PARETIAN SOCIAL WELFARE RELATIONS AND BAIRE PROPERTY

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Abstract. We study the topological and set-theoretical nature of Paretian social welfare relations in a setting with infinite time horizon. Specifically, we answer questions posed in Bowler et al. (2020) about the interplay between total welfare relations satisfying Pareto and anonymity principles with subsets of real numbers not satisfying the Baire property.

1. Introduction and preliminaries

In recent times, several papers have pointed out some interplay between theoretical economics and mathematical logic. More specifically, some connections have arisen between social welfare relations on infinite utility streams and descriptive set theory; such an interaction has gained the attention both from the economic side and from the set-theoretic side.
Notably, regarding the former, the following pioneering results are brought up: Lauwers (2010) proves that the existence of a total social welfare relation satisfying Pareto and anonymity implies the existence of a non-Ramsey set; Zame (2007) proves that the existence of a total social welfare relation satisfying Pareto and anonymity implies the existence of a non-Lebesgue measurable set.
From the set-theoretic side, we refer to Bowler et al. (2020), where the authors investigate particular types of non-Ramsey sets and prove some connections with social welfare relations on infinite utility streams. Moreover, the authors also pose as an open question to investigate whether this interplay with Pareto welfare relations also involves non-Baire sets (Problem 11.15).
Laguzzi (2020) deals with Bowler et al. (2020, Problem 11.15) in the specific case of strong Pareto principle. The proof's method used there is based on some application of Kuratowski-Ulam theorem. Going into the details of that proof reveals that such a method, or some other variant, cannot work in case we weaken the strong Pareto principle to infinite Pareto and weak Pareto (see the technical definitions below).
In this paper we investigate the latter two cases and we introduce a different technique for getting non-Baire sets, relying on a new variant of Mathias and Silver forcing; these results are stated and proved as Theorem 1 and Theorem 2, and they constitute the main scope of this paper, expanding the answer to the open question Bowler et al. (2020, Problem 11.15).
In addition to that, we also briefly discuss, in the concluding section, how to analyze certain combinations of efficiency and equity principles in terms of fragments of AC in a similar fashion as developed on the set-theoretic side for the regularity properties of subsets of real numbers.

We now introduce the basic technical notions. Let Y, a non-empty subset of $\mathbb{R}$, be the set of all possible utilities that any generation can achieve. Then $X \equiv Y^\omega$ is the set of all possible utility streams, with an element $x \in X$ denoted by $x = \langle x(n) : n \in \omega \rangle$. For all y,

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$z \in X$, we write $y \geq z$ if $\forall n \in \omega(y(n) \geq z(n))$; $y > z$ if $y \geq z$ and $y \neq z$; and $y \gg z$ if $\forall n \in \omega(y(n) > z(n))$.
Let $Y^{<\omega}$ be the set of finite sequences of elements from Y. Given $\sigma \in Y^{<\omega}$, the length of σ is denoted by $|\sigma|$. Given σ, $\tau \in Y^{<\omega}$, we write $\sigma \subseteq \tau$ if and only if $|\sigma| \leq |\tau|$ and for all $n < |\sigma|$, $\sigma(n) = \tau(n)$. Analogously, in case $\sigma \in Y^{<\omega}$ and $x \in X$, we write $\sigma \subseteq x$ if and only if $\forall n < |\sigma|$, $\sigma(n) = x(n)$.
A *social welfare relation* (SWR) is a reflexive and transitive binary relation (i.e. a pre-order) on infinite utility streams. We consider binary relations on X, denoted by $\sqsubseteq$, with symmetric and asymmetric parts denoted by $\sim$ and $\sqsubset$ respectively, defined in the usual way. We recall the definition of permutation $\mathcal{P}$, a self map on ω:

$$\mathcal{P} := \{\pi : \omega \to \omega \mid \pi \text{ is a bijection}\}.$$

We denote the set of finite permutations by $\mathcal{F}$, i.e.,

$$\mathcal{F} := \{\pi \in \mathcal{P} \mid \pi(n) = n \text{ for all but finitely many } n \in \omega\}.$$

1.1. **Equity and Pareto principles.** The social welfare relations that we will be concerned with are required to satisfy following equity and Pareto principles.

Definition. Anonymity (AN): If $x, y \in X$, and there exist $i, j \in \omega$ such that $y(j) = x(i)$ and $x(j) = y(i)$, while $y(k) = x(k)$ for all $k \in \omega \setminus \{i, j\}$, then $x \sim y$.

Definition. Strong Pareto (SP): Given $x, y \in X$, if $x \geq y$ and $x(i) > y(i)$ for some $i \in \omega$, then $y \sqsubset x$.

Definition. Infinite Pareto (IP): Given $x, y \in X$, if $x \geq y$ and $x(i) > y(i)$ for infinitely many $i \in \omega$, then $y \sqsubset x$.

Definition. Weak Pareto (WP): Given $x, y \in X$, if $x(i) > y(i)$ for all $i \in \omega$, then $y \sqsubset x$.

1.2. **Mathias-Silver trees.** We recall the standard basic notions and notation about tree-forcings. Let $Y := \{0, 1\}$ or $Y := \omega$. A subset $T \subseteq Y^{<\omega}$ is called a *tree* if and only if for every $t \in T$ every $s \subseteq t$ is in T too, in other words, T is *closed* under initial segments. We call the segments $t \in T$ the *nodes* of T and denote the *length* of the node by $|t|$. A node $t \in T$ is called *splitting* if there are two distinct n, $m \in Y$ such that $t^\frown n$, $t^\frown m \in T$. Given $x \in X \equiv Y^\omega$ and $n \in \omega$, we denote by $x{\restriction}n$ the cut of x of length n, i.e., $x{\restriction}n := \langle x(0), x(1), \cdots, x(n-1)\rangle$. A tree $p \subseteq 2^{<\omega}$ is called *perfect* if and only if for every $s \in p$ there exists $t \supseteq s$ splitting. We define $[p] := \{x \in X : \forall n \in \omega(x{\restriction}n \in p)\}$, and $x \in [p]$ is called a *branch of* p.
A tree $p \subseteq 2^{<\omega}$ is called *Silver* tree if and only if p is perfect and for every s, $t \in p$, with $|s| = |t|$ one has $s^\frown 0 \in p \Leftrightarrow t^\frown 0 \in p$ and $s^\frown 1 \in p \Leftrightarrow t^\frown 1 \in p$. If t is a splitting node of p, we call $|t| + 1$ a splitting level of p and let $S(p)$ denote the set of splitting levels of p. Then set $U(p) := \{n \in \omega : \forall x \in [p](x(n) = 1)\}$ and let $\{n^p_k : k \in \omega\}$ enumerate the set $S(p) \cup U(p)$.
We could also define a Silver tree p and its corresponding set of branches $[p]$ relying on the notion of partial functions. Consider a partial function $f : \omega \to \{0, 1\}$ such that $\mathrm{dom}(f)$ is co-infinite (i.e. the complement of the domain of f is infinite); then define $N_f := \{x \in 2^\omega : \forall n \in \mathrm{dom}(f)(f(n) = x(n))\}$. It easily follows from the definitions that there is a one-to-one correspondence between every Silver tree p and a set N_f. Given any Silver tree p there is a unique partial function $f : \omega \to \{0, 1\}$ such that $[p] = N_f$. In particular, the set of splitting levels $S(p)$ correspond to $\omega \setminus \mathrm{dom}(f)$. Silver trees are extensively studied in the literature, as well as their topological properties (e.g., see Halbeisen (2003) and Brendle et al. (2005).)
We now introduce a variant of Silver trees which perfectly serves for our purpose.

Definition. Let $p \subseteq 2^{<\omega}$ be a Silver tree with $\{n^p_k : k \geq 1\}$ enumeration of $S(p) \cup U(p)$; p is called a *Mathias-Silver* tree ($p \in \mathbb{MV}$) if and only if there are infinitely many triples $(n^p_{m_j}, n^p_{m_j+1}, n^p_{m_j+2})$'s such that:

(1) for all $j \geq 1$, m_j is even;
(2) for all $j \geq 1$, $n^p_{m_j}, n^p_{m_j+1}, n^p_{m_j+2}$ are in $S(p)$ with $n^p_{m_j} + 1 < n^p_{m_j+1}$ and $n^p_{m_j+1} + 1 < n^p_{m_j+2}$;
(3) for all $j \geq 1$, $t \in p$, $i < |t|$ $(n^p_{m_j} < i < n^p_{m_j+1} \vee n^p_{m_j+1} < i < n^p_{m_j+2} \Rightarrow t(i) = 0)$.

We call $(n^p_{m_j}, n^p_{m_j+1}, n^p_{m_j+2})$ satisfying (1), (2) and (3) a *Mathias triple*.

Remark 1. The idea is that a Mathias-Silver tree is a special instance of a Silver tree that mimics infinitely often the feature of a Mathias tree, which is that in between the splitting levels occuring in any Mathias triple $(n^p_{m_j}, n^p_{m_j+1}, n^p_{m_j+2})$ all nodes of the tree p take value 0. In the proof of Theorems 1 and 2 this property will be crucial, and indeed it is not clear how to obtain, if possible, similar results working with Silver trees instead of Mathias-Silver trees.

Definition. A set $X \subseteq 2^\omega$ is called *Mathias-Silver measurable set* (or $\mathbb{MV}$-measurable set) if and only if there exists $p \in \mathbb{MV}$ such that $[p] \subseteq X$ or $[p] \cap X = \emptyset$. A set $X \subseteq 2^\omega$ not satisfying this condition is called a *non-*$\mathbb{MV}$*-measurable set*.

The following lemma is the key step to prove that any set satisfying the Baire property is $\mathbb{MV}$-measurable, or in other words, that a non-$\mathbb{MV}$-measurable set is a particular instance of a non-Baire set. The proof is a variant of the construction developed in Halbeisen (2003) for standard Silver trees.

Lemma 1. *Given any comeager set $C \subseteq 2^\omega$ there exists $p \in \mathbb{MV}$ such that $[p] \subseteq C$.*

Proof. Let $\{D_n : n \in \omega\}$ be a $\subseteq$-decreasing sequence of open dense sets such that $\bigcap_{n \in \omega} D_n \subseteq C$. Given $s \in 2^{<\omega}$, put $N_s := \{x \in 2^\omega : x \supset s\}$. Recall that if D is open dense, then $\forall s \in 2^{<\omega}$ there exists $s' \supseteq s$ such that $N_{s'} \subseteq D$. We construct $p \in \mathbb{MV}$ by recursively building up its nodes as follows: first of all let

$$s_1 = (10000), s_2 = (10001), s_3 = (10100), s_4 = (10101),$$
$$s_5 = (00000), s_6 = (00001), s_7 = (00100), s_8 = (00101).$$

- Pick $t_\emptyset \in 2^{<\omega}$ such that $N_{t_\emptyset} \subseteq D_0$, and then let $F_0 := \bigcup_{k=1}^{8} \left\{ t_\emptyset^\frown s_k \right\}$ and T_0 be the downward closure of F_0, i.e., $T_0 := \{s \in 2^{<\omega} : \exists t \in F_0 (s \subseteq t)\}$;
- Assume F_n is already defined. Let $\{t_j : j \leq J\}$ enumerate all nodes in F_n (note by construction $J = 8^{n+1}$). We proceed inductively as follows: pick $r_0 \in 2^{<\omega}$ such that $N_{t_0^\frown r_0} \subseteq D_{n+1}$; then pick $r_1 \supseteq r_0$ such that $N_{t_1^\frown r_1} \subseteq D_{n+1}$; proceed inductively in this way for every $j \leq J$, so $r_j \supseteq r_{j-1}$ such that $N_{t_j^\frown r_j} \subseteq D_{n+1}$. Finally put $r = r_J$. Then define
$$F_{n+1} := \bigcup \left\{ t^\frown r^\frown s_k : t \in F_n, k = 1, 2, \dots 8 \right\}$$
and
$$T_{n+1} := \{s \in 2^{<\omega} : \exists t \in F_{n+1} (s \subseteq t)\}.$$

Note that by construction, for all $t \in F_{n+1}$ we have $N_t \subseteq D_{n+1}$. Finally put $p := \bigcup_{n \in \omega} T_n$. Then by construction $p \in \mathbb{MV}$ as it is a Silver tree and the use of $s_1, s_2, \cdots, s_8$ ensures that

p contains infinitely many Mathias triples, and so $p \in \mathbb{MV}$. It is left to show $[p] \subseteq \bigcap_{n\in\omega} D_n$. For this, fix arbitrarily $x \in [p]$ and $n \in \omega$. By construction there is $t \in F_n$ such that $t \subset x$. Since $N_t \subseteq D_n$ we then get $x \in N_t \subseteq D_n$. $\square$

Corollary 1. *If $A \subseteq 2^\omega$ satisfies the Baire property, then A is a $\mathbb{MV}$-measurable set.*

Proof. The proof is a simple application of Lemma 1 and the fact that any set satisfying the Baire property is either meager or comeager relative to some basic open set N_t. Indeed, if A is meager, then we apply Lemma 1 to the complement of A and find $p \in \mathbb{MV}$ such that $[p] \cap A = \emptyset$. If there exists $t \in 2^{<\omega}$ such that A is comeager in N_t, then we can use the construction as in Lemma 1 in order to find $p \in \mathbb{MV}$ such that $[p] \subseteq A$, simply by choosing $t_\emptyset \supseteq t$, $t_\emptyset \in D_0$ and then use the same construction as in the proof of Lemma 1. $\square$

2. Pareto, Anonymity and Mathias-Silver trees

In this section we prove our two main results. The social welfare relations on $X = 2^\omega$ satisfying infinite Pareto and anonymity are considered in sub-section 2.1. The social welfare relations on $X = \mathbb{Z}^\omega$ [1] satisfying weak Pareto and anonymity are considered in sub-section 2.2. Note that when dealing with weak Pareto, if Y is well-founded, then one can simply consider the function $f : Y^\omega \to \mathbb{R}$ such that $f(x) := \min\{x(n) : n \in \omega\}$; then define $x \sqsubset y :\Leftrightarrow f(x) < f(y)$ and $x \sim y :\Leftrightarrow f(x) = f(y)$ in order to get a total SWR on Y^ω satisfying WP and AN.

2.1. Infinite Pareto and Anonymity.

Given $x \in 2^\omega$, let $U(x) := \{n \in \omega : x(n) = 1\}$ and $\{n^x_k : k \geq 1\}$ enumerate the numbers in $U(x)$. Define

$$\begin{aligned} o(x) &:= [n^x_1, n^x_2) \cup [n^x_3, n^x_4) \cdots [n^x_{2j+1}, n^x_{2j+2}) \cup \cdots \\ e(x) &:= [n^x_2, n^x_3) \cup [n^x_4, n^x_5) \cdots [n^x_{2j+2}, n^x_{2j+3}) \cup \cdots \end{aligned} \tag{1}$$

As usual we identify subsets of ω with their characteristic functions, so that we can write $o(x)$, $e(x) \in 2^\omega$.

Theorem 1. *Let $\sqsubseteq$ denote a total SWR satisfying IP and AN on $X = 2^\omega$. Then there exists a subset of X which is not $\mathbb{MV}$-measurable.*

Proof. Let $\Gamma := \{x \in 2^\omega : e(x) \sqsubset o(x)\}$. We show Γ is not $\mathbb{MV}$-measurable. Given any $p \in \mathbb{MV}$, let $\{n_k : k \geq 1\}$ enumerate all natural numbers in $S(p) \cup U(p)$ (note that in the enumeration of the n_k's we drop the index p for making the notation less cumbersome, since the tree p we refer to is fixed). To prove our claim, we aim to find x, $z \in [p]$ such that $x \in \Gamma \Leftrightarrow z \notin \Gamma$. We pick $x \in [p]$ such that for all $n_k \in S(p) \cup U(p)$, $x(n_k) = 1$, i.e. for every $k \geq 1$, $n^x_k = n_k$. Let $\left\{\left(n_{m_j}, n_{m_j+1}, n_{m_j+2}\right) : j \geq 1\right\}$ be an enumeration of all Mathias triples in p. We need to consider three cases.

- Case $e(x) \sqsubset o(x)$: We remove n_{m_1+1}, n_{m_j}, n_{m_j+1}, for all $j > 1$ from $U(x)$ to obtain $z \in 2^\omega$ as follows:

$$z(n) := \begin{cases} x(n) & \text{if } n \notin \left\{n_{m_1+1}, n_{m_j}, n_{m_j+1} : j > 1\right\} \\ 0 & \text{otherwise.} \end{cases}$$

[1] In sub-section 2.2, we are going to give a proof with $Y = \mathbb{Z}$, but it will be clear that the argument can be easily generalized in order to work for any set of utilities $Y \subseteq \mathbb{R}$ containing a subset with order type $\mathbb{Z}$.

Note that $z \in [p]$, since $n_{m_1+1}, n_{m_j}, n_{m_j+1}$ are all in $S(p)$. Let

$$\begin{aligned} O(m_1) :=& [n_1, n_2) \cup [n_3, n_4) \cdots [n_{m_1-1}, n_{m_1}), \\ E(m_1) :=& [n_2, n_3) \cup [n_4, n_5) \cdots [n_{m_1}, n_{m_1+1}). \end{aligned}$$

Observe that

- for all $n \in O(m_1)$, $e(z)(n) = 0 < 1 = o(x)(n)$ and $o(z)(n) = 1 > 0 = e(x)(n)$,
- for all $n \in E(m_1)$, $e(z)(n) = 1 > 0 = o(x)(n)$ and $o(z)(n) = 0 < 1 = e(x)(n)$,
- for all $n \in \bigcup_{j>1} [n_{m_j}, n_{m_j+1})$, $e(z)(n) = 1 > 0 = o(x)(n)$ and $o(z)(n) = 0 < 1 = e(x)(n)$, and
- for all remaining $n \in \omega$, $e(z)(n) = o(x)(n)$ and $o(z)(n) = e(x)(n)$.

Let $\{k_1, k_2, \cdots, k_M\}$ enumerate the elements in $O(m_1)$, and let $\{k^1, \cdots, k^M\}$ enumerate the initial M elements of the infinite set $\bigcup_{j>1} [n_{m_j}, n_{m_j+1})$. We permute $e(z)(k_1)$ with $e(z)(k^1)$, $e(z)(k_2)$ with $e(z)(k^2)$, continuing likewise till $e(z)(k_M)$ with $e(z)(k^M)$ to obtain $e^\pi(z)$. Further, $o^\pi(z)$ is obtained by carrying out identical permutation on $o(z)$. Observe that $e^\pi(z)$ and $o^\pi(z)$ are finite permutations of $e(z)$ and $o(z)$ respectively. Then,

- for all $n \in O(m_1)$, $e^\pi(z)(n) = 1 = o(x)(n)$ and $o^\pi(z)(n) = 0 = e(x)(n)$,
- for all $n \in E(m_1)$, $e^\pi(z)(n) = 1 > 0 = o(x)(n)$ and $o^\pi(z)(n) = 0 < 1 = e(x)(n)$,
- for all $n \in \bigcup_{j>1} [n_{m_j}, n_{m_j+1}) \setminus \{k^1, \cdots, k^M\}$, $e^\pi(z)(n) = 1 > 0 = o(x)(n)$ and $o^\pi(z)(n) = 0 < 1 = e(x)(n)$,
- for $n \in \{k^1, \cdots, k^M\}$, $e^\pi(z)(n) = 0 = o(x)(n)$ and $o^\pi(z)(n) = 1 = e(x)(n)$, and
- for all remaining $n \in \omega$, $e^\pi(z)(n) = o(x)(n)$ and $o^\pi(z)(n) = e(x)(n)$.

Observe that AN implies

$$e^\pi(z) \sim e(z) \text{ and } o^\pi(z) \sim o(z). \tag{2}$$

Further, applying IP, we get

$$o(x) \sqsubset e^\pi(z) \text{ and } o^\pi(z) \sqsubset e(x). \tag{3}$$

Combining (2) and (3) and transitivity, we get $o(z) \sim o^\pi(z) \sqsubset e(x) \sqsubset o(x) \sqsubset e^\pi(z) \sim e(z) \to o(z) \sqsubset e(z)$, which implies $z \notin \Gamma$.

- Case $o(x) \sqsubset e(x)$: the argument is similar to the above case and we just need to arrange the details accordingly. We remove n_{m_1}, n_{m_j+1}, n_{m_j+2}, for all $j > 1$ from $U(x)$ to obtain $z \in 2^\omega$ as follows:

$$z(n) := \begin{cases} x(n) & \text{if } n \notin \left\{ n_{m_1}, n_{m_j+1}, n_{m_j+2} : j > 1 \right\} \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$\begin{aligned} O(m_1) :=& [n_1, n_2) \cup [n_3, n_4) \cdots [n_{m_1-1}, n_{m_1}), \\ E(m_1) :=& [n_2, n_3) \cup [n_4, n_5) \cdots [n_{m_1-2}, n_{m_1-1}). \end{aligned}$$

(In case $m_1 = 2$ put $E(m_1) = \emptyset$.)

Then,

- for all $n \in O(m_1)$, $e(z)(n) = 0 < 1 = o(x)(n)$ and $o(z)(n) = 1 > 0 = e(x)(n)$,
- for all $n \in E(m_1)$, $e(z)(n) = 1 > 0 = o(x)(n)$ and $o(z)(n) = 0 < 1 = e(x)(n)$,

- for all $n \in \bigcup_{j>1} [n_{m_j+1}, n_{m_j+2})$, $e(z)(n) = 0 < 1 = o(x)(n)$ and $o(z)(n) = 1 > 0 = e(x)(n)$, and
- for all remaining $n \in \omega$, $e(z)(n) = o(x)(n)$ and $o(z)(n) = e(x)(n)$.

Let $\{k_1, k_2, \cdots, k_M\}$ enumerate the elements in $E(m_1)$, and let $\{k^1, \cdots, k^M\}$ enumerate the initial M elements of the infinite set $\bigcup_{j>1} [n_{m_j+1}, n_{m_j+2})$. We permute $e(z)(k_1)$ with $e(z)(k^1)$, $e(z)(k_2)$ with $e(z)(k^2)$, continuing likewise till $e(z)(k_M)$ with $e(z)(k^M)$ to obtain $e^\pi(z)$. Further, $o^\pi(z)$ is obtained by carrying out identical permutation on $o(z)$. Observe that $e^\pi(z)$ and $o^\pi(z)$ are finite permutations of $e(z)$ and $o(z)$ respectively. Then,

- for all $n \in E(m_1)$, $e^\pi(z)(n) = 0 = o(x)(n)$ and $o^\pi(z)(n) = 1 = e(x)(n)$,
- for all $n \in O(m_1)$, $e^\pi(z)(n) = 0 < 1 = o(x)(n)$ and $o^\pi(z)(n) = 1 > 0 = e(x)(n)$,
- for all $n \in \bigcup_{j>1} [n_{m_j+1}, n_{m_j+2}) \setminus \{k^1, \cdots, k^M\}$, $e^\pi(z)(n) = 0 < 1 = o(x)(n)$ and $o^\pi(z)(n) = 1 > 0 = e(x)(n)$,
- for all $n \in \{k^1, \cdots, k^M\}$, $e^\pi(z)(n) = 1 = o(x)(n)$ and $o^\pi(z)(n) = 0 = e(x)(n)$, and
- for all remaining $n \in \omega$, $e^\pi(z)(n) = o(x)(n)$ and $o^\pi(z)(n) = e(x)(n)$.

Observe that AN implies $e^\pi(z) \sim e(z)$ and $o^\pi(z) \sim o(z)$ and IP gives $e^\pi(z) \sqsubset o(x)$ and $e(x) \sqsubset o^\pi(z)$, and combining them we obtain $e(z) \sim e^\pi(z) \sqsubset o(x) \sqsubset e(x) \sqsubset o^\pi(z) \sim o(z) \to e(z) \sqsubset o(z)$, which implies $z \in \Gamma$.

- Case $e(x) \sim o(x)$: We remove n_{m_j}, n_{m_j+1}, for all $j > 1$ from $U(x)$ to obtain $z \in 2^\omega$ as follows:

$$z(n) = \begin{cases} x(n) & \text{if } n \notin \left\{n_{m_j}, n_{m_j+1} : j > 1\right\} \\ 0 & \text{otherwise.} \end{cases}$$

By construction we obtain $o(z)(n) \geq o(x)(n)$ and $e(z)(n) \leq e(x)(n)$ for all $n \in \omega$. Further, for all $n \in \bigcup_{j\in\omega} \left[n_{m_j}, n_{m_j+1}\right)$, $o(z)(n) = 1 > 0 = o(x)(n)$ and $e(z)(n) = 0 < 1 = e(x)(n)$. Hence, by IP, we get $o(x) \sqsubset o(z)$ and $e(z) \sqsubset e(x)$, and so using transitivity we get $e(z) \sqsubset o(z)$, which implies $z \in \Gamma$.

□

2.2. **Weak Pareto and Anonymity.** Given $x \in 2^\omega$, let $U(x) := \{n \in \omega : x(n) = 1\}$ and $\{n^x_k : k \in \omega\}$ enumerate $U(x)$. As in the case of Theorem 1, define $o(x)$ and $e(x)$; next use the following notation:

- let $\{l_k : k \geq 1\}$ enumerate all elements in $o(x)$ and $\{u_k : k \geq 1\}$ enumerate all elements in $\omega \setminus o(x)$;
- let $\{l'_k : k \geq 1\}$ enumerate all elements in $e(x)$ and $\{u'_k : k \geq 1\}$ enumerate all elements in $\omega \setminus e(x)$;

Note that for every $k \geq 1$, one has $l'_k = u_{n_1+(k-1)}$ and $l_k = u'_{n_1+(k-1)}$. Next we define the following pair of sequences $o(\mathbf{x}), e(\mathbf{x})$ in $\mathbb{Z}^\omega$:

$$o(\mathbf{x})(n) = \begin{cases} k & \text{if } n = l_k, \text{ for some } k \geq 1 \\ -k & \text{if } n = u_k, \text{ for some } k \geq 1, \end{cases} \tag{4}$$

$$e(\mathbf{x})(n) = \begin{cases} k & \text{if } n = l'_k, \text{ for some } k \geq 1 \\ -k & \text{if } n = u'_k, \text{ for some } k \geq 1. \end{cases} \tag{5}$$

Theorem 2. *Let $\sqsubseteq$ denote a total SWR satisfying WP and AN on $X = \mathbb{Z}^\omega$. Then there exists a subset of 2^ω which is not $\mathbb{MV}$-measurable.*[2]

Proof. The structure of the proof is similar to Theorem 1, but the technical details are different. Let $\sqsubseteq$ be a total SWR satisfying WP and AN, and put $\Gamma := \{x \in 2^\omega : e(\mathbf{x}) \sqsubset o(\mathbf{x})\}$. Given any $p \in \mathbb{MV}$, let $\{n_k : k \geq 1\}$ enumerate all natural numbers in $S(p) \cup U(p)$. We aim to find x, $z \in [p]$ such that $x \in \Gamma \Leftrightarrow z \notin \Gamma$. We proceed as follows: pick $x \in [p]$ such that for all $n_k \in S(p) \cup U(p)$, $x(n_k) = 1$. Let $\left\{\left(n_{m_j}, n_{m_j+1}, n_{m_j+2}\right) : j \in \omega\right\}$ be an enumeration of all Mathias triples in p. We need to consider three cases.

- Case $e(\mathbf{x}) \sqsubset o(\mathbf{x})$: We remove n_{m_1+1}, n_{m_j}, n_{m_j+1}, for all $j > 1$ from $U(x)$ to obtain $z \in 2^\omega$ as follows:

$$z(n) = \begin{cases} x(n) & \text{if } n \notin \left\{n_{m_1+1}, n_{m_j}, n_{m_j+1} : j > 1\right\} \\ 0 & \text{otherwise.} \end{cases}$$

  Let

$$\begin{aligned} O(m_1) :=& [n_1, n_2) \cup [n_3, n_4) \cdots [n_{m_1-1}, n_{m_1}), \\ E(m_1) :=& [n_2, n_3) \cup [n_4, n_5) \cdots [n_{m_1}, n_{m_1+1}). \end{aligned}$$

  Then,
  - for all $n \in O(m_1)$, $e(\mathbf{z})(n) < 0 < o(\mathbf{x})(n)$ and $o(\mathbf{z})(n) > 0 > e(\mathbf{x})(n)$,
  - for all $n \in [0, n_1)$, $0 > e(\mathbf{z})(n) = o(\mathbf{x})(n)$ and $0 > o(\mathbf{z})(n) = e(\mathbf{x})(n)$,
  - for all $n \in E(m_1)$, $o(\mathbf{x})(n) < 0 < e(\mathbf{z})(n)$ and $o(\mathbf{z})(n) < 0 < e(\mathbf{x})(n)$,
  - for all $n \geq n_{m_1+1}$ if $0 > e(\mathbf{z})(n)$ then $0 > o(\mathbf{x})(n)$ holds. Also if $o(\mathbf{x})(n) > 0$ then $e(\mathbf{z})(n) > 0$. Similarly, if $0 > e(\mathbf{x})(n)$ then $0 > o(\mathbf{z})(n)$ holds. Also if $o(\mathbf{z})(n) > 0$ then $e(\mathbf{x})(n) > 0$.
  - For all $n \in \bigcup\limits_{j>1} \left[n_{m_j}, n_{m_j+1}\right)$, $o(\mathbf{x})(n) < 0 < e(\mathbf{z})(n)$ and $o(\mathbf{z})(n) < 0 < e(\mathbf{x})(n)$. These are infinitely many elements of the sequence.

  Claim 1. There exists $N \in \omega$ such that $e(\mathbf{x})(n) > o(\mathbf{z})(n)$ holds for all $n > N$.

  Proof. For all $n < n_{m_1+1}$, (a) $e(\mathbf{x})(n)$ contains positive elements at coordinates in $E(m_1)$ and negative elements at coordinates in $O(m_1) \cup [0, n_1)$; and (b) $o(\mathbf{z})(n)$ contains positive elements at coordinates in $O(m_1)$ and negative elements at coordinates in $E(m_1) \cup [0, n_1)$. There are two cases.

  (1) $|O(m_1)| < |E(m_1)|$: Among coordinates $n < n_{m_1+1}$,
    – fewer negative integers have been assigned in $e(\mathbf{x})(n)$ as compared to $o(\mathbf{z})(n)$. Then $0 > e(\mathbf{x})(n_{m_1+1}) > o(\mathbf{z})(n_{m_1+1})$ and for all subsequent coordinates n with both $e(\mathbf{x})(n)$ and $o(\mathbf{z})(n)$ being negative, $0 > e(\mathbf{x})(n) > o(\mathbf{z})(n)$ holds.
    – fewer positive integers have been assigned in $o(\mathbf{z})(n)$ as compared to $e(\mathbf{x})(n)$. Then $e(\mathbf{x})(n_{m_1+2}) > o(\mathbf{z})(n_{m_1+2}) > 0$ and for all subsequent coordinates n with both $e(\mathbf{x})(n)$ and $o(\mathbf{z})(n)$ being positive, $e(\mathbf{x})(n) > o(\mathbf{z})(n) > 0$ holds.

    We take $N = n_{m_1+1}$ in this case.

[2] It is clear from the proof that one could get the same result in an even slightly more general setting, namely with Y any set of utilities with order type $\mathbb{Z}$.

(2) $|O(m_1)| \geq |E(m_1)|$: Among the coordinates $[n_{m_j+1}, n_{m_{j+1}})$ for all $j \in \omega$, $e(\mathbf{x})(n)$ and $o(\mathbf{z})(n)$ contain equally many elements of same sign. Further for the coordinates in $[n_{m_j}, n_{m_j+1})$, $o(\mathbf{z})(n)$ is negative but $e(\mathbf{x})(n)$ is not. Thus for some finite $J \in \omega$,

$$|O(m_1)| < |E(m_1)| + \left| \bigcup_{j \in \{2, \cdots, J\}} \left[n_{m_j}, n_{m_j+1}\right) \right|$$

will be true. In this case, we can apply argument of case (i) above for n_{m_J+1} and therefore obtain $N = n_{m_J+1}$.

Thus we have shown that for all $n > N$, if $e(\mathbf{x})(n)$ and $o(\mathbf{z})(n)$ share the same sign then $e(\mathbf{x})(n) > o(\mathbf{z})(n)$. The remaining situation is $e(\mathbf{x})(n) > 0 > o(\mathbf{z})(n)$. This completes the proof. □

Claim 2. There exists a finite permutation $o^\pi(\mathbf{z})$ of $o(\mathbf{z})$ such that $e(\mathbf{x})(n) > o^\pi(\mathbf{z})(n)$ holds for all $n \in \omega$.

Proof. In claim 1, it has been shown that for all $n > N$ $e(\mathbf{x})(n) > o(\mathbf{z})(n)$. Let $K := \{k^0, k^1, \cdots, k^N\}$ be an increasing enumeration of all elements from the set $\bigcup_{j>J} \left[n_{m_j}, n_{m_j+1}\right)$. We permute $o(\mathbf{z})(0)$ and $o(\mathbf{z})(k^0)$; $o(\mathbf{z})(1)$ and $o(\mathbf{z})(k^1)$ and so on till $o(\mathbf{z})(N)$ and $o(\mathbf{z})(k^N)$ to obtain $o^\pi(\mathbf{z})$. Hence, $o^\pi(\mathbf{z})$ is obtained via a finite permutation of $o(\mathbf{z})$. Observe that if $0 > e(\mathbf{x})(n) \geq o(\mathbf{z})(n)$, then

$$e(\mathbf{x})(n) > o(\mathbf{z})(k^n) = o^\pi(\mathbf{z})(n), \text{ and } e(\mathbf{x})(k^n) > 0 > o(\mathbf{z})(n) = o^\pi(\mathbf{z})(k^n).$$

If $0 > o(\mathbf{z})(n) \geq e(\mathbf{x})(n)$, then

$$e(\mathbf{x})(n) > o(\mathbf{z})(k^n) = o^\pi(\mathbf{z})(n), \text{ and } e(\mathbf{x})(k^n) > 0 > o(\mathbf{z})(n) = o^\pi(\mathbf{z})(k^n).$$

If $e(\mathbf{x})(n) < 0 < o(\mathbf{z})(n)$, then

$$0 > e(\mathbf{x})(n) > o(\mathbf{z})(k^n) = o^\pi(\mathbf{z})(n), \text{ and } e(\mathbf{x})(k^n) > o(\mathbf{z})(n) = o^\pi(\mathbf{z})(k^n) > 0.$$

If $e(\mathbf{x})(n) > o(\mathbf{z})(n) > 0$, then

$$e(\mathbf{x})(n) > 0 > o(\mathbf{z})(k^n) = o^\pi(\mathbf{z})(n), \text{ and } e(\mathbf{x})(k^n) > o(\mathbf{z})(n) = o^\pi(\mathbf{z})(k^n) > 0.$$

□

Applying Claims 1 and 2, we have obtained $o^\pi(\mathbf{z})$ such that $e(\mathbf{x})(n) > o^\pi(\mathbf{z})(n)$ for all $n \in \omega$. AN implies $o(\mathbf{z}) \sim o^\pi(\mathbf{z})$, and by WP we get $e(\mathbf{x}) \sqsupset o^\pi(\mathbf{z})$. By also applying transitivity, we thus obtain, $e(\mathbf{x}) \sqsupset o(\mathbf{z})$.

Notice that arguments of claims 1 and 2 could also be applied to the pair of sequences $e(\mathbf{z})$ and $o(\mathbf{x})$. Thus we are able to obtain $o^\pi(\mathbf{x})$ such that applying AN we get $o(\mathbf{x}) \sim o^\pi(\mathbf{x})$, by WP we get $o^\pi(\mathbf{x}) \sqsubset e(\mathbf{z})$, and finally, by transitivity it follows $o(\mathbf{x}) \sqsubset e(\mathbf{z})$. Combining all together we obtain:

$$o(\mathbf{z}) \sqsubset e(\mathbf{x}) \sqsubset o(\mathbf{x}) \sqsubset e(\mathbf{z}) \to o(\mathbf{z}) \sqsubset e(\mathbf{z}),$$

which implies $z \notin \Gamma$.

- Case $o(\mathbf{x}) \sqsubset e(\mathbf{x})$: Similar to the previous case, only with some different technical details. We remove n_{m_1}, n_{m_j+1}, n_{m_j+2}, for all $j > 1$ from $U(x)$ to obtain $z \in 2^\omega$ as follows:

$$z(n) = \begin{cases} x(n) & \text{if } n \notin \left\{n_{m_1}, n_{m_j+1}, n_{m_j+2} : j > 1\right\} \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$\begin{aligned} O(m_1) :=& [n_1, n_2) \cup [n_3, n_4) \cdots [n_{m_1-1}, n_{m_1}), \\ E(m_1) :=& [n_2, n_3) \cup [n_4, n_5) \cdots [n_{m_1-2}, n_{m_1-1}). \end{aligned}$$

(In case $m_1 = 2$ put $E(m_1) = \emptyset$.)

Then,
  - for all $n \in O(m_1)$, $e(\mathbf{z})(n) < 0 < o(\mathbf{x})(n)$ and $o(\mathbf{z})(n) > 0 > e(\mathbf{x})(n)$,
  - for all $n \in [1, n_1)$, $0 > e(\mathbf{z})(n) = o(\mathbf{x})(n)$ and $0 > o(\mathbf{z})(n) = e(\mathbf{x})(n)$,
  - for all $n \in E(m_1)$, $o(\mathbf{x})(n) < 0 < e(\mathbf{z})(n)$ and $o(\mathbf{z})(n) < 0 < e(\mathbf{x})(n)$,
  - for all $n \geq n_{m_1+1}$ if $0 > o(\mathbf{x})(n)$ then $0 > e(\mathbf{z})(n)$ holds. Also if $e(\mathbf{z})(n) > 0$ then $o(\mathbf{x})(n) > 0$. Similarly, if $0 > o(\mathbf{z})(n)$ then $0 > e(\mathbf{x})(n)$ holds. Also if $e(\mathbf{x})(n) > 0$ then $o(\mathbf{z})(n) > 0$.
  - For all $n \in \bigcup_{j>1} \left[n_{m_j+1}, n_{m_j+2}\right)$, $e(\mathbf{z})(n) < 0 < o(\mathbf{x})(n)$ and $e(\mathbf{x})(n) < 0 < o(\mathbf{z})(n)$.

    These are infinitely many elements of the sequence.

  Applying Claims 1 and 2, we are able to obtain $e^\pi(\mathbf{z})$ and $o^\pi(\mathbf{z})$ such that

$$o(\mathbf{x})(n) > e^\pi(\mathbf{z})(n), \text{ and } o^\pi(\mathbf{z})(n) > e(\mathbf{x})(n) \text{ for all } n \in \omega.$$

  AN implies $o(\mathbf{z}) \sim o^\pi(\mathbf{z})$, and $e(\mathbf{z}) \sim e^\pi(\mathbf{z})$, and by WP we get $e^\pi(\mathbf{z}) \sqsubset o(\mathbf{x})$, and $e(\mathbf{x}) \sqsubset o^\pi(\mathbf{z})$. Hence, by transitivity, it follows:

$$e(\mathbf{z}) \sqsubset o(\mathbf{x}), \text{ and } e(\mathbf{x}) \sqsubset o(\mathbf{z}),$$

  which leads to $z \in \Gamma$.

- Case $e(\mathbf{x}) \sim o(\mathbf{x})$: We remove n_{m_j}, n_{m_j+1}, for all $j > 1$ from $U(x)$ to obtain $z \in 2^\omega$ as follows:

$$z(n) = \begin{cases} x(n) & \text{if } n \notin \left\{n_{m_j}, n_{m_j+1} : j > 1\right\} \\ 0 & \text{otherwise.} \end{cases}$$

  By construction we obtain $o(\mathbf{z})(n) \geq o(\mathbf{x})(n)$ and $e(\mathbf{z})(n) \leq e(\mathbf{x})(n)$ for all $n \in \omega$. Further, for all $n > m_1$, $o(\mathbf{z})(n) > o(\mathbf{x})(n)$ and $e(\mathbf{z})(n) < e(\mathbf{x})(n)$. Applying a similar argument as in the proof of Claim 2, by permuting finitely many elements, we are able to obtain $e^\pi(\mathbf{z})$ and $o^\pi(\mathbf{z})$ such that

$$o(\mathbf{x})(n) < o^\pi(\mathbf{z})(n), \text{ and } e^\pi(\mathbf{z})(n) < e(\mathbf{x})(n), \text{ for all } n \in \omega.$$

  Again, AN implies $o(\mathbf{z}) \sim o^\pi(\mathbf{z})$, and $e(\mathbf{z}) \sim e^\pi(\mathbf{z})$, WP implies $o(\mathbf{x}) \sqsubset o^\pi(\mathbf{z})$, and $e^\pi(\mathbf{z}) \sqsubset e(\mathbf{x})$, and therefore, by transitivity, it follows:

$$e(\mathbf{z}) \sqsubset e(\mathbf{x}), \text{ and } o(\mathbf{x}) \sqsubset o(\mathbf{z}),$$

  which leads to $z \in \Gamma$.

$\square$

3. Concluding remarks

We conclude with some comments on the nature of equitable total SWRs. The non-constructive objects which have played a role so far in the context of SWRs on infinite utility streams are: free ultrafilters, non-Lebesgue measurable sets, non-Ramsey sets, non-Baire sets, and non-Mathias-Silver measurable sets (the latter being particular type of non-Baire sets). In Table 1 below, we summarize the non-constructive objects which emerge as a consequence of the existence of total social welfare relations satisfying anonymity and the Pareto principle mentioned in column (1). These objects could be distinguished on the basis

Table 1.

(1)	(2)	(3)
Axiom	Y	*Corresponding non-constructive set*
Strong Pareto	$\lvert Y\rvert \geq 2$	Non-Lebesgue (Zame (2007)), Non-Baire (Laguzzi (2020))
Infinite Pareto	$\lvert Y\rvert \geq 2$	Non-Ramsey (Lauwers (2010)), Non-Mathias-Silver (Theorem 1)
Weak Pareto	$Y = [0,1]$	Non-Lebesgue (Zame (2007))
Weak Pareto	$Y = \mathbb{Z}$	Non-Ramsey (Dubey (2011)), Non-Mathias-Silver (Theorem 2)

of the corresponding *fragments of AC* needed for existence.[3] For instance, let

$$\mathbf{U} := \text{ There exists a free ultrafilter on } \omega.$$
$$\mathbf{NL} := \text{ There exists a non-Lebesgue measurable set.}$$

On the one hand, well-known Vitali's result shows that $\mathbf{U} \Rightarrow \mathbf{NL}$, but on the other hand, as a consequence of Shelah (1985), one has $\mathbf{NL} \not\Rightarrow \mathbf{U}$. Hence $\mathbf{U}$ corresponds to a strictly larger/stronger fragment of AC than $\mathbf{NL}$. The following diagram shows such fragments of AC when combining AN and IP with set of utilities $Y = \{0,1\}$; moving bottom-up means moving from weaker to stronger fragment of AC.

$$\begin{tikzcd} & \mathbf{U} \arrow[d, dash] & \\ & \mathbf{ANIP} \arrow[dl, dash] \arrow[dr, dash] & \\ \mathbf{NB} & & \mathbf{NR} \end{tikzcd}$$

NB: there exists a non-Baire set
NR: there exists a non-Ramsey set
ANIP: there exists a total SWR satisfying AN and IP
U: there exists a free ultrafilter on ω

Note that we could get other similar diagrams when dealing with SWRs satisfying other combinations of efficiency and equity principles (even with larger utility domains), giving rise to other cases worthy of studying.

[3]Interested reader can consult the following selected list of papers: Brendle and Löwe (1999), Ikegami (2010), Khomskii (2012), and Laguzzi (2014).

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