Solutions Übungsblatt 12

Aufgabe 1: Es sei g eine meromorphe Funktion auf $\mathbb{C}$ mit höchstens einfachen Polen. Wir nehmen an, dass das Residuum an jedem Pol von g eine ganze Zahl ist. Zeigen Sie:

(a) Es existiert eine meromorphe Funktion $f \neq 0$ auf $\mathbb{C}$, so dass $f'/f = g$.

(b) Falls $h \neq 0$ eine weitere meromorphe Funktion mit $h'/h = g$ ist, dann ist h/f konstant.

Solution:
(a) Since g is meromorphic with at most simple poles, and since the residues at these poles are integers, the primitive $\int g(z)dz$ is defined modulo $2\pi i\mathbb{Z}$. This makes

$$f(z) = \exp(\int g(z)dz)$$

well-defined on the entire $\mathbb{C}$. Moreover, f is holomorph and $f(z) \neq 0$ for all $z \in \mathbb{C}$.

(b) Let $w_0, w_1 \in \mathbb{C}$ so that $f(0) = e^{w_0}$ and $h(0) = e^{w_1}$. Define

$$L_f(z) = w_0 + \int_0^z \frac{f'(\zeta)}{f(\zeta)} d\zeta$$

and

$$L_h(z) = w_1 + \int_0^z \frac{h'(\zeta)}{h(\zeta)} d\zeta.$$

L_f and L_h are holomorphic. Moreover

$$\exp(L_f(z)) = f(z) \quad \text{and} \quad \exp(L_h(z)) = h(z).$$

Since $f'/f = g = h'/h$, the difference $L_f - L_h$ is constant; it is in fact equal to $w_0 - w_1$. Therefore

$$f(z)/h(z) = \exp(L_f(z) - L_h(z)) = \exp(w_0 - w_1)$$

for all $z \in \mathbb{C}$.

□

Aufgabe 2: Man definiert die Bernoulli-Zahlen $\{B_n\}_{n\in\mathbb{N}}$ durch die Potenzreihe

$$\frac{z}{e^z-1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}. \tag{1}$$

Zeigen Sie:

(a) (2 Punkte) Es gilt

$$\frac{B_0}{n!\,1!} + \frac{B_1}{(n-1)!\,1!} + \ldots + \frac{B_{n-1}}{1!\,(n-1)!} = \begin{cases} 1 & \text{falls } n = 1 \\ 0 & \text{falls } n > 1. \end{cases}$$

Hinweise: Betrachten Sie die Potenzreihe-Entwicklung von $\frac{z}{e^z-1} \cdot \frac{e^z-1}{z}$ und benutzen Sie Proposition 1.7 (2).

(b) (1 Punkt) $B_n = 0$ falls $n > 1$ und ungerade ist, und $B_n \in \mathbb{Q}$ für alle $n \in \mathbb{N}$.

(c) (1 Punkt) Für alle $n \in \mathbb{N} \setminus \{0\}$ gerade gilt es

$$2\zeta(n) = -B_n \frac{(2\pi i)^n}{n!}.$$

Hinweise: Setzen Sie in (1) $z = 2\pi i w$ und vergleichen Sie die Koeffizienten.

Solution: (a) We have

$$\frac{z}{e^z-1} \cdot \frac{e^z-1}{z} = 1.$$

Write

$$\frac{z}{e^z-1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} \qquad \text{and} \qquad \frac{e^z-1}{z} = \sum_{k=1}^{\infty} \frac{z^{k-1}}{k!}.$$

Multiplying these two series together we obtain

$$\sum_{m=0}^{\infty} \left(\frac{B_0}{m!\,1!} + \frac{B_1}{(m-1)!\,1!} + \ldots + \frac{B_{m-1}}{1!\,(m-1)!} \right) z^m = 1.$$

From here the desired formula follows.

(b) From the formula in (a) for $n = 0$, it follows $B_0 = 1$. By induction on n, we conclude that $B_n \in \mathbb{Q}$ for all $n \in \mathbb{N}$.

Note that $B_1 = -\frac{1}{2}$. To show that $B_{2k+1} = 0$ for all $k \geq 1$, it is enough to show that the function

$$f(z) = \frac{z}{e^z-1} - 1 + \frac{1}{2}z$$

is an even function. This is true, since

$$f(-z) = \frac{-z}{e^{-z}-1} - 1 - \frac{1}{2}z = \frac{ze^z}{e^z-1} - 1 - \frac{1}{2}z = z + \frac{z}{e^z-1} - 1 - \frac{1}{2}z = f(z).$$

The series expansion for $f(z)$ is

$$f(z) = \sum_{n\geq 2} B_n \frac{z^n}{n!}.$$

Since

$$f(-z) = \sum_{n\geq 2} (-1)^n B_n \frac{z^n}{n!}$$

and $f(z)$ even, we must have $B_n = (-1)^n B_n$. This forces $B_n = 0$ for all n odd, $n > 1$.

(c) We have

$$\begin{aligned}\frac{2\pi i w}{e^{2\pi i w} - 1} &= \pi i w \frac{2}{e^{2\pi i w} - 1} \\ &= \pi w \left(\cot(\pi w) - i\right) \\ &= \pi w \cot(\pi w) - i\pi w.\end{aligned}$$

From Beispiel 5.5. in Skript, we have

$$\begin{aligned}\pi w \cot(\pi w) &= \pi w \left(\frac{1}{\pi w} + \sum_{n\geq 1} \frac{2\pi w}{(\pi w)^2 - (\pi n)^2} \right) \\ &= 1 + 2\sum_{n\geq 1} \frac{w^2}{w^2 - n^2} = 1 - 2\sum_{n\geq 1} w^2 \frac{1}{1 - \left(\frac{w}{n}\right)^2} \\ &= 1 - 2\sum_{n\geq 1} w^2 \sum_{k\geq 0} \left(\frac{w}{n}\right)^{2k} = 1 - 2\sum_{k\geq 0} \left(\sum_{n\geq 1} \frac{1}{n^{2k+2}} \right) w^{2k+2} \\ &= 1 - 2\sum_{k\geq 1} \left(\sum_{n\geq 1} \frac{1}{n^{2k}} \right) w^{2k} \\ &= 1 - 2\sum_{k\geq 1} \zeta(2k) w^{2k}\end{aligned}$$

On the other hand, using the power expansion for $\frac{2\pi i w}{e^{2\pi i w}-1}$ given by formula (1), we have

$$\begin{aligned}\pi w \cot(\pi w) &= i\pi w + \frac{2\pi i w}{e^{2\pi i w} - 1} \\ &= i\pi w + \sum_{n\geq 0} \frac{B_n}{n!} (2\pi i w)^n \\ &= i\pi w + \sum_{n\geq 0} \frac{B_n}{n!} (2\pi i)^n w^n \\ &= i\pi w + B_0 + B_1 (2\pi i) + \sum_{n\geq 2} \frac{B_n}{n!} (2\pi i)^n w^n \\ &= i\pi w + 1 - i\pi w + \sum_{k\geq 1} \frac{B_{2k}}{(2k)!} (2\pi i)^{2k} w^{2k}\end{aligned}$$

given that $B_0 = 0$, $B_1 = -\frac{1}{2}$, and $B_{2k+1} = 0$ for all $k \geq 1$. Comparing the coefficients of the two power series, we obtain the desired formula. □

Aufgabe 3: Zeigen Sie (das letzte Gleichheitszeichen in Satz 5.22 im Skript):

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\, n^z}{z(z+1)\dots(z+n)}$$

Solution: Use the first equality in Satz 5.22 in Skript:

$$\frac{1}{z\Gamma(z)} = \lim_{n\to\infty} e^{\gamma z} \prod_{k=1}^{n} \left(1+\frac{z}{k}\right) e^{-z/k}$$

Since by definition

$$\gamma = \lim_{N\to\infty} \left(\sum_{j=1}^{N} \frac{1}{j} - \log N \right)$$

we have

$$\frac{1}{z\Gamma(z)} = \lim_{\substack{n\to\infty \\ N\to\infty}} e^{\sum_{j=1}^{N} \frac{z}{j}} \cdot e^{-\log N \cdot z} \prod_{k=1}^{n} \left(1+\frac{z}{k}\right) e^{-z/k}$$

We can think of this limit as the limit of a double sequence $a_{n,N}$ as $n \to \infty$ and $N \to \infty$. Since it converges, the limit is the same as the limit of the subsequence obtained for $n = N$. Therefore

$$\begin{aligned}
\frac{1}{z\Gamma(z)} &= \lim_{n\to\infty} e^{\sum_{j=1}^{n} \frac{z}{j}} \cdot e^{-\log n \cdot z} \prod_{k=1}^{n} \left(1+\frac{z}{k}\right) e^{-z/k} \\
&= \lim_{n\to\infty} e^{\sum_{j=1}^{n} \frac{z}{j}} \cdot e^{-\log n \cdot z} e^{-\sum_{k=1}^{n} z/k} \prod_{k=1}^{n} \frac{z+k}{k} \\
&= \lim_{n\to\infty} e^{-\log n \cdot z} \prod_{k=1}^{n} \frac{z+k}{k} \\
&= \lim_{n\to\infty} e^{-\log n \cdot z} \prod_{k=1}^{n} \frac{z+k}{k} \\
&= \lim_{n\to\infty} n^{-z} \frac{1}{n!} (z+1)\dots(z+n).
\end{aligned}$$

Multiplying the above by z we obtain the desired formula.

□

Aufgabe 4: Zeigen Sie:

$$(a) \quad \int_{-\infty}^{+\infty} \frac{1}{1+x^6} dx = \frac{2\pi}{3} \quad \text{und} \quad (b) \quad \int_{-\infty}^{+\infty} \frac{1}{1+x^n} dx = 2\frac{\pi/n}{\sin \pi/n}$$

für alle geraden $n \in \mathbb{N} \setminus \{0\}$.
Hinweise: Nehmen Sie den Weg von 0 nach R, dann von R nach $Re^{2\pi i/n}$, und dann zurück nach 0. Oder benutzen Sie den allgemeinen Satz 3.25.

Solution: We are going to use Folgerung 3.25 in Skript to compute these integrals.

The function $R(x) = \frac{1}{1+x^n}$ is a rational function which does not have any singularities along the real axis. Since $n \geq 2$, the order of the zero at ∞ is also $n \geq 2$. Therefore we are in the hypothesis of Folgerung 3.25.

Consider $R(z) = \frac{1}{1+z^n}$ as a function on $\mathbb{C}$. Its singularities are at the points where $1 + z^n = 0$. These are the points

$$\zeta_k = e^{\frac{\pi i}{n}} e^{\frac{2\pi i(k-1)}{n}} = e^{\frac{(2k-1)\pi i}{n}} \quad \text{with } 1 \leq k \leq n.$$

From Folgerung 3.25 we know

$$\int_{-\infty}^{+\infty} \frac{1}{1+x^n} dx = 2\pi i \sum_{\operatorname{Im} \zeta_k > 0} \operatorname{Res}_{\zeta_k} R(z) \tag{2}$$

Since ζ_k is a simple zeroe for $1 + z^n$, Proposition 3.21 (2) gives

$$\operatorname{Res}_{\zeta_k} R(z) = \frac{1}{n\zeta_k^{n-1}} = -\frac{1}{n}\zeta_k. \tag{3}$$

The ζ_k with $\operatorname{Im} \zeta_k > 0$ correspond to those with argument in $(0, \pi)$. This means

$$0 < \frac{2k-1}{n}\pi < \pi$$

which is equivalent to

$$\frac{1}{2} < k < \frac{n+1}{2}.$$

Given that n is even, this corresponds to

$$1 \leq k \leq \frac{n}{2} \tag{4}$$

(a) In this case $n = 6$. Therefore, in formula (2) we only need to consider those k that satisfy $1 \leq k < 3$. This means ζ_1, ζ_2, and ζ_3.

We have

$$\zeta_1 = e^{\pi i/6} = \cos(\pi/6) + i \sin(\pi/6) = \sqrt{3}/2 + i/2,$$

$$\zeta_2 = e^{\pi i/2} = i,$$

and

$$\zeta_3 = e^{5\pi i/6} = -\sqrt{3}/2 + i/2.$$

With this

$$\int_{-\infty}^{+\infty} \frac{1}{1+x^6} dx = 2\pi i \left(-\frac{1}{6}\zeta_1 - \frac{1}{6}\zeta_2 - \frac{1}{6}\zeta_3 \right)$$
$$= -\frac{1}{3}\pi i \left(\zeta_1 + \zeta_2 + \zeta_3 \right)$$
$$= -\frac{1}{3}\pi i (2i) = \frac{2}{3}\pi.$$

(b) In the general case $n = 2m$, the roots of $1 + z^n = 0$ that have strictly positive imaginary part are $\zeta_1, \ldots, \zeta_m$.

From formulas (2) and (3) it follows

$$\int_{-\infty}^{+\infty} \frac{1}{1+x^n} dx = -\frac{2\pi i}{n} \sum_{k=1}^{m} \zeta_k$$

One way to compute $\sum_{k=1}^{m} \zeta_k$ is to observe that $\zeta_k = \zeta_1 \cdot \zeta_1^{2(k-1)}$.

$$\sum_{k=1}^{m} \zeta_k = \zeta_1 \sum_{k=1}^{m} \zeta_1^{2(k-1)} = \zeta_1 \frac{1 - \zeta_1^{2m}}{1 - \zeta_1^2}$$
$$= 2\zeta_1 \frac{1}{1 - \zeta_1^2} = -2 \frac{1}{\zeta_1^{n-1} - \zeta_1^{n+1}}$$
$$= -2 \frac{1}{2i \sin \pi/n} = -\frac{1}{i} \frac{1}{\sin \pi/n}.$$

since

$$\zeta_1^{n-1} = \cos \frac{\pi(n-1)}{n} + i \sin \frac{\pi(n-1)}{n} = -\cos \frac{\pi}{n} + i \sin \frac{\pi}{n}$$

and

$$\zeta_1^{n+1} = \cos \frac{\pi(n+1)}{n} + i \sin \frac{\pi(n+1)}{n}$$
$$= -\cos \frac{\pi}{n} - i \sin \frac{\pi}{n}.$$

Therefore

$$\int_{-\infty}^{+\infty} \frac{1}{1+x^n} dx = 2 \frac{\pi/n}{\sin \pi/n}.$$

□