8^{th} Exercise Sheet, Set Theory of the Real Line, WS 2014/2015

Giorgio Laguzzi

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Exercise 15

Let $f \in \omega^\omega$ be such that $f(n) = 2^n$. Let $S \in ([\omega]^{<\omega})^\omega$ (i.e., S in an infinite sequence of finite subsets of ω) be such that for all $n \in \omega$, $|S(n)| \leq 2^\omega$. We call such a sequence an *f-slalom* and we let $\mathcal{C}$ be the set of all f-slaloms, for $f(n) = 2^n$. We say that a forcing $\mathbb{P}$ satisfies the Sacks property iff

$$\Vdash_{\mathbb{S}} \forall x \in \omega^\omega \exists S \in \mathcal{C} \cap V \forall n (x(n) \in S(n)),$$

where V is the ground model. (Roughly, any real in the extension is "captured" by a slalom in the ground model.)

Show that:

1. the Sacks forcing $\mathbb{S}$ satisfies the Sacks property;
2. the random forcing $\mathbb{B}$ does not satisfy the Sacks property.

Exercise 16

Let $f : \omega \to 2$ be a partial function with $|\text{codom}(f)| = \omega$. Define the tree $p_f \subseteq 2^{<\omega}$ as

$$p_f := \{t \in 2^{<\omega} : \forall n < |t| (n \in \text{dom}(f) \Rightarrow f(n) = t(n))\}.$$

Let $\mathbb{V} := \{p_f : f : \omega \to 2 \text{ partial function with } |\text{codom}(f)| = \omega\}$, ordered by $q \leq p \Leftrightarrow q \subseteq p$. $\mathbb{V}$ is called *Silver forcing*.

Show that $\mathbb{V}$ preserves ω_1 and satisfies the Sacks property.