

On social welfare orders satisfying anonymity and asymptotic density-one Pareto

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Abstract

We study the nature (i.e., constructive as opposed to non-constructive) of social welfare orders on infinite utility streams, and their representability by means of real-valued functions. We assume finite anonymity and introduce a new efficiency concept we refer to as asymptotic density-one Pareto. We characterize the existence of representable and constructive social welfare orders (satisfying the above properties) in terms of easily verifiable conditions on the feasible set of one-period utilities.

Keywords: Anonymity, Non-Ramsey set, Social Welfare Order, Asymptotic Density-One Pareto.

Journal of Economic Literature Classification Numbers: D60, D70, D90.

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1 Introduction

Imagine a social planner (e.g., a policy-maker) performing pairwise ranking of alternative policies that affect also the welfare of generations in the far-off future. Examples of such policies are fiscal policies, issues related to climate, environmental preservation, and sustainable development. Suppose the social planner wants to do so relying on social preferences (i.e., a social welfare order) that satisfy simultaneously both “efficiency” and “equity” properties. The ideal situation would be a social welfare order (SWO, henceforth) that admits a real-valued numerical representation, i.e., a social welfare function (SWF, henceforth). If no SWF exists, it would still be possible to perform pairwise ranking of infinite utility streams, for the purpose of decision-making, as long as the SWO at hand can be described explicitly. The worst-case scenario, wherein the existence of a SWO satisfying a set of desiderata is of no practical use, is that a SWF does not exist, and the SWO under consideration cannot be operationalized. Which of the above three scenarios does the social planner find itself in? With this question in mind, and letting Y be the set of all possible values that each generation’s utility can take, our main goal is to characterize the existence of representable and constructive SWO’s (that treat all generations “equally” and respect some form of the Pareto axiom) in terms of conditions on Y that can be easily checked. The main results of the present paper, i.e., Theorems 1 and 2, serve this purpose. What do we mean by treating all generations equally and the Pareto axiom? We consider the canonical (finite) anonymity axiom (AN, henceforth, which is a procedural equity concept formalized in Diamond (1965)) and define a new efficiency concept, namely asymptotic density-one Pareto (ADP, henceforth). Interestingly, although a constructive SWO need not be representable, yet we find out that there is a representable SWO satisfying AN and ADP if and only if there is a constructive SWO with the same properties. In fact, it turns out that either way existence is equivalent to Y being finite (see Theorems 1 and 2 below).

Our focus on the AN and ADP concepts may be justified as follows. Given Basu and Mitra (2007b) and Demichelis et al. (2010), we know that AN is basically the weakest form of anonymity compatible with strongly Paretian quasi-orderings on the set of infinity utility streams. Now, recall that in this work we deal with SWO’s, and a SWO is indeed a quasi-ordering. Therefore, if we strengthened the inter-generational equity concept to a stronger property than AN, we would risk ending up with a Paretian SWO which does not respect “equity”. So, we use the AN axiom to formalize inter-generational equity, as is standard practice. On the other hand, we learn from Basu and Mitra (2003) that there does not exist any SWF satisfying AN and the strong Pareto axiom when Y contains at least two distinct elements. Moreover, Basu and Mitra (2007a, Theorem 3) show that when Y is a subset of the set of non-negative integers, there is a SWF satisfying AN and weak Pareto. These results suggest that the choice of the efficiency axioms and the properties of Y are both important factors in determining the existence of a numerical representation or, more in general, for establishing a possibility result. Therefore, we are led to pursue the following research strategy: we do not challenge the AN axiom, but we deem it worthwhile to address the following question: in moving from strong Pareto toward weak Pareto, how weak a notion of efficiency can one impose on anonymous SWO’s so as to obtain a possibility theorem that depends on easily verifiable properties of Y ? We would like to argue that the ADP axiom seems suitable to answer the previous question. To see this, observe, first of all, that ADP postulates that, given any two sequences x and y satisfying $x \geq y$, if x strictly dominates y on a subset of the natural numbers having asymptotic density equal to one, then x is socially preferred to y . Clearly, ADP is weaker than strong Pareto and stronger than weak Pareto. Furthermore, ADP is in a way “satisfactorily close” to weak Pareto. To see this, note that ADP is weaker than Weak Upper Asymptotic Pareto which is an efficiency notion introduced in Dubey et al. (2020). In addition, recall that when the asymptotic density of a subset of natural numbers is well-defined, it coincides with the lower asymptotic density which, moreover, is bounded above by one. Hence, ADP is weaker than both Asymptotic Pareto

and d-Asymptotic Pareto¹, and it is also weaker than all conceivable variants of LAP that could be defined by letting the threshold for the value of the lower asymptotic density vary between zero and one.

Theorem 1 is a handy characterization result whose range of applicability may be appreciated by noting that part 1 of Proposition 4.4 in Petri (2019) obtains as an immediate corollary of our Theorem 1. In terms of the analytical techniques employed to prove Theorem 1, we mention, in passing, that the proof of the latter hinges on Lemma 1 below, which asserts that there does not exist any social welfare function satisfying ADP and AN if Y is the set of natural numbers. While it suffices to establish an impossibility result over the set of natural numbers (which is the common practice, in the established literature, we stick to), we stress that the need to exhibit a subset with asymptotic density equal to one prompts us to properly adjust the construction used by Basu and Mitra (2003). More details can be found in section 4.1. As for Theorem 2, we deem it a useful characterization result. Let's discuss why this is the case.

Firstly, in view of part 1 of Theorem 4.1 in Petri (2019), there is no SWF satisfying LAP and AN if Y is infinite, but from Svensson (1980) it follows that a SWO satisfying AN and any efficiency axiom weaker than strong Pareto, like LAP or DAP, is guaranteed to exist. Therefore, one naturally wonders whether a SWO that respects LAP and AN on an infinite Y can be constructed. Incidentally, the constructive nature of social welfare orders, or the lack thereof, was first investigated by Fleurbaey and Michel (2003) who conjectured that “there exists no explicit (that is, avoiding the axiom of choice or similar contrivances) description of an ordering which satisfies weak Pareto and indifference to finite permutations”. In a setting different from the present framework, Zame (2007) and Lauwers (2010) confirmed this conjecture showing the existence of a non-measurable set and a non-Ramsey collection of sets, respectively. Going back to the above query, it turns out that the answer to it, which is in the negative, is a straightforward corollary of Theorem 2 (see Corollary 1 below). We reach this conclusion by proving (see Lemma 2) the existence of a non-Ramsey collection of sets, therefore our approach follows in the footsteps of Lauwers (2010).

Secondly, Dubey (2011) proved that a SWO satisfying AN and weak Pareto is constructive if and only if Y does not contain any subset order isomorphic to the set of integers. Arguably, it is not immediately obvious (without adequate inspection) that a given domain of one-period utilities does not contain a subset order isomorphic to the set of integers. It is easier to figure out whether it is finite or not. That's where our Theorem 2 comes into play. To be more concrete and fix ideas, while one can show, with some work, that neither the set $\{1/n : n \in \mathbb{N}\}$ nor the set $\{n/(n+1) : n \in \mathbb{N}\}$ contains a subset order isomorphic to the set of integers, it is evident that the previous sets both contain infinitely many elements. That said, consider a SWO satisfying ADP and AN. Remember that ADP is stronger than weak Pareto, so one could certainly exploit Dubey (2011) to decide whether such a SWO is susceptible of explicit description if, say, Y is either of the above sets. As a matter of fact, by Dubey (2011) the SWO under consideration is a constructive object. Notice, however, that if one applies directly our Theorem 2 one can make the same determination effortlessly.

The remainder of the paper is organized as follows. Section 2 gathers some preliminary concepts, notation, and presents a brief description of the notion of asymptotic density. In section 3, we define the concept of SWO and various equity and efficiency axioms imposed on social welfare orders. In section 4, we state and prove our main results. Section 5 concludes and presents a table designed to help the reader position our contribution in the established literature. Proofs are relegated to the Appendix.

2 Preliminaries

Let \mathbb{R} , \mathbb{Q} , and \mathbb{N} be the set of real numbers, rational numbers, and natural numbers, respectively. For all $y, z \in \mathbb{R}^{\mathbb{N}}$, we write $y \geq z$ if $y_n \geq z_n$, for all $n \in \mathbb{N}$; $y > z$ if $y \geq z$ and $y \neq z$; and $y \gg z$ if $y_n > z_n$

¹The efficiency concepts of Asymptotic Pareto and d-Asymptotic Pareto (dAP, henceforth) were introduced in Petri (2019). We will refer to Asymptotic Pareto as lower asymptotic Pareto (LAP, henceforth).

for all $n \in \mathbb{N}$. Given any $x \in \mathbb{R}^{\mathbb{N}}$ and $N \in \mathbb{N}$, we denote the vector consisting of the first N components of x by $x(N)$, and the upper tail sequence of x , from the element $N + 1$ onward, by $x[N]$. Formally, $x(N) = (x_1, x_2, \dots, x_N)$ and $x[N] = (x_{N+1}, x_{N+2}, \dots)$.

It is useful to recall the definition of asymptotic density of any $S \subset \mathbb{N}$. As usual, let $|\cdot|$ denote the cardinality of a given finite set. The lower asymptotic density of S is defined as follows:

$$\underline{d}(S) = \liminf_{n \rightarrow \infty} \frac{|S \cap \{1, 2, \dots, n\}|}{n}.$$

Similarly, the upper asymptotic density of S is defined as follows:

$$\bar{d}(S) = \limsup_{n \rightarrow \infty} \frac{|S \cap \{1, 2, \dots, n\}|}{n}.$$

One says that S has asymptotic density $d(S)$ if $\underline{d}(S) = \bar{d}(S)$, in which case $d(S)$ is equal to this common value. Formally,

$$d(S) = \lim_{n \rightarrow \infty} \frac{|S \cap \{1, 2, \dots, n\}|}{n}.$$

3 Equity and efficiency axioms on social welfare orders

Let $Y \subset \mathbb{R}$ be the set of all possible utilities that any generation can achieve. Then, $X \equiv Y^{\mathbb{N}}$ is the set of all feasible utility streams. We denote an element of X by x , or alternately by $\langle x_n \rangle$, depending on the context. If $\langle x_n \rangle \in X$, then $\langle x_n \rangle = (x_1, x_2, \dots)$. For all $n \in \mathbb{N}$, $x_n \in Y$ represents the amount of utility that period- n generation earns.

A binary relation on X is denoted by \succsim . Its symmetric and asymmetric parts, denoted by \sim and \succ , respectively, are defined in the usual manner. A *social welfare order* (SWO) is, by definition, a complete and transitive binary relation on X . Given a SWO \succsim , one says that \succsim can be *represented* by a real-valued function, called a *social welfare function* (SWF), if there is a mapping $W : X \rightarrow \mathbb{R}$ such that for all $x, y \in X$, $x \succsim y$ if and only if $W(x) \geq W(y)$.

We will be dealing with the following equity and efficiency axioms we may want the SWO to satisfy.

Definition 1. Anonymity (AN henceforth): If $x, y \in X$, and there exist $i, j \in \mathbb{N}$ such that $y_j = x_i$ and $x_j = y_i$, while $y_k = x_k$ for all $k \in \mathbb{N} \setminus \{i, j\}$, then $x \sim y$.

Definition 2. Lower Asymptotic Pareto (LAP henceforth): Given $x, y \in X$, if $x \geq y$ and $x_i > y_i$ for all $i \in S \subset \mathbb{N}$ with $\underline{d}(S) > 0$, then $x \succ y$.

Definition 3. Asymptotic Density-one Pareto (ADP henceforth): Given $x, y \in X$, if $x \geq y$ and $x_i > y_i$ for all $i \in S \subset \mathbb{N}$ with $d(S) = 1$, then $x \succ y$.

It's easy to see that ADP is strictly weaker than LAP (i.e., LAP implies ADP but the converse is not true).

4 Results

In this section we state and prove the main results of this paper. The following example exhibits a SWF satisfying ADP and AN if Y is finite. This function, along with the remarks below, will be instrumental for the proof of the two main results of this paper, namely Theorems 1 and 2.

Example 1. Let $X = Y^{\mathbb{N}}$, Y being finite, and let $W : X \rightarrow \mathbb{R}$ be defined as follows:

$$W(x) := \liminf_{n \rightarrow \infty} \frac{\sum_{k=1}^n x_k}{n}. \quad (1)$$

Next, define the following binary relation \succsim on X :

$$x \succsim y \quad \text{if and only if} \quad W(x) \geq W(y). \quad (2)$$

Clearly, (1) is a SWF that represents the SWO given by (2). Petri (2019, p. 860) has shown that the latter satisfies AN and LAP. Since ADP is weaker than LAP, the SWO defined by (2) also satisfies ADP and AN (when Y is finite). Of course, since (1) is an explicit formula for the SWF, the underlying SWO (2) can be explicitly described (it is a constructive object).

4.1 Representation of social welfare orders satisfying asymptotic density-one Pareto and anonymity

In this section we prove that a necessary and sufficient condition for the existence of a SWF satisfying ADP and AN on $X = Y^{\mathbb{N}}$ is that Y be finite. The bulk of Theorem 1 involves showing that there is no SWF satisfying ADP and AN if Y is infinite. To this end, it will be helpful to prove first (in Lemma 1 below) that no SWF satisfying the foregoing properties exists if $Y = \mathbb{N}$. Indeed, as will become apparent in the proof of Theorem 1, Lemma 1 yields almost automatically the desired result. This is because any infinite subset of the real numbers contains a subset which is order isomorphic to \mathbb{N} or to the set of negative integers: (see Petri (2019, Lemma A.3 p. 869) for a proof).

Note that in the proof of Lemma 1 we take advantage of a technique used in Basu and Mitra (2003). More specifically, we begin by noting that in order to invoke the ADP axiom we need to somehow exhibit a subset of the natural numbers (on which a strict Pareto-improvement takes place) that has asymptotic density equal to one. We construct such a set so that it is the complement of a set with zero asymptotic density. To get zero density, in turn, we reproduce the same construction as in Basu and Mitra (2003) except that we transform the natural numbers considered therein into factorials. It turns out that this tweak suffices for our purposes. This is because, loosely put, factorial numbers are far enough apart to yield a ‘‘sparse’’ subset of the natural numbers.

Lemma 1. *There does not exist any social welfare function satisfying ADP and AN on $X = Y^{\mathbb{N}}$, with $Y = \mathbb{N}$.*

Combining Example 1 (and the related remarks) with Lemma 1, yields the following theorem:

Theorem 1. *There exists a social welfare function satisfying ADP and AN on $X = Y^{\mathbb{N}}$ if and only if Y is finite.*

4.2 Construction of social welfare orders satisfying asymptotic density-one Pareto and anonymity

Let T be an infinite subset of \mathbb{N} . We denote by $\Omega(T)$ the collection of all infinite subsets of T , and we let Ω denote the collection of all infinite subsets of \mathbb{N} .

Definition 4. A collection of sets $\Gamma \subset \Omega$ is said to be *non-Ramsey* if for every $T \in \Omega$, the collection $\Omega(T)$ intersects both Γ and its complement $\Omega \setminus \Gamma$.

The bulk of Theorem 2 below involves showing that there is no constructive SWO satisfying ADP and AN on $X = Y^{\mathbb{N}}$ if Y is infinite. We achieve this goal by proving that the existence of a SWO (defined on $X = Y^{\mathbb{N}}$, where Y is an infinite set) satisfying AN and ADP implies the existence of a non-Ramsey collection

of set. Observe, again, that any infinite set of real numbers contains a subset order isomorphic to the set of natural numbers (or to the set of negative integers). Therefore, it will suffice to prove (in Lemma 2 below) that when $Y = \mathbb{N}$, the existence of a SWO satisfying AN and ADP entails the existence of a non-Ramsey collection of set.

The proof of Lemma 2 below is inspired by Lauwers (2010). We start from the same sequence of natural numbers as Lauwers's, but we convert its elements into factorial numbers. Then, following Lauwers's approach, we use that sequence to construct a partition of the set of natural numbers. However, we make the following change to Lauwers's partition: loosely put, we expand and shrink any two consecutive subsets, respectively, that belong to the aforementioned partition. We do that without altering the union of such sets. We resort to this particular partition as it allows us to prove that the asymptotic density of the key sets involved in the proof of Lemma 2 (where we appeal to the ADP axiom) is equal to one.

Lemma 2. *Let $Y = \mathbb{N}$, and assume that there is a social welfare order on $X = Y^{\mathbb{N}}$ satisfying AN and ADP. Then, there exists a non-Ramsey set.*

Combining Example 1 with Lemma 2, we get the following theorem.

Theorem 2. *There exists a constructive social welfare order satisfying ADP and AN on $X = Y^{\mathbb{N}}$ if and only if Y is finite.*

The following corollary is an easy consequence of Theorem 2 and Example 1 (see the comments below (2)):

Corollary 1. *There exists a constructive social welfare order satisfying LAP and AN on $X = Y^{\mathbb{N}}$ if and only if Y is finite.*

5 Concluding Remarks

In this paper we have focused on SWOs, on infinite utility streams, satisfying the ADP and AN axioms. Apart from the considerations already articulated in the introduction, it is also worth considering the following table in order to highlight the scope of our results in relation to the existing literature. The table summarizes some findings (concerning the representability and nature of social welfare orders) obtained when the anonymity axiom is combined with various forms of the Pareto axiom.

Table

Pareto axiom	$ Y $	Representation	Constructive nature
Strong	≥ 2	No (Basu and Mitra (2003))	No (Lauwers (2010)) Zame (2007)
Infinite	≥ 2	No (Crespo et al. (2009))	No (Lauwers (2010))
Upper Asymptotic	≥ 2	No (Dubey et al. (2020))	No (Dubey et al. (2020))
Lower Asymptotic	Finite	Yes (Petri (2019))	Yes (Petri (2019))
Lower Asymptotic	Infinite	No (Petri (2019))	No (Corollary 1)
Asymptotic Density-one	Finite	Yes (Theorem 1)	Yes (Theorem 2)
Asymptotic Density-one	Infinite	No (Theorem 1)	No (Theorem 2)

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6 Appendix

We begin by stating two claims which are useful for the proof of Lemma 1. The easy proof of both of them is left to the reader.

Claim 1. Given the set $A := \{n! : n \in \mathbb{N}\}$, we have that $d(A) = 0$. Therefore, $d(B) = 1$, where $B := \mathbb{N} \setminus A$.

Claim 2. Let $A, B \subseteq \mathbb{N}$ be such that $A \triangle B$ is finite.² Then $d(A) = d(B)$.

Proof of Lemma 1. We establish the claim by contradiction. Let $W : X \rightarrow \mathbb{R}$ be a SWF satisfying ADP and AN. Let q_1, q_2, \dots be any arbitrary enumeration of rational numbers in $(0, 1)$. We keep this enumeration fixed throughout the proof. Pick $r \in (0, 1)$ and let $u_1(r) := \min\{n \in \mathbb{N} : q_n \in [r, 1]\}$. Having defined $u_1(r)$, for every $k \geq 1$ we set

$$u_{k+1}(r) := \min\{n \in \mathbb{N} \setminus \{u_1(r), u_2(r), \dots, u_k(r)\} : q_n \in [r, 1]\}.$$

Note that $u_1(r) < u_2(r) < \dots < u_k(r) < \dots$. Thus, we can define $u(r)$ as follows:

$$u(r) = \{u_1(r), u_2(r), \dots, u_k(r), \dots\}.$$

Let $U(r) := \{u_1(r)!, u_2(r)!, \dots, u_k(r)!, \dots\}$. Define

$$L(r) := \mathbb{N} \setminus U(r) = \{l_1(r), l_2(r), \dots, l_k(r), \dots\},$$

²We denote the symmetric set difference for sets A and B as $A \triangle B := (A \setminus B) \cup (B \setminus A)$.

with

$$l_1(r) < l_2(r) < \cdots < l_k(r) < l_{k+1}(r) < \cdots.$$

We define the utility stream $\langle x(r) \rangle$ as follows:

$$x_t(r) = \begin{cases} 1 & \text{if } t \in \mathbf{U}(r), \\ m+1 & \text{if } t = l_m(r), m \in \mathbb{N}. \end{cases} \quad (3)$$

Observe that the sequence $\langle x(r) \rangle$ takes values $2, 3, \dots$ in increasing order at coordinates in the set $L(r)$ and takes constant value of one for all coordinates in the set $\mathbf{U}(r)$. Thus, $x_{l_1} = 2$, $x_{l_2} = 3$ and so on. At the same time, $x_{u_k(r)!} = 1$ for all $k \in \mathbb{N}$. Next, we define utility stream $\langle z(r) \rangle$ in an identical fashion by taking $\mathbf{U}(r) = \mathbf{U}(r) \setminus \{u_1(r)!\}$, and so $\mathbf{L}(r) = L(r) \cup \{u_1(r)!\}$:

$$z_t(r) = \begin{cases} 1 & \text{if } t \in \mathbf{U}(r), \\ m+1 & \text{if } t = l_m(r), m \in \mathbb{N}, \end{cases} \quad (4)$$

where we consider as above the usual increasing enumeration $\mathbf{U}(r) := \{\mathbf{u}_m(r) : m \in \mathbb{N}\}$ and $\mathbf{L}(r) := \{\mathbf{l}_m(r) : m \in \mathbb{N}\}$. Thus,

$$\begin{aligned} \mathbf{l}_1(r) &= l_1(r), \mathbf{l}_2(r) = l_2(r), \dots, \mathbf{l}_{u_1(r)!-1}(r) = u_1(r)! - 1 = l_{u_1(r)!-1}(r), \text{ and,} \\ \mathbf{l}_{u_1(r)!}(r) &= u_1(r)! = l_{u_1(r)!}(r) - 1, \dots, \mathbf{l}_k(r) = l_k(r) - 1, \text{ for all } k > u_1(r)!. \end{aligned}$$

Hence,

- $x_{u_1(r)!}(r) = 1 < z_{u_1(r)!}(r)$
- $\forall k < u_1(r)! (x_k(r) = z_k(r))$
- $\forall k \in \mathbf{U}(r) (k > u_1(r)! \Rightarrow x_k(r) = 1 = z_k(r))$
- $\forall k \in L(r) (k > u_1(r)! \Rightarrow x_k(r) < z_k(r)).$

Let

$$S_1 := [u_1(r)!, u_2(r)!) \cap \mathbb{N} \text{ and}$$

for all $k \in \mathbb{N}$, $k > 1$, let

$$S_k := (u_k(r)!, u_{k+1}(r)!) \cap \mathbb{N} \text{ and } S := \bigcup_{k=1}^{\infty} S_k$$

It follows from above that for all $t \in S$, we have $z_t(r) > x_t(r)$. Applying Claim 1 and 2 above, one can easily show that $d(S) = 1$. Hence, by ADP we get $x(r) \prec z(r)$, therefore

$$W(x(r)) < W(z(r)). \quad (5)$$

Next, we pick $s \in (r, 1)$ for which $\langle x(s) \rangle$ and $\langle z(s) \rangle$ are defined using the same construction as above. Observe that $\mathbf{U}(s) \subset \mathbf{U}(r)$ and let $\mathbf{U}(rs) := \mathbf{U}(r) \setminus \mathbf{U}(s)$. Note that there are infinitely many natural numbers in $\mathbf{U}(rs)$ and $\mathbf{U}(s)$ since there are infinitely many rational numbers $q_n \in [r, s)$ and $q_n \in [s, 1)$ respectively. Let

$$u_1(rs)! := \min\{n! : n! \in \mathbf{U}(rs)\} \quad \text{and} \quad u_2(rs)! := \min\{n! : n! \in \mathbf{U}(rs) \setminus \{u_1(rs)!\}\}.$$

More generally, it is clear that we can consider the usual enumerations $\mathbf{U}(s) = \{u_k(s) : k \in \mathbb{N}\}$ and $\mathbf{U}(rs) = \{u_k(rs) : k \in \mathbb{N}\}$. Also notice that $\mathbf{U}(s)$ and $\mathbf{U}(rs)$ form a partition of $\mathbf{U}(r)$, which means $\mathbf{U}(s) \cap \mathbf{U}(rs) = \emptyset$ and $\mathbf{U}(s) \cup \mathbf{U}(rs) = \mathbf{U}(r)$. As a consequence, for every $k \in \mathbb{N}$, any $t \in \mathbf{U}(r)$ such that $u_k(rs)! < t < u_{k+1}(rs)!$ is also an element of the set $\mathbf{U}(s)$, and any $t \in \mathbf{U}(r)$ such that $u_k(s)! < t < u_{k+1}(s)!$ is also an element of the set $\mathbf{U}(rs)$. By construction, we only have two possible cases: $u_1(rs)! = u_1(r)!$ or $u_1(s)! = u_1(r)!$. Therefore, there are two cases to consider.

Case (a): $u_1(rs)! = u_1(r)!$, i.e., $q_{u_1(r)} \in [r, s)$. In particular, we have $U(s) \subsetneq U(r)$ and therefore a similar argument to the one above yields $z(r) \prec x(s)$. Details are as follows. Pick $N := \min\{k : u_k(r)! \neq u_k(s)!\}$. Then we obtain:

- $x_{u_N(r)!}(s) > 1 = z_{u_N(r)!}(r)$
- $\forall k < u_N(r)!(x_k(s) = z_k(r))$
- $\forall k \in U(s)(k > u_N(r)! \Rightarrow z_k(r) = 1 = x_k(s))$
- $\forall k \in L(s)(k > u_N(r)! \Rightarrow z_k(r) < x_k(s))$.

By construction, $U(s) \subset U(r)$ which gives $L(s) \supset L(r)$; then $1 = d(L(r)) \leq d(L(s))$ and so by Claim 2 also $d(S) = 1$, where $S := \{n \in L(s) : n > u_N(r)!\}$. By ADP we then get the desired $z(r) \prec x(s)$.

Case (b): $u_1(s)! = u_1(r)!$, i.e., $q_{u_1(r)} \in [s, 1)$. In this case $U(s) \setminus \{u_1(r)!\} \subsetneq U(r)$, which means we can still rely on the above argument, but in this case we have to permute finitely many elements in order to switch the position $k = u_1(r)!$ and for other finitely many k 's, where $x_k(s) < z_k(r)$. More precisely, comparing the streams $x(s)$ and $z(r)$, the following hold:

- $\forall t < u_1(r)!, z_t(r) = x_t(s)$ and $z_{u_1(r)!}(r) > 1 = x_{u_1(r)!}(s)$,
- $\forall t \in (u_1(r)!, u_1(rs)!) \cap U(r), z_t(r) = x_t(s) = 1$, and $z_{u_1(rs)!}(r) = 1 < x_{u_1(rs)!}(s)$,
- $\forall t \in (u_1(r)!, u_1(rs)!) \cap L(r), z_t(r) = x_t(s) + 1 > x_t(s) > 1$,
- $\forall t \in (u_1(rs)!, u_2(rs)!) \cap U(r), z_t(r) = x_t(s) = 1$, and $z_{u_2(rs)!}(r) = 1 < x_{u_2(rs)!}(s)$,
- $\forall t \in (u_1(rs)!, u_2(rs)!) \cap L(r), z_t(r) = x_t(s) > 1$,
- $\forall t > u_2(rs)!, t \in L(s), x_t(s) \geq z_t(r) + 1 > z_t(r)$, and
- $\forall t > u_2(rs)!, t \in U(s), x_t(s) = z_t(r) = 1$.

Hence, we can perform a finite permutation π of coordinates $t \in [u_1(r)!, u_1(rs)!) \cap L(r)$ in sequence $\langle z(r) \rangle$ to obtain $\langle z' \rangle := \langle z_{\pi(t)} : t \in \mathbb{N} \rangle$. More precisely, let $\{a_1, a_2, \dots, a_K\}$ be an increasing enumeration of $[u_1(r)!, u_1(rs)!) \cap L(r)$, then we can define the finite permutation π as follows:

$$\pi(t) := \begin{cases} t & \text{if } t \notin [u_1(r)!, u_1(rs)!) \\ u_1(rs)! & \text{if } t = u_1(r)! \\ a_{j-1} & \text{if } t = a_j \text{ (for } j \geq 2). \end{cases} \quad (6)$$

Then z' satisfies the following properties:

- $z'_{u_1(r)!} = z_{u_1(rs)!}(r) = 1$,
- $\forall t \in (u_1(r)!, u_1(rs)!) \cap U(r), z'_t = z_t(r)$, i.e., no change,
- since $|(u_1(r)!, u_1(rs)!) \cap L(r)| = |[u_1(r)!, u_1(rs)!) \cap L(r)]$, $z'_t = z_{t'}(r)$, where t' and t occupy the same position in the increasing enumerations of the sets $[u_1(r)!, u_1(rs)!) \cap L(r)$ and $(u_1(r)!, u_1(rs)!) \cap L(r)$ respectively,
- $z'_t = z_t(r)$ for remaining $t \in \mathbb{N}$.

Hence, by AN, we have $z' \sim z(r)$, and $W(z') = W(z(r))$. Also $x_t(s) \geq z'_t$ for all $t \in \mathbb{N}$ and furthermore

- $\forall t < u_2(rs)!, z'_t \leq x_t(s)$,
- $z'_{u_2(rs)!} = 1 < x_{u_2(rs)!}(s)$.

- $\forall t > u_2(rs)!, t \in L(s), x_t(s) \geq z'_t + 1 > z'_t$, and
- $\forall t > u_2(rs)!, t \in U(s), x_t(s) = z'_t = 1$.

It is useful to note that similarly to case (a) above, in particular with $S := \{n \in L(s) : n > u_2(rs)!\}$ we have that for all $t \in S, x_t(s) > z'_t$ and $d(S) = 1$. Therefore, by ADP $z' \prec x(s)$, hence

$$W(z') < W(x(s)). \quad (7)$$

Combining $z(r) \sim z'$ and $z' \prec x(s)$, we get $z(r) \prec x(s)$, hence

$$W(z(r)) < W(x(s)). \quad (8)$$

Consequently, in both cases, we obtain

$$W(z(r)) < W(x(s)). \quad (9)$$

Therefore, (5) and (9) imply that $(W(x(r)), W(z(r)))$ and $(W(x(s)), W(z(s)))$ are non-empty and disjoint open intervals. Hence, because r and s , with $r < s$, were arbitrary, by density of \mathbb{Q} in \mathbb{R} we conclude that we have found a one-to-one mapping from $(0, 1)$ to \mathbb{Q} , which is impossible as the latter set is countable. \square

Proof of Theorem 1. Suppose that there is a social welfare function W satisfying ADP and AN on $X = Y^{\mathbb{N}}$. We must show that Y is finite. To see this, suppose, by way of contradiction, that Y is infinite, and let A be a subset of Y which is order isomorphic to \mathbb{N} . Then, the restriction of W to $A^{\mathbb{N}}$, say \bar{W} , is still a SWF that satisfies ADP and AN. Next, observe that there is an order-preserving bijection between A and \mathbb{N} , let's say $f : \mathbb{N} \rightarrow A$. Therefore, it's routine matter to verify that a suitable composition of f with \bar{W} maps $\mathbb{N}^{\mathbb{N}}$ into \mathbb{R} and satisfies ADP and AN, which contradicts Lemma 1.

Conversely, assume that Y is finite. Then, it readily follows from Example 1 and the relative remarks that the mapping defined by 1) is a SWF that satisfies ADP and AN. \square

The following results (regarding the lower asymptotic density of subsets of \mathbb{N}) will be used in the proof of Lemma 2. Let N be a sequence of natural numbers such that $n_1 < n_2 < n_3 < \dots$ denoted as $N := \{n_k : k \in \mathbb{N}\}$. We partition the set of natural numbers, \mathbb{N} , in two infinite subsets based on N .

$$U_k(N) := \left(n_{2k-1}!, n_{2k+1}! - \frac{n_{2k+1}!}{n_{2k}!} \right] \cap \mathbb{N}, k \in \mathbb{N}, U(N) := \bigcup_{k=1}^{\infty} U_k(N),$$

$$L_1(N) := [1, n_1!] \cap \mathbb{N}, L_{k+1}(N) := \left(n_{2k+1}! - \frac{n_{2k+1}!}{n_{2k}!}, n_{2k+1}! \right] \cap \mathbb{N}, k \in \mathbb{N}, \text{ and}$$

$$L(N) := \bigcup_{k=1}^{\infty} L_k(N).$$

Note that for all $k \in \mathbb{N}$, we have,

$$n_{2k+1}! - \frac{n_{2k+1}!}{n_{2k}!} > n_{2k}!.$$

To see this, observe that the inequality above is equivalent to the following inequality,

$$1 - \frac{1}{n_{2k}!} > \frac{n_{2k}!}{n_{2k+1}!}.$$

Since, $n_{2k} \geq 2$, $n_{2k+1} \geq n_{2k} + 1$ and $n_{2k} + 1 \geq 3$, we obtain

$$1 - \frac{1}{n_{2k}!} \geq 1 - \frac{1}{2!} = \frac{1}{2} > \frac{1}{3} \geq \frac{1}{n_{2k} + 1} \geq \frac{n_{2k}!}{n_{2k+1}!}.$$

Furthermore,

$$|\mathcal{U}(k)| = \left| \left(n_{2k-1}!, n_{2k+1}! - \frac{n_{2k+1}!}{n_{2k}!} \right) \cap \mathbb{N} \right| = n_{2k+1}! - \frac{n_{2k+1}!}{n_{2k}!} - n_{2k-1}!.$$

Hence,

$$\begin{aligned} \frac{|\mathcal{U}(k)|}{n_{2k+1}!} &= \frac{n_{2k+1}! - \frac{n_{2k+1}!}{n_{2k}!} - n_{2k-1}!}{n_{2k+1}!} \\ &= 1 - \frac{1}{n_{2k}!} - \frac{n_{2k-1}!}{n_{2k+1}!}. \end{aligned}$$

Observe that $n_{2k+1} \geq n_{2k} + 1 \geq n_{2k-1} + 2$. Thus,

$$\frac{n_{2k-1}!}{n_{2k+1}!} \leq \frac{n_{2k-1}!}{(n_{2k-1} + 2)(n_{2k-1} + 1)(n_{2k-1}!)} = \frac{1}{(n_{2k-1} + 2)(n_{2k-1} + 1)} \rightarrow 0, \text{ when } k \rightarrow \infty.$$

Therefore,

$$\lim_{k \rightarrow \infty} \frac{|\mathcal{U}(k)|}{n_{2k+1}!} = 1.$$

Also, it can be verified that for all $l < k$,

$$\frac{|\mathcal{U}(l)|}{n_{2k+1}!} = \frac{n_{2l+1}! - \frac{n_{2l+1}!}{n_{2l}!} - n_{2l-1}!}{n_{2k+1}!} \rightarrow 0, \text{ when } k \rightarrow \infty.$$

Therefore,

$$\underline{d}(\mathcal{U}(\mathbb{N})) = \liminf_{m \rightarrow \infty} \frac{\left| \bigcup_{k=1}^m \mathcal{U}_k(\mathbb{N}) \right|}{n_{2m+1}!} = \lim_{m \rightarrow \infty} \left(\frac{|\mathcal{U}_m(\mathbb{N})|}{n_{2m+1}!} \right) = 1.$$

We also consider sequence $\mathbb{N} \setminus \{n_1\}$. Corresponding partitions the set of natural numbers \mathbb{N} are

$$\hat{\mathcal{U}}_k(\mathbb{N}) := \left(n_{2k}!, n_{2k+2}! - \frac{n_{2k+2}!}{n_{2k+1}!} \right) \cap \mathbb{N}, k \in \mathbb{N}, \hat{\mathcal{U}}(\mathbb{N}) := \bigcup_{k=1}^{\infty} \hat{\mathcal{U}}_k(\mathbb{N}),$$

$$\hat{\mathcal{L}}_1(\mathbb{N}) := [1, n_2!] \cap \mathbb{N}, \hat{\mathcal{L}}_{k+1}(\mathbb{N}) := \left(n_{2k+2}! - \frac{n_{2k+2}!}{n_{2k+1}!}, n_{2k+2}! \right) \cap \mathbb{N}, k \in \mathbb{N}, \text{ and}$$

$$\hat{\mathcal{L}}(\mathbb{N}) := \bigcup_{k=1}^{\infty} \hat{\mathcal{L}}_{k+1}(\mathbb{N}).$$

By using a similar argument as for $\mathcal{U}(\mathbb{N})$ above, one can obtain

$$\underline{d}(\hat{\mathcal{U}}(\mathbb{N})) = 1. \tag{10}$$

Proof of Lemma 2. Given sequence N defined above, we describe the set $U(N)$ as

$$U(N) = \{u_k : u_k < u_{k+1}, \text{ for all } k \in \mathbb{N}\}.$$

We define the utility stream $x(N)$ whose components are,

$$x_t(N) = \begin{cases} 1 & \text{if } t \in L(N), \\ 2 & \text{if } t = u_1 \\ x_{u_l} + 1 & \text{if } t = u_{l+1}, \text{ for } l \in \mathbb{N}. \end{cases} \quad (11)$$

We also construct the sequence $y(N)$ using the subset $N \setminus \{n_1\}$, with its components being defined as follows

$$y_t(N) = \begin{cases} 1 & \text{if } t \in \widehat{L}(N), \\ 2 & \text{if } t = \widehat{u}_1 \\ y_{\widehat{u}_l} + 1 & \text{if } t = \widehat{u}_{l+1}, \text{ for } l \in \mathbb{N}. \end{cases} \quad (12)$$

Let \succsim be a social welfare order satisfying AN and ADP. We claim that the collection of sets $\Gamma \equiv \{N \in \Omega : x(N) \prec y(N)\}$ is non-Ramsey. We must show that for each $T \in \Omega$, the collection $\Omega(T)$ intersects both Γ and $\Omega \setminus \Gamma$. To this end, it is sufficient to show that for every $T \in \Omega$ (1) if $T \in \Gamma$, then there exists $S \in \Omega(T)$ such that $S \notin \Gamma$, and (2) if $T \notin \Gamma$, then there exists $S \in \Omega(T)$ such that $S \in \Gamma$. As the binary relation is assumed to be complete, one of the following cases must be true: (a) $x(T) \prec y(T)$; (b) $y(T) \prec x(T)$; and (c) $x(T) \sim y(T)$. We now consider each of these three cases.

(a) Let $x(T) \prec y(T)$ or $T \in \Gamma$. Take $S = T - \{t_1\} = \{t_2, t_3, t_4, \dots\}$. Note that $S \in \Omega(T)$.

(i) Since $y_t(T) = x_t(T - \{t_1\}) = x_t(S) \forall t \in \mathbb{N}$, we get

$$y(T) \sim x(S). \quad (13)$$

(ii) $y_t(S) = x_t(T - \{t_1, t_2\})$ yields

- $x_t(T) \geq y_t(S)$ for all $t \in \mathbb{N}$,
- $y_t(S) = x_t(T) = 1$ for all $t \in L(T)$,
- $x_t(T) > 1 = y_t(S)$ for all $t \in U_1(T)$ which implies $x_t(T) - y_t(S) \geq 1$, and
- $x_t(T) > y_t(S) \geq 1$, for all $t \in U(T)$.

Since $d(U(T)) = 1$, using ADP, we obtain

$$y(S) \prec x(T) \quad (14)$$

Therefore, by (13) and (14) we get $y(S) \prec x(T) \prec y(T) \sim x(S)$. By transitivity, $y(S) \prec x(S)$, which establishes that $S \notin \Gamma$, as was to be proven.

(b) Let $y(T) \prec x(T)$ or $T \notin \Gamma$. We drop t_1 and $t_4, t_5, \dots, t_{2m}, t_{2m+1}$ to obtain the set $S = \{t_2, t_3, t_{2m+2}, t_{2m+3}, \dots\}$ such that

$$|U_1(T)| < |L_3(T)| + |L_4(T)| + \dots + |L_{m+1}(T)|.$$

Note that $S \in \Omega(T)$. We claim

$$\left| \widehat{L}_2(T) \right| + \left| \widehat{L}_3(T) \right| + \dots + \left| \widehat{L}_{m+1}(T) \right| < |L_2(S)|. \quad (15)$$

To see this, observe that

$$\left| \widehat{L}_2(T) \right| = \frac{t_4!}{t_3!}, \left| \widehat{L}_3(T) \right| = \frac{t_6!}{t_5!}, \dots, \left| \widehat{L}_{m+1}(T) \right| = \frac{t_{2m+2}!}{t_{2m+1}!}, |L_2(S)| = \frac{t_{2m+2}!}{t_3!}.$$

Also the desired inequality is equivalent to the following condition

$$\frac{t_{2m+2}!}{t_3!} > \frac{t_{2m+2}!}{t_{2m+1}!} + \dots + \frac{t_4!}{t_3!}. \quad (16)$$

The right hand side member of the above inequality (16) contains m terms. We rearrange (16) as the sum of m terms

$$\left(\frac{t_{2m+2}!}{t_3!} - m \frac{t_{2m+2}!}{t_{2m+1}!} \right) + \left(\frac{t_{2m+2}!}{t_3!} - m \frac{t_{2m}!}{t_{2m-1}!} \right) + \dots + \left(\frac{t_{2m+2}!}{t_3!} - m \frac{t_4!}{t_3!} \right) > 0. \quad (17)$$

To prove the above inequality, it suffices to show that each of the m expressions inside the parentheses are positive. The expression in the first parenthesis is

$$\left(\frac{t_{2m+2}!}{t_3!} - m \frac{t_{2m+2}!}{t_{2m+1}!} \right) = \left(\frac{t_{2m+2}!}{t_{2m+1}!} \right) \left(\frac{t_{2m+1}!}{t_3!} - m \right). \quad (18)$$

Since $t_{2m+1} \geq t_3 + (2m - 2)$, and $t_3 \geq 3$,

$$\frac{t_{2m+1}!}{t_3!} \geq \frac{(t_3 + 2m - 2) \cdots (t_3 + 1) t_3!}{t_3!} \geq (3 + 2m - 2)(3 + 2m - 3) \cdots (3 + 1) = (2m + 1)(2m) \cdots (4) > m.$$

Thus, each of the two terms on the right hand side member of (18) is positive, hence the expression in the first parenthesis of (17) is positive. The argument to show that the expressions inside the other parentheses goes as follows (we focus on the the second parenthesis).

$$\left(\frac{t_{2m+2}!}{t_3!} - m \frac{t_{2m}!}{t_{2m-1}!} \right) = \left(\frac{t_{2m}!}{t_{2m-1}!} \right) \left(\frac{t_{2m+2}!}{t_{2m}!} \frac{t_{2m-1}!}{t_3!} - m \right). \quad (19)$$

Since $t_{2m+2} \geq t_{2m} + 2$, $t_{2m-1} \geq t_3$, and $t_{2m} \geq 2m$, we get

$$\frac{t_{2m+2}!}{t_{2m}!} \frac{t_{2m-1}!}{t_3!} \geq (t_{2m} + 2)(t_{2m} + 1)(1) \geq (2m + 2)(2m + 1) > m.$$

(i) For $y(T)$ and $x(S)$, we get

- $[1, t_{2m+2}!] = \widehat{L}_1(T) \cup \widehat{U}_1(T) \cup \dots \cup \widehat{L}_m(T) \cup \widehat{U}_m(T) \cup \widehat{L}_{m+1}(T) = L_1(S) \cup U_1(S) \cup L_2(S)$, $\widehat{L}_1(T) = L_1(S)$ and therefore, $\widehat{U}_1(T) \cup \dots \cup \widehat{L}_m(T) \cup \widehat{U}_m(T) \cup \widehat{L}_{m+1}(T) = U_1(S) \cup L_2(S)$,
- $y_t(T) = x_t(S) = 1$ for all $t \in \widehat{L}_1(T)$ and $y_t(T) = x_t(S) > 1$ for all $t \in \widehat{U}_1(T)$,
- $|\widehat{L}_2(T) \cup \widehat{L}_3(T) \cup \widehat{L}_{m+1}(T)| < |L_2(S)|$ and therefore, $|\widehat{U}_1(T) \cup \widehat{U}_2(T) \cup \widehat{U}_m(T)| > |U_1(S)|$. In other words, there are fewer coordinates in $[t_2!, t_{2m+2}!] \cap \mathbb{N}$, with $y_t(T) = 1 < x_t(S)$ and a strictly larger number of coordinates with $y_t(T) > x_t(S) \geq 1$,
- $y_t(T) - x_t(S) \geq 1$ for all $\bigcup_{k=m+1}^{\infty} \widehat{U}_k(T)$, and $y_t(T) = x_t(S) = 1$ for all $\bigcup_{k=m+2}^{\infty} \widehat{L}_k(T)$.

We switch those coordinates of $x(S)$ in $[t_2!, t_{2m+2}!]$ having $y_t(T) = 1 < x_t(S)$ with an equal number of elements from the remaining coordinates satisfying $x_t(S) = 1 < y_t(T)$ to obtain a

sequence x' such that $x'_t \leq y_t(T)$ for all $t \in \mathbb{N}$. Since only finitely many terms of $x(S)$ have been permuted to obtain x' , AN implies

$$x' \sim x(S). \quad (20)$$

Furthermore, for all coordinates in $\bigcup_{k=3}^{\infty} \widehat{U}_k(T)$, $y_t(T) - x'_t \geq 1$. Since by (10), $d(\widehat{U}(T)) = 1$, using ADP, we have obtained

$$x' \prec y(T). \quad (21)$$

Thus, it follows from (20), (21), and transitivity that

$$x(S) \prec y(T). \quad (22)$$

(ii) For $x(T)$ and $y(S)$, we note that

$$\bullet [1, 2t_{2t+3}] = L_{m+2} \cup \bigcup_{k=1}^{m+1} L_k(T) \cup U_k(T) = \widehat{L}_1(S) \cup \widehat{U}_1(S) \cup \widehat{L}_2(S),$$

$$\widehat{L}_1(S) = L_1(T) \cup U_1(T) \cup L_2(T)$$

and therefore,

$$\widehat{U}_1(S) = U_{m+1} \cup \bigcup_{k=3}^{m+1} L_k(T) \cup U_{k-1}(T).$$

- $y_t(T) = x_t(S) = 1$ for all $t \in L_1(T) \cup L_2(T)$, for all $t \in U_1(T)$, $y_t(S) = 1 < x_t(T)$, and for all $t \in L_3(T) \cup \dots \cup L_{m+1}(T)$, $y_t(S) > 1 = x_t(T)$.
- since

$$|U_1(T)| < |L_3(T)| + |L_4(T)| + \dots + |L_{m+1}(T)|.$$

we get

$$|\widehat{U}_1(S)| > |U_1(T) \cup U_2(T) \cup \dots \cup U_{m+1}(T)|.$$

In other words, there are fewer coordinates in $[(t_1!), 2t_{2t+1}] \cap \mathbb{N}$, with $y_t(S) = 1 < x_t(T)$ and a strictly larger number of coordinates with $y_t(S) > x_t(T) \geq 1$,

- therefore, $y_t(S) - x_t(T) \geq 1$ for all $\bigcup_{k=m+2}^{\infty} U_k(T)$, and $y_t(S) = x_t(T) = 1$ for all $\bigcup_{k=m+2}^{\infty} L_k(T)$.

We switch elements of $y(S)$ from coordinates $\widehat{L}_1(S) \cup \widehat{U}_1(S)$ to obtain a sequence y' such that $y'_t \geq x_t(T)$ for all $t \in \mathbb{N}$. As in (i) above, we can implement a finite permutation of $y(S)$ and invoke AN and ADP to obtain

$$x(T) \prec y(S). \quad (23)$$

Therefore, by (23) and (22) we get $x(S) \prec y(T) \prec x(T) \prec y(S)$. By transitivity, $x(S) \prec y(S)$, which yields $S \in \Gamma$, as was to be proven.

(c) Let $x(T) \sim y(T)$, i.e., $T \notin \Gamma$. We drop $t_1, t_2, t_3, t_6, t_7, \dots, t_{2m}$ and t_{2m+1} to obtain the set $S = \{t_4, t_5, t_{2m+2}, t_{2m+3}, \dots\}$. Note that $S \in \Omega(T)$ and

$$|L_4(T)| + \dots + |L_{m+1}(T)| > |U_1(T)| + |U_2(T)|. \quad (24)$$

Using the technique relied upon in case (b) above, we can show

$$\left| \widehat{L}_3(T) \right| + \left| \widehat{L}_4(T) \right| + \dots + \left| \widehat{L}_{m+1}(T) \right| < |L_2(S)|. \quad (25)$$

(i) For $x(S)$ and $y(T)$, we get

$$|\bigcup_{k=1}^{m+1} \widehat{L}_k(T)| + |\bigcup_{k=1}^m \widehat{U}_k(T)| = |L_1(S) \cup U_1(S) \cup L_2(S)|.$$

- $y_t(T) = x_t(S)$ for all coordinates in $\widehat{L}_1(T) \cup \widehat{L}_2(T)$,
- $y_t(T) > x_t(S) = 1$ for all $t \in \widehat{U}_1(T)$,
- in view of (25) there is a strictly larger number of coordinates in $\bigcup_{k=1}^m \widehat{U}_k(T)$, such that $y_t(T) > 1$ than the number of coordinates in $U_1(S)$ for which $x_t(S) > 1$, and
- for all elements in $\widehat{U}_k(T)$, with $k > m$, $y_t(T) > x_t(S)$.

As in part (i) of case (b) above, we can perform a finite permutation of $x(S)$ and use AN and ADP to obtain

$$x(S) \prec y(T). \quad (26)$$

(ii) For $x(T)$ and $y(S)$, we note that

- for all $t \in L_1(T) \cup L_2(T) \cup L_3(T)$, $y_t(S) = 1 = x_t(T)$,
- for all $t \in U_1(T) \cup U_2(T)$, $y_t(S) = 1 < x_t(T)$, and
- for all $t \in U_3(T)$, $1 < y_t(S) < x_t(T)$,
- furthermore, for all $t \in L_4(T) \cup L_5(T) \cup \dots \cup L_{m+1}(T)$, $1 = x_t(T) < y_t(S)$.

We switch elements of $y(S)$ from coordinates in $U_3(T) \cup L_4(T)$ with elements from coordinates $U_1(T) \cup U_2(T)$ satisfying $y_t(S) < x_t(T)$ to obtain a sequence y' such that $y'_t \geq x_t(T)$ for all $t \in \mathbb{N}$. As in case (b)-(i) above, we can apply a finite permutation of $y(S)$ and invoke AN and ADP to obtain

$$x(T) \prec y(S). \quad (27)$$

Therefore, by (26) and (27) we get $x(S) \prec y(T) \sim x(T) \prec y(S)$. By transitivity, $x(S) \prec y(S)$, which yields $S \in \Gamma$, as was to be proven. □

Proof of Theorem 2. Assume that Y is infinite. We must show that there does not exist any constructive SWO satisfying ADP and AN on $Y^{\mathbb{N}}$. To this end, suppose, by way of obtaining a contradiction, that there is a constructive SWO satisfying ADP and AN on $Y^{\mathbb{N}}$. Let $A \subset Y$ be order isomorphic to \mathbb{N} , thus, let $f : \mathbb{N} \rightarrow A$ be an order-preserving bijection. Consider the restriction of the foregoing SWO to $A^{\mathbb{N}}$, and call it \succeq . Next, we define an induced binary relation on $\mathbb{N}^{\mathbb{N}}$, denoted by \succeq_f , as follows: for all $\langle n_s \rangle$ and $\langle \bar{n}_s \rangle$ in $\mathbb{N}^{\mathbb{N}}$,

$$\langle n_1, n_2, n_3, \dots \rangle \succeq_f \langle \bar{n}_1, \bar{n}_2, \bar{n}_3, \dots \rangle \text{ if and only if } (f(n_1), f(n_2), f(n_3), \dots) \succeq (f(\bar{n}_1), f(\bar{n}_2), f(\bar{n}_3), \dots).$$

Since f is increasing, it's easy to show that also \succeq_f is a constructive SWO that satisfies ADP and AN, which contradicts Lemma 2.

For the converse, suppose that Y is finite. Then, it follows from Example 1 that the SWO defined by (2) is constructive and satisfies ADP and AN, as was to be proven. □