

On non-constructive nature of ethical social welfare orders

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April 14, 2020

Abstract

In this paper we study the non-constructive nature of ethical social welfare orders on infinite utility streams. Relying on the existence of a set without Baire property, we show that any social welfare order satisfying anonymity and strong equity axioms requires the axiom of choice, and so it is non-constructive, for every non-trivial domain Y . We further prove that the social welfare orders satisfying (a) anonymity and infinite Pareto (for all non-trivial domain Y); or (b) anonymity and weak Pareto (for $Y =$ set of integers) imply the existence of a non-special Silver set, which is a specific type of non-Baire set. Since non-special Silver sets are identified as stronger non-constructive objects in the mathematical logic literature than non-Ramsey and non-Lebesgue sets, we provide even more compelling evidence that ethical social welfare orders are not useful policy tools.

Journal of Economic Literature Classification Numbers: D60, D70, D90.

Keywords and Phrases: Anonymity, Baire Property, Infinite Pareto, Non-Special Silver Set, Strong Equity, Weak Pareto.

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1 INTRODUCTION

This paper studies the nature of social welfare orders satisfying inter-generational equity and efficiency axioms along the lines of Fleurbaey and Michel (2003). The literature on inter-generational equity deals with the social choices on the distribution of resources among current and (infinite) future generations, which seek to include the welfare of all generations. The infinite welfare streams are known as collection of utility sequences and take the form of $X \equiv Y^{\mathbb{N}}$ with an element x denoted by (x_1, x_2, \dots) where $x_i \in Y$ for all $i \in \mathbb{N}$, and $Y \subset \mathbb{R}$. In a seminal paper, Ramsey (1928) observed that discounting one generation's utility relative to another's is "ethically indefensible", and something that "arises merely from the weakness of the imagination". Diamond (1965) formalized this idea as a notion of "equal treatment" of all generations (present and future) in the form of an *Anonymity* axiom on social preferences. The axiom stipulates that a society should be indifferent between two utility streams, if one is obtained from the other by interchanging the levels of well-being of any two generations. It is an example of *procedural* equity, i.e., the changes involved in the utilities of generations do not alter the distribution of utilities in the utility stream.

A social planner concerned with the welfare of all generations, present and infinite future, has to deal with the question of evaluating infinite utility streams consistently with social preferences which respect the chosen equity axioms. If the equity principle involved is the anonymity axiom, then ranking infinite utility streams presents no challenge as a social welfare relation which evaluates all infinite utility streams as indifferent satisfies the anonymity axiom trivially. However, such social preference relations are of no practical interest. Clearly, we would like the social preference relation to exhibit *some* sensitivity (in addition to indifference generated by anonymity axiom) to individual utility levels in the infinite utility streams. This sensitivity is usually captured in some form of the efficiency (Pareto) principle: society should prefer one infinite utility stream to another if at least one generation is better off and no generation is worse off in the former compared to the latter. However, as soon as we add any sensitivity requirement (in the nature of Pareto condition) to the anonymity axiom, fundamental difficulties emerge in the consistent evaluation of infinite utility streams which have been well documented in the literature.

Diamond (1965) showed that no real valued representation (called social welfare function) of the utility streams satisfying anonymity, strong Pareto and sup-norm continuity axioms exists. Subsequent researches refined as well as expanded this result by either eliminating the continuity axiom which need not be merely a technical assumption in infinite generations framework or weakening the sensitivity requirements. Basu and Mitra (2003) showed the negative outcome persists even when we drop the continuity axiom and require the set $Y \subset \mathbb{R}$ to be the smallest non-trivial set (containing only two distinct elements, 0 and 1). Crespo et al. (2009) showed weakening the efficiency condition of strong Pareto to infinite Pareto does not alter the outcome. Dubey and Mitra (2011) replaced the sensitivity condition to weak Pareto and characterized the set Y for the existence of social welfare functions satisfying anonymity and weak Pareto axioms.

Negative result on the real valued representation front called for another approach in the consistent evaluation of the infinite utility streams satisfying principles of inter-generational equity, namely, devising pair-wise ranking rules i.e., complete binary ranking rules (called social welfare orders).

Svensson (1980), the first paper to initiate this approach, proved that a social welfare order satisfying anonymity, strong Pareto and a continuity axiom exists for infinite utility streams with $Y = [0, 1]$. Fleurbaey and Michel (2003) explored the possibility of social planners using this social welfare order for making the welfare decisions. After a careful analysis of the mathematical assumption and techniques employed in the proof of the existence result, they conjectured that “there exists no explicit description (that is, avoiding the axiom of choice or similar contrivances) of an ordering which satisfies the anonymity and weak Pareto axioms”. Thus if the conjecture is found to be true, then these social welfare orders would turn out to be “non-constructive”.

The conjecture was confirmed in several contributions. Zame (2007) proved the non-constructive nature of the social welfare orders satisfying both anonymity and strong Pareto on every non-trivial domain Y ; and anonymity and weak Pareto on the standard domain $Y = [0, 1]$, based on existence on a non-measurable set. Lauwers (2010) showed the non-constructive character of social welfare orders satisfying anonymity and intermediate Pareto by relying on the idea of non-Ramsey sets. Dubey (2011) refined the result of Zame (2007) for social welfare order satisfying anonymity and weak Pareto by characterizing the sets $Y \subset [0, 1]$ for which the social welfare orders are non-constructive following the approach introduced in Lauwers (2010). Laguzzi (2020) employed the existence of a non-Baire set to establish the non-constructive character of social welfare orders satisfying anonymity and strong Pareto for all non-trivial domains. The first contribution of this paper adds to the pool of social welfare orders with no explicit description. We examine social welfare orders satisfying anonymity and equity condition known as strong equity (SE). A brief description of the equity concept is called for before proceeding further.

The strong equity axiom belongs to the class of *consequentialist* equity notions. It was introduced by d’Aspremont and Gevers (1977), who referred to it as an *extremist equity* axiom and is a strong form of the equity axiom of Hammond (1976). It deals with situations in which the *distribution* of utilities of generations has changed in specific ways and involves comparisons between two utility streams (x and y) in which all generations except two have the same utility levels in both utility streams. Regarding the two remaining generations (say, i and j), one of the generations (say i) is better off in utility stream x , and the other generation (j) is better off in utility stream y , thereby setting up a conflict. The axiom states that if for both utility streams, it is generation i which is worse off than generation j (this, of course, requires us to make inter-generational comparisons of utilities), then generation i should be allowed (on behalf of the society) to choose between x and y . That is, x is socially preferred to y , since generation i is better off in x than in y .

In this example, sensitivity is introduced via strong equity, i.e., the social welfare relation is required to show indifference to permutation of the welfare of a pair of generations and strict preference in case of re-distribution of welfare of a pair of generations in the particular manner described earlier. Given the choice of equity conditions, any non-trivial Y must contain at least four distinct elements, since strong equity comparison is not possible with three or less distinct welfare levels.

Our choice of the equity principles (anonymity and strong equity) was considered earlier in Bossert et al. (2007). They have shown that there exists a social welfare order satisfying equity preference, anonymity and strong Pareto axioms (see Bossert et al. (2007, Theorem 2)). In the presence of strong Pareto, their equity preference is equivalent to the strong equity condition.

Thus, we conclude that there exists a social welfare order satisfying strong equity, anonymity and strong Pareto axioms. However, the result has been established with the aid of a variant of Szpilrajn's lemma (Szpilrajn (1930)) given in Arrow (1963). Szpilrajn's Lemma is usually established using Zorn's Lemma, which is equivalent to the axiom of choice, hence they have utilized a non-constructive technique. Hence, whether the social welfare order satisfying anonymity and strong equity admits an explicit description remains a valid question. In Theorem 1, we show that the existence of a social welfare order for every non-trivial Y satisfying strong equity and anonymity implies the existence of a set without Baire property, which is a non-constructive object.

We next introduce the notion of a non-special Silver set, yet another non-constructive object, which we prove to be a particular type of a non-Baire set. In Theorem 2, we show that social welfare orders for every non-trivial Y satisfying anonymity and infinite Pareto imply existence of a non-special Silver set. The non-special Silver set could appear in the context of other social welfare orders as well. This is demonstrated in Theorem 3, where we prove that the social welfare orders for $Y = \mathbb{Z}$ satisfying anonymity and weak Pareto imply the existence of a non-special Silver set.

Laguzzi (2020) has examined if the existence of a non-measurable set (non-Ramsey set) imply the existence of social welfare order satisfying anonymity and weak (intermediate) Pareto axioms. It is the reverse implication of the results of Zame (2007) and Lauwers (2010). He has proved that this implication does not hold. Based on this result one could infer that the ethical social welfare orders require a strictly larger fragment of the axiom of choice as compared to non-measurable or non-Ramsey sets. In other words, ethical social welfare orders are in a sense more robust non-constructive objects as compared to non-measurable or non-Ramsey sets. It helps us in learning the non-constructive nature of equitable social welfare orders vis-vis non-measurable sets, non-Ramsey sets, non-Baire sets, non-special Silver sets and free ultrafilters in a precise sense.

The remainder of the paper is organized as follows. In section 2, we introduce notations and definitions and describe the features of non-Baire sets. In section 3 we demonstrate that the social welfare order satisfying strong equity and anonymity axioms is a non-constructive object. In section 4 relying on the idea of a non-special Silver set we show the non-constructive nature of Pareto social welfare orders satisfying anonymity axioms. We explain the structure of social welfare orders as non-constructive objects in Section 5. We conclude in section 6.

2 PRELIMINARIES

Let \mathbb{N} , \mathbb{Z} and \mathbb{R} be the set of natural numbers, integers and real numbers respectively. Let Y , a non-empty subset of \mathbb{R} , be the set of all possible utilities that any generation can achieve. Then $X \equiv Y^{\mathbb{N}}$ is the set of all possible utility streams, with an element $x \in X$ denoted by $x = (x_1, x_2, \dots)$ where $x_n \in Y$ for each $n \in \mathbb{N}$. For all $y, z \in X$, we write $y \geq z$ if $y_n \geq z_n$, for all $n \in \mathbb{N}$; $y > z$ if $y \geq z$ and $y \neq z$; and $y \gg z$ if $y_n > z_n$ for all $n \in \mathbb{N}$.

Let $Y^{<\mathbb{N}}$ be the set of finite sequences of elements from Y . Given $\sigma \in Y^{<\mathbb{N}}$, the length of σ is denoted by $|\sigma|$. Given $\sigma, \tau \in Y^{<\mathbb{N}}$, we write $\sigma \subseteq \tau$ if and only if $|\sigma| \leq |\tau|$ and for all $n < |\sigma|$, $\sigma_n = \tau_n$. Analogously, in case $\sigma \in Y^{<\mathbb{N}}$ and $x \in X$, we write $\sigma \subseteq x$ if and only if $\forall n < |\sigma|$, $\sigma_n = x_n$.

We consider binary relations on X , denoted by \succsim , with symmetric and asymmetric parts denoted

by \sim and \succ respectively, defined in the usual way. A *social welfare order* (SWO) is a complete and transitive binary relation. We recall the definition of permutation \mathcal{P} , a self map on the set of natural number:

$$\mathcal{P} := \{\pi: \mathbb{N} \rightarrow \mathbb{N} \mid \pi \text{ is a bijection}\}.$$

We denote the set of finite permutations by \mathcal{F} , i.e.,

$$\mathcal{F} := \{\pi \in \mathcal{P} \mid \pi(n) = n \text{ for all but finitely many } n \in \mathbb{N}\}.$$

2.1 Equity and Pareto Axioms

The social welfare orders that we will be concerned with are required to satisfy following equity and Pareto axioms.

Definition 1. Anonymity (AN henceforth): If $x, y \in X$, and there exist $i, j \in \mathbb{N}$ such that $y_j = x_i$ and $x_j = y_i$, while $y_k = x_k$ for all $k \in \mathbb{N} \setminus \{i, j\}$, then $x \sim y$.

Definition 2. Strong Equity (SE henceforth): If $x, y \in X$, and there exist $i, j \in \mathbb{N}$, such that $y_j > x_j > x_i > y_i$ while $y_k = x_k$ for all $k \in \mathbb{N} \setminus \{i, j\}$, then $x \succ y$.

Definition 3. Infinite Pareto (IP henceforth): Given $x, y \in X$, if $x \geq y$ and $x_i > y_i$ for infinitely many $i \in \mathbb{N}$, then $x \succ y$.

Definition 4. Weak Pareto (WP henceforth): Given $x, y \in X$, if $x_i > y_i$ for all $i \in \mathbb{N}$, then $x \succ y$.

2.2 Non-Baire Set

In this sub-section, we describe some useful notions from topology and descriptive set theory relevant to our result in section 3. Recall that the symmetric difference between two sets, A and B is $A \triangle B := (A \setminus B) \cup (B \setminus A)$.

Definition 5. Let X be a topological space.

- (a) $N \subseteq X$ is called *nowhere dense* if the interior of its closure is empty.
- (b) $M \subseteq X$ is called *meager* if it is a countable union of nowhere dense sets. The complement of a meager set is called *comeager*.
- (c) $S \subseteq X$ has the *Baire property* if there is an open set $O \subseteq X$ such that $S \triangle O$ is meager.

We refer to a set satisfying the Baire property as a *Baire set*. A set that does not satisfy condition (c) above is called a *non-Baire set*. Well-known arguments prove that the Axiom of Choice (AC) implies the existence of a non-Baire set, while Solovay (1970) established that there exists a model of ZF theory without AC where every set has the Baire property. Hence a non-Baire set is considered a non-constructive object. In the proof below we implicitly use the following well-known result. See Kechris (1995, Proposition 8.26) for a proof.

Lemma 1. Assume $S \subseteq X$ has the Baire property. Then either S is meager or there is a basic open set $U \subseteq X$ such that S is comeager in U , i.e., $S \cap U$ is comeager with respect to the relative topology \upharpoonright_U .

In section 3 we will often use the following well-known result. See Kechris (1995, Theorem 8.41) for a detailed proof.

Kuratowski-Ulam Theorem. Let X, Y be second countable topological spaces, i.e., spaces whose topology has a countable base. Let $A \subseteq X \times Y$ have the Baire property. Denote $A_y := \{x \in X : (x, y) \in A\}$ and $A^x := \{y \in Y : (x, y) \in A\}$. Then the following hold:

1. $\{y : A_y \text{ has the Baire property}\}$ is comeager in Y and $\{x : A^x \text{ has the Baire property}\}$ is comeager in X .
2. A is meager $\Leftrightarrow \{y : A_y \text{ is meager}\}$ is comeager in $Y \Leftrightarrow \{x : A^x \text{ is meager}\}$ is comeager in X .
3. A is comeager $\Leftrightarrow \{y : A_y \text{ is comeager}\}$ is comeager in $Y \Leftrightarrow \{x : A^x \text{ is comeager}\}$ is comeager in X .

2.3 Non-special Silver set

We introduce the notion of a tree from combinatorial set theory in order to define the notion of a special Silver set. Let $Y := \{0, 1\}$ or $Y := \mathbb{N}$. A subset $T \subseteq Y^{<\mathbb{N}}$ is called a *tree* if and only if for every $t \in T$ every $s \subseteq t$ is in T too, in other words, T is *closed* under initial segments. We call the segments $t \in T$ the *nodes* of T and denote the *length* of the node by $|t|$. A node $t \in T$ is called *splitting* if there are two distinct $n, m \in Y$ such that $t \frown n, t \frown m \in T$. Given $x \in X \equiv Y^{\mathbb{N}}$ and $n \in \mathbb{N}$, we denote by $x \upharpoonright n$ the cut of x of length n , i.e., $x \upharpoonright n := \langle x_1, x_2, \dots, x_n \rangle$. A tree $p \subseteq 2^{<\mathbb{N}}$ is called *perfect* if and only if for every $s \in p$ there exists $t \supseteq s$ splitting. Hence, any perfect tree is infinite and has an infinite height, i.e., there is no $n \in \mathbb{N}$ such that for every $t \in p$ one has $|t| < n$. We define $[p] := \{x \in X : \forall n \in \mathbb{N} (x \upharpoonright n \in p)\}$, and $x \in [p]$ is called a *branch* of p .

A tree $p \subseteq 2^{<\mathbb{N}}$ is called *Silver tree* if and only if p is perfect and for every $s, t \in p$, with $|s| = |t|$ one has $s \frown 0 \in p \Leftrightarrow t \frown 0 \in p$ and $s \frown 1 \in p \Leftrightarrow t \frown 1 \in p$. We could also define Silver tree p and its corresponding set of branches $[p]$ relying on the notion of partial functions. Consider a partial function $f : \mathbb{N} \leftrightarrow \{0, 1\}$ such that $\text{dom}(f)$ is co-infinite (i.e. the complement of the domain of f is infinite); then define $N_f := \{x \in 2^{\mathbb{N}} : \forall n \in \text{dom}(f) (f(n) = x(n))\}$. It easily follows from the definitions that there is a one-to-one correspondence between every Silver tree p and a set N_f . Given any Silver tree p there is a unique partial function $f : \mathbb{N} \leftrightarrow \{0, 1\}$ such that $[p] = N_f$. In particular, the set of splitting levels $S(p)$ correspond to $\mathbb{N} \setminus \text{dom}(f)$.¹ Let $p \subseteq 2^{<\mathbb{N}}$ be a Silver tree, let $S(p)$ denote the set of splitting levels of p , let $U(p) := \{n \in \mathbb{N} : \forall x \in [p] (x(n) = 1)\}$ and let $\{n_k : k \in \mathbb{N}\}$ enumerate the set $S(p) \cup U(p)$.

¹Both notations and definitions are common in the set theory literature. Choice between the two usually depends on which one is more convenient for the technical details. We have opted for the tree-like notation as it is more appropriate to develop our proofs. It is important to note that our arguments could easily be replaced with the definition involving the N_f instead of Silver tree p .

Definition 6. A Silver tree $p \subseteq 2^{<\mathbb{N}}$ is called a *special Silver tree* ($p \in \mathbb{S}\mathbb{V}$) if and only if there are infinitely many even $(n_{k_i}, n_{k_i+1}, n_{k_i+2})$'s such that:

(1) for all $i \in \mathbb{N}$, $n_{k_i}, n_{k_i+1}, n_{k_i+2}$ are splitting levels with $n_{k_i} + 1 < n_{k_i+1}$ and $n_{k_i+1} + 1 < n_{k_i+2}$;

(2) for all $i \in \mathbb{N}, j \in \mathbb{N}, t \in p$ ($n_{k_i} < j \leq n_{k_i+1} \vee n_{k_i+1} < j \leq n_{k_i+2} \Rightarrow t(j) = 0$).

We call $(n_{k_i}, n_{k_i+1}, n_{k_i+2})$ satisfying (1) and (2) a *Mathias triple*.

Definition 7. A set $X \subseteq 2^{\mathbb{N}}$ is called *special Silver set* if and only if there exists a special Silver tree ($p \in \mathbb{S}\mathbb{V}$) such that $[p] \subseteq X$ or $[p] \cap X = \emptyset$. A set $X \subseteq 2^{\mathbb{N}}$ not satisfying this condition is called a *non-special Silver set*.

The following lemma is a first step in order to prove that any special Silver set satisfies the Baire property.

Lemma 2. *Given any comeager set $C \subseteq 2^{\mathbb{N}}$ there exists $p \in \mathbb{S}\mathbb{V}$ such that $[p] \subseteq C$.*

Proof. Let $D_n : n \in \mathbb{N}$ be a \subseteq -decreasing sequence of open dense sets such that $\bigcap_{n \in \mathbb{N}} D_n \subseteq C$.

Recall that if D is open dense, then $\forall s \in 2^{<\mathbb{N}}$ there exists $s' \supseteq s$ such that $[s'] \subseteq D$. We construct $p \in \mathbb{S}\mathbb{V}$ by recursively building up its nodes as follows: first of all let

$$\begin{aligned} s_1 &= (10100), s_2 = (10101), s_3 = (10000), s_4 = (10001), \\ s_5 &= (00000), s_6 = (00100), s_7 = (00101), s_8 = (00001). \end{aligned}$$

- Pick $t_\emptyset \in 2^{<\mathbb{N}}$ such that $[t_\emptyset] \subseteq D_0$, and then let $F_0 := \bigcup_{k=1}^8 \{t_\emptyset \hat{\ } s_k\}$ and T_0 be the downward closure of F_0 , i.e., $T_0 := \{s \in 2^{<\mathbb{N}} : \exists t \in F_0 (s \subseteq t)\}$;
- Assume F_n is already defined. Let $\{t_j : j \leq J\}$ enumerate all nodes in F_n (note by construction $J = 8^{n+1}$). We proceed inductively as follows: pick $r_0 \in 2^{<\mathbb{N}}$ such that $[t_\emptyset \hat{\ } r_0] \subseteq D_{n+1}$; then pick $r_1 \supseteq r_0$ such that $[t_1 \hat{\ } r_1] \subseteq D_{n+1}$; proceed inductively in this way for every $j \leq J$, so $r_j \supseteq r_{j-1}$ such that $[t_j \hat{\ } r_j] \subseteq D_{n+1}$. Finally put $r = r_J$. Then define

$$F_{n+1} := \bigcup \left\{ t \hat{\ } r \hat{\ } s_k : t \in F_n, k = 1, 2, \dots, 8 \right\} \text{ and } T_{n+1} := \left\{ s \in 2^{<\mathbb{N}} : \exists t \in F_{n+1} (s \subseteq t) \right\}.$$

Note that by construction, for all $t \in F_{n+1}$ we have $[t] \subseteq D_{n+1}$. Finally put $p := \bigcup_{n \in \mathbb{N}} T_n$. Then by construction $p \in \mathbb{S}\mathbb{V}$ as it is a Silver tree and the usage of s_1, s_2, \dots, s_8 ensures that p contains infinitely many Mathias triples. It is left to show $[p] \subseteq \bigcap_{n \in \mathbb{N}} D_n$. For this, fix arbitrarily $x \in [p]$ and $n \in \mathbb{N}$. By construction there is $t \in F_n$ such that $t \subset x$. Since $[t] \subseteq D_n$ we then get $x \in [t] \subseteq D_n$. \square

Corollary 1. *If $A \subseteq 2^{\mathbb{N}}$ satisfies the Baire property, then A is a special Silver set.*

Proof. The proof is a simple application of Lemma 1. If A is meager, then apply lemma 2 to the complement of A and find $p \in \mathbb{S}\mathbb{V}$ such that $[p] \cap A = \emptyset$. If there exists $t \in 2^{<\mathbb{N}}$ such that A is comeager in $[t]$, then we can use the construction as in lemma 2 in order to find $p \in \mathbb{S}\mathbb{V}$ such that $[p] \subseteq A$. \square

3 STRONG EQUITY, ANONYMITY AND BAIRE PROPERTY

In this section, we show that existence of any social welfare order satisfying strong equity and anonymity implies existence of a set without Baire property. Therefore, the social welfare order is non-constructive in nature. We take set $Y = \{a, b, c, d\}$ with $a < b < c < d$ to be endowed with the discrete topology and on the space $X = Y^{\mathbb{N}}$ we set the product topology, i.e., the topology generated by the basic open sets of the form

$$N_{\sigma} := \{x \in X : \sigma \subseteq x\}, \text{ where } \sigma \in Y^{<\mathbb{N}}.^2 \quad (1)$$

Theorem 1. *Let \succsim denote a social welfare order satisfying strong equity and anonymity on $X = Y^{\mathbb{N}}$ where $Y = \{a, b, c, d\}$ with $a < b < c < d$. Then there exists a subset of X without Baire property.*

Proof. We consider the following subsets of X .

$$E := \{(x, y) \in X \times X : x \succ y\},$$

$$D := \{(x, y) \in X \times X : y \succ x\},$$

and

$$A := \{(x, y) \in X \times X : x \not\succeq y \wedge y \not\succeq x\}.$$

Note that the function $\gamma : E \rightarrow D$, $\gamma(x, y) := (y, x)$ maps meager (comeager) sets into meager (comeager) sets. Hence E is meager $\Leftrightarrow D$ is meager. Since $E \cap D = \emptyset$, both of them together cannot be comeager. We prove the claim in two steps.

1. If E has the Baire property, then E is meager.
2. Given the social welfare order \succsim , the set E or D do not have the Baire property.

Step 1. We show that E is meager, and note that an analogous argument works for D as well. Since we have assumed that E has the Baire property, we can therefore find a Borel set $B \subseteq E$ such that $E \setminus B$ is meager. Moreover, for every $\pi, \pi' \in \mathcal{F}$, in the family of finite permutations, we can define

$$B(\pi, \pi') := \{(f_{\pi}(x), f_{\pi'}(y)) : (x, y) \in B\}.$$

Let

$$B^* := \bigcup \{B(\pi, \pi') : \pi, \pi' \in \mathcal{F}\}$$

²Observe that this notation is in line with the definition of the set N_f presented in the previous section, in particular by viewing a finite sequence $\sigma \in Y^{<\mathbb{N}}$ as a partial function $\sigma : \mathbb{N} \rightarrow Y$ with finite domain.

and note that $B^* \subseteq E$. Finally note also that $E \setminus B^*$ is meager. Let

$$B_y^* := \{x \in X : (x, y) \in B^*\},$$

$$I_0 := \{y : B_y^* \text{ is meager}\} \text{ and } I_1 := \{y : B_y^* \text{ is comeager}\}.$$

Note that each B_y^* is invariant under finite permutations, i.e.,

$$x \in B_y^* \Leftrightarrow f_\pi(x) \in B_y^*,$$

where $\pi \in \mathcal{F}$. We claim B_y^* is either meager or comeager. If not, then there are $\sigma, \tau \in Y^{<\mathbb{N}}$ such that B_y^* is comeager in N_σ and B_y^* is meager in N_τ . Without loss of generality, we can assume $|\sigma| = |\tau| = k$. Let $\pi \in \mathcal{F}$ be as follows:

$$\pi(n) := \begin{cases} |\sigma| + n & \text{if } n \leq |\sigma|, \\ n - |\sigma| & \text{if } |\sigma| < n \leq |\sigma| + |\tau|, \\ n & \text{otherwise.} \end{cases} \quad (2)$$

Put $\rho_0 = \sigma \hat{\ } \tau := \langle \sigma_1, \sigma_2, \dots, \sigma_k, \tau_1, \tau_2, \dots, \tau_k \rangle$ and $\rho_1 = \tau \hat{\ } \sigma := \langle \tau_1, \tau_2, \dots, \tau_k, \sigma_1, \sigma_2, \dots, \sigma_k \rangle$. Note that π provides a meager-preserving homeomorphism between N_{ρ_0} and N_{ρ_1} , i.e., $f_\pi[N_{\rho_0}] = N_{\rho_1}$ and both (a) meager sets are mapped into meager sets, and (b) pre-images of meager sets are meager. Moreover, since $\rho_0 \supseteq \sigma$ and $\rho_1 \supseteq \tau$, we obtain

$$B_y^* \text{ is comeager in } N_{\rho_0} \text{ and } B_y^* \text{ is meager in } N_{\rho_1}.$$

But then we should get

- on the one hand, $f_\pi[B_y^*] = B_y^*$ is meager in N_{ρ_1} ;
- on the other hand, $f_\pi[B_y^*] = B_y^*$ is comeager in $f_\pi[N_{\rho_0}] = N_{\rho_1}$,

providing us with a contradiction. Hence $I_0 \cup I_1 = X$. We also observe that both I_0 and I_1 are invariant under $\pi \in \mathcal{F}$. In fact, it is straightforward to check that if $\pi \in \mathcal{F}$ and B_y^* is meager, then $B_{f_\pi(y)}^*$ is meager too (and the same holds for comeager sets). By Kuratowski-Ulam theorem, only two options are possible: either I_0 is comeager or I_1 is comeager. By previous observation, we know E cannot be comeager and so it cannot be the case that I_1 is comeager. Hence I_0 has to be comeager, and that implies E is meager (follows by applying Kuratowski-Ulam theorem once more).

Step 2. We give a proof for $Y := \{a, b, c, d\}$ with the ordering $a < b < c < d$. In order to reach a contradiction assume both E and D have the Baire property. Hence, relying on result in Step 1 and since we are assuming \succsim be total, we know that $A = \{(x, y) : x \sim y\}$ has to be comeager.

Let $A_y := \{x \in X : (x, y) \in A\}$, with $y \in A$ chosen in such a way that A_y be comeager.

Consider the set

$$H := \{x \in X : (x_1 = x_2 = a \wedge x_3 = x_4 = d) \vee (x_1 = a \wedge x_2 = b \wedge x_3 = c \wedge x_4 = d)\}.$$

Then define the function $\phi : H \rightarrow X$ as follows:

$$\phi(x)_n := \begin{cases} a & \text{if } n = 1 \text{ and } x_1 = a \text{ and } x_2 = a, \\ b & \text{if } n = 1 \text{ and } x_1 = a \text{ and } x_2 = b, \\ b & \text{if } n = 2, \\ c & \text{if } n = 3, \\ c & \text{if } n = 4 \text{ and } x_3 = c \text{ and } x_4 = d, \\ d & \text{if } n = 4 \text{ and } x_3 = d \text{ and } x_4 = d, \\ x_n & \text{if } n \geq 5. \end{cases} \quad (3)$$

Roughly speaking ϕ maps the initial 4-tuple $\langle a, a, d, d \rangle$ into $\langle a, b, c, d \rangle$ and the initial 4-tuple $\langle a, b, c, d \rangle$ into $\langle b, b, c, c \rangle$. Since the remaining tail of any given $x \in X$ is not changed by ϕ , it follows $x \prec \phi(x)$. Since

$$\phi[H] := \{x \in X : (x_1 = a \wedge x_2 = b \wedge x_3 = c \wedge x_4 = d) \vee (x_1 = x_2 = b \wedge x_3 = x_4 = c)\}, \quad (4)$$

we get,

$$H \cap \phi[H] = \{x \in X : x_1 = a \wedge x_2 = b \wedge x_3 = c \wedge x_4 = d\}$$

and so in particular it is not meager and it is actually the basic open set N_σ , with $\sigma = \langle a, b, c, d \rangle$. Recall that if $x \in A_y$, then $x \sim y$; but also for every $x \in H$, $x \prec \phi(x)$, by definition of strong equity. Hence, we have the following two mutually contradictory consequences.

- On the one side, $H \cap A_y \cap \phi[H \cap A_y] = \emptyset$; indeed if there exists $z \in H \cap A_y \cap \phi[H \cap A_y]$, then there is $x \in H \cap A_y$ such that $z := \phi(x)$; then on the one hand we have $z \sim y$, but on the other hand we have $x \in H \cap A_y$ that in turn gives $x \sim y$ and so together with $x \prec \phi(x) = z$ we would get $y \prec z$; contradiction.
- On the other side, $H \cap A_y \cap \phi[H \cap A_y]$ cannot be meager, since $H \cap \phi[H]$ is a basic open set, $H \cap A_y$ is comeager in H and $\phi[H \cap A_y]$ is comeager in $\phi[H]$.

□

4 Pareto, Anonymity and non-special Silver sets

In this section we examine the non-constructive property of social welfare orders satisfying Pareto and anonymity axioms. The non-constructive object we rely on for the analysis is the non-special Silver set. The social welfare orders on $X = 2^{\mathbb{N}}$ satisfy infinite Pareto and anonymity are considered

in the sub-section 4.1. The social welfare orders on $X = \mathbb{Z}^{\mathbb{N}}$ satisfy weak Pareto and anonymity are considered in the sub-section 4.2.

4.1 Infinite Pareto and Anonymity

Given $x \in 2^{\mathbb{N}}$, let $U(x) := \{n \in \mathbb{N} : x(n) = 1\}$ and $\{n_k : k \in \mathbb{N}\}$ enumerate $U(x)$. Define

$$\begin{aligned} o(x) &:= [n_1, n_2] \cup [n_3, n_4] \cdots [n_{2k+1}, n_{2k+2}] \cup \cdots \\ e(x) &:= [n_2, n_3] \cup [n_4, n_5] \cdots [n_{2k+2}, n_{2k+3}] \cup \cdots \end{aligned} \quad (5)$$

As usual we identify subsets of \mathbb{N} with their characteristic functions, so that we can write $o(x), e(x) \in 2^{\mathbb{N}}$.

Theorem 2. *Let \sqsubseteq denote a social welfare order satisfying infinite Pareto and anonymity on $X = 2^{\mathbb{N}}$. Then there exists a subset of X which is a non-special Silver set.*

Proof. Let $\Gamma := \{x \in 2^{\mathbb{N}} : e(x) \sqsubseteq o(x)\}$. We show Γ is a non-special Silver set. Given any $p \in \mathbb{S}\mathbb{V}$, let $\{n_k : k \in \mathbb{N}\}$ enumerate all natural numbers in $S(p) \cup U(p)$. To prove our claim, we aim to find $x, z \in [p]$ such that $x \in \Gamma \Leftrightarrow z \notin \Gamma$. We pick $x \in [p]$ such that for all $n_k \in S(p) \cup U(p)$, $x(n_k) = 1$. Let $\left\{ \left(n_{m_j}, n_{m_j+1}, n_{m_j+2} \right) : j \in \mathbb{N} \right\}$ list all Mathias triples in p . Note that all elements in the Mathias triples are splitting levels, and so they can be dropped from x . We need to consider three cases.

(1) $e(x) \sqsubseteq o(x)$: We remove $n_{m_1+1}, n_{m_j}, n_{m_j+1}$, for all $j > 1$ from $U(x)$ to obtain $z \in 2^{\mathbb{N}}$, i.e.

$$z(n) = \begin{cases} x(n) & \text{if } n \notin \left\{ n_{m_1+1}, n_{m_j}, n_{m_j+1}, \text{ for all } j > 1 \right\} \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$\begin{aligned} O(m_1) &:= [n_1, n_2] \cup [n_3, n_4] \cdots [n_{m_1-1}, n_{m_1}], \\ E(m_1) &:= [n_2, n_3] \cup [n_4, n_5] \cdots [n_{m_1}, n_{m_1+1}]. \end{aligned}$$

Observe that

- for all $n \in O(m_1)$, $e(z)(n) = 0 < 1 = o(x)(n)$ and $o(z)(n) = 1 > 0 = e(x)(n)$,
- for all $n \in E(m_1)$, $e(z)(n) = 1 > 0 = o(x)(n)$ and $o(z)(n) = 0 < 1 = e(x)(n)$,
- for all $n \in \bigcup_{j>1} [n_{m_j}, n_{m_j+1})$, $e(z)(n) = 1 > 0 = o(x)(n)$ and $o(z)(n) = 0 < 1 = e(x)(n)$,
- and
- for all remaining $n \in \mathbb{N}$, $e(z)(n) = o(x)(n)$ and $o(z)(n) = e(x)(n)$.

Denote $O(m_1) : \{k_1, k_2, \dots, k_M\}$ and the initial M elements of the infinite set $\bigcup_{j>1} [n_{m_j}, n_{m_j+1})$ by $\{k^1, \dots, k^M\}$. We permute $e(z)(k_1)$ with $e(z)(k^1)$, $e(z)(k_2)$ with $e(z)(k^2)$, continuing likewise till $e(z)(k_M)$ with $e(z)(k^M)$ to obtain $e^\pi(z)$. Further, $o^\pi(z)$ is obtained by carrying out

identical permutation on $o(z)$. Observe that $e^\pi(z)$ and $o^\pi(z)$ are finite permutations of $e(z)$ and $o(z)$ respectively. Then,

- for all $n \in O(m_1)$, $e^\pi(z)(n) = 1 = o(x)(n)$ and $o^\pi(z)(n) = 0 = e(x)(n)$,
- for all $n \in E(m_1)$, $e^\pi(z)(n) = 1 > 0 = o(x)(n)$ and $o^\pi(z)(n) = 0 < 1 = e(x)(n)$,
- for all $n \in \bigcup_{j>1} [n_{m_j}, n_{m_j+1}] \setminus \{k^1, \dots, k^M\}$, $e^\pi(z)(n) = 1 > 0 = o(x)(n)$ and $o^\pi(z)(n) = 0 < 1 = e(x)(n)$,
- for $n \in \{k^1, \dots, k^M\}$, $e^\pi(z)(n) = 0 = o(x)(n)$ and $o^\pi(z)(n) = 1 = e(x)(n)$, and
- for all remaining $n \in \mathbb{N}$, $e^\pi(z)(n) = o(x)(n)$ and $o^\pi(z)(n) = e(x)(n)$.

Observe that anonymity implies

$$e^\pi(z) \sim e(z) \text{ and } o^\pi(z) \sim o(z). \quad (6)$$

Further, applying infinite Pareto axiom, we get

$$o(x) \sqsubset e^\pi(z) \text{ and } o^\pi(z) \sqsubset e(x). \quad (7)$$

Combining (6) and (7) and transitivity, we get

$$o(z) \sim o^\pi(z) \sqsubset e(x) \sqsubset o(x) \sqsubset e^\pi(z) \sim e(z) \rightarrow o(z) \sqsubset e(z),$$

which implies

$$z \notin \Gamma.$$

(2) $o(x) \sqsubset e(x)$: We remove $n_{m_1}, n_{m_j+1}, n_{m_j+2}$, for all $j > 1$ from $U(x)$ to obtain $z \in 2^{\mathbb{N}}$, i.e.,

$$z(n) = \begin{cases} x(n) & \text{if } n \notin \{n_{m_1}, n_{m_j+1}, n_{m_j+2}, \text{ for all } j > 1\} \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$\begin{aligned} O(m_1) &:= [n_1, n_2] \cup [n_3, n_4] \cdots [n_{m_1-1}, n_{m_1}], \\ E(m_1) &:= [n_2, n_3] \cup [n_4, n_5] \cdots [n_{m_1-2}, n_{m_1-1}]. \end{aligned}$$

Then,

- for all $n \in O(m_1)$, $e(z)(n) = 0 < 1 = o(x)(n)$ and $o(z)(n) = 1 > 0 = e(x)(n)$,
- for all $n \in E(m_1)$, $e(z)(n) = 1 > 0 = o(x)(n)$ and $o(z)(n) = 0 < 1 = e(x)(n)$,
- for all $n \in \bigcup_{j>1} [n_{m_j+1}, n_{m_j+2}]$, $e(z)(n) = 0 < 1 = o(x)(n)$ and $o(z)(n) = 1 > 0 = e(x)(n)$,
- and

for all remaining $n \in \mathbb{N}$, $e(z)(n) = o(x)(n)$ and $o(z)(n) = e(x)(n)$.

Denote $E(m_1) : \{k_1, k_2, \dots, k_M\}$ and the initial M elements of the infinite set $\bigcup_{j>1} [n_{m_j+1}, n_{m_j+2})$ by $\{k^1, \dots, k^M\}$. We permute $e(z)(k_1)$ with $e(z)(k^1)$, $e(z)(k_2)$ with $e(z)(k^2)$, continuing likewise till $e(z)(k_M)$ with $e(z)(k^M)$ to obtain $e^\pi(z)$. Further, $o^\pi(z)$ is obtained by carrying out identical permutation on $o(z)$. Observe that $e^\pi(z)$ and $o^\pi(z)$ are finite permutations of $e(z)$ and $o(z)$ respectively. Then,

for all $n \in E(m_1)$, $e^\pi(z)(n) = 0 = o(x)(n)$ and $o^\pi(z)(n) = 1 = e(x)(n)$,

for all $n \in O(m_1)$, $e^\pi(z)(n) = 0 < 1 = o(x)(n)$ and $o^\pi(z)(n) = 1 > 0 = e(x)(n)$,

for all $n \in \bigcup_{j>1} [n_{m_j+1}, n_{m_j+2}) \setminus \{k^1, \dots, k^M\}$, $e^\pi(z)(n) = 0 < 1 = o(x)(n)$ and $o^\pi(z)(n) = 1 > 0 = e(x)(n)$,

for all $n \in \{k^1, \dots, k^M\}$, $e^\pi(z)(n) = 1 = o(x)(n)$ and $o^\pi(z)(n) = 0 = e(x)(n)$, and

for all remaining $n \in \mathbb{N}$, $e^\pi(z)(n) = o(x)(n)$ and $o^\pi(z)(n) = e(x)(n)$.

Observe that anonymity implies

$$e^\pi(z) \sim e(z) \text{ and } o^\pi(z) \sim o(z). \quad (8)$$

Further, applying infinite Pareto axiom, we get

$$e^\pi(z) \sqsubset o(x) \text{ and } e(x) \sqsubset o^\pi(z). \quad (9)$$

Combining (8) and (9) and transitivity, we get

$$e(z) \sim e^\pi(z) \sqsubset o(x) \sqsubset e(x) \sqsubset o^\pi(z) \sim o(z) \rightarrow e(z) \sqsubset o(z),$$

which implies

$$z \in \Gamma.$$

(3) $e(x) \sim o(x)$: We remove n_{m_j}, n_{m_j+1} , for all $j > 1$ from $U(x)$ to obtain $z \in 2^{\mathbb{N}}$, i.e.,

$$z(n) = \begin{cases} x(n) & \text{if } n \notin \{n_{m_j}, n_{m_j+1}, \text{ for all } j > 1\} \\ 0 & \text{otherwise.} \end{cases}$$

By construction we obtain $o(z)(n) \geq o(x)(n)$ and $e(z)(n) \leq e(x)(n)$ for all $n \in \mathbb{N}$. Further, for all $n \in \bigcup_{j \in \mathbb{N}} [n_{m_j}, n_{m_j+1})$, $o(z)(n) = 1 > 0 = o(x)(n)$ and $e(z)(n) = 0 < 1 = e(x)(n)$.

Hence, applying infinite Pareto axiom, we get

$$o(x) \sqsubset o(z) \text{ and } e(z) \sqsubset e(x). \quad (10)$$

Transitivity and (10) yields

$$e(z) \sqsubset o(z).$$

Therefore,

$$z \in \Gamma.$$

□

4.2 Weak Pareto and Anonymity

Given $x \in 2^{\mathbb{N}}$, let $U(x) := \{n \in \mathbb{N} : x(n) = 1\}$ and $\{n_k : k \in \mathbb{N}\}$ enumerate $U(x)$. Define

$$\begin{aligned} o(x) &:= [n_1, n_2) \cup [n_3, n_4) \dots [n_{2k+1}, n_{2k+2}) \cup \dots \\ e(x) &:= [n_2, n_3) \cup [n_4, n_5) \dots [n_{2k+2}, n_{2k+3}) \cup \dots \end{aligned}$$

We use notation

$$\begin{aligned} o_L(x) &= \{l_1, l_2, \dots : l_k < l_{k+1}, \forall k \in \mathbb{N}\} \\ o_U(x) &= \mathbb{N} \setminus o_L(x) = \{u_1, u_2, \dots : u_k < u_{k+1}, \forall k \in \mathbb{N}\}. \end{aligned}$$

and analogous notation $e_L(x)$ and $e_U(x)$. Next we define following pair of sequences of positive and negative integers.

$$o(\mathbf{x})(n) = \begin{cases} k & \text{if } n = l_k, l_k \in o_L(x) \text{ for some } k \in \mathbb{N} \\ -k & \text{if } n = u_k, l_k \in o_U(x) \text{ for some } k \in \mathbb{N}, \end{cases} \quad (11)$$

$$e(\mathbf{x})(n) = \begin{cases} k & \text{if } n = l_k, l_k \in e_L(x) \text{ for some } k \in \mathbb{N} \\ -k & \text{if } n = u_k, l_k \in e_U(x) \text{ for some } k \in \mathbb{N}, \end{cases} \quad (12)$$

Observe that $o(\mathbf{x}), e(\mathbf{x}) \in \mathbb{Z}^{\mathbb{N}}$ consists of every integer in \mathbb{Z} , with natural numbers appearing in increasing order and the negative integers appearing in decreasing order.

Example 1. Consider for illustration $x = \{0, 0, 1, 0, 1, 0, 1, \dots\}$. Then $U(x) = \{3, 5, 7, 9, \dots\}$, $o(x) = [3, 5) \cup [7, 9) \cup [11, 13) \dots$, $e(x) = [5, 7) \cup [9, 11) \cup [13, 17) \dots$. Hence

$$\begin{aligned} o^L(x) &= \{3, 4, 7, 8, 11, 12, \dots\}, \quad o^U(x) = \{1, 2, 5, 6, 9, 10, \dots\}, \\ e^L(x) &= \{5, 6, 9, 10, 13, 14, \dots\}, \quad \text{and } e^U(x) = \{1, 2, 3, 4, 7, 8, 11, 12, \dots\}. \end{aligned}$$

The corresponding integer valued sequences are

$$\begin{aligned} o(\mathbf{x}) &= \{-1, -2, 1, 2, -3, -4, 3, 4, -5, -6, 5, 6, \dots\}, \\ e(\mathbf{x}) &= \{-1, -2, -3, -4, 1, 2, -5, -6, 3, 4, -7, -8, \dots\}. \end{aligned}$$

Theorem 3. Let \sqsubseteq denote a social welfare order satisfying weak Pareto and anonymity on $X = \mathbb{Z}^{\mathbb{N}}$. Then there exists a subset of X which is a non-special Silver set.

Proof. Let \sqsubseteq be a SWO satisfying WP and AN, and put $\Gamma := \{x \in 2^{\mathbb{N}} : e(\mathbf{x}) \sqsubseteq o(\mathbf{x})\}$. Given any $p \in \mathbb{S}\mathbb{V}$, let $\{n_k : k \in \mathbb{N}\}$ enumerate all natural numbers in $S(p) \cup U(p)$. We aim to find $x, z \in [p]$ such that $x \in \Gamma \Leftrightarrow z \notin \Gamma$. We proceed as follows: pick $x \in [p]$ such that for all $n_k \in S(p) \cup U(p)$, $x(n_k) = 1$. Let $\left\{ (n_{m_j}, n_{m_j+1}, n_{m_j+2}) : j \in \mathbb{N} \right\}$ list all Mathias triples in p . As in the previous proof, note that all elements in the Mathias triples are splitting levels, and so they can be dropped from x . We need to consider three cases.

(1) $e(\mathbf{x}) \sqsubseteq o(\mathbf{x})$: We remove $n_{m_1+1}, n_{m_j}, n_{m_j+1}$, for all $j > 1$ from $U(x)$ to obtain $z \in 2^{\mathbb{N}}$ i.e.,

$$z(n) = \begin{cases} x(n) & \text{if } n \notin \{n_{m_1+1}, n_{m_j}, n_{m_j+1}, \text{ for all } j > 1\} \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$\begin{aligned} O(m_1) &:= [n_1, n_2) \cup [n_3, n_4) \cdots [n_{m_1-1}, n_{m_1}), \\ E(m_1) &:= [n_2, n_3) \cup [n_4, n_5) \cdots [n_{m_1}, n_{m_1+1}). \end{aligned}$$

Then,

for all $n \in O(m_1)$, $e(\mathbf{z})(n) < 0 < o(\mathbf{x})(n)$ and $o(\mathbf{z})(n) > 0 > e(\mathbf{x})(n)$,

for all $n \in [1, n_1)$, $0 > e(\mathbf{z})(n) = o(\mathbf{x})(n)$ and $0 > o(\mathbf{z})(n) = e(\mathbf{x})(n)$,

for all $n \in E(m_1)$, $o(\mathbf{x})(n) < 0 < e(\mathbf{z})(n)$ and $o(\mathbf{z})(n) < 0 < e(\mathbf{x})(n)$,

for all $n \geq n_{m_1+1}$ if $0 > e(\mathbf{z})(n)$ then $0 > o(\mathbf{x})(n)$ holds. Also if $o(\mathbf{x})(n) > 0$ then $e(\mathbf{z})(n) > 0$. Similarly, if $0 > e(\mathbf{x})(n)$ then $0 > o(\mathbf{z})(n)$ holds. Also if $o(\mathbf{z})(n) > 0$ then $e(\mathbf{x})(n) > 0$.

For all $n \in \bigcup_{j>1} [n_{m_j}, n_{m_j+1})$, $o(\mathbf{x})(n) < 0 < e(\mathbf{z})(n)$ and $o(\mathbf{z})(n) < 0 < e(\mathbf{x})(n)$. These are infinitely many elements of the sequence.

Claim 1. *There exists $N \in \mathbb{N}$ such that $e(\mathbf{x})(n) > o(\mathbf{z})(n)$ holds for all $n > N$.*

Proof. For all $n < n_{m_1+1}$, (a) $e(\mathbf{x})(n)$ contains positive elements at coordinates in $E(m_1)$ and negative elements at coordinates in $O(m_1) \cup [1, n_1)$; and (b) $o(\mathbf{z})(n)$ contains positive elements at coordinates in $O(m_1)$ and negative elements at coordinates in $E(m_1) \cup [1, n_1)$. There are two cases.

(i) $|O(m_1)| < |E(m_1)|$: Among coordinates $n < n_{m_1+1}$,

- fewer negative integers have been assigned in $e(\mathbf{x})(n)$ as compared to $o(\mathbf{z})(n)$. Then $0 > e(\mathbf{x})(n_{m_1+1}) > o(\mathbf{z})(n_{m_1+1})$ and for all subsequent coordinates n with both $e(\mathbf{x})(n)$ and $o(\mathbf{z})(n)$ being negative, $0 > e(\mathbf{x})(n) > o(\mathbf{z})(n)$ holds.
- fewer positive integers have been assigned in $o(\mathbf{z})(n)$ as compared to $e(\mathbf{x})(n)$. Then $e(\mathbf{x})(n_{m_1+2}) > o(\mathbf{z})(n_{m_1+2}) > 0$ and for all subsequent coordinates n with both $e(\mathbf{x})(n)$ and $o(\mathbf{z})(n)$ being positive, $e(\mathbf{x})(n) > o(\mathbf{z})(n) > 0$ holds.

We take $N = n_{m_1+1}$ in this case.

- (ii) $|O(m_1)| \geq |E(m_1)|$: Among the coordinates $[n_{m_j+1}, n_{m_{j+1}}]$ for all $j \in \mathbb{N}$, $e(\mathbf{x})(n)$ and $o(\mathbf{z})(n)$ contain equally many elements of same sign. Further for the coordinates in $[n_{m_j}, n_{m_{j+1}}]$, $o(\mathbf{z})(n)$ is negative but $e(\mathbf{x})(n)$ is not. Thus for some finite $J \in \mathbb{N}$,

$$|O(m_1)| < |E(m_1)| + \left| \bigcup_{j \in \{2, \dots, J\}} [n_{m_j}, n_{m_{j+1}}] \right|$$

will be true. In this case, we can apply argument of case (i) above for n_{m_J+1} and therefore obtain $N = n_{m_J+1}$.

Thus we have shown that for all $n > N$, if $e(\mathbf{x})(n)$ and $o(\mathbf{z})(n)$ share the same sign then $e(\mathbf{x})(n) > o(\mathbf{z})(n)$. The remaining situation is $e(\mathbf{x})(n) > 0 > o(\mathbf{z})(n)$. This completes the proof. \square

Claim 2. *There exists a finite permutation $o^\pi(\mathbf{z})$ of $o(\mathbf{z})$ such that $e(\mathbf{x})(n) > o^\pi(\mathbf{z})(n)$ holds for all $n \in \mathbb{N}$.*

Proof. In claim 1, it has been shown that for all $n > N$ $e(\mathbf{x})(n) > o(\mathbf{z})(n)$. Let $K := \{k^1, k^2, \dots, k^N\}$ be elements (listed in increasing order of magnitude) from the set $\bigcup_{j > J} [n_{m_j}, n_{m_{j+1}}]$. We permute $o(\mathbf{z})(1)$ and $o(\mathbf{z})(k^1)$; $o(\mathbf{z})(2)$ and $o(\mathbf{z})(k^2)$ and so on till $o(\mathbf{z})(N)$ and $o(\mathbf{z})(k^N)$ to obtain $o^\pi(\mathbf{z})$. Hence, $o^\pi(\mathbf{z})$ is obtained via a finite permutation of $o(\mathbf{z})$. Observe that if $0 > e(\mathbf{x})(n) \geq o(\mathbf{z})(n)$, then

$$e(\mathbf{x})(n) > o(\mathbf{z})(k^n) = o^\pi(\mathbf{z})(n), \text{ and } e(\mathbf{x})(k^n) > 0 > o(\mathbf{z})(n) = o^\pi(\mathbf{z})(k^n).$$

If $0 > o(\mathbf{z})(n) \geq e(\mathbf{x})(n)$, then

$$e(\mathbf{x})(n) > o(\mathbf{z})(k^n) = o^\pi(\mathbf{z})(n), \text{ and } e(\mathbf{x})(k^n) > 0 > o(\mathbf{z})(n) = o^\pi(\mathbf{z})(k^n).$$

If $e(\mathbf{x})(n) < 0 < o(\mathbf{z})(n)$, then

$$0 > e(\mathbf{x})(n) > o(\mathbf{z})(k^n) = o^\pi(\mathbf{z})(n), \text{ and } e(\mathbf{x})(k^n) > o(\mathbf{z})(n) = o^\pi(\mathbf{z})(k^n) > 0.$$

If $e(\mathbf{x})(n) > o(\mathbf{z})(n) > 0$, then

$$e(\mathbf{x})(n) > 0 > o(\mathbf{z})(k^n) = o^\pi(\mathbf{z})(n), \text{ and } e(\mathbf{x})(k^n) > o(\mathbf{z})(n) = o^\pi(\mathbf{z})(k^n) > 0.$$

\square

Applying Claims 1 and 2, we have obtained $o^\pi(\mathbf{z})$ such that $e(\mathbf{x})(n) > o^\pi(\mathbf{z})(n)$ for all $n \in \mathbb{N}$. Anonymity axiom implies

$$o(\mathbf{z}) \sim o^\pi(\mathbf{z}). \tag{13}$$

By weak Pareto axiom we get

$$e(\mathbf{x}) \sqsupset o^\pi(\mathbf{z}). \quad (14)$$

By transitivity, (13) and 14) imply

$$e(\mathbf{x}) \sqsupset o(\mathbf{z}). \quad (15)$$

Notice that arguments of claims 1 and 2 could also be applied to the pair of sequences $e(\mathbf{z})$ and $o(\mathbf{x})$. Thus we are able to obtain $o^\pi(\mathbf{x})$ such that applying anonymity axiom we get

$$o(\mathbf{x}) \sim o^\pi(\mathbf{x}). \quad (16)$$

By weak Pareto axiom we get

$$o^\pi(\mathbf{x}) \sqsubset e(\mathbf{z}). \quad (17)$$

By transitivity, (16) and 17) imply

$$o(\mathbf{x}) \sqsubset e(\mathbf{z}). \quad (18)$$

Combining (15) and (18) and transitivity, we get

$$o(\mathbf{z}) \sqsubset e(\mathbf{x}) \sqsubset o(\mathbf{x}) \sqsubset e(\mathbf{z}) \rightarrow o(\mathbf{z}) \sqsubset e(\mathbf{z}),$$

which implies

$$z \notin \Gamma.$$

(2) $o(x) \sqsubset e(x)$: We remove $n_{m_1}, n_{m_j+1}, n_{m_j+2}$, for all $j > 1$ from $U(x)$ to obtain $z \in 2^{\mathbb{N}}$, i.e.,

$$z(n) = \begin{cases} x(n) & \text{if } n \notin \{n_{m_1}, n_{m_j+1}, n_{m_j+2}, \text{ for all } j > 1\} \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$\begin{aligned} O(m_1) &:= [n_1, n_2) \cup [n_3, n_4) \cdots [n_{m_1-1}, n_{m_1}), \\ E(m_1) &:= [n_2, n_3) \cup [n_4, n_5) \cdots [n_{m_1-2}, n_{m_1-1}). \end{aligned}$$

Then,

for all $n \in O(m_1)$, $e(\mathbf{z})(n) < 0 < o(\mathbf{x})(n)$ and $o(\mathbf{z})(n) > 0 > e(\mathbf{x})(n)$,

for all $n \in [1, n_1)$, $0 > e(\mathbf{z})(n) = o(\mathbf{x})(n)$ and $0 > o(\mathbf{z})(n) = e(\mathbf{x})(n)$,

for all $n \in E(m_1)$, $o(\mathbf{x})(n) < 0 < e(\mathbf{z})(n)$ and $o(\mathbf{z})(n) < 0 < e(\mathbf{x})(n)$,

for all $n \geq n_{m_1+1}$ if $0 > o(\mathbf{x})(n)$ then $0 > e(\mathbf{z})(n)$ holds. Also if $e(\mathbf{z})(n) > 0$ then $o(\mathbf{x})(n) > 0$. Similarly, if $0 > o(\mathbf{z})(n)$ then $0 > e(\mathbf{x})(n)$ holds. Also if $e(\mathbf{x})(n) > 0$ then $o(\mathbf{z})(n) > 0$.

For all $n \in \bigcup_{j>1} [n_{m_j+1}, n_{m_j+2})$, $e(\mathbf{z})(n) < 0 < o(\mathbf{x})(n)$ and $e(\mathbf{x})(n) < 0 < o(\mathbf{z})(n)$.

These are infinitely many elements of the sequence.

Applying Claims 1 and 2, we would be able to obtain $e^\pi(\mathbf{z})$ and $o^\pi(\mathbf{z})$ such that

$$o(\mathbf{x})(n) > e^\pi(\mathbf{z})(n), \text{ and } o^\pi(\mathbf{z})(n) > e(\mathbf{x})(n) \text{ for all } n \in \mathbb{N}.$$

Anonymity axiom implies

$$o(\mathbf{z}) \sim o^\pi(\mathbf{z}), \text{ and } e(\mathbf{z}) \sim e^\pi(\mathbf{z}). \quad (19)$$

By weak Pareto axiom we get

$$e^\pi(\mathbf{z}) \sqsubset o(\mathbf{x}), \text{ and } e(\mathbf{x}) \sqsubset o^\pi(\mathbf{z}). \quad (20)$$

By transitivity, (19) and 20) imply

$$e(\mathbf{z}) \sqsubset o(\mathbf{x}), \text{ and } e(\mathbf{x}) \sqsubset o(\mathbf{z}),$$

which leads to

$$z \notin \Gamma.$$

(3) $e(\mathbf{x}) \sim o(\mathbf{x})$: We remove n_{m_j+1}, n_{m_j+2} , for all $j > 1$ from $U(x)$ to obtain $z \in 2^{\mathbb{N}}$, i.e.,

$$z(n) = \begin{cases} x(n) & \text{if } n \notin \{n_{m_j+1}, n_{m_j+2}, \text{ for all } j > 1\} \\ 0 & \text{otherwise.} \end{cases}$$

By construction we obtain $o(\mathbf{z})(n) \leq o(\mathbf{x})(n)$ and $e(\mathbf{z})(n) \geq e(\mathbf{x})(n)$ for all $n \in \mathbb{N}$. Further, for all $n \in \bigcup_{j \in \mathbb{N}} [n_{m_j+1}, n_{m_j+2})$, $o(\mathbf{z})(n) < 0 < o(\mathbf{x})(n)$ and $e(\mathbf{z})(n) > 0 > e(\mathbf{x})(n)$. Applying Claims 1 and 2, we would be able to obtain $e^\pi(\mathbf{z})$ and $o^\pi(\mathbf{z})$ such that

$$e(\mathbf{x})(n) > o^\pi(\mathbf{z})(n), \text{ and } e^\pi(\mathbf{z})(n) > o(\mathbf{x})(n) \text{ for all } n \in \mathbb{N}.$$

Anonymity axiom implies

$$o(\mathbf{z}) \sim o^\pi(\mathbf{z}), \text{ and } e(\mathbf{z}) \sim e^\pi(\mathbf{z}). \quad (21)$$

By weak Pareto axiom we get

$$o^\pi(\mathbf{z}) \sqsubset e(\mathbf{x}), \text{ and } o(\mathbf{x}) \sqsubset e^\pi(\mathbf{z}). \quad (22)$$

By transitivity, (21) and 22) imply

$$o(\mathbf{z}) \sqsubset e(\mathbf{x}), \text{ and } o(\mathbf{x}) \sqsubset e(\mathbf{z}),$$

which leads to

$$z \in \Gamma.$$

□

5 STRUCTURE OF NON-CONSTRUCTIVE EGALITARIAN SOCIAL WELFARE ORDERS

In this section, we describe the nature of equitable social welfare orders relative to the non-constructive objects from mathematical logic literature. The non-constructive objects which have played a role so far in the context of SWOs on infinite utility streams are: free ultrafilters, non-Lebesgue sets, non-Ramsey sets, non-Baire sets, and non-special Silver sets (we consider these as a particular type of the non-Baire sets). The non-Baire set was introduced in the study of egalitarian social welfare order in Laguzzi (2020). Further, Theorems 1 - 3 together with the results in Lauwers (2010), Lauwers (2012), Dubey and Mitra (2014a), Dubey and Mitra (2014b), Banerjee and Dubey (2014), Dubey (2016b), Dubey (2016a), and Laguzzi (2020) show the non-constructive nature of SWOs satisfying combinatorial principles. These objects could be distinguished on the basis of either *consistency strength*; or *fragments of AC* needed for existence. In the following sub-sections, we analyze these two aspects in more detail.

5.1 Consistency strength

The seminal work of Solovay (1970) was the corner-stone to assert that non-Lebesgue sets and non-Baire sets are non-constructible. Meanwhile the same idea was used in Mathias (1977) concerning non-Ramsey sets. The conclusion relies on the strengthening of the usual axiom of Zermelo-Fraenkel set theory by using a so-called large cardinal axiom, in the specific case, the axiom IN asserting that an *inaccessible cardinal* κ exists, i.e, a cardinal κ such that for all $\alpha < \kappa$, $(2^\alpha < \kappa)$. In the 1980s, Shelah introduced a refinement of Solovay-Mathias technique, called *amalgamation* and obtained two important results marking a clear distinction between Lebesgue measurability and Baire property.³

Theorem 4 (Shelah (1985a) and Shelah (1985b)).

- In ZFC without IN, there exists a model of set-theory (without AC) where every set of reals has the Baire property.
- In ZFC without IN, there exists no model of set-theory (without AC) where every set of reals is Lebesgue measurable.

In the hierarchy of large cardinals (see Kanamori (2009)), inaccessible cardinals appear on the first level. In other words they are considered small types of large cardinals. However, they are strong enough to increase the consistency strength of the theory. Thus, ZFC + IN is a theory strictly stronger than ZFC in the sense that in ZFC + IN we can construct a model for ZFC, but not *vice-versa*. Alternatively, ZFC is too weak a theory for building/proving the existence of a model for

³Shelah's amalgamation technique has also been used to establish results about the existence of similar types of non-regular sets in Laguzzi (2014) and Friedman and Schritterser (2020).

ZFC + IN. Also IN is accepted as a perfectly reasonable axiom, and a large majority of set theorists believes in its consistency. So, in a sense, we can say that it is considered *socially* accepted as a consistent axiom.

Hence, we could infer that the non-constructive nature of a non-Baire set employed in Theorem 1 is more robust than the non-constructive nature of a non-Lebesgue measurable set. This is because the way of removing all non-Lebesgue measurable sets (i.e., construct a model where all sets are Lebesgue measurable) relies on the existence of an inaccessible cardinal, whereas a model in which we can remove all non-Baire sets can be obtained just working in ZFC without any need of IN. In other words, if in future, ZFC + IN is found to be inconsistent, while ZFC continues to remain consistent, the proof about the non-constructive nature of non-Lebesgue measurable sets will no longer hold, but the non-constructive nature of non-Baire sets would be intact. In this point of view, non-Baire sets can be considered to have a higher degree of non-constructive nature compared to non-Lebesgue measurable sets. Hence, proving implications from a SWO to a non-Baire set highlights the non-constructive nature of such order in a more robust manner than implications to a non-Lebesgue set.⁴ Let us see a specific consequence in our context. In Theorem 2, we have proved that the existence of SWO satisfying (infinite or weak) Pareto and anonymity implies the existence of a non-special Silver set, which in turn is a non-Baire set. Hence, our argument provides more robust result about the non-constructive nature of the order itself compared to non-Lebesgue and non-Ramsey sets. Under the meta-theoretical side, it therefore strengthens the previous results in Zame (2007) and Lauwers (2010).

5.2 Fragments of AC

In economic theory several non-constructive mathematical objects, viz. non-Baire sets, non-Lebesgue sets, non-Ramsey sets, free ultrafilters have been used. The study of the non-constructive degree of these objects has been widely developed in mathematical logic and descriptive set theory through the past several years.⁵ They are considered non-constructive since the AC is involved in the construction. However the existence of these objects are not all at the same *level of non-constructiveness*. For instance, let

$$\begin{aligned} U &:= \text{There exists a free ultrafilter on } \mathbb{N}. \\ NL &:= \text{There exists a non-Lebesgue measurable set.} \end{aligned}$$

On the one hand, well-known Vitali's result shows that $U \Rightarrow NL$, but on the other Shelah (1985a) proved $NL \not\Rightarrow U$. Hence U corresponds to a strictly larger/stronger fragment of AC than NL , since the *amount* of AC needed to build an object satisfying U is strictly larger/stronger than the one for NL . The following diagram shows such fragments of AC; moving bottom-up means moving from weaker to stronger fragment of AC.

⁴We also remark that the issue for non-Ramsey sets are still in a *limbo* (in between), as it is an open question whether an inaccessible cardinal is necessary or not to build a model where all non-Ramsey sets are removed.

⁵Interested reader can consult the following selected list of papers: Brendle and Löwe (1999), Bartoszyński and Judah (1999), Ikegami (2010), Khomskii (2012), Laguzzi (2012) and Laguzzi (2014).

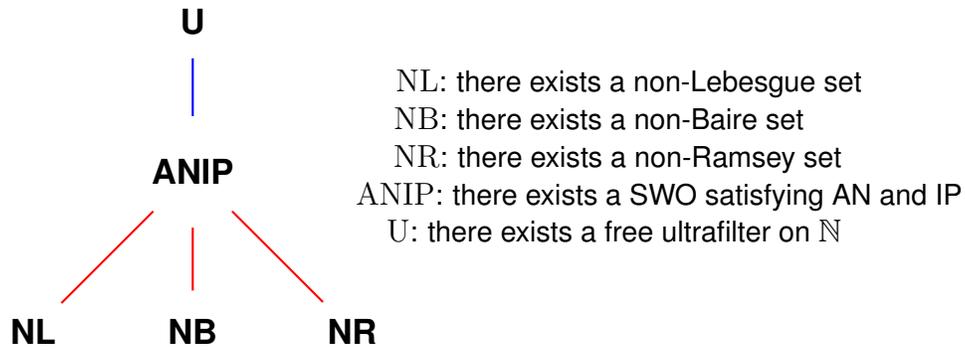


Figure 1: Fragments of AC

The red lines in the diagram indicate an existing implication between the statement above and the one below, and furthermore that the reverse implication is provably false in ZF. For instance, $ANIP \Rightarrow NB$ but $NB \not\Rightarrow ANIP$. In other words, ANIP requires strictly larger fragment of AC than the objects at the bottom level. On the contrary, the blue line must be understood as an existing implication between the statement above and the one below, but such that the reverse implication is not yet known to hold. This means that in the diagram above $U \Rightarrow ANIP$ but we do not know whether the reverse implication holds or not. Put differently, the free ultrafilters correspond to a larger fragment of AC than ANIP, but whether this is strictly stronger or equivalent is still unknown. Similarly we can get a similar table when dealing with SWOs satisfying Strong Equity and Anonymity.

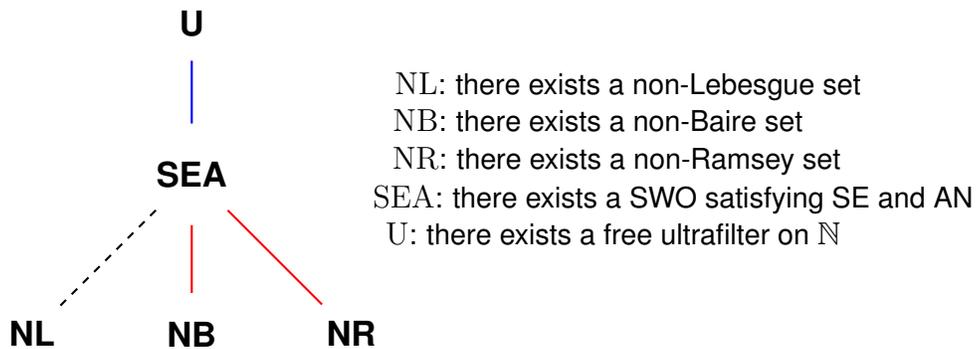


Figure 2: Fragments of AC

The dashed lines mean that none of the directions of the implication is known. We consider it for our future work to better understand these cases in an even more general setting.

6 CONCLUDING REMARKS

In this paper, we have shown that there exists inherent conflict in combining anonymity and strong equity axioms while seeking explicit description of social welfare orders. The conflict emerges as

soon as we consider any non-trivial domain Y , i.e., it contains at least four distinct elements, by using as non-constructive objects the non-Baire sets. Moreover in the second part we prove that non-special Silver set are also implied by SWOs satisfying infinite Pareto and anonymity with utility domain $Y = \{0, 1\}$ and by SWOs satisfying weak Pareto and anonymity with utility domain $Y = \mathbb{Z}$.⁶ In the Table 1 below, we list the non-constructive objects which emerge as a consequence of existence of social welfare orders satisfying anonymity and the equity/efficiency axiom mentioned in column (1).

Table 1:

(1)	(2)	(3)
Axiom	Y	Non-constructive set
Strong Pareto	$ Y \geq 2$	Non-measurable (Zame (2007)), Non-Baire (Laguzzi (2020))
Infinite Pareto	$ Y \geq 2$	Non-Ramsey (Lauwers (2010)), Non-special Silver (Theorem 2)
Weak Pareto	$Y = [0, 1]$	Non-measurable (Zame (2007))
Weak Pareto	$Y = \mathbb{Z}$	Non-Ramsey (Dubey (2011)), Non-special Silver (Theorem 3)
Strong Equity	$ Y \geq 4$	Non-Baire (Theorem 1)

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⁶Interested reader may note that the theorems 2 and 3 resolve the points raised in Bowler et al. (2017, Problem 11.14).

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