Abteilung für Mathematische Logik
Giorgio Laguzzi

Infinite Games

Sommersemester 2019

Exercise sheet 1, 02.05.2019

We first fix a notation: given two sequences $s, t \in X^{<\omega}$ (where X can be any set, usually in our case we deal with $X = 2$ or $X = \omega$) we define

$$s^\frown t := \langle s(0), s(1), \dots, s(|s|-1), t(0), t(1), \dots t(|t|-1)\rangle.$$

1. Given $s \in \omega^{<\omega}$ we define

   if k is even, then $a^k_n := \langle 1, 1, , \dots, 1, 1, 0\rangle$, with $|a^k_n| = n + 1$
   if k is odd, then $a^k_n := \langle 0, 0, \dots, 0, 0, 1\rangle$, with $|a^k_n| = n + 1$

   For every $x \in \omega^\omega$, let $j(x) := a^0_{x(0)}{}^\frown a^1_{x(1)}{}^\frown \dots {}^\frown a^n_{x(n)}{}^\frown \dots$.

   Let $C := \{x \in 2^\omega : \forall n \exists m_0, m_1 > n(x(m_0) \neq x(m_1))\}$, i.e., C consists of those binary sequences which are not eventually constant.

   Show that $j : \omega^\omega \to 2^\omega \setminus C$ is a homeomorphism.

2. Given $s \in 2^{<\omega}$, let $w(s) := \frac{|\{i<|s|:s(i)=1\}|}{|s|}$ (intuitively $w(s)$ counts the density of 1's in the sequence). Let $G_{<\infty}(A_I, A_{II})$ be the following finite-unbounded game, for some given $\delta_1, \delta_2, \epsilon_1, \epsilon_2 \in [0, 1]$:

$$A_I := \{s \in 2^{<\omega} : |s| > 10 \wedge \delta_1 < w(s) < \epsilon_1\}$$
$$A_{II} := \{s \in 2^{<\omega} : |s| > 10 \wedge \delta_2 < w(s) < \epsilon_2\}.$$

   Analyse these games for different $\delta_1, \epsilon_1, \delta_2, \epsilon_2 \in [0, 1]$ and investigate in which cases the game is decided or not, and which player has a winning strategy.

3. Given a finite unbounded game $G_{<\infty}(A_I, A_{II})$ and $s \in 2^{<\omega}$, let $G_\infty(A_I, A_{II}, s)$ denote the game in which player I starts playing x_0, II replies with y_0, then I plays x_1, II replies with y_1 and so on, and player I wins the game iff there is $n \in \omega$ such that $s^\frown\langle x_0, y_0, \dots x_n, y_n\rangle \in A_I$, and player II wins the game iff $s^\frown\langle x_0, y_0, \dots x_n, y_n\rangle \in A_{II}$. Prove that the game $G_{<\infty}(A_I, A_{II}, s)$ is the same as the game $G_{<\infty}(B_I, B_{II})$ where

$$B_I := \{t \in 2^{<\omega} : s^\frown t \in A_I\}$$
$$B_{II} := \{t \in 2^{<\omega} : s^\frown t \in A_{II}\}.$$