SOCIAL WELFARE RELATIONS AND IRREGULAR SETS

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Abstract. Total social welfare relations satisfying Pareto and equity principles on infinite utility streams has revealed a non-constructive nature. In this paper we study more deeply the needed fragment of AC. In particular, we show that such relations need a strictly larger fragment of AC than non-Lebesgue and non-Ramsey sets. We also prove a connection with the Baire property, answering Problem 11.14 posed in [4].

1. Introduction and basic definitions

In recent years, various papers have shown some interplay between theoretical economics and mathematical logic. More specifically, some connections have risen between social welfare relations on infinite utility streams and descriptive set theory. In particular the following results have been proven:

- in [15] Lauwers proves that the existence of a total social welfare relation satisfying Pareto and finite anonymity implies the existence of a non-Ramsey set.
- in [20] Zame proves that the existence of a total social welfare relation satisfying Pareto and finite anonymity implies the existence of a non-Lebesgue measurable set.

(For a precise definition of these combinatorial concepts from economic theory see Definition 1 and 2 below.)

So in terms of set-theoretical considerations, these results mean that the existence of these specific relations satisfying certain combinatorial principles from economic theory are connected to a fragment of the axiom of choice, AC. As a consequence, from the set-theoretical point of view, it is natural and interesting to understand more deeply the *exact* fragment of AC they correspond to, in particular in connection with other objects coming from measure theory, topology and infinitary combinatorics, extensively studied in the set-theoretic literature (for a detailed overview see [10],[11],[5]). More precisely we provide negative answers to the following questions (which are the reverse implications of Lauwers and Zame's results):

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Q1: Does the existence of a non-Lebesgue measurable set imply the existence of a total social welfare relation satisfying Pareto and finite anonymity?

Q2: Does the existence of a non-Ramsey set imply the existence of a total social welfare relation satisfying Pareto and finite anonymity?

Our negative answers imply that total social welfare relations satisfying Pareto and finite anonymity need a strictly larger fragment of AC than non-Lebesgue measurable and non-Ramsey sets. The main tool from forcing theory we are going to use is Shelah's amalgamation. A detailed approach and excellent exposition of this forcing tool is given in [12]; in the Appendix at the end of this paper we give a brief introduction in as much detail as needed for our purpose. In the proof we give to negatively answer **Q1**, we provide an answer to [4, Problem 11.14].

Since the motivation of this paper comes from the study of some combinatorial concepts studied in economic theory, we briefly remind the basic notions about social welfare relations and infinite utility streams, in as much detail as required for our scope.

We consider a *set of utility levels* Y (or *utility domain*) endowed with some topology and totally ordered, and we call $X := Y^\omega$ the corresponding *space of infinite utility streams*, endowed with the product topology. Given $x, y \in X$ we write $x \leq y$ iff $\forall n \in \omega(x(n) \leq y(n))$, and $x < y$ iff $x \leq y \wedge \exists n \in \omega(x(n) < y(n))$. Furthermore we set $\mathcal{F} := \{\pi : \omega \to \omega : \text{finite permutation}\}$, and we define, for $x \in X$, $f_\pi(x) := \langle x(\pi(n)) : n \in \omega \rangle$.

We say that $R \subseteq X \times X$ is a *social welfare relation* (SWR) on X iff R is reflexive and transitive.

Definition 1. A social welfare relation R is called *Paretian* (P) (or satisfying Pareto principle) iff

$$\forall x, y \in X(x \leq y \Rightarrow xRy)$$
$$\forall x, y \in X(x < y \Rightarrow xRy \wedge \neg yRx).$$

This property is sometimes referred to with the term "strongly Paretian", due to other two concepts which have also been studied in the economic literature: *intermediate Paretian relations* obtained by replacing the second condition with $(x \leq y \wedge \exists^\infty n(x(n) < y(n)) \Rightarrow xRy \wedge \neg yRx)$, and *weakly Paretian relations* by using $(x \leq y \wedge \forall n(x(n) < y(n)) \Rightarrow xRy \wedge \neg yRx)$. In our paper we focus on the original Pareto principle as in Definition 1.

Beyond the Pareto principle, the combinatorial concepts studied from theoretical economists include also the following equity principle.

Definition 2. A social welfare relation R is called *finitely anonymous* (FA) (or satisfying finite anonymity) iff for every $\pi \in \mathcal{F}$ we have $f_\pi(x) \sim x$.

Throughout the paper, we use the following notation:

FAP :=“There exists a total SWR satisfying FA and P”

NL :=“There exists a non-Lebesgue (measurable) set”

NR :=“There exists a non-Ramsey set”

NB :=“There exists a non-Baire set”

Remark 3. The nature of these social welfare relations depends on the set theoretic structure of Y. In our paper, we will mainly focus on the case $Y := \{0, 1\}$ endowed with the discrete topology or $Y := [0, 1]$ endowed with the standard archimedian topology. We also provide more comments on the study with other utility domains in the last section about the concluding remarks.

Remark 4. The study on infinite populations and these combinatorial principles has been rather extensively developed in the economic literature. Summarizing the reasons and analysing the intepretations in the context of economic theory is away from the aim of this paper, which should be meant as a contribution to a set-theoretic question coming from the study of combinatorial objects introduced in economic theory, rather than an effective application of set theory to economic theory. The reader interested in a more economic background may consult the following selected list of papers: [2], [3], [7], [6], [17], [15], [16], [20].

On the forcing theoretic side, throughout the paper we use the following well-known forcing notions:

- Random forcing $\mathbb{B} := \{C \subseteq 2^\omega : C \text{ closed } \wedge \mu(C) > 0\}$, where μ is the standard Lebesgue measure on 2^ω. The order is given by: $C' \leq C \Leftrightarrow C' \subseteq C$.
- Mathias forcing $\mathbb{M}$ consisting of pairs (s, x) such that $x \in [\omega]^\omega$, $s \in [\omega]^{<\omega}$ and $\max s < \min x$, ordered by $(t, y) \leq (s, x)$ iff $t \supseteq s$, $t{\restriction}|s| = s$ and $y \subseteq x$. Moreover we denote

$$[s, x] := \{y \in [\omega]^\omega : y \supset s \wedge y{\restriction}|s| = s \wedge y \subseteq x\}.$$

- Given $\kappa > \omega$ cardinal, let

$$\mathbf{Fn}(\omega, \kappa) := \{f : f \text{ is a function} \wedge |\mathrm{dom}(f)| < \omega \wedge \mathrm{dom}(f) \subseteq \omega \wedge \mathrm{ran}(f) \subseteq \kappa\},$$

ordered by: $f' \leq f \Leftrightarrow f' \supseteq f$. Note $\mathbf{Fn}(\omega, \kappa)$ is the standard poset adding a surjection $f_G : \omega \to \kappa$, i.e., the forcing collapsing κ to ω.

Finally we remind that we are also going to use a tool called *amalgamation* introduced by Shelah in the 1980s to build very homogeneous Boolean algebras. More details and references about it will be given in section 3 and in the Appendix.

2. SWRs, Baire property and non-Lebesgue sets

In this section we investigate the connection between total SWRs satisfying Pareto and finite anonymity, and non-Baire sets, answering Problem 11.14 posed in [4]. Remind that, in a topological space X, a subset $A \subseteq X$ satisfies the *Baire property* (or A is a *Baire set*) iff there is $O \subseteq X$ open such that $A \bigtriangleup O$ is meager.

We study the case $X := [0,1]^\omega$. In order to show that **FAP** $\Rightarrow$ **NB**, we start with a basic example. Let $\rhd$ (usually called Suppes-Sen principle) be defined as follows: for every $x, y \in X$, we say

$$x \rhd y \text{ iff there exists } \pi \in \mathcal{F} \text{ such that } f_\pi(x) > y.$$
$$x \sim y \text{ iff there exists } \pi \in \mathcal{F} \text{ such that } f_\pi(x) = y.$$

Let $\mathsf{supp}(y) := \{n \in \omega : y(n) \neq 0\}$ denote the support of y.

It is clear that $\rhd$ is a SWR satisfying P and FA.

Remark 5. The Suppes-Sen principle is rather coarse from the topological point of view, as many pairs $x, y \in X$ are incompatible w.r.t. $\rhd$. More precisely, $A := \{(x,y) \in X \times X : x \not\rhd y \wedge y \not\rhd x \wedge x \not\sim y\}$ is comeager.

Let A' be the complement of A. We show that A' is meager. First partition A' into three pieces: $E := \{(x,y) \in X \times X : x \rhd y\}$, $D := \{(x,y) \in X \times X : y \rhd x\}$ and $C := \{(x,y) \in X \times X : y \sim x\}$. We prove that E is meager and then note that similar arguments work for D and C as well. Fix $y \in X$ so that $\mathsf{supp}(y)$ is infinite (i.e., y is not eventually 0) and consider $E^y := \{x \in X : (x,y) \in E\}$. Let $H^y := \{x \in X : x > y\}$. Note that $E^y := \bigcup_{\pi \in \mathcal{F}} H^{f_\pi(y)}$. Since $\mathcal{F}$ is countable it is enough to prove that for each $\pi \in \mathcal{F}$, $H^{f_\pi(y)}$ is meager. Actually we show that H^y is nowhere dense, for every $y \in X$ with $|\mathsf{supp}(y)| = \omega$. Indeed, fix $U := \prod_{n \in \omega} U_n \subseteq X$ basic open set, and let $k \in \omega$ be sufficiently large that for all $n \geq k$, $U_n = [0,1]$. Then pick $n^* > k$ such that $n^* \in \mathsf{supp}(y)$ and pick $U' \subseteq U$ so that: $\forall n \neq n^*$, $U_n = U'_n$ and $U'_{n^*} := [0, y(n^*))$. Then it is clear that $U' \cap H^y = \emptyset$. This concludes the proof that each H^y is nowhere dense, when $|\mathsf{supp}(y)| = \omega$. Note that if $\pi \in \mathcal{F}$ we get $|\mathsf{supp}(f_\pi(y))| = \omega$ as well, and so $H^{f_\pi(y)}$ is nowhere dense too.

By Ulam-Kuratowski theorem, we conclude the proof if we show that the set $\{y \in X : |\mathsf{supp}(y)| = \omega\}$ is comeager. So let B be the complement of such a set, i.e., B consists of those y that are eventually 0. Define $B_n := \{y \in B : |\mathsf{supp}(y)| \leq n\}$. Clearly $B := \bigcup_{n \in \omega} B_n$. Moreover each B_n in nowhere dense. Indeed, let U be a basic open set and pick $k > n$ so that for all $m \geq k$, $U_m = [0,1]$. Then define $U' \subseteq U$ by replacing the kth cartesian piece of U with $(0,1]$. It is clear that $U' \cap B_n = \emptyset$. Hence, we have proved that for comeager many y, E^y is meager, and that implies E is meager by Kuratowski-Ulam theorem.

Under this point of view the Suppes-Sen principle can then be considered rather "poor", as we aim at finding SWRs able to compare as many elements as possible. Part 1 of the following proposition shows that this is unfortunately not only specific for Suppes-Sen principle, but it holds for any "regular" SWR satisfying FA and P. Moreover, in part 2 we show that when assuming **FAP**, the price to pay is to get a set without Baire property. This answers Problem 11.14 posed in [4]. In the following we consider $X = [0,1]^\omega$. Note that the existence of a SWR on $[0,1]^\omega$ satisfying FA and P implies the existence of a total SWR on 2^ω satisfying FA and P ([15, pg.37]).

Proposition 6. *Let $Y = [0,1]$ and $X := Y^\omega$. Then the following hold:*

(1) Let $\succ$ be a SWR satisfying FA and P on X, $E := \{(x,y) \in X \times X : x \succ y\}$ and $D := \{(x,y) \in X \times X : y \succ x\}$. If both E and D have the Baire property, then $E \cup D$ is meager.
(2) Let $\succ$ be a total SWR satisfying FA and P on X, and let E,D as above. Then either E or D does not have the Baire property.

Proof. Under the assumption E, D both satisfying the Baire property, we show that E is meager, and remark that the argument for D is essentially the same. Since we assume E has the Baire property, we can find a Borel set $B \subseteq E$ such that $E \setminus B$ is meager; moreover for every $\pi, \pi' \in \mathcal{F}$ we can define $B(\pi, \pi') := \{(f_\pi(x), f_{\pi'}(y)) : (x,y) \in B\}$. Put $B^* := \bigcup\{B(\pi,\pi') : \pi, \pi' \in \mathcal{F}\}$ and note that $B^* \subseteq E$, as E is closed under finite permutations. Moreover, $E \setminus B^*$ is meager too. Let $I_0 := \{y \in X : B^*_y \text{ is meager}\}$ and $I_1 := \{y \in X : B^*_y \text{ is comeager}\}$. Note that each B^*_y is by definition invariant under finite permutations, i.e., $x \in B^*_y \Leftrightarrow f_\pi(x) \in B^*_y$, where $\pi \in \mathcal{F}$.

Hence by the [13, Theorem 8.46] with G being the group on X induced by finite permutations, we have that each B^*_y is either meager or comeager, and hence $I_0 \cup I_1 = X$. We also observe that both I_0 and I_1 are invariant under $\pi \in \mathcal{F}$. In fact, it is straightforward to check that if $\pi \in \mathcal{F}$ and B^*_y is meager, then $B^*_{f_\pi(y)}$ is meager too. So, if I_1 is comeager, by Kuratowski-Ulam theorem we get E is comeager. But since an analogous argument could be done for D too, we would have that also D is comeager, but $E \cap D = \emptyset$, which is a contradiction. As a consequence, we get I_0 is comeager, which implies E (and D as well) is meager.

2. Note that in this case the SWR is total and so, if E and D both satisfy the Baire property, the set $A := \{(x,y) \in X \times X : x \sim y\}$ is comeager. Thus, by Kuratowski-Ulam's theorem there is $y \in X$ such that A_y is comeager. Pick $0 < r < \frac{1}{2}$ and define

$$H := [0, 1-r] \times \prod_{i \in \omega} [0,1],$$

and define the injective function $\phi : X' \to X$ such that $i(x(0)) := x(0) + r$. Note that

$$\phi[H] := [r, 1] \times \prod_{n \in \omega} [0,1],$$

Note also that for every $x \in H$, $\phi(x) \succ x$ by Pareto, and so in particular $x \sim y \Leftrightarrow x \not\sim \phi(y)$. Hence, we have the following two mutually contradictory consequences.

- On the one side, $H \cap A_y \cap \phi[H \cap A_y] = \emptyset$; indeed if there exists $z \in H \cap A_y \cap \phi[H \cap A_y]$, then there is $x \in H \cap A_y$ such that $z := \phi(x)$; then on the one hand we have $z \in A_y$ which gives $z \sim y$, but on the other hand we have $x \in H \cap A_y$ that in turn gives $x \sim y$ and so together with $x \prec \phi(x) = z$ we would get $y \prec z$; contradiction.
- On the other side, $H \cap A_y \cap \phi[H \cap A_y]$ cannot be meager, since $H \cap \phi[H]$ is a non-empty open set, $H \cap A_y$ is comeager in H and $\phi[H \cap A_y]$ is comeager in $\phi[H]$.

□

Let N be Shelah's model constructed in [19], where every set of reals has the Baire property. Note that such a model is obtained via Shelah's amalgamation and it does not need the use of inaccessible cardinals. Since again in [19] it is shown that to get a model where every set of reals is Lebesgue measurable we need an inaccessible, we can then deduce that in N there is a set that is not Lebesgue measurable.

Corollary 7. *Let N be Shelah's model constructed in* [19]*. Then $N \models \neg$**FAP** $\wedge$ **NL**. (Hence the answer to **Q1** is negative.) As a consequence, we furthermore obtain*

$$Con(ZF \wedge \neg\mathbf{FAP}) \Leftrightarrow Con(ZFC).$$

Proof. In N every set of reals has the Baire property and then, by Proposition 6, there is no total SWR satisfying P and FA. But in N there is a non-Lebesgue measurable set. □

3. SWRs and non-Ramsey sets

We consider the case of utility domain $Y = \{0, 1\}$, and thus $X = 2^\omega$. In [15], Lauwers proves the a total SWR satisfying FA and P on utility domain $Y = \{0, 1\}$ implies the existence of a non-Ramsey set, i.e. **FAP** $\Rightarrow$ **NR**. Remind that a set $X \subseteq [\omega]^\omega$ is *non-Ramsey* iff for every $F \in [\omega]^\omega$ one has $[F]^\omega \nsubseteq X$ and $[F]^\omega \cap X \neq \emptyset$. Note that we can identify in a standard way $[\omega]^\omega$ with 2^ω and so the definition makes perfectly sense for subsets of 2^ω as well.

In this section we answer negatively to **Q2**. More precisely we are going to prove that the implication above does not reverse, i.e. **NR** $\not\Rightarrow$ **FAP**, and thus total SWRs satisfying P and FA need a strictly larger fragment of AC than non-Ramsey sets. For doing that we also need Zame's result [20] proving that the existence of a SWR satisfying FA and P implies the existence of a non-Lebesgue set, in other words $\neg$**NL** $\Rightarrow$ $\neg$**FAP**.

Theorem 8. *There is a model N for ZF satisfying DC such that*

$$N \models \mathbf{NR} \wedge \neg\mathbf{FAP}$$

Hence the answer to **Q2** *is negative.*

The model N is going to be the inner model of a certain forcing extension that we are going to define in the proof of Theorem 8 below. The key idea to obtain such a forcing-extension is to use Shelah's amalgamation over random forcing with respect to a certain name Y for sets of elements in 2^ω in order to get a complete Boolean algebra B such that, if G is B-generic over V, in $V[G]$ the following hold:

(1) every subset of 2^ω in $L(\mathbb{R}, \{Y\})$ is Lebesgue measurable
(2) Y is non-Ramsey.

Hence, we obtain that in $L(\mathbb{R}, \{Y\})^{V[G]}$ every subset of 2^ω is Lebesgue measurable (and so by Zame's result there cannot be any total SWR satisfying FA and P), but there is a non-Ramsey set.

Shelah's amalgamation ([19]) is the main tool we need for our forcing construction. Since it is a rather demanding machinery, we refer the reader to the Appendix for a more detailed approach and an exposition of the main properties. The reader already familiar with Shelah's amalgamation can proceed with no need of such Appendix.

Before going to the detailed and technical proof, we just give a short overview of the key-idea. Starting from a Boolean algebra B, two complete subalgebras $B_0, B_1 \lessdot B$ isomorphic to the random algebra with ϕ isomorphism between them, the amalgamation process provides us with the pair (B^*, ϕ^*) such that $B \lessdot B^*$ and $\phi^* \supseteq \phi$ such that ϕ^* is an automorphism of B^*. We will denote this amalgamation process by $\mathsf{Am}^\omega(B, \phi)$, so that $B^* = \mathsf{Am}^\omega(B, \phi)$.

Since the process itself generates more and more copies of random algebras, we have to iterate this process as long as we run out all of the copies of such random algebras. For doing that a recursion of length κ inaccessible will be sufficient (and necessary to ensure the final construction satisfy κ-cc).

The idea to obtain 1 and 2 above is based on the following main parts:

(a) The Boolean algebra B is built via a recursive construction, alternating the amalgamation, iteration with Levy collapse, iteration with Mathias forcing and picking direct limits at limit steps.
(b) The set Y is also recursively build by carefully adding Mathias reals cofinally often in order to get a non-Ramsey set.
(c) In order to obtain that all sets of reals in $L(\mathbb{R}, \{Y\})$ be Lebesgue measurable, we have to amalgamate over random forcing, and we also need to recursively close Y under the isomorphisms between copies of the random algebras generated by the amalgamation process, in order to get $\Vdash \phi[Y] = Y$, for every such isomorphism ϕ.

In particular to get (c) the algebra B we are going to construct is going to satisfy $(\mathbb{B}, Y)$-*homogeneity*, i.e., for every pair of random algebras $\mathbb{B}_0, \mathbb{B}_1 \lessdot B$ with $\phi : \mathbb{B}_0 \to \mathbb{B}_1$ isomorphism, there exists $\phi^* \supseteq \phi$ automorphism of B such that $\Vdash_B \phi^*[Y] = Y$. (Roughly speaking: any isomorphism between copies of random algebra can be extendend to an automorphism which fixes Y). See [12, Theorem 6.2.b] for a proof that $(\mathbb{B}, Y)$-homogeneity implies that all sets in $L(\mathbb{R}, \{Y\})$ are Lebesgue measurable.

We now see the construction of the complete Boolean algebra B and the proof of Theorem 8 in detail.

Proof of Theorem 8. Start from a ground model V we are going to recursively define $\{B_\alpha : \alpha < \kappa\}$ sequence of complete Boolean algebras such that $B_\alpha \lessdot B_\beta$, for $\alpha < \beta$, and $\{Y_\alpha : \alpha < \kappa\}$ $\subseteq$-increasing sequence of sets of names for reals, and then put $B := \lim_{\alpha<\kappa} B_\alpha$ and $Y := \bigcup_{\alpha<\kappa} Y_\alpha$. The construction follows the line of the one presented in [12], even if it sensitively differs in the construction of the set Y, which in this framework is forced to be a non-Ramsey set, instead of a set without the Baire property. We also use the forcing **Fn** instead of the amoeba for measure, as it also serves the scope of collapsing the additivity of the null ideal and to ensure the inaccessible κ be gently collapsed to ω_1 in the forcing-extension. We start with $B_0 = \{0\}$ and $Y_0 = \emptyset$.

(1) In order to obtain the $(\mathbb{B}, \dot{Y})$-homogeneity we use a standard bookkeeping argument to hand us down all possible situations of the following type: if $B_\alpha \lessdot B' \lessdot B$ and $B_\alpha \lessdot B'' \lessdot B$ are such that B_α forces $(B'/B_\alpha) \approx (B''/B_\alpha) \approx \mathbb{B}$ and $\phi_0 : B' \to B''$ an isomorphism s.t. $\phi_0 \restriction B_\alpha = \mathrm{Id}_{B_\alpha}$, then there exists a sequence of functions in order to extend the isomorphism ϕ_0 to an automorphism $\phi : B \to B$, i.e., $\exists\langle\alpha_\eta : \eta < \kappa\rangle$ increasing, cofinal in κ, with $\alpha_0 = \alpha$, and $\exists\langle\phi_\eta : \eta < \kappa\rangle$ such that
   - for $\eta > 0$ successor ordinal, $B_{\alpha_\eta+1} = \mathsf{Am}^\omega(B_{\alpha_\eta}, \phi_{\eta-1})$, and ϕ_η be the automorphism on $B_{\alpha_\eta+1}$ generated by the amalgamation;
   - for η limit ordinal, let $B_{\alpha_\eta} = \lim_{\xi<\eta} B_{\alpha_\xi}$ and $\phi_\eta = \lim_{\xi<\eta} \phi_\xi$, in the obvious sense;
   - for every $\eta < \kappa$, we have $B_{\alpha_\eta+1} \lessdot B_{\alpha_{\eta+1}}$.

   In order to fix the set of names by each automorphism ϕ_η, one then sets
   - successor case $\eta > 0$:
   $$B_{\alpha_\eta+1} \Vdash Y_{\alpha_\eta+1} := Y_{\alpha_\eta} \cup \{\phi^j_\eta(\dot{y}), \phi^{-j}_\eta(\dot{y}) : \dot{y} \in Y_{\alpha_\eta}, j \in \omega\},$$
   - limit case: $B_{\alpha_\eta} \Vdash Y_{\alpha_\eta} := \bigcup_{\xi<\eta} Y_{\alpha_\xi}$.

(2) In order to get Y being non-Ramsey, for cofinally many α's, put $B_{\alpha+1} = B_\alpha * \dot{\mathbb{M}}$ and
$$B_{\alpha+1} \Vdash Y_{\alpha+1} := Y_\alpha \cup \{\dot{y}_{(s,x)} : (s,x) \in \mathbb{M}\},$$
where $\dot{y}_{(s,x)}$ is a name for a Mathias real over V^{B_α} such that $(s,x) \Vdash s \subset \dot{y}_{(s,x)} \subseteq x$.

(3) For cofinally many α's pick a cardinal $\lambda_\alpha < \kappa$ such that $B_\alpha \Vdash \lambda_\alpha > \omega$, put $B_{\alpha+1} = B_\alpha * \mathbf{Fn}(\omega, \lambda_\alpha)$, and let $B_{\alpha+1} \Vdash Y_{\alpha+1} := Y_\alpha$, where $\mathbf{Fn}(\omega, \lambda_\alpha)$ is the forcing adding a surjection $F_\alpha : \omega \to \lambda_\alpha$.

(4) For any limit ordinal, put $B_\lambda = \lim_{\alpha<\lambda} B_\alpha$ and $B_\lambda \Vdash Y_\lambda = \bigcup_{\alpha<\lambda} Y_\alpha$.

Let G be B-generic over V. As mentioned above, the proof of "every set of reals in $L(\mathbb{R}, Y)$ is Lebesgue measurable" is a standard Solovay-style argument, and can be found in [12]. The only difference we adopt is the use of $\mathbf{Fn}(\omega, \lambda_\alpha)$. i.e. the poset "collapsing" λ_α to ω instead of the amoeba for measure. The property needed for our purpose, which is to turn the union of all Borel null sets coded in the "ground model" $V[G{\restriction}\alpha+1]$ into a null set, is fulfilled by $\mathbf{Fn}(\omega, \lambda_\alpha)$ as well, i.e.

$$\mathbf{Fn}(\omega, \lambda_\alpha) \Vdash \bigcup\{N_c : c \text{ is a Borel code for a null set in } V[G{\restriction}\alpha+1]\} \text{ is null,}$$

where $N_c \subseteq 2^\omega$ is the Borel null set coded by c.

What is left to show is that

$$B \Vdash \text{“}Y \text{ is non-Ramsey”}. \tag{1}$$

For proving that, pick arbitrarily $(s, x) \in \mathbb{M}$; we have to show

$$B \Vdash Y \cap [s, x] \neq \emptyset \text{ and } [s, x] \not\subseteq Y.$$

For the former, Let $(\dot{s,x})$ be a B-name for a Mathias condition. By κ-cc and part (2) of the recursive construction, there is $\alpha < \kappa$ such that $(\dot{s,x})$ is a B_α-name, $B_{\alpha+1} = B_\alpha * \dot{\mathbb{M}}$ and $B_{\alpha+1} \Vdash Y_{\alpha+1} = Y_\alpha \cup \{\dot{y}_{(s,x)} : (s, x) \in \mathbb{M}^{B_\alpha}\}$. Consider $\dot{y}_{(s,x)}$ name for a Mathias real over V^{B_α} such that $B_{\alpha+1} \Vdash \dot{y}_{(s,x)} \in [s, x]$. Thus,

$$B \Vdash \dot{y}_{(s,x)} \in Y \cap [s, x].$$

On the other hand, by part (3) of the construction, there is also $\alpha < \kappa$, such that $(\dot{s,x})$ is a B_α-name, $B_{\alpha+1} = B_\alpha * \mathbf{Fn}(\omega, \lambda_\alpha)$ and $B_{\alpha+1} \Vdash Y_{\alpha+1} = Y_\alpha$. Let $\dot{y}$ be a $B_{\alpha+1}$-name for a Mathias real over V^{B_α} such that $B \Vdash \dot{y} \in [s, x]$. Obviously, $B \Vdash \dot{y} \notin Y_\alpha$ (since "the real y is added at stage $\alpha + 1$"), and hence

$$B \Vdash \dot{y} \in [s, x] \setminus Y_{\alpha+1},$$

since $B \Vdash Y_{\alpha+1} = Y_\alpha$. So it is left to show that also for every $\beta > \alpha + 1$, $B \Vdash y \notin Y_\beta \setminus Y_\alpha$, which means, roughly speaking, y cannot fall into Y at any later stage $\beta > \alpha + 1$. For proving that we show the following Claim 9. Fix the notation: given $x \in 2^\omega$, we denote by f_x the increasing enumeration of the set $\{n \in \omega : x(n) = 1\}$. It is well-known (and straightforward to check) that if x is a Mathias real over V, then f_x is dominating over V.

Claim 9. *For $\beta < \kappa$, $\beta > \alpha + 1$ and $\dot{y} \in Y_\beta \setminus Y_{\alpha+1}$, one has*

$$B \Vdash \text{“}f_{\dot{y}} \textit{ is dominating over } V^{B_{\alpha+1}}\text{”}.$$

For β limit the proof is trivial. For $\beta + 1$, we have two cases:

Case as in part (2) of the recursive construction, i.e. $Y_{\beta+1} = Y_\beta \cup \{\dot{y}_{(s,x)} : (s, x) \in \mathbb{M}\}$. In this case $\dot{y}$ has to be a Mathias real over $V^{B_{\alpha+1}}$ and therefore $f_{\dot{y}}$ is dominating over $V^{B_{\alpha+1}}$.

Case as in part (1) of the construction, i.e.

$$B_{\beta+1} \Vdash Y_{\beta+1} := Y_\beta \cup \{\phi^j(\dot{y}), \phi^{-j}(\dot{y}) : \dot{y} \in Y_\beta, j \in \omega\},$$

where ϕ's are the associated automorphisms generated by the amalgamation.

The aim is to show that the property of "being dominating" is preserved through the construction unfolded in part (1), both by the amalgamation process and by iteration of random forcing. More precisely, we need the following lemma.

Lemma 10. *Let $\eta > 0$ be a successor ordinal. Let $B', B'' \lessdot B_{\alpha_\eta}$ and $\dot{x} \in V^{B_{\alpha_\eta}} \cap 2^\omega$ such that*

$$B_{\alpha_\eta} \Vdash \text{“ } f_{\dot{x}} \text{ is dominating over both } V^{B'} \text{ and } V^{B''} \text{”},$$

and $\psi : B' \to B''$ isomorphism.

Then, for every $j \in \omega$,

$$B_{\alpha_\eta+1} \Vdash \text{“}f_{\phi_\eta^j(\dot{x})} \text{ and } f_{\phi_\eta^{-j}(\dot{x})} \text{ are dominating over } V^{B_{\alpha_\eta}} \text{”}.$$

where $B_{\alpha_\eta+1} = \mathsf{Am}^\omega(B_{\alpha_\eta}, \psi)$, and ϕ_η is the automorphism extending ψ, generated by the amalgamation.

Sublemma 1 (Preservation by one-step amalgamation)**.** *Let B, B_1, B_2, ϕ_0, e_1, e_2 as in the Appendix and $\dot{x}$ a B-name for an element of 2^ω such that B forces $f_{\dot{x}}$ is dominating over V^{B_1} and V^{B_2}. Then*

$$\mathsf{Am}(B, \phi_0) \Vdash \text{“}f_{e_1(\dot{x})} \text{ is dominating over } V^{e_2[B]} \text{”}. \tag{2}$$

(And analogously $\mathsf{Am}(B, \phi_0) \Vdash$ “ $f_{e_2(\dot{x})}$ is dominating over $V^{e_1[B]}$ ”.)

Proof of Sublemma 1. By Lemma 11 in Appendix, putting $V = N[H]$, $A_1 = (B/B_1)^H$, $A_2 = (B/B_2)^H$, it is sufficient to prove that given A_1, A_2 complete Boolean algebras and $\dot{f}$ a A_1-name for an element of ω^ω, if

$$A_1 \Vdash \text{“}\dot{f} \text{ is dominating over } V\text{”},$$

then

$$A_1 \times A_2 \Vdash \text{“}\dot{f} \text{ is dominating over } V[G]\text{”} ,$$

where G is A_2-generic over V. To reach a contradiction, assume there is $z \in \omega^\omega \cap V[G]$, $(a_1, a_2) \in A_1 \times A_2$ such that $(a_1, a_2) \Vdash \exists^\infty n \in \omega(z(n) > f(n))$. Let $\{n_j : j \in \omega\}$ enumerate all such n's, and for every $j \in \omega$ pick $b_j \in A_2$, $b_j \leq a_2$ and $k_j \in \omega$ such that $(a_1, b_j) \Vdash z(n_j) = k_j$; note that this can be done since $z \in V[G]$ and G is A_2-generic over V; hence z can be seen as an A_2-name and so it is suffcient to strengthen conditions in A_2 in order to decide its values. Since A_1 forces f be dominating over V, one can pick $a \leq a_1$ such that $(a, a_2) \Vdash \exists m \forall j \geq m(k_j \leq f(n_j))$. Pick $j' > m$; then

- on the one side, since $(a, b_{j'}) \leq (a_1, a_2)$, it follows $(a, b_{j'}) \Vdash f(n_{j'}) < k_{j'} = z(n_{j'})$
- on the other side, since $(a, b_{j'}) \leq (a, a_2)$, it follows $(a, b_{j'}) \Vdash f(n_{j'}) \geq k_{j'} = z(n_{j'})$,

which is a contradiction. $\square$

Sublemma 2 (Preservation by ω-step amalgamation)**.** *Let B be a complete Boolean algebra, $B', B'' \lessdot B$ and $\dot{x} \in V^B \cap 2^\omega$ such that*

$$B \Vdash \text{“}f_{\dot{x}} \textit{ is dominating over both } V^{B'} \textit{ and } V^{B''}\text{”},$$

with $\psi : B' \to B''$ isomorphism.

Then, for every $j \in \omega$,

$$\mathsf{Am}^\omega(B, \psi) \Vdash \text{“}f_{\phi^j(\dot{x})} \textit{ and } f_{\phi^{-j}(\dot{x})} \textit{ are dominating over } V^B\text{”}.$$

where $\phi : \mathsf{Am}^\omega(B, \psi) \to \mathsf{Am}^\omega(B, \psi)$ is the automorphism extending ψ, generated by the amalgamation.

The proof simply consists of a recursive application of Sublemma 1 following the line of the proof of [12, Lemma 3.4] by replacing the notion of “unbounded” with “dominating”.

Note that 2 is enough to show Lemma 10 when $\eta \geq 2$ successor, by considering $B = B_{\alpha_\eta}$, $\mathsf{Am}^\omega(B, \phi) = B_{\alpha_\eta+1}$, $B' = B_{\alpha_{\eta-1}}$, $B'' = \phi_{\eta-1}[B_{\alpha_{\eta-1}}]$ and $\psi = \phi_{\eta-1}$.

It is only left to show the case $\eta = 1$, which is: $B_{\alpha_0} \lessdot B', B'' \lessdot B_{\alpha_1}$ such that B_{α_0} forces $(B'/B_{\alpha_0}) \approx (B''/B_{\alpha_0}) \approx \mathbb{B}$, and $\phi_0 : B' \to B''$ isomorphism such that $\phi_0 \restriction B_{\alpha_0} = Id_{B_{\alpha_0}}$. Then for every $\dot{x} \in V^{B_{\alpha_1}} \cap 2^\omega$ such that $B_{\alpha_1} \Vdash$ “$f_{\dot{x}}$ is dominating over $V^{B_{\alpha_0}}$”, one has, for every $j \in \omega$,

$$B_{\alpha_1+1} \Vdash \text{“ } f_{\phi_1^j(\dot{x})} \text{ and } f_{\phi_1^{-j}(\dot{x})} \text{ are dominating over } V^{B_{\alpha_1}}\text{”}.$$

But, since B_{α_0} forces both $(B'/B_{\alpha_0}) \approx (B''/B_{\alpha_0}) \approx \mathbb{B}$, and then, by Sublemma 1 and the fact that random forcing is ω^ω-bounding and thus it preserves dominating reals, we obtain $\mathsf{Am}^\omega(B_{\alpha_1}, \phi_0) = B_{\alpha_1+1}$ and

$$B_{\alpha_1+1} \Vdash \text{“}f_{\dot{x}} \text{ is dominating over both } V^{B_{\alpha_0} * (B'/B_{\alpha_0})} \text{ and } V^{B_{\alpha_0} * (B''/B_{\alpha_0})}\text{”}.$$

$\square$

4. Concluding remarks

The aim of this paper was first motivated by answering Problem 11.14 in [4], but we have also elaborated on this type of issues and applied other forcing techniques. These results might just be the tip of the iceberg of a potentially rather interesting research project, in order to use tools from infinitary combinatorics, forcing theory and descriptive set theory, to give a theoretical structure to the several social welfare relations on infinite utility streams defined in economic theory. We mention that in another working paper in preparation [8], together with Ram Sewak Dubey we also analyse the SWRs concerning intermediate and weak Pareto and their relation with the Baire property, by using a different method using a variant of Lauwers' technique and introducing a notion of a special Mathias-Silver tree.

Other economic combinatorial principles which can be investigated are those *á la Hammond*: given infinite utility streams $x, y \in X = Y^\omega$, we say that $x \leq_H y$ whenever there are $i \neq j$ such that $x(i) < y(i) < y(j) < x(j)$ and for all $k \neq i, j$, $x(k) = y(k)$. Intuitively this type of pre-orders assert that a stream is better than another one if the distribution is less unequal (think of the interpretation of a stream as a distribution of income or wealth).

So we can distinguish two different paths to elaborate on.

A first path, comparing different types of social welfare relations, in particular with respect to the following three categories: equity principles (e.g. finite anonymity), efficiency principles (e.g. Pareto), egalitarian principles (e.g. Hammond), and describe a hierarchy of such relations based on the associated fragment of AC. From a pure theoretical point of view, this may suggest a ranking-method among combinations of the three kinds of principles, analysing a degree of compatibility between them.

A second path, deserving further investigations is represented by the case of uncountable streams. Indeed, papers in theoretical economics about the study of free markets with infinitely many traders, were motivated by the idea of considering (from the macroeconomic point of view) a market with *uncountably many* traders (see the pioneering [3]). Hence, it is worth elaborating on the study of SWRs on utility streams with length $\kappa > \omega$.

Appendix: on Shelah's amalgamation

Let B be a complete Boolean algebra and $A \lessdot B$. The *projection* map $\pi : B \to A$ is defined by $\pi(b) = \prod\{b \leq a : a \in A\}$.

Let B be a complete Boolean algebra and B_1, B_2 two isomorphic complete subalgebras of B and ϕ_0 the isomorphism between them. One defines the *amalgamation of B over ϕ_0*, say $\mathsf{Am}(B, \phi_0)$, as the complete Boolean algebra generated by the following set: $\{(b', b'') \in B \times B : \phi_0(\pi_1(b')) \cdot \pi_2(b'') \neq \mathbf{0}\}$, where $\pi_j : B \to B_j$ is the projection, for $j = 1, 2$. Consider on $\mathsf{Am}(B, \phi_0)$ simply the product order. One can easily see that $e_j : B \to \mathsf{Am}(B, \phi_0)$ such that

$$e_1(b) = (b, \mathbf{1}) \text{ and } e_2(b) = (\mathbf{1}, b)$$

are both complete embeddings ([12], lemma 3.1), and for any $b_1 \in B_1$, one can show that $(b_1, \mathbf{1})$ is equivalent to $(\mathbf{1}, \phi_0(b_1))$; indeed, assume $(a', a'') \leq (b_1, \mathbf{1})$ and (a', a'') incompatible with $(\mathbf{1}, \phi_0(b_1))$ (in $\mathsf{Am}(B, \phi_0)$). The former implies $\pi_1(a') \leq b_1$, while the latter implies $\pi_2(a'') \cdot \phi_0(b_1) = \mathbf{0}$, and hence one obtains $\phi_0(\pi_1(a')) \cdot \pi_2(a'') = \mathbf{0}$, which means that the pair (a', a'') does not belong to the amalgamation.

Moreover, if one considers $f_1 : e_2[B] \to e_1[B]$ such that, for every $b \in B$, $f_1(\mathbf{1}, b) = (b, \mathbf{1})$, one obtains an isomorphism between two copies of B into $\mathsf{Am}(B, \phi_0)$, such that f_1 is an extension of ϕ_0 (since for every $b_1 \in B_1$, $e_1(b_1) = (b_1, \mathbf{1}) = (\mathbf{1}, \phi_0(b_1)) = e_2(\phi_0(b_1))$, which means $e_1 \restriction B_1 = e_2 \circ \phi_0$).

We can thus consider $e_1[B], e_2[B]$ as two isomorphic complete subalgebras of $\mathsf{Am}(B, \phi_0)$, and then repeat the same procedure to construct

$$\mathsf{Am}^2(B, \phi_0) := \mathsf{Am}(\mathsf{Am}(B, \phi_0), f_1)$$

and f_2 the isomorphism between two copies of $\mathsf{Am}(B, \phi_0)$ extending f_1. It is clear that one can continue such a construction, in order to define, for every

$n \in \omega$,

$$\mathsf{Am}^{n+1}(B, \phi_0) := \mathsf{Am}(\mathsf{Am}^n(B, \phi_0), f_n)$$

and f_{n+1} the isomorphism between two copies of $\mathsf{Am}^n(B, \phi_0)$ extending f_n. Finally, let $\mathsf{Am}^\omega(B, \phi_0)$ be the Boolean completion of direct limit of $\mathsf{Am}^n(B, \phi_0)$'s, and $\phi = \lim_{n \in \omega} f_n$. One obtains $B_1, B_2 \lessdot \mathsf{Am}^\omega(B, \phi_0)$ and ϕ automorphism of $\mathsf{Am}^\omega(B, \phi_0)$ extending ϕ_0. (N.B.: it is common in this framework to abuse terminology by referring to the *Boolean completion* of the direct limit of a sequence of Boolean algebras simply as their *direct limit*, and thus we write $\lim_{\alpha<\lambda} B_\alpha$ for the direct limit understood in this way.)

Note that the *one-step amalgamation* $\mathsf{Am}(B, \phi_0)$ is forcing equivalent to a two step iteration $B_1 * (B/B_1 \times B/B_2)$, where remind $B_2 := \phi_0[B_1]$ and $B/B_1, B/B_2$ denote the quotient-algebras. More precisely we have

Lemma 11. *Let H be a B_1-generic filter over the ground model V. Then*

$$V[H] \models (B/B_1)^H \times (B/B_2)^H \text{ densely embeds into } (\mathsf{Am}(B, \phi_0)/e_1[B_1])^H$$

For a proof one can see [12, Lemma 3.2].

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