \|\nabla \mathbf{v}\|_p^p\big)^{-\mu} \|\nabla^2 \mathbf{v}\|_{\frac{6}{p+1}}^{\frac{2}{p-1}}\,dt \le c(\mathbf{f}, \mathbf{v}_0)\,. \tag{5.20}$$

Now we proceed analogously as in the *Problem (NS-Per)$_p$* (cf. [11], Chapter 5, pp. 237–238, or [9]) to obtain from (5.20) $\nabla \mathbf{v}^\varepsilon \to \nabla \mathbf{v}$ strongly in $L^1(Q_T)$. The proof of Theorem 1.17 is complete.

6. Appendix. Here we first collect some general auxiliary definitions and assertions used in the previous text, and then we prove more technical assertions concerning the potentials Φ and Φ_A.

Proposition 6.1 (Korn's inequality). *Let $1 < p < \infty$ and let $\Omega \subset \mathbb{R}^d$ be of class $\mathcal{C}^1$. Then there exists a constant $K_p = K_p(\Omega)$ such that the inequality*

$$K_p \|\mathbf{v}\|_{1,p} \leq \|\mathbf{D}(\mathbf{v})\|_p \tag{6.2}$$

is fulfilled for all $\mathbf{v} \in W_0^{1,p}(\Omega)^d$.

Proof.
See e.g. [14].

Lemma 6.3. *Let $1 < p < \infty$ and let $\Omega \subset \mathbb{R}^d$ be a domain of class $\mathcal{C}^1$. Then there exists a constant $c_p = c_p(\Omega)$ such that for all $\mathbf{v} \in W_0^{1,p}(\Omega)^d \cap W^{2,p}(\Omega)^d$*

$$c_p \|\nabla^{(2)} \mathbf{v}\|_p \leq \|\mathbf{D}(\nabla \mathbf{v})\|_p \,. \tag{6.4}$$

Proof. The assertion follows immediately from the algebraic identity

$$\frac{\partial^2 v_i}{\partial x_j \partial x_k} = \frac{\partial D_{ik}(\mathbf{v})}{\partial x_j} + \frac{\partial D_{ij}(\mathbf{v})}{\partial x_k} - \frac{\partial D_{jk}(\mathbf{v})}{\partial x_i} \,.$$

Lemma 6.5. *Let $\Omega \subset \mathbb{R}^d$ be a domain, $\partial\Omega \in \mathcal{C}^1$ and let $\mathbf{v} \in W^{1,2}(\Omega)^d$, $\xi \in \mathcal{D}(\Omega)$. Then*

$$\int_\Omega |\mathbf{D}(\mathbf{v})|^2 \xi^2 \, dx \geq C_{15} \int_\Omega |\nabla \mathbf{v}|^2 \xi^2 \, dx - C_{16} \int_\Omega |\mathbf{v}|^2 |\nabla \xi|^2 \, dx \,. \tag{6.6}$$

Proof. We have

$$\int_\Omega \mathbf{D}(\mathbf{v}) \cdot \mathbf{D}(\mathbf{v}) \xi^2 \, dx = \frac{1}{2} \int_\Omega \frac{\partial v_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} \xi^2 \, dx + \frac{1}{2} \int_\Omega \frac{\partial v_i}{\partial x_j} \frac{\partial v_j}{\partial x_i} \xi^2 \, dx = I_1 + I_2 \,.$$

The integration by parts yields

$$I_2 = \frac{1}{2} \int_\Omega |\operatorname{div} \mathbf{v}|^2 \xi^2 \, dx + \int_\Omega v_i \frac{\partial v_j}{\partial x_j} \xi \frac{\partial \xi}{\partial x_i} \, dx - \int_\Omega v_i \frac{\partial v_j}{\partial x_i} \xi \frac{\partial \xi}{\partial x_j} \, dx \,.$$

Thus we obtain

$$\begin{aligned}
&\frac{1}{2} \int_\Omega |\nabla \mathbf{v}|^2 \xi^2 \, dx + \frac{1}{2} \int_\Omega |\operatorname{div} \mathbf{v}|^2 \xi^2 \, dx \\
&\quad = \int_\Omega |\mathbf{D}(\mathbf{v})|^2 \xi^2 \, dx - \int_\Omega v_i \frac{\partial v_j}{\partial x_j} \xi \frac{\partial \xi}{\partial x_i} \, dx + \int_\Omega v_i \frac{\partial v_j}{\partial x_i} \xi \frac{\partial \xi}{\partial x_j} \, dx \\
&\quad \leq \int_\Omega |\mathbf{D}(\mathbf{v})|^2 \xi^2 \, dx + \frac{1}{4} \int_\Omega |\operatorname{div} \mathbf{v}|^2 \xi^2 \, dx + \frac{1}{4} \int_\Omega |\nabla \mathbf{v}|^2 \xi^2 \, dx + c \int_\Omega |\mathbf{v}|^2 |\nabla \xi|^2 \, dx,
\end{aligned}$$

and the assertion follows.

Theorem 6.7 (On negative norms). *Let $1 < p < \infty$ and let $\mathbf{v} \in W_0^{1,p}(\Omega)^d$. Then there exists a constant such that*

$$c \|\mathbf{v}\|_p \leq \|\mathbf{v}\|_{-1,p} + \|\nabla \mathbf{v}\|_{-1,p} \,. \tag{6.8}$$

Proof. See e.g. [12].

In the rest of this section we assume that the potential Φ satisfies (1.9)–(1.11). Then, by Lemma 2.22, the potential Φ_A satisfies (2.23)–(2.26).

We denote $I_{p,q}(\mathbf{v}) \equiv \int_\Omega (1 + |\mathbf{D}(\mathbf{v})|)^{q(p-2)} |\mathbf{D}(\nabla \mathbf{v})|^q \, dx$.

Lemma 6.9. *Let $p \geq 2$ and $q \in [1,\infty)$. Then there exist constants C_{17}, C_{18} and C_{19} such that*

$$C_{17} I_{p,q}(\mathbf{v}) \leq \int_\Omega |\nabla \partial \Phi(|\mathbf{D}(\mathbf{v})|^2)|^q \, dx \leq C_{18} I_{p,q}(\mathbf{v}) \tag{6.10}$$

$$\|\nabla \mathbf{v}\|^{q(p-1)}_{(p-1)\frac{3q}{3-q}} \leq C_{19}\big(1 + \|\nabla \mathbf{v}\|_p^{q(p-1)} + I_{p,q}(\mathbf{v})\big)\,. \tag{6.11}$$

Proof. We have

$$\begin{aligned}
I_{p,q}(\mathbf{v}) &= \int_\Omega \big[(1+|\mathbf{D}(\mathbf{v})|)^{p-2}|\mathbf{D}(\nabla \mathbf{v})|^2\big]^{\frac{q}{2}} (1+|\mathbf{D}(\mathbf{v})|)^{(p-2)\frac{q}{2}} \, dx \\
&\overset{(1.10)}{\leq} \int_\Omega \big[\partial_{ij}\partial_{kl}\Phi(|\mathbf{D}(\mathbf{v})|^2) D_{ij}(\nabla \mathbf{v}) \cdot D_{kl}(\nabla \mathbf{v})\big]^{\frac{q}{2}} (1+|\mathbf{D}(\mathbf{v})|)^{(p-2)\frac{q}{2}} \, dx \\
&\leq \int_\Omega |\nabla \partial \Phi(|\mathbf{D}(\mathbf{v})|^2)|^{\frac{q}{2}} |\mathbf{D}(\nabla \mathbf{v})|^{\frac{q}{2}} (1+|\mathbf{D}(\mathbf{v})|)^{(p-2)\frac{q}{2}} \, dx \\
&\overset{\text{Young}}{\leq} \frac{1}{2} I_{p,q}(\mathbf{v}) + c\int_\Omega |\nabla \partial \Phi(|\mathbf{D}(\mathbf{v})|^2)|^q \, dx\,,
\end{aligned}$$

which immediately gives the first inequality in (6.10). The second one follows easily from the chain rule and (1.11). Further, we have

$$\begin{aligned}
I_{p,q}(\mathbf{v}) &\geq \int_\Omega |\nabla(1+|\mathbf{D}|)|^q (1+|\mathbf{D}|)^{q(p-2)} \, dx = \frac{1}{(p-1)^q}\int_\Omega |\nabla(1+|\mathbf{D}|)^{p-1}|^q \, dx \\
&\geq c\Big(\int_\Omega (1+|\mathbf{D}|)^{(p-1)\frac{3q}{3-q}} \, dx\Big)^{\frac{3-q}{3}} - \|(1+|\mathbf{D}|)^{p-1}\|^q_{p'} \\
&\overset{(6.2)}{\geq} c\|\nabla \mathbf{v}\|^{q(p-1)}_{(p-1)\frac{3q}{3-q}} - c\|\nabla \mathbf{v}\|_p^{q(p-1)} - c\,,
\end{aligned}$$

which is (6.11).

Lemma 6.12. *Let $p \geq 2$ and $q \in [1,\infty)$ and let χ_A be the characteristic function of the set $\{x \in \Omega : |\mathbf{D}(\mathbf{v}(x))| \leq A\}$. Then there exist constants C_{20}, C_{21}, C_{22} such that*

$$\begin{aligned}
C_{20}\int_\Omega &\big((1+|\mathbf{D}(\mathbf{v})|)^{q(p-2)}\chi_A + (1+A)^{q(p-2)}(1-\chi_A)\big)|\mathbf{D}(\nabla \mathbf{v})|^q \, dx \\
&\leq \int_\Omega |\nabla \partial \Phi_A(|\mathbf{D}(\mathbf{v})|^2)|^q \, dx \qquad\qquad (6.13)\\
&\leq C_{21}\int_\Omega \big((1+|\mathbf{D}(\mathbf{v})|)^{q(p-2)}\chi_A + (1+A)^{q(p-2)}(1-\chi_A)\big)|\mathbf{D}(\nabla \mathbf{v})|^q \, dx,
\end{aligned}$$

and therefore

$$\int_\Omega |\mathbf{D}(\nabla \mathbf{v})|^q \, dx \leq C_{22}\int_\Omega |\nabla \partial \Phi_A(|\mathbf{D}(\mathbf{v})|^2)|^q \, dx\,. \tag{6.14}$$

Proof. We proceed analogously to the proof of Lemma 6.9 just using (2.24) instead of (1.10) and (2.26) instead of (1.11).

Acknowledgment. The authors have been partially supported by grants of SFB256 at Institute of Applied Mathematics of the University of Bonn. J. Nečas and J. Málek were partially supported by the grant 201/96/0228 of the Grant Agency of the Czech Republic, and by CEZ:J13/98113200007. M. Růžička was partially supported by fellowships from DFG and CNR. Finally the authors thank the referees for valuable remarks and suggestions.

REFERENCES

[1] H. Amann, *Stability of the rest state of a viscous incompressible fluid*, Arch. Rat. Mech. Anal. Vol. 126 (1994), 231–242.

[2] H. Bellout, F. Bloom, and J. Nečas, *Solutions for incompressible non–Newtonian fluids*, C. R. Acad. Sci. Paris Vol. 317 Série I (1993), 795–800.

[3] H. Bellout, F. Bloom, and J. Nečas, *Young measure-valued solutions for non-Newtonian incompressible fluids*, Comm. in PDE Vol. 19 (11 & 12) (1994), 1763–1803.

[4] A. Kufner, O. John, and S. Fučík, "Function Spaces", Academia, Praha, 1977.

[5] O.A. Ladyzhenskaya, *On some new equations describing dynamics of incompressible fluids and on global solvability of boundary value problems to these equations*, Trudy Steklov's Math. Institute Vol. 102 (1967), 85–104.

[6] O.A. Ladyzhenskaya, *On some modifications of the Navier-Stokes equations for large gradients of velocity*, Zapiski Naukhnych Seminarov LOMI Vol. 7 (1968), 126–154.

[7] O.A. Ladyzhenskaya, "The Mathematical Theory of Viscous Incompressible Flow", Gordon and Breach, New York, 1969.

[8] J.L. Lions, "Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires", Dunod, Paris, 1969.

[9] J. Málek, J. Nečas, and M. Růžička, *On the non–Newtonian incompressible fluids*, Math. Models Methods Appl. Sci. Vol. 3 (1993), 35–63.

[10] J. Málek, J. Nečas, and M. Růžička, *On weak solutions to a class of non-Newtonian incompressible fluids in bounded bomains. The case* $p \geq 2$., Preprint no. 481 SFB 256 of the University of Bonn, October 1996.

[11] J. Málek, J. Nečas, M. Rokyta, and M. Růžička, "Weak and Measure–Valued Solutions to Evolutionary Partial Differential Equations", Applied Mathematics and Mathematical Computation, vol. 13, Chapman and Hall, 1996.

[12] J. Málek, and D. Pražák, *Finite fractal dimension of the global attractor for a class of non-Newtonian fluids*, (to appear), Applied Math. Letters **13** (2000), 105–110.

[13] J. Málek, K.R. Rajagopal, and M. Růžička, *Existence and regularity of solutions and stability of the rest state for fluids with shear dependent viscosity*, Math. Models Methods Appl. Sci. Vol. 5 (6) (1995), 789–812.

[14] J. Nečas, *Sur le normes équivalentes dans* $W_p^k(\Omega)$ *et sur la coercivité des formes formellement positives*, in Séminaire Equations aux Dérivées Partielles, Montreal (1966), 102–128.