PROJECT PROPOSAL

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My research project regards the interconnection between two burgeoning areas of mathematical logic: generalized descriptive set theory and set theory of the κ-reals.

1 State of the art and preliminary work

State of the art. Set theory is a branch of mathematical logic, which has been extensively developed during the 20th century as a foundation of mathematics. Two important subbranches are the set theory of the real numbers and the descriptive set theory, which concern a carefully investigation of definable subsets of the real line. In particular, one of the main purposes was to analyze how much *regularity* one could obtain. Obviously, the notion of regularity is not uniquely determined; the most common ones are certainly the Lebesgue measurability and the Baire property, which are not only of interest from the set-theoretical viewpoint, and indeed they came from measure theory and topology, respectively. A very old result coming from the beginning of 20th century, due to Vitali, showed that the axiom of choice (AC) implied the existence of a set without Baire property and non-Lebesgue measurable. One of the main questions was then to understand the exact role of AC in the construction of such negligible sets, and in particular to comprehend whether AC was strictly necessary. In the 1960s, Solovay gave a positive answer to such a question, and the method to resolve this measure theoretical and topological issue used a tool coming from set theory: the method of *forcing*. After that, a large array of interesting issues have been studied along these decades about regularity properties, tree forcings, cardinal characteristics and ultrafilters. All of these investigations have constituted what is commonly known as *set theory of the reals*, with a certain interconnection with *descriptive set theory*.

The protagonists are the *real numbers*, which from our point of view can be considered as elements of the Cantor space 2^ω or of the Baire space ω^ω. From the set-theoretical point of view it is rather natural to be intellectually attracted from the generalization of these spaces obtained by considering sequences of uncountable length, i.e., the so-called generalized Baire space κ^κ and the generalized Cantor space 2^κ, and then to investigate the analogous problems concerning regularity properties, cardinal characteristic and so on in this generalized framework. This hint of moving from the standard setting (i.e., 2^ω and ω^ω) to the generalized one (i.e., $2^\kappa$and κ^κ) has been started in the 1980s by Kanamori, who first studied a type of Sacks forcing in the context of 2^κ, then almost forgotten for several years, and then resurrected by several people in the 1990s and later: by Väänänen, Mekler and Hyttinen from the model theoretical side; by Friedman, Kulikov, Motto Ros and Schlicht from the descriptive set theoretical side; by Shelah, Halko, Roslanowski and Blass from the side of set theory of the κ-reals.

My research project is meant to lie in the interconnected area between the generalized descriptive set theory and the set theory of the κ-reals. In particular our interest will be mainly focused on the investigations of the regularity properties in κ^κ and 2^κ, such as the Baire property and the other notions coming from tree-like forcings, such as Sacks, Silver, Miller, Laver, Hechler and Mathias forcing.

The topology we choose for our investigation is the *bounded topology*, generated by clopen sets $[s] := \{x \in \kappa^\kappa : x \supset s\}$, for $s \in \kappa^{<\kappa}$. Even if some basic standard results extend to κ^κ, such as the Baire category theorem and the Kuratowski-Ulam theorem, many other concepts strictly connected to category behave much differently. As a well-known example, one can mention the fact that the Baire property fails for $\mathbf{\Sigma}^1_1$-sets, as it is proven by Halko and Shelah in [25]. This result is just the tip of the iceberg of the differences between the standard setting and the generalized one. In particular, many differences are due to the type of uncountable cardinal κ one deals with.

Preliminary work. In this section we present a selected list of works on this subject, which are crucial as a starting point of my research project. The subject is essentially divided in three main branches.

Tree forcings, κ-ideals and amoeba. A key point when dealing with set theory of the κ-reals is the study of tree-like forcings. This kind of forcing notions are crucial for understanding and proving most of the results about both cardinal characteristics and regularity properties. As a precursor, one can consider [30], where Kanamori presents a detailed study of club-Sacks forcing, proving some interesting results concerning specific preservation theorems, and showing for this purpose $\diamondsuit$ as an essential ingredient to get the appropriate combinatorial structure needed in the generalized context. In this spirit, Friedman and Zdomskyy ([19]) do a similar work for club-Miller forcing and the combination with club-Sacks forcing. Another generalized version of Miller forcing is also studied by Brown and Groszek in [9], while a basic study around trees is constituted by [42] due to Mekler and Väänänen. A detailed study of products of extended version of Cohen forcing is presented in [36] and [37], where Landver proves some preservation theorems about these forcing notions and investigates the connections with the Baire number. Finally, in [60], Shelah presents a random-like forcing for κ weakly compact, while in the last chapter of [32], I try to approach the issue by directly defining a sort of generalized measure. Concerning the amoeba forcing, the literature about the generalized case is almost empty, except for few cases fallen into the range of the abstract approach of Roslanovski and Shelah using generalized creature forcing, [53]. One of the aims of this project will also be to fill this lack, since amoeba forcings play a strategically indispensable role around the investigation of cardinal characteristics and regularity properties. Obviously, one of the key results that one needs when dealing with this kind of issues are the so-called *preservation theorems.* A pretty large casuistry has been developed in [51], [52], [55], [62] and [63], where Roslanovski and Shelah mostly focus on the case κ inaccessible. Nevertheless, such a work has not been completed yet.

Regularity properties and uncountable games. This part in constituted by the generalization of the large variety of results obtained in the standard case by several authors, of which we present a selected list: Solovay ([66]), Todorcevic and Di Prisco ([12]), Shelah ([58], [59]), Friedman and Schrittesser ([57]), Brendle, Löwe and Halbeisen ([5], [4], [6], [7]), Friedman, Fisher and Khomskii ([14]), Ikegami ([28]), Laguzzi ([34]). Some results have been already found in the generalized case as well: in [24], Halko studies the behaviours and pathologies of the generalized Baire property, and he also investigates some types of κ-ideals; in [25], Shelah and Halko improve this work showing that there is a $\mathbf{\Sigma}^1_1$ set without Baire property; in [15], Friedman, Hyttinen and Kulikov prove how to force that all $\mathbf{\Delta}^1_1$ sets have the Baire property; in [56], Schlicht presents a method to partially obtain the factoring lemma, and as a consequence he finds a way to force that all On^κ-definable sets have the perfect set property; in the last chapter of [32], I present some implication bewteen the Baire property for $\mathbf{\Delta}^1_1$ and trascendental principle over L. In [20], Friedman, Khomskii and Kulikov start a systematic investigation of notions of regularity coming from tree forcings; in [35], I introduce some variant of Schlicht's method, showing how to force all On^κ-definable sets to be stationary-Silver measurable for κ inaccessible and how to get projective Miller measurability by collapsing an inaccessible to κ^+, together with some other combinatorial results about generic trees. The topic regarding games of uncountable length is extremely wide, and it can be approached from completely different sides. On the model-theoretical side we mention [15], where the authors investigate the Ehrenfeucht-Fraïssé game, in particular for coding isomorphisms between models. However, my interest is mostly focused on the descriptive set-theoretical side, in particular the connections with regularity properties and topological issues: in [24], Halko introduces a generalization of Banach-Mazur game for the Baire property; in [10], Coskey and Schlicht deal with a natural generalization of Choquet games and the related Choquet spaces; in [35], I analyze the difficulties in introducing a game associated with Silver measurability.

Combinatorial cardinal characteristics. The source of inspiration of this part of the project is obviously to generalize the rich field of studies summarized in the splendid exposition [2] by Blass. The state of the art of this branch is constituted by some investigations mainly led by Shelah and others: in [11], Cummings and Shelah analyze the generalized versions of $\mathfrak{b}(\kappa)$

and $\mathfrak{d}(\kappa)$, showing that they are stricly closed with their akin club versions $\mathfrak{b}_{\mathsf{club}}(\kappa)$ and $\mathfrak{d}_{\mathsf{club}}(\kappa)$; in [72], on the wave of the previous results of Suzuki ([71]), Zapletal analyzes the generalized version of $\mathfrak{s}$, showing that already the inequality $\mathfrak{s}(\kappa) > \kappa^+$ requires large cardinal assumptions; in [65], Shelah and Spasojevic study $\mathfrak{b}(\kappa)$ and $\mathfrak{t}(\kappa)$; in [61], Shelah shows some inequalities between $\mathfrak{d}(\kappa)$ and cardinal invariants associated to the meager ideal. About generalized mad families and the corresponding cardinal $\mathfrak{a}(\kappa)$, [3] and [26] are certainly worthy of mention: Blass, Hyttinen and Zhang focus on the relations between $\mathfrak{a}(\kappa)$ and other cardinal characteristics, showing some crucial differences between the inaccessible and accessible cases. Quite surprisingly, the inequality $\mathfrak{d}(\kappa) < \mathfrak{a}(\kappa)$ is provably false for κ successor. Furthermore, in [3], the authors show that the usual combinatorial characterization of comeager sets in terms of matching families fails for $\kappa > \omega$ successor. Finally, a subject strictly related to cardinal characterisics is that of ultrafilters on κ: for this purpose, we mention [64] and [54]. Moreover, in [21] and [8], Shelah, Garti and Brooke-Taylor, respectively, deal with the cardinal characteristic $\mathfrak{u}(\kappa)$.

We also would like to mention that a very detailed and exhaustive introduction to the topic can be found in [15]. Other papers worthy of mention are [43], [27] and [48].

2 Objectives and working program

Beyond the failure of the Baire property for $\mathbf{\Sigma}^1_1$, there is a wide variety of issues and obstacles making the situation deeply different from the classical setting. Here follows a selected list of those we are going to focus mostly.

Part 1. When dealing with tree forcings, one of the main properties that we want to extend from the classical setting is the so-called *fusion*. For this goal, it seems reasonable to require the tree-conditions to have, in a sense, club splitting. However, such a requirement seems to be too strong to have a good notion of regularity related to these forcings, because the club filter itself turns out to be non-regular in most cases. On the contrary, by dropping the requirement about club splitting, one obtains a more flexible notion of regularity, which in some cases can be forced to hold for all On^κ-definable sets. Hence, there seems to be a sort of tension between these two points of view, since club splitting, which is reasonable from the forcing viewpoint, looks too strong from the viewpoint of projective regularity, and viceversa. We aim at finding the right compromise to run both points of view.

Part 2. Solovay's method to force that all projective sets are regular requires the so-called "factoring lemma", stating that the quotient of the Levy collapse $\mathsf{Coll}(\omega, \kappa)$ by reals is forcing equivalent to $\mathsf{Coll}(\omega, \kappa)$, when κ is inaccessible. In the generalized setting this fails even for κ-Cohen forcing. In [56], Philipp Schlicht has recently found a way to partially recover the factoring lemma, in order to prove a generalized version of the perfect set property. As mentioned in the previous section, in [35] I have then introduced some variants of Schlicht's method to obtain results about Silver and Miller measurability. However, except for these few cases, a full and exhaustive investigation is far to be done; in fact it is not clear how to extend (whether possible) classical results about Laver and Mathias forcing as well. The situation seems to be highly delicate, since we know that the club filter (which is a $\mathbf{\Sigma}^1_1$ set without Baire property) marks a sort of impassable line. We aim at enlarging the casuistry in order to shed more light on that issue. Furthermore, one of the main lacks in the theory of κ-reals is certainly a reasonable notion of random-like forcing with its related notion of measurability. As we said above, in [60] Shelah introduced a possible generalization for κ weakly compact, while in [32] I analyze a notion of generalized measure. However, the investigation is far away to be satisfactory, and it seems that a bunch of natural issues remain open. We refer to the next section for a more detailed discussion.

Part 3. Concerning cardinal invariants, I will investigate the relation between $\mathfrak{t}(\kappa)$, $\mathfrak{h}(\kappa)$, $\mathfrak{s}(\kappa)$, $\mathfrak{g}(\kappa)$, $\mathfrak{b}(\kappa)$ and $\mathfrak{d}(\kappa)$ (I expect most of the usual inequalities to hold in the generalized context as well). A first goal would be to extend Suzuki's lower bound for $\mathfrak{s}(\kappa)$ also to $\mathfrak{t}(\kappa)$ and $\mathfrak{h}(\kappa)$, and hopefully to find a generalized abstract version of Suzuki's theorem depending on some syntactical form of the cardinal invariant. A final goal will be to force strict inequalities between these cardinal invariants. I expect that these results strongly depend on the uncountable regular cardinal that we choose (for instance, my conjecture is that if κ is inaccessible and not weakly compact, then $\mathfrak{h}(\kappa) = \mathfrak{t}(\kappa) = \mathfrak{s}(\kappa) = \kappa$).

My working program will be organized as follows:

- months 1-14: Part 1 (Tree-forcings, κ-ideals and amoeba);
- months 15-28: Part 2 (Regularity properties and uncountable games);
- months 29-36: Part 3 (Combinatorial cardinal characteristics).

1. Tree forcings, κ-ideals and amoeba (expected duration: 14 months). Our *dramatis personae* are the notions of forcing with tree conditions, briefly called tree forcings. In the standard setting these posets play an essential role for proving results about regularity properties, cardinal characteristics and ultrafilters. These kind of posets are of interest in set theory of the reals because they add particular types of reals, by which one is able to make certain statements true and some others false, by carefully building the appropriate forcing. Moreover, each of these posets is naturally associated with a notion of measurability and with an ideal of small sets. When working with trees $T \subseteq \kappa^{<\kappa}$ or $T \subseteq 2^{<\kappa}$, there are different ways for generalizing the common tree forcing like Sacks, Silver, Miller, Hechler and so on. In the previous section, we have mentioned that from the viewpoint of forcing, it is more convenient to consider trees with club splitting. The main reasons are: firstly, we would like that the forcing be $< \kappa$-closed (so not collapsing κ); and secondly, in many cases, we would like that the forcing satisfy fusion. Hence, it seems to be natural to demand the splitting node to occur *often* and to have *sufficiently many* successors, in a sense. The technical answer to the previous informal italic-style words given so far has been the club filter (see [30], [53], and [19]). Moreover, it seems that to obtain some basic preservation theorems (such as preserving κ^+) the use of a certain version of $\Diamond$ is necessary.

Given a tree forcing $\mathbb{P}$, one can introduce an associated κ-ideal $\mathcal{I}_\mathbb{P}$ as follows: first define X to be $\mathbb{P}$-null iff $\forall T \in \mathbb{P} \exists T' \in \mathbb{P}(T' \subseteq T \wedge X \cap [T'] = \emptyset)$ and then put $\mathcal{I}_\mathbb{P}$ to consist of all $\leq \kappa$-unions of such $\mathbb{P}$-null sets. The investigation of these ideals has been deeply and extensively developed in the standard setting, and one of the main papers is the splendid [5]. Nevertheless, an analogous profound study in the generalized case has not been done so far. Some results about notions of smallness have been presented in [24], but in some different flavour. In my research project I aim at filling the lack, by trying to understand which results of the standard setting generalize, and which are the sensible differences. Here follow two concrete examples that I consider worthy of mention, and on which I am going to focus the attention. We remark that many similar issues come out naturally, and they will certainly be part of our investigation during the research project.

Q.1.1 When κ is inaccessible, one can show that $\mathcal{I}_\mathbb{C} \subseteq \mathcal{I}_\mathbb{V}$. Nevertheless for κ successor the situation is different and we do not have an analogous proof. My conjecture is that the inclusion is no longer true. More generally, we want to investigate the relations between $\mathcal{I}_\mathbb{P}$ and $\mathcal{I}_\mathbb{Q}$ for different tree-forcings $\mathbb{P}$ and $\mathbb{Q}$.

Q.1.2 In the standard case, if we assume AD, then one can show that $i_\mathbb{S} \subseteq i_\mathbb{M}$ and $i_\mathbb{M} \subseteq i_\mathbb{L}$ (where $i_\mathbb{P}$ denotes the ideal related to the version of $\mathbb{P}$ in the classical setting). This essentially follows from the fact that under AD all sets are Miller and Laver measurable. However, we know that for our generalized case that can happen only rarely. So the question is: can one build a model where $I_\mathbb{S} \subseteq I_\mathbb{M}$ or $I_\mathbb{M} \subseteq I_\mathbb{L}$?

Other important issues concern the cardinal characteristics associated with these ideals, such as $\mathsf{add}(\mathcal{I})$, $\mathsf{cov}(\mathcal{I})$, $\mathsf{cof}(\mathcal{I})$ and $\mathsf{non}(\mathcal{I})$. About them, few results have been proven so far, such as the consistency of $\mathfrak{d}(\kappa) > \mathsf{cov}(\mathcal{I}_{\mathbb{C}})$ in [61]. Another important cardinal characteristic directly related to a tree forcing $\mathbb{P}$ is the $\mathfrak{a}(\mathbb{P})$ number, i.e., the size of the smallest antichain in $\mathbb{P}$ of size $> \kappa$. This part of the work follows a line of research that in the standard setting has been developed recently for Silver forcing by Spinas and Wyszkowski in [70].

Another key point of this part of the project will be the study of the amoeba forcings. As one can realize from the standard setting, the investigation of such notions is as important as the study of the tree-forcings themselves. In fact their applications are crucial for proving results about regularity properties and cardinal invariants. The general question is then: given a tree-forcing $\mathbb{P}$ can one find a forcing $\mathbb{AP}$ such that in any ZFC-model $\mathrm{M} \supseteq \mathrm{V}^{\mathbb{AP}}$ one has

$$\mathrm{M} \models \exists T \in \mathbb{P} \forall x \in [T](x \text{ is } \mathbb{P}\text{-generic over V})?$$

Note that in the standard case, the method for proving the existence of such an object uses absoluteness, and so it does not trivially generalize. Furthermore, another point that I aim at solving is to find *nice* versions of amoeba satisfying certain particular features, such as not adding κ-Cohen reals, not adding κ-random reals, not collapsing κ^+. The following questions naturally arise by looking at what happen with the classical amoebas; in particular, Q.1.3 and Q.1.4 follow a line of research developed by Spinas ([67], [68]) and myself ([34]).

Q.1.3 Can one find amoeba-Sacks $\mathbb{AS}$, amoeba-Miller $\mathbb{AM}$, amoeba-Laver $\mathbb{AL}$ not collapsing κ^+?

Q.1.4 Can one find finer versions not adding κ-Cohen reals?

We remark that the amoeba-Silver should be treated separately. Indeed, very recently Spinas has proved that any amoeba-Silver necessarily adds Cohen reals (see [69]).

2. Regularity properties and uncountable games (expected duration: 14 months). In [25], Halko and Shelah prove that the club filter does not have the Baire property. This example shows that the usual argument used for the standard Cantor space to force that all projective sets have the Baire property does not work in this generalized case. As we mentioned before, the main reason for which Solovay's proof does not work for the generalized regularities is that the version of factoring lemma for κ-sequences fails, i.e., considering $\mathrm{Coll}(\kappa, \lambda)$ the Levy forcing collapsing an inaccessible λ to κ^+, there exists $x \in 2^\kappa \cap \mathrm{V}^{\mathrm{Coll}(\kappa,\lambda)}$ such that $\mathrm{Coll}(\kappa, \lambda)/x$ is not equivalent to $\mathrm{Coll}(\kappa, \lambda)$ itself, and hence such x has bad quotient. Indeed, even the κ-Cohen forcing $\mathbb{C}_\kappa$ has bad quotients, i.e., $\mathbb{C}_\kappa \cong P * S$ with P forcing equivalent to $\mathbb{C}_\kappa$ and S shooting a club through the κ-sequence added by P and so killing the stationarity of the complement of the Cohen κ-sequence.

As mentioned above, Philipp Schlicht has recently found a way to partially avoid this problem (see [56]), and so to obtain projective perfect set property. Inspired by this result, I have found a variant of this method applied to Silver and Miller forcing as well. In particular, one of the results in [35] shows that, for λ inaccessible, $\mathsf{Coll}(\kappa, \lambda)$ forces all On^κ-definable sets to be Miller measurable (here Miller trees are meant without club splitting requirement). The interesting issues arising from that is to understand whether the use of the inaccessible is strictly necessary or not. Since Miller measurability follows from the Baire property, in the classical setting this issue did not play a direct role, because of Shelah's construction to get projective Baire property without using inaccessible (see [58]). But, in our generalized setting, the Baire property and the non-club version of Miller measurability behave very differently, since we know the former to be provably false for $\mathbf{\Sigma}^1_1$ sets. Hence, we really need a completely different approach and an original direct argument. I conjecture a positive answer to such a question, i.e., one can build a model for Miller measurability without using inaccessible cardinals. The main point here is to adapt Shelah's amalgamation in our context.

Q.2.1 Investigate the consistency strength of "all On^κ-definable sets are Miller measurable". Try to adapt Shelah's amalgamation for the generalized setting.

Another intriguing issue is how to suitably generalize the random forcing. In fact, the notion of a measure does not extend trivially to the generalized case, and the problem of defining a reasonable measure so that the corresponding generalized random forcing satisfied the properties of being κ^κ-bounding and κ^+-cc has been open for a while. In the third chapter of [32], I tried to fix such a problem, but it turned out that the forcing that I introduced was not κ^κ-bounding; the reason of that almost certainly depends on the fact that the perfect trees in 2^κ, when κ is a successor cardinal, are *fat*, i.e., there are many levels $\alpha < \kappa$ such that $2^\alpha \geq \kappa$, and that gives rise to the unbounded κ-sequence. A posteriori one can say that my intention to find a solution for every $\kappa > \omega$ was probably too ambitious (and maybe even impossible), and I should have tried to start from the simplest case of inaccessible cardinals. In a recent paper ([60]), Shelah solved the problem for κ weakly compact, but the question remains open without this assumption. One of my aims is to improve Shelah's result, by defining a random like forcing for κ inaccessible, but without any need of weak compactness. For that the idea to attack the problem is to define a refined version of Sacks forcing for κ inaccessible, considering only perfect trees with pseudo

Cohen branches, in order to ensure the forcing to be κ^+-cc, without affecting the κ^κ-boundedness. I acknowledge that such an idea was inspired by a stimulating and fruitful discussion with Sy David Friedman, and that some progress about this problem has been done during the Freiburg set theory workshop in June ([18]).

Q.2.2 Can one introduce a random-like forcing (i.e., κ^κ-bounding and satisfying κ^+-cc) for κ inaccessible and not necessarily weakly compact?

Q.2.3 Does $\mathsf{Coll}(\kappa, \lambda)$ force that all On^κ-definable sets are measurable w.r.t. Shelah's measure?

Q.2.4 Introduce a kind of generalization of Mycielsky's game of length κ and investigate it.

Remark that some of these questions are explicitely asked by Shelah himself in [60] (for instance our Q.2.3 was explicitly asked by Shelah as question (C), on page 31). I conjecture a negative answer to Q.2.3.

Another part of my project will be to extend the investigation around club version of Miller-, Laver- and Silver-measurablity in this generalized case for $\mathbf{\Delta}^1_1$ sets. The complication coming out is that we do not have sufficiently many preservation theorems to ensure that certain types of κ-reals are not added, and so the main effort will be to fill this gap. Note that the situation is completely different from the standard setting, since in most cases one can provably find a $\mathbf{\Sigma}^1_1$ non-regular set (i.e., the club filter). However, some partial result can be recovered; we remark that the main difference in our generalized case is that we do not have an anologous of Shoenfield's absoluteness theorem, and this seems to be somehow caused by the fact that "being well-founded" for a tree is a Borel property, and not only Π^1_1 as in the classical setting. As a consequence, one obtains a $\mathbf{\Delta}^1_1$ well ordering of κ^κ, and so even $\mathbf{\Delta}^1_1$ non-regular sets in L. Hence, some implications (which in the standard setting occur at the second level) occur for $\mathbf{\Delta}^1_1$ sets in the generalized context (see [20] and the third chapter of [32]).

Q.2.4 Investigate which separations of regularity properties one can prove for $\mathbf{\Delta}^1_1$.

This investigation has been recently continued by Friedman, Khomskii and Kulikov in [20]. We remark that this work plays a central role in the current research. In particular, some of these questions have been presented and discussed within the "Young Set Theory Workshop 2013", during the tutorial of Sy David Friedman, one of the leaders in this field.

3. Combinatorial cardinal characteristics (expected duration: 8 months). As we have seen in the preliminary section, some works about cardinal invariants in this generalized setting have been developed by Shelah, Cummings, Spasojevic, Landver, Zapletal, Blass, Hyttinen, Zhang and Brooke-Taylor. The leading paper from the classical setting is the survey [2] by Blass; the first main aim would be to fully understand which of the inequalities in the diagram of the standard setting lift to our generalized context as well, and which differences occur by picking different large cardinals.

For instance, [65] concerns the inequality $\mathfrak{t}(\kappa) \leq \mathfrak{b}(\kappa)$ and other related results. A popular cardinal invariants strictly close to $\mathfrak{t}(\kappa)$ is the distributivity number $\mathfrak{h}(\kappa)$, i.e., the smallest size of a family of open dense subsets of $[\kappa]^\kappa$ with non-empty intersection. In the standard setting $\mathfrak{t} \leq \mathfrak{h}$ is provable in ZFC, and a result due to Dordal shows that one can force $\mathfrak{t} < \mathfrak{h}$. A first important goal will be to understand the full picture involving the following cardinal invariants: $\mathfrak{h}(\kappa)$, $\mathfrak{t}(\kappa)$, $\mathfrak{g}(\kappa)$, $\mathfrak{s}(\kappa)$, $\mathfrak{b}(\kappa)$ and $\mathfrak{d}(\kappa)$. We expect that the usual inequalities hold in our generalized setting. We will focus on the following two issues.

Q.3.1 Generalize Suzuki's theorem for other cardinal invariants, such as $\mathfrak{h}(\kappa)$ and $\mathfrak{t}(\kappa)$.

Q.3.2 Try to force strict inequalities between the cardinal invariants above mentioned, when possible. In particular investigate the consistency of $\mathfrak{s}(\omega_2) = \aleph_1 \wedge \mathfrak{t}(\omega_2) = \aleph_0$, $\mathfrak{s}(\omega_2) = \aleph_1 \wedge \mathfrak{h}(\omega_2) = \aleph_0$, $\mathfrak{h}(\omega_2) = \aleph_1 \wedge \mathfrak{t}(\omega_2) = \aleph_0$, and $\mathfrak{s}(\omega_3) = \aleph_2 \wedge \mathfrak{h}(\omega_3) = \aleph_1 \wedge \mathfrak{t}(\omega_3) = \aleph_0$; otherwise, find particular types of cardinals for which some cardinal characteristics are provably equal.

Another interesting issue might be to investigate the cardinal characteristics associated with the ideal of the so-called combinatorially meager sets; in the standard setting, this notion comes out as a combinatorial characterization of meager sets by using a simple combinatorial tool called matching families. In [3], Blass, Hyttinen and Zhang prove that for $\kappa > \omega$ successor, the characterization fails, and so it makes perfectly sense to study this combinatorial notion of meager sets independently. I plan to analyze this difference as well.

4. Final considerations. The final expected output will consist of 4-7 (submitted) journal articles and several seminar talks to present step by step the developments and the progress of our research, and to finally present the concluding results. I also remark that to fully answer all of the questions presented in this project more than 36 months are required. I strongly believe that such a period will be perfect for covering most of the problems that we mentioned, but in the end still many issues will remain worthy of interest, and moreover many other questions will certainly come out along our studies.

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