SEMINAR/READING COURSE ON “THE HOMOTOPY TYPE OF THE COBORDISM CATEGORY” BY GALATIUS, MADSEN, TILLMANN, AND WEISS

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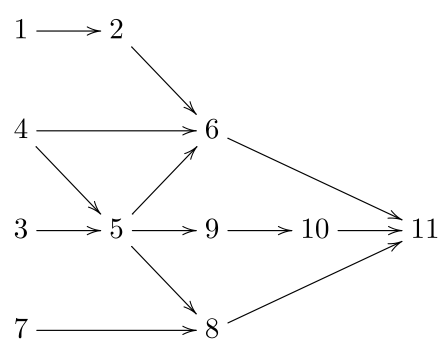
We want to read the paper in the title. It contains an important technical result about “moduli spaces of manifolds” that has found applications in different areas of Riemannian geometry, differential topology, and theoretical physics. The Main Theorem of [4] says that a certain map

$$\alpha\colon B\mathcal{C}_d \longrightarrow \Omega^{\infty-1} MTO(d) \tag{1}$$

is a weak homotopy equivalence.

Taking π_0, that is, the set of connected components, on the left hand side, we get the set of cobordism classes of $(d-1)$-manifolds. On the right hand side, we get the homotopy group $\pi_{d-1}MO$ of the unoriented Thom spectrum. Thus, after applying π_0, we obtain the Pontryagin-Thom isomorphism in dimension $d-1$. Thus, (1) is a refinement of the Pontryagin-Thom theorem to the level of spaces. There are similar results for the oriented cobordism category, and more generally, for cobordisms with tangential structure.

During the talk, we will also learn several important tools from algebraic and differential topology, among them simplicial sets and their realisations, and various different constructions of classifying spaces for bundle of manifolds and cobordisms, sometimes also called “moduli spaces”. The diagram below shows the interdependence of the talks.



If one considers the oriented case in dimension 2, one recovers [5, Theorem 1.1]. They consider the stable mapping class group Γ_∞ of Riemann surfaces, where “stable” here means “of arbitrarily high genus”. Applying Quillen’s plus construction to its classifying space, one obtains a space $B\Gamma_\infty^+$ that is weakly homotopy equivalent to an infinite loop space $\Omega^\infty\mathbb{C}\mathbf{P}^\infty_{-1}$. This is one of the key steps in the proof of the Mumford conjecture, which describes the rational cohomology ring of the stable moduli space of oriented surfaces. If there is time, we can give an overview of the proof of the Mumford conjecture, or explain some other applications of (1).

1. Spaces of Manifolds

The classifying space of a diffeomorphism group has a concrete description. Let M, N be smooth manifolds. Let M be *closed,* that is, M is compact without boundary. Recall the compact-open topology on $C(M, N) = C^0(M, N)$. Then describe the Whitney topologies on $C^k(M, N)$ and on $C^\infty(M, N)$ [2, Section 2.1].

Recall the Whitney embedding theorem. Then show that the space

$$\operatorname{Emb}(M) = \operatorname{colim}_{n\to\infty}\bigl\{\, \varphi \in C^\infty(M, \mathbb{R}^n) \bigm| \varphi \text{ is an embedding} \,\bigr\}$$

is contractible [1, Theorem 20.7]. The action of $\operatorname{Diff}(M)$ on $\operatorname{Emb}(M)$ is free, and we call

$$B_\infty(M) = B\operatorname{Diff}(M) = \operatorname{Emb}(M)/\operatorname{Diff}(M)$$

the *classifying space* of the diffeomorphism group. Show that as a set,

$$\begin{aligned} B_\infty(M) \cong \bigl\{\, A \subset \mathbb{R}^n \subset \mathbb{R}^\infty \bigm| n \in \mathbb{N} \\ \text{and } A \text{ is a } C^\infty\text{-submanifold diffeomorphic to } M \,\bigr\}\ , \end{aligned}$$

and describe its quotient topology as "C^∞-Gromov-Hausdorff topology". One can show that $E\operatorname{Diff}(M) \to B_\infty(M)$ is a $\operatorname{Diff}(M)$-principal bundle.

Consider the map

$$E_\infty(M) = E\operatorname{Diff}(M) \times_{\operatorname{Diff}(M)} M \longrightarrow B_\infty(M)\ .$$

Using the Whitney embedding theorem, show that for each fibre bundle $E \to B$ with fibre M and structure group $\operatorname{Diff}(M)$, there exists a pullback diagram

$$\begin{array}{ccc} E & \longrightarrow & E_\infty(M) \\ \downarrow & & \downarrow \\ B & \xrightarrow{\ f\ } & B_\infty(M)\ , \end{array}$$

where f is unique up to homotopy. In other words, $E_\infty(M) \to B_\infty(M)$ is a universal M-family.

2. The Cobordism Category

We define the cobordism category $\mathcal{C}_d$ as a topological category. Since this a (small) topological category (a.k.a. a category object in $\mathcal{T}$), recall first that the category $\mathcal{T}$ of topological spaces is a monoidal category. Then define topological categories.

Recall talk 1 and repeat the construction of $E_\infty(W) \to B_\infty(W)$ for cobordisms W as explained in [4, Section 2.1]. Let $\mathbb{R}^2_+ = \{\, (a, b) \in \mathbb{R}^2 \mid a < b \,\}$ and define the cobordism category with

$$\operatorname{ob}\mathcal{C}_d \cong \mathbb{R} \times \coprod_{[M]} B_\infty(M) ,$$
$$\operatorname{mor}\mathcal{C}_d \cong \operatorname{ob}\mathcal{C}_d \sqcup \mathbb{R}^2_+ \times \coprod_{[W]} B_\infty(W) .$$

Here $[M]$ runs over diffeomorphism classes of $(d-1)$-dimensional closed manifolds, and $[W]$ runs over diffeomorphism classes of d-dimensional cobordisms. See also [1, Definition 20.19]

Finally, also explain the "oriented version" $B\mathcal{C}_d^+$.

3. Simplicial Sets and Geometric Realisations

The passage from categories and other abstract constructs to topological spaces capturing some of their properties is one of the central methods in the paper [4]. The purpose of this talk is to introduce simplicial sets and their classifying spaces. They will be needed in some of the later talks, and they also forms one of the more classical ways to turn categories into spaces. Milnor's article [6] is one of the main sources for this talk. We want to rephrase his results in modern language, as outlined in the following.

Define the category Δ with

$$\operatorname{ob}\Delta = \mathbb{N} ,$$
$$\operatorname{mor}_\Delta(m,n) = \big\{ \text{nondecreasing maps } \{0,\dots,m\} \to \{0,\dots,n\} \big\} .$$

Introduce the category $s\mathcal{C}$ of simplicial objects in some category $\mathcal{C}$ as contravariant functors from Δ to $\mathcal{C}$. One often denotes simplicial objects by $X_\bullet$. Introduce the *face maps* $\partial_i \colon X_{m+1} \to X_m$ and the *degeneracies* $s_i \colon X_m \to X_{m+1}$.

Explain the *singular functor* $\operatorname{Sing}\colon \mathcal{T} \to s\mathcal{S}et$ and the *geometric realisation* $|\cdot| : s\mathcal{S}et \to \mathcal{T}$ as in [6]. Then we have the classical adjunction of functors $|\cdot| \dashv \operatorname{Sing}$. For a topological space X, one gets a natural map $|\operatorname{Sing} X| \to X$. Show that this a weak homotopy equivalence. As a consequence, singular (co-) homology is the unique (co-) homology on the category $\mathcal{T}$ that is invariant under weak homotopy equivalences.

The categorical product on simplicial sets is the *diagonal simplicial set.* Because of the adjunction $|\cdot| \dashv \operatorname{Sing}$, it is clear (but still somewhat counterintuitive) that $|K_\bullet \times L_\bullet|$ is homeomorphic to $|K_\bullet| \times |L_\bullet|$.

Finally, we can extend geometric realisation also to simplicial objects in other categories. For a simplicial space, the construction of its geometric realisation is almost the same as above. The realisation of a simplicial category is a topological category, and so on.

4. Sheaves on the Category of Manifolds
"Einfach alle Pfeile umdrehen und Yoneda-Lemma anwenden"

For the construction of the classifying space $B\mathcal{C}_d$ of the topological category $\mathcal{C}_d$, we will use sheaves on the category $\mathcal{X}$ of smooth manifolds. This talk is meant as a warm-up.

A sheaf on $\mathcal{X}$ with values in some category $\mathcal{C}$ is a contravariant functor $\mathcal{X} \to \mathcal{C}$ that satisfies the usual gluing property for open coverings of individual manifolds [5, Definition 2.1]. Smooth maps to a given manifold Y form a typical example $C(\,\cdot\,, Y)$. Indeed, this gives the embedding of the category $\mathcal{X}$ into the category of sheaves on $\mathcal{X}$ that we alluded to in the subtitle.

As another typical example, we want to consider a sheaf that assigns to each manifold all fibre bundles (say, with a given fibre and structure group) over it. There are some problems with the naïve approach that are solved by introducing *graphical sheaves* [5, Definition 2.2]. Let G be a topological group, for example $G = U(k)$ or $G = \mathrm{Diff}(M)$. Then principal G-bundles form a graphical sheaf $\mathcal{G}$. Its concordance classes are represented by $|\mathcal{G}|$, which is a model for the classifying space BG.

5. Classifying Spaces of Sheaves and Weak Equivalences

Let $\mathcal{F}$ be a sheaf on $\mathcal{X}$ and let X be a manifold Two elements of $\mathcal{F}(X)$ are called *concordant* if they extend to an element of $\mathcal{F}(X \times \mathbb{R})$. Concordance classes of a sheaf $\mathcal{F}$ define a contravariant functor $X \mapsto \mathcal{F}[X]$. In general, it is too coarse to be a sheaf. In this talk, we will construct a space representing $\mathcal{F}[\,\cdot\,]$. This allows us to define a notion of weak equivalence of sheaves. We follow [4, Section 2.2], which summarises [5, Section 2.4 and Appendix A.1]

Because extended simplices are manifolds, each sheaf $\mathcal{F}$ with values in $\mathcal{C}$ defines a simplicial object $\mathcal{F}_\bullet$ in $s\mathcal{C}$, and one can take its geometric realisation $|\mathcal{F}|$. The space $|\mathcal{F}|$ represents $\mathcal{F}[X]$, more precisely

$$\mathcal{F}[X] = [X, |\mathcal{F}|] \,,$$

see [5, Appendix A.1].

A map v of sheaves induces a map $|v|$ between their classifying spaces, and we call v a *weak equivalence* if $|v|$ is a homotopy equivalence. To check that v is a weak equivalence, we use the *Surjectivity Criterion* [5, Proposition 2.18]: It suffices to show that v induces surjective maps between relative concordance classes. This is an important step in the proof of the main result (1).

6. Cobordism Sheaves

To apply the methods of talk 5 to the classifying space of the cobordism category $\mathcal{C}_d$ of talk 2, we need to give sheaf model for it. To each manifold X, we associate a set of bundles of cobordisms over X that are embedded in $\mathbb{R}^\infty$. We regard such cobordisms as morphisms between their boundaries, obtaining a topological category. We consider two such sheaves of topological categories, $C_d^\pitchfork$ and $C_d^\perp$, which differ in the way the collars are treated. The sheaf $C_d^\perp$ is isomorphic to the sheaf represented by the topological category $\mathcal{C}_d$; this connects the present talk to the left hand side of (1). This is explained in [4, Section 2.3].

The two sheaves $C_d^\pitchfork$ and $C_d^\perp$ are weakly equivalent. This proved by combining a geometric construction with the surjectivity criterion introduced in talk 5, see [4, Proposition 4.4].

7. Madsen-Tillmann spectra and the Pontryagin-Thom construction

We first recall the Pontryagin-Thom construction, which turns cobordism classes of manifolds into elements of higher homotopy groups of Thom spaces. Any closed smooth d-manifold M can be identified with a submanifold of $\mathbb{R}^{d+n}$ using the Whitney embedding theorem if n is sufficiently large. The Gauß map of this embedding maps M to the Grassmannian $G_{d,n}$ of d-planes in $\mathbb{R}^{d+n}$. Let $U_{d,n}^{\perp} \to G_{d,n}$ denote the complementary n-plane bundle, then its pullback can be identified with a tubular neighbourhood of $M \subset \mathbb{R}^{d+n}$. Collapsing everything else, one obtains a map from the Alexandrov compactification S^{d+n} of $\mathbb{R}^{n+d}$ to the Thom space $MTO_{d+n}(d) = \mathrm{Th}(U_{d,n}^{\perp})$. By the results of Pontryagin and Thom, is establishes an isomorphism from the group Ω_d^O of d-manifolds up to cobordism to

$$\pi_{n+d} MTO_{d+n}(d) .$$

There are natural maps $\Sigma MTO_{d+n}(d) \to MTO_{d+n+1}(d)$ that turns these space into a spectrum $MTO(d)$. To $MTO(d)$, we associate infinite loop spaces

$$\Omega^{\infty+k} MTO(d) = \mathrm{colim}_{n\to\infty} \Omega^{n+d} MTO_{d+n-k}(d) .$$

The right hand side of (1) is of this form. If one does not restrict to a fixed dimension d, one can construct the classical Thom spectrum MO. One should note that the group $O = \mathrm{colim}_{n\to\infty} O(n)$ classically refers to the structure group of the normal bundle, whereas here, we fix the structure group $O(d)$ of the tangent bundle. Hence the extra “T” in the notation. The relation between these structures is explained in [4, Section 3.1].

The oriented Madsen-Tillmann spectrum $MTSO(d)$ is constructed similarly. More generally, we can consider arbitrary *tangential structures* and define their Madsen-Tillmann spectra as well as their cobordism categories. If there is time, this could be explained in this talk.

8. A Sheaf Model for the Madsen-Tillmann Spectrum

As in talk 6, we construct a sheaf D_d of cobordisms over the category $\mathcal{X}$ of manifolds. By a bundle version of the Pontryagin-Thom construction, one shows that $|D_d|$ is weakly homotopy equivalent to the infinite loop space $\Omega^{\infty-1} MT(d)$ on the right hand side of (1), see [4, Theorem 3.4].

9. Cocycle Sheaves and Classifying Spaces I

To a small category $\mathcal{C}$ one associates a simplicial set, the *nerve* $N_\bullet\mathcal{C}$. It is a simplicial set, where the set of k-simplices consists of all collections of k composable arrows. Faces and degeneracies are given by composing two adjacent arrows or inserting identities, respectively. Its geometric realisation is called the *classifying space* $B\mathcal{C} = |N_\bullet\mathcal{C}|$.

If $\mathcal{F}$ is a sheaf of categories, one can consider a bisimplicial set where one simplicial direction describes the nerve as above and the other one describes the classifying space of a sheaf as in talk 5. Taking the geometric realisation, we can construct the classifying space $B\,|\mathcal{F}|$ of a sheaf of categories. Later, we will turn the cobordism category $\mathcal{C}_d$ of talks 2 into a sheaf C_d, such that $B\mathcal{C}_d = B\,|C_d|$.

The *cocycle sheaf* $\beta\mathcal{F}$ is a different construction with the same purpose, see [4, Section 2.4] for an overview and [5, Section 4.1] for the details. In the next talk, we show that $|\beta\mathcal{F}|$ and $B\,|\mathcal{F}|$ are homotopy equivalent [5, Theorem 4.2]. Elements of $\beta\mathcal{F}(X)$ are pairs of locally finite open coverings $\mathcal{U}$ of X and systems of compatible morphisms in $\mathcal{F}$ for each inclusion of finite intersections of the sets of $\mathcal{U}$.

If one considers again the sheaf $\mathcal{G}$ of principal G-bundles for a topological group G, then $\beta\mathcal{G}$ can be interpreted as a "sheaf of gluing data" for principal G-bundles. Then [5, Theorem 4.2] gives another proof of the well-known fact that BG classifies G-bundles, see talk 4.

10. Cocycle Sheaves and Classifying Spaces II

The purpose of this talk is to prove [5, Theorem 4.2], that is, to show that $|\beta\mathcal{F}|$ and $B\,|\mathcal{F}|$ are homotopy equivalent. This proof is somewhat technical and contained in [5, Appendix A.3]. If one talk is not enough, we can split it further.

11. Equivalence of Cobordism Sheaves

In this talk, we complete the proof of (1). By combining methods from talks 6 and 8, we define yet another sheaf of cobordisms $D_d^{\pitchfork}$. Then there is a zig-zag of functors

$$C_d \longrightarrow C_d^{\pitchfork} \longleftarrow D_d^{\pitchfork} \longrightarrow D_d\,,$$

see [4, Chapter 4]. Each of these functors is a weak equivalence. For the leftmost, we have shown this in talk 6. For the rightmost functor, we first have to replace $D_d^{\pitchfork}$ by its cocycle sheaf $\beta D_d^{\pitchfork}$. In each case, the argument is reduced to the surjectivity criterion from talk 5.

Literatur

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