Questions on Generalised Baire Spaces

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Abstract. We provide a list of open problems in the research area of generalised Baire spaces, compiled with the help of the participants of two workshops held in Amsterdam (2014) and Hamburg (2015).

1 Introduction

When studying questions about real numbers, it is common practice in set theory to investigate the closely related Baire space $\omega^\omega$ and Cantor space $2^\omega$. These spaces have been extensively studied by set theorists from various points of view, e.g., questions about cardinal characteristics of the continuum, descriptive set theory and other combinatorial questions. Furthermore, the investigation of $2^\omega$ and $\omega^\omega$ has played a role in model theory as well, since countable structures can be coded as elements in these spaces (e.g., Scott’s and Lopez-Escobar’s theorems). Various motivations from the different areas mentioned resulted in interest in systematically studying the uncountable analogues $2^\kappa$ and $\kappa^\kappa$. In recent years, this subject has developed in its own right, with internally motivated open questions and a rich overarching theory. Moreover, unexpected applications to other areas of set theory and mathematics have been discovered (e.g., connections to large cardinals and forcing axioms).

This survey paper is the output of two workshops on generalised Baire spaces, the first (Amsterdam Workshop on Set Theory 2014) held in Amsterdam in 2014 (3 & 4 November 2014), and the second (Hamburg Workshop on Set Theory 2015) in Hamburg in 2015 (20 & 21 September 2015). During both meetings, a group of set theorists met and presented some of the recent developments in this area. This compilation is based on the open questions raised in the talks and the discussions during these two workshops, and it aims to provide a structured overview of the current state of this field.

2 Background and preliminary notions

2.1 Basic definitions

Let $\kappa$ be an uncountable regular cardinal satisfying $\kappa^{<\kappa} = \kappa$.\footnote{Cf., e.g., [17, Section II.2.1.] for a discussion of the condition $\kappa^{<\kappa} = \kappa$.} We consider the spaces $\kappa^\kappa$ and $2^\kappa$ with the following topological structure:

**Definition 2.1.** The bounded topology on $\kappa^\kappa$ is the one generated by basic open sets of the form

$$[s] = \{ f \in \kappa^\kappa \mid f \upharpoonright |s| = s \}$$

with $s \in \kappa^{<\kappa}$. The bounded topology on $2^\kappa$ is defined analogously. We call $\kappa^\kappa$ and $2^\kappa$ with this topology the generalised Baire space and generalised Cantor space, respectively.

**Definition 2.2.** The family of Borel sets is defined as the smallest collection containing basic open sets and closed under complements and unions of size $\leq \kappa$. The projective hierarchy is defined in the same way as for the classical Baire/Cantor space.
We note that in contrast to the classical setting, the class of generalised Borel sets does not coincide with the class of universally Baire sets in $2^\kappa$ (due to Ikegami and Viale). Also, whenever we refer to topological notions, we always assume that $\kappa$ is a regular uncountable cardinal satisfying $\kappa^{< \kappa} = \kappa$. On the other hand, questions concerning cardinal characteristics or combinatorics are usually formulated for arbitrary $\kappa$.

**Definition 2.3.** A tree is a partially ordered set $(T, <)$ such that for all $t \in T$ the set $\{ s \in T \mid s < t \}$ is well-ordered. Terminology such as rank, height of a tree are standard.

**Remark 2.4.** As in classical descriptive set theory, it is often useful to consider “descriptive set theoretic” trees as subsets of $\kappa^{< \kappa}$ or $2^{< \kappa}$ closed under initial segments. For such trees, $[T]$ denotes the set of branches through $T$ (i.e., $x \in \kappa^{< \kappa}$ or $2^{< \kappa}$ such that for every $\alpha < \kappa$, $x \upharpoonright \alpha \in T$). It is clear that $[T]$ is a closed set and every closed set has the form $[T]$ for some tree $T$ in the above sense. We shall use trees in this sense and in the more general above sense interchangeably.

**Definition 2.5.** A $\kappa$-Kurepa tree is a tree $T$ such that
1. $\text{height}(T) = \kappa$,
2. $T$ has strictly more than $\kappa$ branches,
3. for each $\alpha < \kappa$, $\{ t \in T \mid \text{height}(t) = \alpha \} \leq |\alpha| + \aleph_0$.

**Definition 2.6.** An ideal $I \subseteq \wp(\kappa)$ is $\kappa$-complete iff for every $X \subseteq I$, $|X| < \kappa$, one has $\bigcup X \in I$. We put $I^+ := \wp(\kappa) \setminus I$.

In the following definitions, we always refer to $\kappa$-complete ideals.

**Definition 2.7.**
1. An ideal $I \subseteq \wp(\kappa)$ is called a weak $P$-point iff given $A \in I^+$ and $f \in \kappa^A$ with $\{ f^{-1}(\{ \alpha \}) : \alpha < \kappa \} \subseteq I$, there is $B \in I^+ \cap \wp(A)$ such that $f$ is $(< \kappa)$-to-one on $B$.
2. An ideal $I \subseteq \wp(\kappa)$ is called a local $Q$-point iff given $g \in \kappa^\kappa$ there is $B \in I^+$ such that for every $(\alpha, \beta) \in B \times B$ with $\alpha < \beta$ holds $g(\alpha) < \beta$. $I$ is a weak $Q$-point iff $I \upharpoonright A$ is a local $Q$-point for every $A \in I^+$.

**Definition 2.8.** Let $\kappa < \lambda$ be cardinals. An ideal $I \subseteq \wp([\lambda]^{< \kappa})$ is called a weak $\chi$-point iff given $A \in I^+$ and $g \in ([\lambda]^{< \kappa})^\kappa$, there exists $B \in I^+ \cap \wp(A)$ such that $g(\bigcup (a \cap \kappa)) \subseteq b$, for all $a, b \in B$ such that $\bigcup (a \cap \kappa) < \bigcup (b \cap \kappa)$.

**Definition 2.9.** Let $\mathbb{P}$ be a forcing partial order and $\lambda$ any cardinal. We say that $\mathbb{P}$ has the $\lambda$-c.c. iff every antichain has size $< \lambda$. We say that $\mathbb{P}$ is $< \lambda$-closed iff for every $\gamma < \lambda$ and every decreasing sequence $\langle p_\beta \mid \beta < \gamma \rangle$ there is $p \in \mathbb{P}$ with $p \leq p_\beta$ for all $\beta < \gamma$. We say that $\mathbb{P}$ is $\kappa^\kappa$-bounding iff for every condition $p \in \mathbb{P}$ and every $\mathbb{P}$-name $f$ for an element of $\kappa^\kappa$ in the generic extension, there is $q \leq p$ and a ground model $g \in \kappa^\kappa$ such that $q \not\Vdash f(\alpha) \leq g(\alpha)$ for all $\alpha$.\[ \]
2.2 Cardinal characteristics.

Classically, cardinal characteristics of the continuum have been studied extensively in recent decades, with the questions considered there being a primary motivation for the development of sophisticated forcing iteration and preservation techniques. We refer the reader to the expositions in [1] and [2] for a detailed overview. The former focuses mainly on the cardinal characteristics occurring in Cichoń’s diagram, i.e., those associated with the null and meager ideals, as well as the bounding number $b$ and the dominating number $d$. [2] presents many cardinal characteristics associated with combinatorial aspects of $\omega^\omega$, such as the splitting number $s$, the reaping number $r$, the ultrafilter number $u$, the tower number $t$ and the pseudointersection number $p$, which are equal by [44], the distributivity number $h$, the groupwise density number $g$, the almost disjointness number $a$, the independence number $i$ and the evasion number $e$.

All relations provable in ZFC are summarized in Cichoń’s diagram and the diagram of combinatorial cardinal characteristics (Figure 1).²

Definition 2.10. Let $\kappa$ be regular uncountable with $\kappa^{<\kappa} = \kappa$. A set $A$ in $\kappa^\kappa$ or $2^\kappa$ is nowhere dense if the interior of its closure is empty, and a set $A$ is $\kappa$-meager if it is a $\leq \kappa$-sized union of nowhere dense sets. The $\kappa$-ideal of $\kappa$-meager sets is denoted by $M_\kappa$. This lets us define $\text{cov}(M_\kappa)$, $\text{add}(M_\kappa)$, $\text{non}(M_\kappa)$ and $\text{cof}(M_\kappa)$ in the standard way.

Definition 2.11. Let $f, g \in \kappa^\kappa$. Then we define $f \leq^* g$ iff $\exists \alpha < \kappa \forall \beta > \alpha : f(\beta) \leq g(\beta)$ and say that $g$ dominates $f$. A family $B \subseteq \kappa^\kappa$ is called unbounded if for all $g \in \kappa^\kappa$ there is $f \in B$ such that $f \not\leq^* g$. A family $D \subseteq \kappa^\kappa$ is called dominating if for all $g \in \kappa^\kappa$ there is $f \in D$ such that $g \leq^* f$. The cardinals

$$
\begin{align*}
b(\kappa) &= \min\{|B| \mid B \subseteq \kappa^\kappa \text{ is an unbounded family}\} \\
d(\kappa) &= \min\{|D| \mid D \subseteq \kappa^\kappa \text{ is a dominating family}\}
\end{align*}
$$

are called the bounding number and the dominating number, respectively. We also define $\overline{d(\kappa)}$ as the least size of a family $D \subseteq \kappa^\kappa$ such that for every $g \in \kappa^\kappa$ there is $X \in [D]^{<\kappa}$ such that for all $\alpha < \kappa$, $g(\alpha) \in \bigcup_{f \in X} f(\alpha)$.

² In Cichoń’s diagram, the lines from left to right and from bottom to top represent $\leq$, provable in ZFC. Additionally the equalities $\text{add}(M_\kappa) = \min(b, \text{cov}(M_\kappa))$ and $\text{cof}(M_\kappa) = \max(\text{non}(M_\kappa), d)$ hold. In the combinatorial diagram, lines from bottom to top represent $\leq$. These two diagrams are complete in the sense that any implications missing from these diagrams are consistently false.
There is currently no agreement on the right generalisation of the Lebesgue-null ideal on the generalised Baire space: indeed, the search for a suitable generalisation is an important open problem—cf. Question 3.20. Nevertheless, one can instead consider generalisations of certain combinatorial statements which are equivalent to $\text{add}(\mathcal{N})$ and $\text{cof}(\mathcal{N})$ in the classical setting.

**Definition 2.12.**

1. A slalom is a function $F : \kappa \to [\kappa]^{<\kappa}$ such that $F(\alpha) \in [\kappa]|^{\alpha}$ for all $\alpha < \kappa$.

2. A partial slalom a partial function $F : \text{dom}(F) \subseteq \kappa \to [\kappa]^{<\kappa}$ such that $F(\alpha) \in [\kappa]|^{\alpha}$ for all $\alpha \in \text{dom}(F)$.

3. If $f \in \kappa^\kappa$ and $F$ a slalom, then $f \in^* F$ holds iff $\exists \beta \forall \alpha > \beta \ (f(\alpha) \in F(\alpha))$. If $F$ is a partial slalom, then $f \in^*_p F$ holds iff $\exists \beta \forall \alpha > \beta, \alpha \in \text{dom}(F) \ (f(\alpha) \in F(\alpha))$.

4. Then we can define

\[ b(\in^*(\kappa)) := \min \{|F| \mid \text{for all slaloms } F \text{ there is an } f \in F \text{ such that } f \notin^* F\}, \]

and analogously $\delta(\in^*_p(\kappa))$ and $\delta(\in^*(\kappa))$.

**Theorem 2.13.** (Brooke-Taylor, Brendle, Friedman, Montoya [7]) If $\kappa$ is strongly inaccessible then all the implications in Figure 2 hold.

![Cichoń’s diagram for strongly inaccessible $\kappa$](image)

Fig. 2 Cichoń’s diagram for strongly inaccessible $\kappa$

In [7], several models are produced to separate cardinal invariants in this diagram (e.g., $\kappa$-Cohen forcing increases $\text{cov}(\mathcal{M}_\kappa)$ but leaves $\text{non}(\mathcal{M}_\kappa)$ small) although many questions remain, cf. Question 3.1.

The combinatorial cardinal characteristics are, in general, easy to generalise. In particular, the following numbers are defined by a direct replacement of $\omega$ by $\kappa$ and “finite” by “$<\kappa$”: $a(\kappa), c(\kappa), g(\kappa), i(\kappa), t(\kappa), s(\kappa), \text{ and } u(\kappa)$.

**Remark 2.14.** A special remark concerns the generalisations of the cardinal characteristics $t, p$ and $h$. A straightforward generalisation as above yields a cardinal number which is always equal to $\omega$ if $\kappa$ has uncountable cofinality and $\omega_1$ if $\kappa$ is singular of countable cofinality. Therefore, a more appropriate definition of $p(\kappa), t(\kappa)$ and $h(\kappa)$ is to require that the family in question has size at least $\kappa$. In [55], it was shown that under this assumption for $t(\kappa)$, it follows that $t(\kappa) \geq \kappa^+$, and a similar argument works for $p(\kappa)$. We do not know whether this argument also works for $h(\kappa)$. We also do not know about the relationship between $p(\kappa)$ and $t(\kappa)$.

We introduce one new characteristic (which is equal to $\text{cov}(\mathcal{M})$ in the classical case):

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3 We would like to mention that Galeotti developed the basic theory of an generalised analogue $\mathbb{E}_\kappa$ of the real numbers in [22, 23] on the basis of Conway’s surreal numbers [12]. The space $\mathbb{E}_\kappa$ allows us to define appropriate notions of $\kappa$-metric and $\kappa$-Polish spaces and gives hope for a generalisation of measure theory that might shed some light on the question of the generalisation of random forcing.
Definition 2.15. Let \( f, g \in \kappa^\kappa \). We say that \( f, g \) are eventually different iff there is \( \beta < \kappa \) such that for all \( \alpha \geq \beta \), \( f(\alpha) \neq g(\alpha) \). A family \( \mathcal{E} \subseteq \kappa^\kappa \) is called eventually different iff for every \( g \in \kappa^\kappa \) there is \( f \in \mathcal{E} \) which is eventually different from \( g \). The cardinal

\[
\text{in}(\kappa) = \min\{|B| \mid B \subseteq \kappa^\kappa \text{ is an eventually different family}\}
\]

is called the inequality number (cf. Question 3.9).

The currently known relations between these cardinal characteristics are summarized in Figure 3, (cf. Question 3.2). As in Figure 1, lines from bottom to top signify \( \leq \), however, this diagram is far from complete, in the sense that for many implications, it is not known whether they are consistent or not. Moreover, we should note that by results of Zapletal [56], the consistency of such relations can have substantial large cardinal strength (unlike the classical situation). Most notably, the following holds: if \( s(\kappa) > \kappa^+ \) is consistent, then so is \( \text{ZFC} + o(\kappa) \geq \kappa^{++} \) (where \( o(\kappa) \) refers to the Mitchell order). This has direct consequences for relations between other cardinal characteristics as well, for example: if \( t(\kappa) < s(\kappa) \) is consistent, then so is \( \text{ZFC} + o(\kappa) \geq \kappa^{++} \) (because \( \kappa^+ \leq t(\kappa) \) is provable in \( \text{ZFC} \)).

Cardinal characteristics for uncountable \( \kappa \) have been studied by various researchers. Some of the more significant contributions include [14, 56, 55, 24, 10, 25, 54, 11, 48].

\[
\begin{array}{c}
\text{u}(\kappa) & \text{i}(\kappa) \\
\text{a}(\kappa) & \text{r}(\kappa) & \text{d}(\kappa) \\
\text{b}(\kappa) & \text{e}(\kappa) & \text{g}(\kappa) \\
\text{s}(\kappa) \\
\text{t}(\kappa) & \text{p}(\kappa) \\
\kappa^+ \\
\end{array}
\]

Fig. 3 Diagram of the generalised combinatorial cardinal characteristics.

2.3 Regularity properties

The three classical properties that have played a crucial role in the development of descriptive set theory are the Baire property, Lebesgue-measurability and the perfect set property. All analytic sets satisfy these properties, the Axiom of Choice allows us to construct counterexamples, and in the Solovay model (obtained by collapsing an inaccessible to \( \omega_1 \) using the Levy collapse) all projective sets satisfy these properties. Moreover, the statement “all \( \Pi^1_1 \) sets have the perfect set property” is equivalent to \( \omega_1^{\aleph_0} < \omega_1 \) for all \( a \in \omega^\omega \). The Baire property and Lebesgue measurability for \( \Sigma^1_2 \)-sets is false if \( V=L \) but holds in generic extensions of \( L \). By [51], an inaccessible is necessary to prove the consistency of “all projective sets are Lebesgue measurable”, whereas the strength of “all projective sets have the property of Baire” is just \( \text{ZFC} \). Several people studied various generalisations of these
properties, e.g., the Ramsey property, the \( \kappa \)-dichotomy, and various properties naturally related to definable forcing notions.

In the generalised setting, we do not have an adequate notion of Lebesgue-measurability, but we do have natural definitions for the other properties.

**Definition 2.16.** A set \( A \subseteq 2^\kappa \) has the \( \kappa \)-Baire-property iff \( A \Delta O \) is \( \kappa \)-meager for some open set \( O \subseteq 2^\kappa \) (here “open” refers to the bounded topology, cf. Definition 2.1).

All generalised Borel sets have the \( \kappa \)-Baire-property, but by results of Halko and Shelah [28], the club filter on \( \kappa \) is a generalised-\( \Sigma^1_1 \) set without the Baire property, in stark contrast to the classical setting. On the other hand, it is independent whether all generalised \( \Delta^1_1 \) sets satisfy the Baire property, see [17] (recall that in the generalised setting, the class of Borel sets is smaller than the class of \( \Delta^1_1 \) sets).

The Baire property can also be generalised to measurability properties generated by tree-like forcing notions, in a way similar to the classical results [9, 8, 33].

**Definition 2.17.** Let \( \mathbb{P} \) be a forcing notion whose conditions are trees on \( \kappa \leq \kappa \) or \( 2^\leq \kappa \), ordered by inclusion. Let \( \mathcal{N}_\mathbb{P} \) be the ideal of subsets \( A \) such that for every \( T \in \mathbb{P} \) there is \( S \leq T \) with \( |S| \cap A = \emptyset \). Let \( \mathcal{I}_\mathbb{P} \) be the \( \sigma \)-ideal generated by \( \mathcal{N}_\mathbb{P} \). Finally, a subset \( A \) of \( 2^\kappa \) or \( \kappa ^\kappa \) is called \( \mathbb{P} \)-measurable if for every \( T \in \mathbb{P} \) there is \( S \leq T \) such that \( |S| \leq^* A \) or \( |S| \cap A =^* \emptyset \), where \( \leq^* \) and \( =^* \) refers to “modulo \( \mathcal{I}_\mathbb{P} \)”.

In this setting, the Baire property is the same as \( \mathbb{P} \)-measurability for \( \mathbb{P} \) being the \( \kappa \)-Cohen forcing on \( 2^\kappa \). A first systematic study of such regularity properties, where \( \mathbb{P} \) was a suitably generalised version of Cohen, Sacks, Miller, Laver, Mathias and Silver forcing, was conducted in [19], where it was established that (1) all Borel sets satisfy \( \mathbb{P} \)-measurability for all \( \mathbb{P} \); (2) \( \Sigma^1_1 \) sets do not satisfy \( \mathbb{P} \)-measurability for any \( \mathbb{P} \), and (3) \( \mathbb{P} \)-measurability for \( \Delta^1_1 \) sets is independent, and the implications between statements of the form “all \( \Delta^1_1 \) sets are \( \mathbb{P} \)-measurable” follows the pattern shown in Figure 4, in parallel to the situation on the \( \Delta^1_1 \) level in the classical setting.

**Fig. 4** Diagram of implications for \( \mathbb{P} \)-measurability of \( \Delta^1_1 \) sets.

Another classical property that has interesting generalisations is the perfect set property and the related Hurewicz dichotomy.

**Definition 2.18.** A set \( A \) satisfies the \( \kappa \)-perfect set property if either \( |A| \leq \kappa \) or \( A \) contains a closed homeomorphic copy of \( 2^\kappa \) (alternatively, a perfect subset). A set \( A \) satisfies the Hurewicz dichotomy if \( A \) is either a \( \kappa \)-union of \( \kappa \)-compact sets, or \( A \) contains a closed homeomorphic copy of \( \kappa ^\kappa \).

Here, the situation diverges even more from the classical setting: if there exists a \( \kappa \)-Kurepa tree \( T \), then \( [T] \) cannot have the perfect set property, so it is consistent for the perfect set property to fail even for closed sets. In [50], Schlicht constructed a model where all projective sets satisfy the generalised perfect set property. The related Hurewicz dichotomy was first studied in [40]. It consistently fails for closed sets and consistently holds for \( \Sigma^1_1 \)
Another central topic in classical descriptive set theory is the study of definable equivalence relations on Polish spaces. Important contributions to these and related questions include [47, 50, 17, 19, 40, 38].

For projective sets, the Silver dichotomy is consistent (cf. Questions 3.36 and 3.38). Two essential results in this are the Silver dichotomy: if $E \subseteq 2^\omega \times 2^\omega$ is a $\Pi^1_1$ equivalence relation, then either it has countably many equivalence classes or there are perfectly many $E$-inequivalent points (equivalently: either $E \leq_B \text{id}_E$ or $\text{id}_{2^\omega} \leq_B E$); and the Harrington-Kechris-Louveau (or Glimm-Effros) dichotomy: if $E \subseteq 2^\omega \times 2^\omega$ is a Borel equivalence relation, then either $E \leq_B \text{id}_{2^\omega}$ or $\text{E}_0 \leq E$, where $x \in \text{E}_0 \iff \exists m \forall n \geq m(x(n) = y(n))$.

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Important contributions to these and related questions include [47, 50, 17, 19, 40, 38].

### 2.4 Borel equivalence relations.

Another central topic in classical descriptive set theory is the study of definable equivalence relations on Polish spaces. Given an equivalence relation $E$ on a Polish space $X$ and an equivalence relation $F$ on a Polish space $Y$, one says that $E$ is *Borel reducible* to $F$ ($E \leq_B F$) if there exists a Borel function $\varphi : X \to Y$ such that $xEy \iff \varphi(x)F\varphi(y)$ (analogously, one can define *continuous reducibility* $\leq_c$ by replacing “Borel” with “continuous”).

Two essential results in this are the Silver dichotomy: if $E \subseteq 2^\omega \times 2^\omega$ is a $\Pi^1_1$ equivalence relation, then either it has countably many equivalence classes or there are perfectly many $E$-inequivalent points (equivalently: either $E \leq_B \text{id}_E$ or $\text{id}_{2^\omega} \leq_B E$); and the Harrington-Kechris-Louveau (or Glimm-Effros) dichotomy: if $E \subseteq 2^\omega \times 2^\omega$ is a Borel equivalence relation, then either $E \leq_B \text{id}_{2^\omega}$ or $\text{E}_0 \leq E$, where $x \in \text{E}_0 \iff \exists m \forall n \geq m(x(n) = y(n))$.

Analytic equivalence relations of particular interest include the *isomorphism relation* and the *bi-embeddability relation* on countable structures. We refer the reader to [37, Section 16.C] for a good introduction, and to [29] for a more extensive survey. Cf. also [39, 21] for results about the *bi-embeddability relation* on countable structures.

Given a theory $T$, put $M^T_\omega := \{x \in 2^\omega : x \models T\}$ (where the map $x \mapsto M_x$ is a bijection), and $\equiv^T_{\leq_T} := \{(x, y) \in M^T_\omega \times M^T_\omega : M_x \equiv M_y\}$. One can then define $T \leq_T T'$ iff $\equiv^T_{\leq_T} \leq_B \equiv^{T'}_{\leq_T}$. A natural question is whether $T \leq_T T'$ gives us information about the relationship between $T$ and $T'$. When dealing with countable structures, the answer is negative (e.g., $T = \text{Th}(\mathbb{Q}, \leq)$ and $T'$ be the theory of the vector space over the field of rational numbers, cf. [17]). But if one moves to uncountable structures and analogously defines $M^T_\kappa$, $\equiv^T_{\leq_T}$ and $\equiv^T_{\leq_T}$ for uncountable $\kappa$, it turns out that there is a strict relation between $\leq^\kappa$ and the classification in stability theory. For a detailed exposition of this topic, we refer the reader to [17].

We note that results revealing this type of connection between uncountable structures and generalised descriptive set theory represent perhaps the strongest motivation behind the study of generalised Baire spaces.

### 2.5 Universally Baire sets

Finally we include a recent topic due to Ikegami and Viale [35]. Recall that a set $A \subseteq 2^\omega$ is called *universally Baire* if for every complete Boolean algebra $B$ and every continuous function $f : \text{St}(B) \to 2^\omega$ (where $\text{St}(B)$ is the Stone space of $B$), the pre-image $f^{-1}[A]$ has the Baire property in $\text{St}(B)$. This notion is due to Feng, Magidor and Woodin [15] and plays a crucial role in bridging diverse areas of set theory: descriptive set theory, large cardinals, determinacy and inner model theory.

**Definition 2.19.** (Ikegami & Viale) Let $T$ be a first-order extension of $\text{ZFC}$. A set $A \subseteq 2^\omega$ is *universally Baire in $T$* if for every complete Boolean algebra $B$ satisfying $\models_B T$ and any continuous function $f : \text{St}(B) \to 2^\omega$, where $2^\omega$ is endowed with the *product topology*, the set $f^{-1}[A]$ has the Baire property in the Stone space of $B$.

In this definition, it is essential to consider the product rather than the bounded topology on $2^\omega$ as we have done in the other sections. This is necessary to achieve the correspondence between names for $\kappa$-reals (i.e., elements of $2^\kappa$) and continuous functions from the corresponding Stone space to $2^\kappa$.

Ikegami and Viale study to what extent this concept resembles the classical one under suitable large cardinal axioms and forcing axioms. Cf. [35] and Section 3.3.2.
3 The list of open questions

The list of open questions is organized according to the three categories below.

3.1. Set theory and combinatorics of the generalised reals: This section deals with questions on generalisations of results concerning cardinal characteristics, filters and ideals, tree combinatorics, properties of forcings, and related questions. Many of these questions can be viewed as attempts at generalising the theory summarized in [1] and [2].

3.2. Generalised descriptive set theory: This includes questions about $2^\kappa$ and $\kappa^\kappa$ from the topological perspective, as well as questions concerning definable subsets of these spaces.

3.3. Model theory and other topics: This includes questions on the application of descriptive set theoretic methods to model theory, (e.g., complexity of embeddability relations) and applications to stability theory. In the last part of this section we also consider questions concerning universally Baire sets and connections with forcing axioms.

Each category is further subdivided into sub-categories (although there will invariably be some overlap between the subcategories). We have attempted to list closely related questions next to one another, and in some cases the questions are preceded by a short introduction. Some questions are followed up by “Further background”: this is intended to provide background material for a better understanding or motivation of the question at hand (e.g., references to undefined notions, explanation of what is already known, potential applications etc.) We have tried to reference the authors who posed the questions whenever this is known, and we give exact references if the corresponding question has appeared in written form.

3.1 Set theory and combinatorics of the generalised reals

3.1.1 Cardinal characteristics on $\kappa$

The most general question (asked by many people) is to what extent the classical Cichoń’s diagram and the diagram of combinatorial characteristics (cf. Figure 1) lift to higher $\kappa$. In fact, the next two questions can be understood as a summary of a vast array of questions concerning a better understanding of cardinal characteristics for uncountable $\kappa$.

**Question 3.1** (Brendle, Brooke-Taylor, Friedman, Montoya; [7]). To what extent does Cichoń’s diagram generalise? Specifically, are the implications shown in Figure 2 the only possible ones or are there any additional implications?

**Question 3.2.** To what extent does the diagram of combinatorial cardinal characteristics generalise? Are the known implications in Figure 3 the only ones or are there any additional implications? Which consistency statements do generalise? If they do, can they be proved in ZFC or do they have large cardinal strength? Which assumptions on $\kappa$ are necessary? Does $b(\kappa)$ have a canonical generalisation? If so, how does it relate to other combinatorial cardinal characteristics?

It is a long-standing open question whether $d = \omega_1 < a$ is consistent (Roitman’s problem). For regular uncountable $\kappa$ this problem has a negative solution, i.e., if $d(\kappa) = \kappa^+$ then $a(\kappa) = \kappa^+$, cf. [3, Theorem 2.1]. However, not much is known about singular $\kappa$.

**Question 3.3.** What is the status of the “Roitman-style” problem for singular $\kappa$, i.e., can one consistently have $d(\kappa) = \kappa^+ < a(\kappa)$?

**Question 3.4** (Blass, Hyttinen, Zhang; [3], Brendle, Brooke-Taylor; [5, 6]). Is $b(\kappa) < a(\kappa)$ consistent?

**Question 3.5** (Brendle; [5]).

1. Is $a(\aleph_\omega) = \aleph_\omega$ consistent?
2. Is $b = a(\aleph_\omega)$ consistent?
3. Is it consistent for some singular \( \kappa \) to have \( a(\kappa) < a(\text{cf}(\kappa)) \)?

One of the obstacles on the way to solving the above questions concerns Canjar ultrafilters on \( \kappa \). A Canjar ultralayer \( U \) on \( \kappa \) is an ultrafilter on \( \kappa \) for which the generalised Mathias forcing \( \text{MA}^U(\kappa) \) does not add dominating reals.

**Question 3.6** (Brooke-Taylor). Is there a \( \kappa \)-complete Canjar ultrafilter on \( \kappa \), at least for measurable \( \kappa \)? Do Canjar ultrafilters have a characterization using P-points?

**Question 3.7** (Brooke-Taylor). What is the consistency strength of \( u(\kappa) < 2^\kappa \)?

In the definition of \( \text{d}(\kappa) \) one could replace the ordering \( \leq^\kappa \) with \( \leq_{\text{club}}^\kappa \), i.e., \( f \leq_{\text{club}}^\kappa g \iff \{ \alpha < \kappa : f(\alpha) < g(\alpha) \} \) contains a club. Let \( \text{d}_{\text{club}}(\kappa) \) be the dominating number for this relation. Cummings and Shelah [14] proved some analogies between these two different versions, but the following remained open:

**Question 3.8** (Cummings, Shelah; [14]). Is it consistent that \( \text{d}_{\text{club}}(\kappa) < \text{d}(\kappa) \)?

Recall the cardinal invariant \( \text{in}(\kappa) \) from Definition 2.15. In the classical setting, this is always equal to \( \text{cov}(\mathcal{M}) \), and the same holds for strongly inaccessible \( \kappa \). It is also known that if \( \kappa \) is successor and \( 2^{<\kappa} = \kappa \) then \( \text{in}(\kappa) = \text{d}(\kappa) \).

**Question 3.9** (Matet, Shelah; [46]).

1. Is \( \text{in}(\kappa) < \text{d}(\kappa) \) consistent, for \( \kappa \) successor (assuming \( 2^{<\kappa} \neq \kappa \))?
2. Is it consistent that \( \kappa \) is a limit cardinal and \( \text{cov}(\mathcal{M}_\kappa) < \text{in}(\kappa) \)?

**Question 3.10** (Matet, Shelah; [46]). Let \( I \subseteq \varphi(\kappa) \) be a \( \kappa \)-complete ideal on \( \kappa \). Let \( \text{cov}(I) \) be the least size of \( X \subseteq I \) such that for every \( A \in I \) there is \( B \in [X]^{<\kappa} \) such that \( A \subseteq \bigcup B \). Let \( P \) be a property on \( I \). We define \( \text{non}(P) \) (respectively, \( \text{non}(\mathcal{M}(P)) \)) as the least cardinal \( \lambda \) for which there exists an ideal \( I \subseteq \varphi(\kappa) \) such that \( \text{cof}(I) = \lambda \) (respectively, \( \text{cof}(I) = \lambda \)) and \( I \) does not satisfy \( P \).

1. Is \( \text{d}(\kappa) > \text{non}(\text{weak P-point}) \) consistent?
2. Is \( \kappa^+ < \text{non}(\text{weak Q-point}) \) consistent, for \( \kappa \) limit cardinal with \( 2^{<\kappa} > \kappa \)?

### 3.1.2 Borel conjecture, filters and ideals

Halko and Shelah have given a definition of the concept of strong measure zero subsets of \( 2^\kappa \) in [28, Section 2]. The definition uses straightforward combinatorics which do not require the concept of a measure on \( 2^\kappa \). The Borel conjecture on \( \kappa \), abbreviated by \( \text{BC}(\kappa) \), is the statement “every strong measure zero set in \( 2^\kappa \) has size \( \leq \kappa \).” In [28] it is proved that \( \text{BC}(\kappa) \) is false if \( \kappa \) is a successor satisfying \( \kappa = \kappa^{<\kappa} \).

**Question 3.11** (Halko, Shelah; [28]). Is it consistent that \( \text{BC}(\kappa) \) holds for inaccessible \( \kappa \)?

**Question 3.12** (Matet, Shelah; [46]). Given cardinals \( \kappa < \lambda \), let \( I_{\kappa,\lambda} \) be the ideal of all subsets of \( \varphi([\lambda]^{<\kappa}) \) which are not cofinal in \( ([\lambda]^{<\kappa}, \subseteq) \). Is it consistent that \( \kappa < \kappa^{<\kappa} \) and \( I_{\kappa,\kappa^+} \) is a weak-\( \chi \)-point?

Shelah has introduced the concepts reasonable ultrafilter, very reasonable ultrafilter and super-reasonable filter in an attempt to generalise the notion of P-points on \( \omega \). The technical definitions can be found in [52, Definition 2.5 (4)-(5)] and [49, Definition 1.11 (4)].

**Question 3.13** (Shelah; [52]). Let \( \kappa \) be regular. Is it provable in ZFC that there exist reasonable ultrafilters? Is it provable in ZFC that there exist very reasonable ultrafilters?

**Question 3.14** (Shelah; [52]). Let \( D \subseteq \varphi(\kappa) \) be a filter on \( \kappa \) and \( f \in \kappa^ \). Let \( D/f := \{ A \subseteq \kappa : f^{-1}[A] \in D \} \).

1. Is it consistent that there exists a very reasonable ultrafilter \( D \) on \( \kappa \) such that for every very reasonable ultrafilter \( D' \) on \( \kappa \) there exists a non-decreasing and unbounded \( f \in \kappa^ \) such that \( D/f = D'/f \)?
2. Is it consistent that for every ultrafilter \( D \) on \( \kappa \) there is a non-decreasing unbounded function \( f \in \kappa^ \) such that either \( D/f \) is normal or \( D/f \) is reasonable (or even very reasonable)?

**Question 3.15** (Shelah; [49]). Is it consistent that there is no super-reasonable filter?
3.1.3 Trees and tree forcings

Every closed subset of $\omega^\omega$ is a continuous image of the Baire-space $\omega^\omega$ (in fact it is even a retract of the whole space) [37, Proposition 2.8]. Every closed subset of $\kappa^\kappa$ can be written as $[T]$ for some tree $T \subseteq \kappa^\kappa$, however, by results from [41] there is always a tree $T$ such that $[T]$ is not a continuous image of $\kappa^\kappa$. Therefore it is interesting to ask whether the closed sets induced by trees with certain special properties (e.g. Kurepa trees) can be continuous images of $\kappa^\kappa$.

**Question 3.16** (Holy, Lücke, Schlicht). Suppose that $\kappa \geq \omega_2$. Is it consistent that there are $\kappa$-Kurepa trees, and for every $\kappa$-Kurepa tree $T$, $[T]$ is a continuous image of $\kappa^\kappa$?

**Further background.** It is known to be consistent [42] that

1. there are $\kappa$-Kurepa trees, but for every $\kappa$-Kurepa tree $T$, $[T]$ is not the continuous image of $\kappa^\kappa$, and
2. there are $\kappa$-Kurepa trees, and for some $\kappa$-Kurepa tree $T$, $[T]$ is a continuous image of $\kappa^\kappa$ while for some other $\kappa$-Kurepa tree $S$, $[S]$ is not a continuous image of $\kappa^\kappa$.

Also, (1) holds when $\kappa = \omega_1$.

The class of countably infinite trees without infinite branches has a universal family of size $\omega_1$, i.e., there is a family $\mathcal{U}$ of size $\omega_1$ of countably infinite trees without infinite branches such that every such tree can be mapped into some $T \in \mathcal{U}$ preserving the tree-order. It is natural to ask of how small a universal family can be considering the analogue of such trees on $\omega_1$.

Consider the class $\mathcal{T}$ of all trees of size $\aleph_1$ without branches of length $\geq \omega_1$. We can order these trees by the relation $T \leq T'$ if and only if there is a strict order preserving map from $T$ to $T'$. We say that $T \in \mathcal{T}$ is a largest tree if it is the largest element in $\mathcal{T}$ with respect to $\leq$.

**Question 3.17** (Väänänen). Is it consistent that there is a largest tree in $\mathcal{T}$?

**Further background.** If CH holds, then there is no such tree by [47].

The following question refers to a generalisation of the following well-known classical fact: if there is a non-constructible real, the for every perfect set $P$ there is a non-constructible real $x \in P$.

**Question 3.18** (Woodin. Groszek, Slaman; [26]). Assume that there is a non-constructible subset of $\omega_1$. Does every countably closed perfect tree on $\omega_1$ have a non-constructible branch?

In [4], Jörg Brendle considered “Marczewski-style” ideals associated to various combinatorial tree forcing notions (Sacks, Matthias, Miller, Laver, Silver and others) and determined the inclusion and orthogonality relations between these ideals.

**Question 3.19.** Investigate the relations between ideals generated by tree forcing notions in the generalised context, similarly to [4].

Since there is no adequately generalisation of the concepts Lebesgue measure and Lebesgue null, there is also no adequate generalisation of random forcing (see Section 2.2). One approach is to try to find a forcing which at least satisfies some of the properties of random forcing.

**Question 3.20** (Shelah; [54]; Friedman, Laguzzi; [20]). Is there a (non-trivial) tree forcing notion $\mathbb{P}$ satisfying the following properties?

1. $\mathbb{P}$ is $\kappa^+\text{-c.c.}$,
2. $\mathbb{P}$ is $\textsc{<}\kappa\text{-closed}$,
3. $\mathbb{P}$ is $\kappa^\kappa\text{-bounding}$,
4. $\mathbb{P}$ does not have the generalised Sacks property.
Is there a non-trivial tree forcing notion \( \mathbb{P} \) fulfilling conditions (1)–(3) for a successor cardinal \( \kappa \)?

**Further background.**

1. Here, the generalised Sacks property is the following statement: for every name \( \dot{f} \) and \( T \models \dot{f} \in \kappa^\kappa \), there is a slalom \( \dot{F} \) in the ground model (cf. Definition 2.12) and an \( S \leq T \) such that \( S \models \forall \alpha (\dot{f}(\alpha) \in \dot{F}(\alpha)) \).

2. A tree forcing satisfying conditions (1)–(3) for weakly compact \( \kappa \) was constructed by Shelah in [53], and for inaccessible \( \kappa \) assuming \( \Box^{\kappa^+} \) (where \( S^\kappa_{\kappa^+} \) is the set of \( \kappa \)-cofinal ordinals below \( \kappa^+ \)) by Friedman and Laguzzi in [20].

An “amoeba” for a forcing poset is a forcing adding a specific tree of generic branches. Amoeba forcings play a central role in increasing additivity numbers and other properties of the ideals.

**Question 3.21.** Investigate the status of amoebas for tree forcings in the generalised context. In particular:

1. Is there an amoeba for Sacks, Miller and Laver which does not add Cohen reals?

2. Can we prove that any reasonable amoeba for Silver forcing necessarily adds Cohen reals?

### 3.2 Generalised descriptive set theory

#### 3.2.1 Topology and Silver-Dichotomy

**Question 3.22** (Holy, Lücke, Schlicht; [41]). Is it consistent that the club filter on \( \kappa \) is an injective continuous image of \( \kappa^\kappa \)?

**Further background.** It is consistent that the club filter on \( \kappa \) is not a continuous injective image of any closed subset of \( \kappa^\kappa \) (Lücke, Schlicht; [41]).

**Question 3.23** (Lücke, Schlicht; [41]). Let \( C^{\kappa,\kappa^+} \) be the class of all continuous images of subsets of \( (\kappa^+)^\kappa \). Is it consistent that every set in \( C^{\kappa,\kappa^+} \) is a continuous injective image of a closed subset of \( \kappa^\kappa \)?

**Question 3.24** (Holy, Lücke, Schlicht). Is it consistent that every closed relation on \( \kappa^\kappa \) has a definable uniformization and there is no definable wellorder of \( \kappa^\kappa \)?

**Further background.** It is known to be consistent that there is a closed relation with no definable uniformization [43].

**Question 3.25** (Holy, Lücke, Schlicht). Is it consistent that \( \varphi(\kappa) \not\subseteq L \) and there is a wellorder of \( \kappa^\kappa \) definable over \( H_{\kappa^+} \) by a \( \Sigma_1 \)-formula without parameters?

**Further background.** Note that such a wellorder exists if \( V=L \). Also, recall that in the classical setting, the existence of such a wellorder of the reals implies that the reals are constructible.

**Question 3.26** (Holy, Lücke, Schlicht; [41]). Let \( S_1^L \) be the class of subsets \( A := \{ x \in \kappa^\kappa : L[x, y] \models \varphi(x, y) \} \), for some \( y \in \kappa^\kappa \) and some formula \( \varphi \). If \( \Sigma_1^1 = S_1^L \), is there an \( x \subseteq \kappa \) such that \( \kappa^\kappa \subseteq L[x] \)?

**Further background.** This question is motivated by [41, Proposition 1.13], which shows that \( \Sigma_1^1 = S_1^L \) in models of the form \( L[x] \), with \( x \subseteq \kappa \).

**Question 3.27** (Coskey, Schlicht; [13]). Suppose that \( X \) is a regular strong \( \kappa \)-Choquet space of size \( > \kappa \) and weight \( \leq \kappa \). Is there a closed nonempty subset of \( \kappa^\kappa \) which is not a continuous image of \( \kappa^\kappa \)?

**Further background.** For a definition of strong \( \kappa \)-Choquet spaces one can check [13, Definition 2.1]. Roughly speaking, it is obtained by considering the Choquet game of length \( \kappa \) instead of \( \omega \).

**Question 3.28.** (Coskey, Schlicht; [13]) Is there a universal space for regular strong \( \kappa \)-Choquet spaces of weight \( \leq \kappa \)?
Question 3.29. (Friedman; [16]) The Silver dichotomy at $\kappa$ is the statement that if a Borel equivalence relation $E$ on $\kappa^\kappa$ has more than $\kappa$ classes, then there is a continuous reduction of the identity relation on $2^\kappa$ to $E$. Is the Silver dichotomy for $\kappa$ consistent without assuming the consistency of $0^{\#}$?

Further background. It is known that at least the consistency of an inaccessible is needed [17]. It is shown in [16] that the general Silver dichotomy for Borel equivalence relations is consistent assuming the consistency of $0^{\#}$.

Question 3.30 (Holy, Lücke, Schlicht). Does the Silver dichotomy for closed sets imply the Silver dichotomy for Borel sets in $\kappa^\kappa$?

Question 3.31 (Friedman, Hyttinen, Kulikov; [17]). Is it consistent that $\Delta^1_1 = \text{Borel}^*$?

Further background. Cf. [17, 32], or the original source [27] for the definition of Borel* on the generalised Baire space.

3.2.2 Regularity properties

Recall the regularity properties generalising the Baire property from Definition 2.17 and

Question 3.32 (Friedman, Khomskii, Kulikov; [19]). Complete the diagram of implications (Figure 4) for regularity properties related to forcing notions at the $\Delta^1_1$ level.

Since the club filter is a counterexample to all the properties considered above, a natural question is the following:

Question 3.33 (Friedman, Khomskii, Kulikov; [19]). Are there adequate generalisations of classical regularity properties related to forcing notions for which the club filter is regular?

Further background. Note that the notions of “$\kappa$-Miller measurability”, “$\kappa$-Sacks measurability” and “$\kappa$-Silver measurability” considered in [40] and [38] are potential candidates for such properties; however, they are not generated by $<\kappa$-closed forcing notions on $\kappa^\kappa$.

Recall the perfect set property and Hurewicz dichotomy from Definition 2.18.

Question 3.34 (Holy, Lücke, Schlicht). Does the perfect set property for closed subsets of $\kappa^\kappa$ imply the perfect set property for $\Delta^1_1$ subsets of $\kappa^\kappa$?

Question 3.35 (Holy, Lücke, Schlicht). Is a $\Sigma^1_1$ wellorder of $\kappa^\kappa$ compatible with the perfect set property for $\Sigma^1_1$ subsets of $\kappa^\kappa$?

Let $\lambda > \kappa$ be inaccessible and let $\text{Coll}(\kappa, < \lambda)$ be the Levy forcing collapsing $\lambda$ to $\kappa^+$. In [50] it was shown that in $V^{\text{Coll}(\kappa, < \lambda)}$ (the Silver model) all sets definable from ordinals and subsets of $\kappa$, and therefore all generalised projective sets, satisfy the perfect set property. If $\kappa$ is not weakly compact, then the same sets satisfy the Hurewicz dichotomy (see Section 2.3). But it is not clear what happens in the weakly compact case.

Question 3.36 (Lücke, Motta Ros, Schlicht; [40]). Let $\kappa$ be weakly compact.

1. Does $\text{Coll}(\kappa, < \lambda)$ force that all sets definable from ordinals and subsets of $\kappa$ satisfy the Hurewicz dichotomy?

2. If the Hurewicz dichotomy holds for $\kappa$-coanalytic sets, is there an inner model with an inaccessible cardinal?

In the next two questions, “$\kappa$-Miller measurability” and “$\kappa$-Silver measurability” refer to weaker notion than those from Question 3.32.

Question 3.37 (Lücke, Motta Ros, Schlicht; [40]). Can we force $\kappa$-Miller measurability for all sets definable from ordinals and subsets of $\kappa$, without assuming an inaccessible above $\kappa$? Can we do the same for $\kappa$-Silver measurability, for $\kappa$ successor?

Further background. By [38], $\kappa$-Miller measurability can be forced to hold for all sets definable from ordinals and subsets of $\kappa$ in the Silver model (which requires an inaccessible $\lambda > \kappa$), and if $\kappa$ is inaccessible, then $\kappa$-Silver measurability for all sets definable from ordinals and subsets of $\kappa$ holds in the $\kappa$-Cohen model.
Question 3.38 (Lücke, Motta Ros, Schlicht; [40]).
1. Is it consistent that for a weakly compact $\kappa$, all $\kappa$-analytic sets have the $\kappa$-perfect set property but there is a closed set not satisfying the Hurewicz dichotomy?
2. Is it consistent that all $\kappa$-analytic sets are $\kappa$-Miller measurable but there is a $\kappa$-analytic (closed?) set that does not satisfy the Hurewicz dichotomy? Can such a $\kappa$ be weakly compact?
3. Can we separate the $\kappa$-Miller measurability from the $\kappa$-perfect set property in the non-weakly compact case?

Question 3.39 (Friedman, Laguzzi). Is there a version of generalised Silver forcing $\mathbb{V}^*(\kappa)$ which is $\leq\kappa$-closed and such that for the corresponding regularity property, one can force that all generalised projective sets are regular?

3.3 Model theory and other topics

3.3.1 Isomorphism relations

One of the more remarkable points in studying generalised Baire spaces is the relationship between Borel reducibility of isomorphism relations of structures and the classification of theories in Shelah’s stability theory. Arguably, this represents the most solid motivation for the study of these spaces.

As we are talking about uncountable structures and their classification, we consider infinitary logics of the form $M_{\kappa^+\kappa}$. See, e.g., [17, 32], or the original source [36], for the definition of these infinitary languages.

Question 3.40 (Hyttinen, Kulikov; [32]). Is it consistent that the sets $B \subseteq \kappa^\kappa$ definable in $M_{\kappa^+\kappa}$ are precisely the Borel sets closed under isomorphism?

Question 3.41. Is there $\varphi \in M_{\kappa^+\kappa}$ such that for all $\psi \in M_{\kappa^+\kappa}$ for some model $M$ of size $\kappa$, $M \models \neg(\psi \Leftrightarrow \varphi)$?

Question 3.42 (Friedman, Hyttinen, Kulikov; [17]). Is it consistent that $\equiv^\kappa_T$ is $\Delta^1_1$ for some complete first-order non-classifiable theory $T$?

Further background. Here $\equiv^\kappa_T$ denotes the isomorphism relation on the class of models of $T$ with domain $\kappa$.

Question 3.43 (Friedman, Hyttinen, Kulikov; [17]). Under which assumptions on $T$ and $\kappa$ does it hold that if the number of equivalence classes of $\equiv_T$ is greater than $\kappa$, then $\text{Id}_B \leq_B \equiv^\kappa_T$?

Further background. By [17], this holds if $\kappa$ is (strongly) inaccessible.

Question 3.44 (Friedman, Hyttinen, Kulikov; [17]). How much can we do without the assumption $\kappa^{<\kappa} = \kappa$? In particular, does it hold (without $\kappa^{<\kappa} = \kappa$) that if a set $A \subseteq \kappa^\kappa$ is closed under permutations, then there is a sentence $\varphi$ in $L_{\kappa^+\kappa}$ such that $A = \{\eta \mid A_\eta \models \varphi\}$?

Question 3.45 (Friedman, Hyttinen, Kulikov; [17]). Suppose $T$ is a classifiable, and $T'$ a not classifiable theory. Is it true that $\equiv^\kappa_T \leq_B \equiv^\kappa_{T'}$? What about other relations between isomorphisms of theories?

Question 3.46 (Friedman, Hyttinen, Kulikov; [18]). Let $E^\lambda_\mu$ for $\lambda \in \{2, \kappa\}$ and $\mu < \kappa$ regular be the equivalence relation on $\lambda^\kappa$ where $(\eta, \xi) \in E^\lambda_\mu$ iff the set $\{\alpha \mid \eta(\alpha) = \xi(\alpha)\}$ contains a $\mu$-cub, i.e., an unbounded set which contains all the limits of its increasing $\mu$-long sequences. Is $E^\kappa_\mu \leq_B E^{2\kappa}_\mu$?

Further background. If the answer is “yes”, then a partial answer to Question 3.45 is obtained (cf. [18]): if $T_1$ is classifiable and shallow, $T_2$ is non-classifiable and $\kappa = \lambda^+ = 2^\lambda > 2^\omega$ where $\lambda^{<\lambda} = \lambda$, then $\equiv^\kappa_{T_1} \leq_B \equiv^\kappa_{T_2}$.

Question 3.47 (Friedman, Hyttinen, Kulikov; [17]). Assuming $\kappa = \omega_2$ and using the notation of Question 3.46, is it consistent that $E^{2\omega}_\omega$, Borel reduces to $E^{2\omega}_\omega$?

Question 3.48 (Holy, Lücke, Schlicht). Is it consistent that $\kappa > \omega_1$ there is a supercompact cardinal $\lambda > \kappa$, and there is a $\Sigma^1_1(\kappa)$ wellorder of $H_{\kappa^+}$?

Further background. Results of Woodin on the $\Pi^1_2$-maximality of the $\mathbb{P}_{\text{max}}$-extension of $L(\mathbb{R})$ directly imply that this is not possible for $\kappa = \omega_1$ (cf. [30]).
3.3.2 Universally Baire sets

Here we consider the setting from Section 2.5. All the questions in this subsection refer to [35]. Recall that a cardinal \( \kappa \) is called super almost huge iff for every \( \gamma \) there is \( j : V \prec M \) with critical point \( \kappa \), such that \( \gamma < j(\kappa) \) and \( \langle j(\kappa) \rangle^M \subseteq M \). We say that a set \( A \subseteq 2^\kappa \) is homogeneously Suslin if it is the projection of an homogeneous tree \( S \), i.e., a tree for which there is a system of measures \( \{ \mu_s \mid s \in Y^{<\omega} \} \) satisfying the following properties:

1. \( \mu_s \) is a countably additive measure on \( \{ t \mid (s, t) \in S \} \),
2. if \( s \subseteq t \), then \( \mu_s(A) = 1 \iff \mu_t(\{ r \mid r \upharpoonright |s| \in A \}) = 1 \),
3. if \( x \in p[S] \), then the ultrapower by \( \{ \mu_{x^n} \mid n \in \omega \} \) is well-founded.

Working in a first order theory \( T \) extending ZFC, we say that a formula \( \varphi(x) \) is \( (\omega_1, T) \)-generically absolute if for any \( Y \subseteq \omega_1 \) in \( V \), and any preorder \( P \) with \( \text{FA}_{\omega_1}(P) \) and \( \Vdash_T \varphi \), the statement \( \varphi[Y] \) is absolute between \( V \) and \( V^P \). We say that it is \( (\omega_1, T) \)-honestly absolute if for every model \( W \) of \( T \) satisfying \( (\omega_1, \in)^V = (\omega_1, \in)^W \), for any \( Y \subseteq \omega_1 \) in \( V \), the statement \( \varphi[Y] \) is absolute between \( V \) and \( W \), whenever \( W \) satisfies: for any set \( A \in V \) which is universally Baire in \( 2^\omega \) in \( T \), there is a set \( A' \subseteq W \) also universally Baire in \( 2^\omega \) over \( T \), such that the structure \( (\text{H}_{\omega_2}, \in, \text{NS}_{\omega_1}, A')^W \) is a \( \Sigma_2 \)-elementary substructure of \( (\text{H}_{\omega_2}, \in, \text{NS}_{\omega_1}, A')^V \).

**Question 3.49** (Ikegami, Viale; [35]). Let \( T \) be ZFC + MM++ “there are proper class many super almost huge cardinals” and work in \( T \). Let \( \varphi(x) \) be a formula which is \( \Delta_2 \) in \( T \). Assume \( \varphi(x) \) is \( (\omega_1, T) \)-generically absolute. Is \( (\omega_1, T) \)-honestly absolute?

**Further background.** For a definition of \( M^{++} \), cf. [34, Definition 5.11].

**Question 3.50** (Ikegami, Viale; [35]). Can methods of inner model theory be used in such a way that, without assuming the “generic nice UBH”, one can obtain simple iteration strategies for transitive structures \( M \) of size \( \omega_1 \) with a proper class of Woodin cardinals such that \( \text{NS}^M_{\omega_1} = \text{NS}_{\omega_1} \cap M \) and that for any stationary set preserving forcing \( P \) in \( M \), there is an \( (M, P) \)-generic filter in \( V' \)?

**Further background.** For a definition of iteration strategies and generic nice UBH (unique branches hypothesis), see [45].

**Question 3.51** (Ikegami, Viale; [35]). Can one develop the theory of homogeneously Suslin sets to characterize universally Baire sets in the presence of large cardinals?

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