

On the Representation and Construction of Equitable Social Welfare Orders

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Abstract

This paper examines the representation and explicit description of social welfare orders on infinite utility streams. It is assumed that the social welfare orders under investigation satisfy upper asymptotic Pareto and anonymity axioms. We prove that there exists no real-valued representation of such social welfare orders. In addition, we establish that the existence of a social welfare order satisfying the anonymity and upper asymptotic Pareto axioms implies the existence of a non-Ramsey set, which is a non-constructive object. Thus, we conclude that the social welfare orders under study do not admit explicit description.

Keywords: Anonymity, Non-Ramsey set, Social Welfare Function, Social Welfare Order, Upper Asymptotic Pareto.

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1 Introduction

This paper deals with efficiency and intergenerational equity in the setting of policies that affect present and future generations. Relevant questions are: how should a social planner weigh the welfare of the present generation against the well-being of future generations? Is there a conflict (and in what sense) between intergenerational equity and efficiency in the evaluation of infinite utility streams? This subject has received wide attention in the economics, philosophy and political science literature in recent years. In this paper we investigate preference relations, on the space of infinite utility streams, that are complete, transitive, invariant to finite permutations, and respect some version of the Pareto ordering: equitable preferences, for short. We stick to the standard framework which concerns the problem of defining a social welfare order on the set X of infinite utility streams, where X is of the form $X = Y^{\mathbb{N}}$, Y denotes a non-empty subset of real numbers, and \mathbb{N} is the set of natural numbers. There is a vast body of literature on the subject matter. In what follows we will briefly overview it in order to highlight and put our own contribution in context.

In a pioneering paper, Ramsey (1928) observed that discounting one generation's utility relative to another's is "ethically indefensible", and something that "arises merely from the weakness of the imagination". Following in Ramsey's footsteps, Diamond (1965) introduced the concept of *anonymity* (as an axiom imposed on preferences over infinite utility streams) to formalize the principle of equitable preferences ("equal treatment" of present and future generations). This axiom requires that two infinite utility streams be indifferent if one is obtained from the other by interchanging the utility level of any two generations. There is also broad consensus among scholars on another desirable attribute that preferences should possess, namely the Pareto criteria. In its strongest form, the Pareto principle asserts that one utility stream must be deemed strictly better than another if at least one generation is better off and no generation is worse off. Therefore, a question that naturally arises is whether one can aggregate infinite utility streams with a social welfare function,¹ and consistently evaluate them while respecting anonymity and some form of the Pareto axiom. This question was first approached formally by Diamond (1965) who showed that, if the possible range of utilities in each period is the closed interval $[0, 1]$, a social welfare order that displays anonymity and the strong Pareto ordering cannot be continuous in the topology induced by the supremum norm. Hence, there does not exist any continuous (in the topology induced by the sup norm) social welfare function satisfying the anonymity and strong Pareto axioms. Basu and Mitra (2003) refined Diamond's result by showing that the non-representability result still holds when continuity is dispensed with, and even for subsets Y of the real numbers containing only two elements. The bottom line is that when Y contains at least two elements, there exists no representable social welfare order satisfying the anonymity and strong Pareto axioms. Hence, one cannot exploit canonical constrained-maximization techniques to figure out an optimal policy. One potential way out consists in weakening the strong Pareto condition while still demanding that the social welfare order be representable. However, Crespo et al. (2009) established that if Y contains at least two elements, there is no social welfare function satisfying anonymity and the infinite Pareto axiom.² On the other hand, Basu and Mitra (2007a) provided an example of a social welfare function that satisfies anonymity and the weak Pareto principle.³ Petri (2019)

¹A real-valued function representing a given social welfare order is referred to as a social welfare function.

²According to the infinite Pareto axiom, one utility stream is strictly better than another if infinitely many generations are better off and no generation is worse off. So, infinite Pareto is weaker than strong Pareto.

³The weak Pareto principle states that an infinite utility stream is preferred to another if with the former every

examined a version of the Pareto axiom, namely lower asymptotic Pareto, which is weaker than infinite Pareto and stronger than weak Pareto. In a nutshell, given any pair x and y of infinite utility streams, if x dominates y and the lower asymptotic density of the subset of natural numbers such that $x_n > y_n$ is positive, the lower asymptotic Pareto ordering requires x to be preferred to y . Petri exhibited an explicit formula for a social welfare function satisfying lower asymptotic Pareto and anonymity under the assumption that Y is finite. This result raises the question of whether there exists a social welfare function if one considers a Pareto ordering that is weaker than infinite Pareto but stronger than lower asymptotic Pareto. We address this issue by focusing on a version of the Pareto ordering that we term upper asymptotic Pareto as it hinges on the upper asymptotic density of the subset of natural numbers over which a welfare improvement occurs. It is easy to see that upper asymptotic Pareto is weaker than infinite Pareto. Moreover, as will become clear later, upper asymptotic Pareto is stronger than lower asymptotic Pareto. Therefore, is the existence of a social welfare function still guaranteed if Y is any non-trivial domain and one postulates upper asymptotic Pareto together with anonymity? Proposition 1 below provides a negative answer to the preceding question.⁴

Petri (2019) found a social welfare function satisfying lower asymptotic Pareto and anonymity on a domain Y containing finitely many elements, but we know from Proposition 1 below that there is no numerical representation of a social welfare order satisfying weak upper asymptotic Pareto and anonymity. Although a real-valued representation of an underlying social welfare order can be very useful, yet pairwise ranking of utility streams would suffice for the purpose of policy-making as long as the binary relation at hand exists and can be operationalized. Therefore, we wish to know if it is possible to describe explicitly (for the purpose of economic policy) a social welfare order satisfying weak upper asymptotic Pareto and anonymity. To provide some background on this line of inquiry, before we preview our own result, recall that Svensson (1980) established the existence of a social welfare order that satisfies the anonymity and strong Pareto axioms, assuming the set Y of possible range of utilities to be the closed interval $[0, 1]$. However, his possibility result relies on Szpilrajn's Lemma whose proof depends on the axiom of choice. Consequently, this social welfare order cannot be used by policy makers for social decision-making. In the wake of Svensson's result, Fleurbaey and Michel (2003) conjectured that "there exists no explicit description (that is, avoiding the axiom of choice or similar contrivances) of an ordering which satisfies the Anonymity and Weak Pareto axioms". As shown by Lauwers (2010) and Zame (2007), it turns out that the axiom of choice is unavoidable for the existence of a social welfare order satisfying the anonymity and Pareto axioms. The proof of their result relies on the existence of non-Ramsey sets and non-measurable sets, respectively. Similar to the above findings, in Proposition 2 of the present paper we show that the existence of a social welfare order satisfying anonymity and weak upper asymptotic Pareto (on a domain Y containing at least two elements) entails the existence of a non-Ramsey set.

In order to highlight the scope of our results within the existing literature, it is worth considering the following table. It summarizes some results on the representation and constructive nature of anonymous social welfare orders satisfying various forms of the Pareto axiom.⁵ We defer further

generation is better off than under the latter. So, infinite Pareto is stronger than weak Pareto.

⁴As a matter of fact, in the proof of proposition 1 we use the concept of weak upper asymptotic Pareto (see definitions 1 through 4 below) which is weaker than upper asymptotic Pareto. Arguably, this makes our impossibility result more compelling.

⁵In the table below $|Y|$ denotes the cardinality of the set Y .

discussion on the question mark appearing in Table 1 to the concluding remarks.

Table 1:

Pareto axiom	Y	Representation	Constructive nature
Strong	≥ 2	No (Basu and Mitra (2003))	No (Lauwers (2010)) Zame (2007)
Infinite	≥ 2	No (Crespo et al. (2009))	No (Lauwers (2010))
Upper Asymptotic	≥ 2	No (Proposition 1)	No (Proposition 2)
Lower Asymptotic	Finite	Yes (Petri (2019))	Yes (Petri (2019))
Lower Asymptotic	Infinite	No (Petri (2019))	?

The remainder of the paper is organized as follows. In section 2 we introduce the basic notation which will be used throughout the paper and gather all the definitions. In section 3, we state and prove our main results (Propositions 1 and 2). Section 4 concludes.

2 Preliminaries

Let \mathbb{R} , \mathbb{Q} , and \mathbb{N} be the set of real numbers, rational numbers, and natural numbers, respectively. For $y, z \in \mathbb{R}^{\mathbb{N}}$, we write $y \geq z$ if $y_n \geq z_n$, for all $n \in \mathbb{N}$; $y > z$ if $y \geq z$ and $y \neq z$; and $y \gg z$ if $y_n > z_n$ for all $n \in \mathbb{N}$.

2.1 Social Welfare Orders

Let $Y \subset \mathbb{R}$ be the set of all possible utilities that any generation can achieve. Then, $X \equiv Y^{\mathbb{N}}$ is the set of all feasible utility streams. We denote an element of X by x , or alternately by $\langle x_n \rangle$, depending on the context. If $\langle x_n \rangle \in X$, then $\langle x_n \rangle = (x_1, x_2, \dots)$, where $x_n \in Y$ represents the amount of utility earned by the n^{th} generation.

A binary relation on X is denoted by \succsim . Its symmetric and asymmetric parts, denoted by \sim and \succ , respectively, are defined in the usual way. A social welfare order (SWO henceforth) is by definition a complete and transitive binary relation. Given a SWO \succsim on X , one says that \succsim can be represented by a real-valued function, called a social welfare function (SWF henceforth), if there is a mapping $W: X \rightarrow \mathbb{R}$ such that for all $x, y \in X$, $x \succsim y$ if and only if $W(x) \geq W(y)$.

It is useful to recall the definitions of lower and upper asymptotic density of a set $S \subset \mathbb{N}$. As usual, we will let $|\cdot|$ denote the cardinality of a given finite set. The lower asymptotic density of S is defined as follows:

$$\underline{d}(S) = \liminf_{n \rightarrow \infty} \frac{|S \cap \{1, 2, \dots, n\}|}{n}.$$

Similarly, the upper asymptotic density of S is defined as follows:

$$\overline{d}(S) = \limsup_{n \rightarrow \infty} \frac{|S \cap \{1, 2, \dots, n\}|}{n}.$$

2.2 Equity and Efficiency Axioms

We will be dealing with the following equity and efficiency axioms that we may want the SWO to satisfy.

Definition 1. Anonymity (AN henceforth): If $x, y \in X$, and there exist $i, j \in \mathbb{N}$ such that $y_j = x_i$ and $x_j = y_i$, while $y_k = x_k$ for all $k \in \mathbb{N} \setminus \{i, j\}$, then $x \sim y$.

Definition 2. Upper Asymptotic Pareto (UAP henceforth): Given $x, y \in X$, if $x \geq y$ and $x_i > y_i$ for all $i \in S \subset \mathbb{N}$ with $\bar{d}(S) > 0$, then $x \succ y$.

Definition 3. Weak Upper Asymptotic Pareto (WUAP henceforth): Given $x, y \in X$, if $x \geq y$ and $x_i > y_i$ for all $i \in S \subset \mathbb{N}$ with $\bar{d}(S) = 1$, then $x \succ y$.

Definition 4. Lower Asymptotic Pareto (LAP henceforth): Given $x, y \in X$, if $x \geq y$ and $x_i > y_i$ for all $i \in S \subset \mathbb{N}$ with $\underline{d}(S) > 0$, then $x \succ y$.

Of course, WUAP is weaker than UAP. Moreover, it is easy to see that UAP is stronger than LAP.

2.3 Non-Ramsey collection of sets

Let T be an infinite subset of \mathbb{N} . We denote by $\Omega(T)$ the collection of all infinite subsets of T , and we will refer to $\Omega(\mathbb{N})$ simply as Ω . Thus, any infinite subset T of \mathbb{N} belongs to Ω . A collection of sets $\Gamma \subset \Omega$ is called Ramsey if there exists $T \in \Omega$ such that either $\Omega(T) \subset \Gamma$ or $\Omega(T) \subset \Omega \setminus \Gamma$. Next, we define a collection of sets known as non-Ramsey.

Definition 5. A collection of sets $\Gamma \subset \Omega$ is said to be non-Ramsey if for every $T \in \Omega$, the collection $\Omega(T)$ intersects both Γ and its complement $\Omega \setminus \Gamma$.

We refer the reader to Fleurbaey and Michel (2003), Zame (2007, Section 4), Lauwers (2010, Section 4), Dubey and Mitra (2014, Section 2.2.5), Laguzzi (2018, Sections 2 and 3) and Dubey and Laguzzi (2019, Section 4) for a detailed account of the relevance of non-constructive objects (e.g., non-Ramsey sets, non-measurable sets, non-Baire sets etc.) to economics.

3 Results

In this section we state and prove the main results of this paper. Define $f : \mathbb{N} \rightarrow \mathbb{N}$ by

$$f(1) := 1, \text{ and } f(n+1) := (n+1)f(n) = n+1! \text{ for all } n > 1. \quad (1)$$

Consider the following partition of \mathbb{N} which will play an important role in the proof of Propositions 1 and 2 below.

$$I_1 := f(1) = \{1\}, \text{ and } I_n := (f(n-1), f(n)] \cap \mathbb{N} \text{ for } n \geq 2. \quad (2)$$

Note that $|I_1| = 1$, and

$$|I_n| = f(n) - f(n-1) = nf(n-1) - f(n-1) = (n-1)f(n-1), \text{ for } n > 1, \text{ and}$$

$$\sum_{m=1}^n |I_m| = f(1) + \sum_{m=2}^n [f(m) - f(m-1)] = f(n).$$

Also, note that

$$\alpha_n := \frac{|I_n|}{\sum_{m=1}^n |I_m|} = \frac{(n-1)f(n-1)}{f(n)} = \frac{(n-1)f(n-1)}{nf(n-1)} = \frac{n-1}{n} = 1 - \frac{1}{n}. \quad (3)$$

Observe that $\alpha_n \rightarrow 1$ as $n \rightarrow \infty$.

3.1 No social welfare function satisfies upper asymptotic Pareto and anonymity

We first prove that there is no social welfare function satisfying UAP and AN. We exploit techniques used in Basu and Mitra (2003) and Crespo et al. (2009) together with the partition of the set of natural numbers introduced above (see (2)).

Proposition 1. *There does not exist any social welfare function satisfying UAP and AN on $X = Y^{\mathbb{N}}$, with $Y = \{a, b\}$ and $a < b$.*

Proof. We establish the claim by contradiction. In the following proof we employ WUAP instead of UAP in order to stress that our result is robust to a weaker specification of the Pareto axiom. Let $W : X \rightarrow \mathbb{R}$ be a SWF satisfying WUAP and AN. We let $a = 0$ and $b = 1$ without any loss of generality. Let q_1, q_2, \dots be an enumeration of rational numbers in $(0, 1)$. We keep this enumeration fixed throughout the proof. Let $r \in (0, 1)$. Based on the above enumeration of rational numbers, we construct a sequence $x(r)$ as detailed below. Let $l_1(r) = \min\{n \in \mathbb{N} : q_n \in (0, r)\}$. Having defined $l_1(r), l_2(r), \dots, l_k(r)$, we set

$$l_{k+1}(r) = \min\{n \in \mathbb{N} \setminus \{l_1(r), l_2(r), \dots, l_k(r)\} : q_n \in (0, r)\}.$$

Note that $l_1(r) < l_2(r) < \dots < l_k(r) < \dots$. Thus, we can define $L(r)$ as follows:

$$L(r) = \{l_1(r), l_2(r), \dots, l_k(r), \dots\}.$$

Now, let $U(r) = \{u_1(r), u_2(r), \dots, u_k(r), \dots\}$ denote the set $\mathbb{N} \setminus L(r)$, with

$$u_1(r) < u_2(r) < \dots < u_k(r) < u_{k+1}(r) < \dots.$$

We are ready to define the utility stream $\langle x(r) \rangle$ as follows:⁶

$$x_n(r) = \begin{cases} 1 & \text{if } n \in I_l \text{ and } l \in L(r), \\ 0 & \text{otherwise.} \end{cases} \quad (4)$$

Next, we select from the (fixed) enumeration of rational numbers a strictly decreasing sequence $\langle q_{n_k}(r) \rangle \in (r, 1)$ which is convergent to r . Consider the subsequence $\{n_k : k \in \mathbb{N}\}$. Let $\Delta(z(r), x(r)) := \bigcup_{k \in \mathbb{N}} I_{n_k}$. We define another utility stream $\langle z(r) \rangle$ as follows:

⁶It is defined in blocks of $f(n) - f(n-1)$ elements at a time (for $n \geq 2$). The initial I_1 element of $\langle x(r) \rangle$ equals 1 if $1 \in L(r)$, and 0 otherwise. The next I_2 element of $\langle x(r) \rangle$ equal 1 if $2 \in L(r)$, and 0 otherwise, and so on. Observe that $\langle x(r) \rangle$ is well-defined.

$$z_n(r) = \begin{cases} 1 & \text{if } n \in \Delta(z(r), x(r)), \\ x_n(r) & \text{otherwise.} \end{cases} \quad (5)$$

Observe that for each term in the subsequence $\{n_k : k \in \mathbb{N}\}$, we have

$$\alpha_{n_k} = \frac{|I_{n_k}|}{\sum_{m=1}^{n_k} |I_m|} = 1 - \frac{1}{n_k}.$$

Also, notice that $\Delta(z(r), x(r)) \cap \{1, 2, \dots, f(n_k)\} \supset I_{n_k}$ for each $k \in \mathbb{N}$. Therefore, since $\alpha_{n_k} \rightarrow 1$ as $k \rightarrow \infty$, we get $\bar{d}(\Delta(z(r), x(r))) = 1$. Hence, by WUAP $x(r) \prec z(r)$, therefore

$$W(x(r)) < W(z(r)). \quad (6)$$

Next, we pick $s \in (r, 1)$ and define the utility stream $\langle x(s) \rangle$ using the same construction as (4). Observe that, if $x_n(r) = 1$, then by construction $z_n(r) = x_n(s) = 1$. Also, $q_{n_k} < s$ for all but finitely many n_k . Pick K , K being finite, such that $q_{n_1} \geq s, \dots, q_{n_K} \geq s$. Then, for every n belonging to $I_{n_1}, I_{n_2}, \dots, I_{n_K}$ (there are finitely many such n), we have $z_n(r) = 1 > 0 = x_n(s)$. There exist infinitely many $l_m(s) \in \mathbb{N} \setminus \{n_1, n_2, \dots, n_K\}$, with $l_m(s) > n_K$, that are distinct from the subsequence $\{n_k\}$ and are such that $q_{l_m(s)} \in [r, q_{n_K}) \cap [r, s)$. For every $l_m(s)$ there are $|I_{l_m(s)}|$ elements of the utility stream $\langle x(s) \rangle$ such that $x_n(s) = 1 > 0 = z_n(r)$. We interchange the I_{n_1}, \dots, I_{n_K} coordinates of $\langle z(r) \rangle$ (having value 1) with equally many elements (having value 0) from the $I_{l_m(s)}, I_{l'_m(s)}, \dots$ so as to obtain the utility stream $\langle z' \rangle$. It follows from AN that $z' \sim z(r)$, hence

$$W(z') = W(z(r)). \quad (7)$$

Compare $\langle z' \rangle$ to $\langle x(s) \rangle$ and observe that $z'_n = 0 = x_n(s)$ for every k and $n \in I_{u_k(s)}$. Also, $z'_n = 1 = x_n(s)$ for every k and $n \in I_{l_k(r)}$. Moreover, $z'_n = 1 = x_n(s)$ for every $k > K$ and $n \in I_{n_k}$. Let $\Delta(z(r), x(s)) := \bigcup_{k' \in \mathbb{N}} I_{v_{k'}}$ where $v_{k'} \in (U(r) \cap L(s)) \setminus \{n_K, n_{K+1}, \dots\}$. Thus, $z'_n = 0 < 1 = x_n(s)$ for every $n \in \Delta(z(r), x(s))$. Let

$$\alpha_{v_{k'}} = \frac{|I_{v_{k'}}|}{\sum_{m=1}^{v_{k'}} |I_m|} = 1 - \frac{1}{v_{k'}}.$$

Observe that $v_{k'} \rightarrow \infty$ as $k' \rightarrow \infty$, therefore $\alpha_{v_{k'}} \rightarrow 1$. By WUAP, $z' \prec x(s)$, consequently

$$W(z') < W(x(s)). \quad (8)$$

Therefore, (6), (7), and (8) imply that $(W(x(r)), W(z(r)))$ and $(W(x(s)), W(z(s)))$ are non-empty and disjoint open intervals. Hence, because r and s , with $r < s$, were arbitrary, by density of \mathbb{Q} in \mathbb{R} we conclude that we have found a one-to-one mapping from $(0, 1)$ to \mathbb{Q} , which is impossible as the latter set is countable. ■

3.2 The non-constructive nature of a social welfare order satisfying upper asymptotic Pareto and anonymity

Proposition 1 leaves us wondering if one can describe the SWO under consideration despite the latter having no real-valued representation. Therefore, in Proposition 2 below we restrict ourselves to

the same setting as in Proposition 1 and we show that no social welfare order satisfying anonymity and weak upper asymptotic Pareto is susceptible of an explicit description. To this end, it will suffice to prove that the existence of such a social welfare order, when Y contains only two elements, entails the existence of a non-Ramsey set. The proof of the following proposition is inspired by Lauwers (2010).

Proposition 2. *Let $Y = \{a, b\}$, with $a < b$, and assume that there is a social welfare order on $X = Y^{\mathbb{N}}$ satisfying UAP and AN. Then, there exists a non-Ramsey set.*

Proof. We already know that UAP is stronger than WUAP. Thus, it will be enough to prove the claim assuming that the given SWO satisfies WUAP. Given any $N := \{n_1, n_2, \dots, n_k, \dots\}$ that belongs to Ω , using (1) above we define recursively the following partition of natural numbers:

$$I_1(N) := [0, f(n_1)) \cap \mathbb{N} \text{ and } I_k(N) := [f(n_{k-1}), f(n_k)) \cap \mathbb{N} \text{ for } k > 1.$$

Next, we define $x(N), y(N) \in X$ as follows:

$$x_t(N) = \begin{cases} a & \text{if } t \in I_k(N) \text{ and } k \text{ is odd} \\ b & \text{if } t \in I_k(N) \text{ and } k \text{ is even,} \end{cases} \quad (9)$$

$$y_t(N) = \begin{cases} a & \text{if } t \in I_1(N), \text{ or } I_k(N) \text{ and } k \text{ is even} \\ b & \text{if } t \in I_k(N) \text{ and } k \text{ is odd, } k > 1. \end{cases} \quad (10)$$

Let $\Gamma := \{N \in \Omega : x(N) \prec y(N)\}$. We claim that Γ is a non-Ramsey set. According to Definition 5, we must show that for every $T \in \Omega$ there exists $S \in \Omega(T)$ such that $T \in \Gamma \Leftrightarrow S \notin \Gamma$. Pick any arbitrary $T := \{t_1, t_2, \dots, t_k, \dots\}$. We distinguish three cases.

(1) $x(T) \prec y(T)$: In this case, let $S := T \setminus \{t_1, t_{4k+1}, t_{4k+2} : k \in \mathbb{N}\} = \{t_2, t_3, t_4, t_7, t_8, \dots\}$. Note that $I_1(S) = I_1(T) \cup I_2(T)$, $I_2(S) = I_3(T)$, $I_{2k+1}(S) = I_{4k}(T)$, for all $k \geq 1$, and $I_{2k}(S) = I_{4k-3}(T) \cup I_{4k-2}(T) \cup I_{4k-1}(T)$ for all $k \geq 2$. Therefore, by (9) and (10) we have $y_t(S) = a < b = x_t(T)$ for $t \in I_2(T) \cup I_{4k+2}(T)$ and $y_t(S) = x_t(T)$ for all remaining $t \in \mathbb{N}$. Define

$$\Delta := \{t \in \mathbb{N} : y_t(S) < x_t(T)\} \text{ and } \delta_k := \frac{|\Delta \cap [0, f(t_{4k+2}))|}{f(t_{4k+2})}.$$

Then, $I_2(T) \subset \Delta$, and $I_{4k+2}(T) \subset \Delta$ for all $k \in \mathbb{N}$. For every $k \geq 1$ we have

$$\Delta \cap [0, f(t_{4k+2})) \supseteq I_{4k+2}(T) \quad (11)$$

and $[f(t_{4k+2}-1), f(t_{4k+2})) \cap \mathbb{N} \subset I_{4k+2}(T)$. Therefore, $|I_{4k+2}(T)| \geq f(t_{4k+2}) - f(t_{4k+2}-1) = (t_{4k+2}-1)f(t_{4k+2}-1)$, and

$$\sum_{j \leq 4k+2} |I_j(T)| = f(t_{4k+2}). \quad (12)$$

Also,

$$\frac{|I_{4k+2}(T)|}{\sum_{j \leq 4k+2} |I_j(T)|} \geq \frac{(t_{4k+2}-1)f(t_{4k+2}-1)}{f(t_{4k+2})} = \frac{(t_{4k+2}-1)f(t_{4k+2}-1)}{(t_{4k+2})f(t_{4k+2}-1)} = 1 - \frac{1}{t_{4k+2}}.$$

Hence, by (11) and (12) and the above inequality, we have

$$\delta_k := \frac{|\Delta \cap [0, f(t_{4k+2})]|}{f(t_{4k+2})} \geq \frac{|I_{4k+2}(T)|}{\sum_{j \leq 4k+2} |I_j(T)|} \geq 1 - \frac{1}{t_{4k+2}}.$$

Hence, $\bar{d}(\Delta) = 1$. This is because given $\langle n_k : k \geq 1 \rangle$, with $n_k := f(t_{4k+2})$, δ_k is a subsequence such that

$$\bar{d}(\Delta) = \limsup_{n \rightarrow \infty} \frac{|\Delta \cap \{1, \dots, n\}|}{n} \geq \lim_{k \rightarrow \infty} \delta_k = 1.$$

Thus, we have found a set $\Delta \in \Omega$ such that $\bar{d}(\Delta) = 1$ and $y_t(S) = a < b = x_t(T)$ for $t \in \Delta$, and $y_t(S) = x_t(T)$ for all remaining $t \in \mathbb{N}$. Therefore, it follows from WUAP that

$$y(S) \prec x(T). \quad (13)$$

Since $y_t(T) = a < b = x_t(S)$ for $t \in I_{4k+2}(T)$, and $y_t(T) = x_t(S)$ for all remaining $t \in \mathbb{N}$, by the same logic one can prove that WUAP implies

$$y(T) \prec x(S). \quad (14)$$

Therefore, by (13) and (14) we get $y(S) \prec x(T) \prec y(T) \prec x(S)$. By transitivity, $y(S) \prec x(S)$, which establishes that $S \notin \Gamma$, as was to be proven.

- (2) $y(T) \prec x(T)$: Let $S := T \setminus \{t_1, t_{4k}, t_{4k+1} : k \in \mathbb{N}\} = \{t_2, t_3, t_6, t_7, t_{10}, \dots\}$. Note that $I_1(S) = I_1(T) \cup I_2(T)$, $I_{2k+1}(S) = I_{4k}(T) \cup I_{4k+1}(T) \cup I_{4k+2}(T)$, and $I_{2k}(S) = I_{4k-1}(T)$, for all $k \in \mathbb{N}$. Therefore, by (9) and (10) we have $x_t(S) = a < b = y_t(T)$ for $t \in I_{4k+1}(T)$, and $x_t(S) = y_t(T)$ for all remaining $t \in \mathbb{N}$. As in case (1) above, one can prove that WUAP implies

$$x(S) \prec y(T). \quad (15)$$

Furthermore, $x_t(T) = a < b = y_t(S)$ if $t \in I_{4k+1}(T)$, $y_t(S) = a < b = x_t(T)$ if $t \in I_2(T)$, and $x_t(S) = y_t(T)$ for all remaining $t \in \mathbb{N}$. Interchanging finitely many coordinates of $y(S)$ lying in $I_2(T)$ with an equal number of coordinates in $I_5(T)$ yields the auxiliary sequence $y'(S)$. Hence, AN implies

$$y'(S) \sim y(S). \quad (16)$$

Also, $x_t(T) = a < b = y'_t(S)$ if $t \in I_{4k+1}(T)$, with $k \geq 2$, and $x_t(S) = y'_t(S)$ for all remaining $t \in \mathbb{N}$. As in case (1) above, using WUAP one can prove that

$$x(T) \prec y'(S). \quad (17)$$

Thus, it follows from (16), (17), and transitivity that

$$x(T) \prec y(S). \quad (18)$$

Therefore, by (15) and (18) we get $x(S) \prec y(T) \prec x(T) \prec y(S)$. By transitivity, $x(S) \prec y(S)$, which yields $S \in \Gamma$, as was to be proven.

(3) $x(T) \sim y(T)$: Let $S := T \setminus \{t_{4k-1}, t_{4k} : k \in \mathbb{N}\} = \{t_1, t_2, t_5, t_6, t_9, \dots\}$. Note that $I_1(S) = I_1(T)$, $I_{2k+1}(S) = I_{4k-1}(T) \cup I_{4k}(T) \cup I_{4k+1}(T)$, and $I_{2k}(S) = I_{4k-2}(T)$ for all $k \in \mathbb{N}$. Then, by (9) and (10) we have $x_t(S) = a < b = x_t(T)$ for $t \in I_{4k}(T)$, and $x_t(S) = x_t(T)$ for all remaining $t \in \mathbb{N}$. As in case (1) above, WUAP implies

$$x(S) \prec x(T). \quad (19)$$

Further, $y_t(T) = a < b = y_t(S)$ for $t \in I_{4k}(T)$, and $y_t(T) = y_t(S)$ for all remaining $t \in \mathbb{N}$. As in case (1) above, one can prove that WUAP implies

$$y(T) \prec y(S). \quad (20)$$

Therefore, by (19) and (20) $x(S) \prec x(T) \sim y(T) \prec y(S)$. By transitivity, $x(S) \prec y(S)$. Therefore, $S \in \Gamma$, as was to be proven.

■

In sum, it is worth pointing out that upper asymptotic Pareto is strictly stronger than lower asymptotic Pareto. Consequently, the above propositions offer a novel result that can be contrasted with Petri's: while there exists a SWF satisfying anonymity and lower asymptotic Pareto if Y is finite (Petri (2019)), there is neither explicit description (Proposition 2 above) nor a real-valued representation (Proposition 1 above) of a SWO that satisfies anonymity and upper asymptotic Pareto.

Remark 1. In what follows we substantiate our claim that upper asymptotic Pareto is indeed strictly stronger than lower asymptotic Pareto. We accomplish this by showing first that the lower asymptotic density of the set Δ constructed in the proof of case (1) of Proposition 2 is zero (while its upper asymptotic density is one, as we already know from that proof), and then by sketching the proof that the lower asymptotic density of $\Delta(z(r), x(r))$ and $\Delta(z(r), x(s))$ used in the proof of Proposition 1 is zero as well.

We know that $I_2(T) \subset \Delta$ and $I_{4k+2}(T) \subset \Delta$ for all $k \in \mathbb{N}$. Observe that $|I_2(T)| = f(t_2) - f(t_1) < f(t_2)$. Similarly, $|I_{4k-2}(T)| = f(t_{4k-2}) - f(t_{4k-3}) < f(t_{4k-2})$ for each $k \in \mathbb{N}$. Therefore, $\Delta \cap [0, f(t_{4k-2})) = I_2(T) \cup I_6(T) \cup \dots \cup I_{4k-2}(T)$, and

$$|\Delta \cap [0, f(t_{4k-2}))| = |I_2(T) \cup I_6(T) \cup \dots \cup I_{4k-2}(T)| < kf(t_{4k-2}). \quad (21)$$

Also,

$$t_{4k+1} \geq t_{4k-2} + 3. \quad (22)$$

Moreover, notice that $|\Delta \cap [0, f(t_{4k-2}))| = |\Delta \cap [0, f(t_{4k+1}))|$. Therefore, it follows from (21) and (22) that

$$\begin{aligned} \frac{|\Delta \cap [0, f(t_{4k+1}))|}{f(t_{4k+1})} &= \frac{|\Delta \cap [0, f(t_{4k-2}))|}{f(t_{4k+1})} \leq \frac{kf(t_{4k-2})}{f(t_{4k-2} + 3)} \\ &= \frac{kf(t_{4k-2})}{(t_{4k-2} + 3)(t_{4k-2} + 2)(t_{4k-2} + 1)f(t_{4k-2})} = \frac{k}{(t_{4k-2} + 3)(t_{4k-2} + 2)(t_{4k-2} + 1)} \\ &\leq \frac{k}{(4k - 2 + 3)(4k - 2 + 2)(4k - 2 + 1)} = \frac{1}{4(4k + 1)(4k - 1)} \rightarrow 0 \text{ as } k \rightarrow \infty. \end{aligned}$$

Hence, if we let $n_k := t_{4k+1}$, for $k \in \mathbb{N}$, we have proven that

$$\lim \frac{|\Delta \cap [0, f(n_k)]|}{f(n_k)} = 0. \quad (23)$$

Since $\frac{|\Delta \cap [0, f(n_k)]|}{f(n_k)}$ is a subsequence of $\frac{|\Delta \cap \{1, \dots, n\}|}{n}$, (23) above establishes that $\liminf \frac{|\Delta \cap \{1, \dots, n\}|}{n} = 0$, as desired.

Next we sketch the proof that the lower asymptotic density of $\Delta(z(r), x(r))$ is zero. To this end, observe that for the subsequence $\{n_k\}$ used in the construction of $\langle z(r) \rangle$ the following holds from some $k \in \mathbb{N}$ onward: $n_{k+1} > 3 + n_k > k$. Therefore,

$$\begin{aligned} \frac{|\Delta(z(r), x(r)) \cap [0, f(n_k)]|}{f(n_{k+1} - 1)} &\leq \frac{kf(n_k)}{f(n_k + 2)} = \frac{kf(n_k)}{(n_k + 2)(n_k + 1)f(n_k)} \\ &= \frac{k}{(n_k + 2)(n_k + 1)} \leq \frac{k}{(k + 2)(k + 1)} \\ &\leq \frac{1}{k + 2} \rightarrow 0 \text{ as } k \rightarrow \infty. \end{aligned}$$

A similar argument applies to $\Delta(z(r), x(s))$.

4 Concluding Remarks

We close this paper with a remark on further research we plan to undertake in the future. Petri (2019) proved that a social welfare order satisfying AN and LAP on an infinite domain Y admits no real-valued representation. This leaves open the question of whether such a social welfare order can be described explicitly. We are currently working on this open question in a companion paper.

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