Abteilung für Mathematische Logik
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Infinitary Combinatorics
Wintersemester 2019/20

Übungsblatt 3

Abgabe am 12.11.2019 vor der Vorlesung

Consider the poset $\mathbb{A} = \{(n, T) \mid n \in \omega\ ,\ T \subseteq 2^{<\omega}$ is a closed nowhere dense tree$\}$ with

$$(k, T') \leq (n, T) \Leftrightarrow k \geq n,\ T' \supseteq T \text{ and } T' \cap {}^{n}2 = T \cap {}^{n}2$$

Given a filter $G \subseteq \mathbb{A}$, put $T_G := \bigcap\{T : \exists n (n, T) \in G\}$.

Exercise 1. (3 points) Show that for $k \in \omega$ the set

$$D_k := \{(n, T) \in \mathbb{A} \mid n \geq k\}$$

is dense. Then prove that for a filter G in $\mathbb{A}$ with $G \cap D_k \neq \emptyset$ for all $k \in \omega$, the tree T_G is infinite.

Exercise 2. (5 points) Show that for $k \in \omega$ the set

$$E_k := \Big\{(n, T) \in \mathbb{A} \mid \forall t \in (T \cap {}^{k}2)\ \exists s \supseteq t\ :\ s^\frown 0, s^\frown 1 \in (T \cap {}^{n}2)\Big\}$$

is dense.

Exercise 3. (2 points) Let G be a filter in $\mathbb{A}$ with $G \cap E_k \neq \emptyset$ for all $k \in \omega$. Show that T_G is a perfect tree.

Exercise 4. (2 points) Show that for $k \in \omega$ the set

$$H_k := \Big\{(n, T) \in \mathbb{A} \mid \forall s \in {}^{k}2\ \exists \tau_s \supseteq s\ :\ |\tau_s| < n \wedge \tau_s \notin T\Big\}$$

is dense. If G is a filter in $\mathbb{A}$ with $G \cap H_k \neq \emptyset$ for all $k \in \omega$, show that the tree T_G is (closed) nowhere dense.

Bonus. Let $\{S_\alpha \mid \alpha < \kappa\}$ be a family of nowhere dense trees. Then apply $\mathsf{MA}(\kappa)$ to the poset $\mathbb{A}$ in order to find a suitable closed nowhere dense set B covering all S_α's.